Symmetry groups of the integro-differential equations and an approach for solutions of nonlocal determining equations

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Abstract

In this study we introduce the general theory of Lie group analysis of integro-differential equations. A generalized version of the direct methods of determination of symmetry group of the point transformations is presented for the equations with nonlocal structure. First, the symmetry group definition of point transformations for the integro-differential equations is discussed and then a new approach for solving of nonlocal determining equations is presented.

1. Introduction

The classical techniques for investigation of symmetry groups of differential equations have been developed and applied to the many problems in the engineering, mathematics, and physics in the last century. Sophus Lie, Ovsiannikov, Olver, Bluman, and Ibragimov are some of the mathematicians who have carried out extensive studies in applications of Lie groups to differential equations. In the case of nonlocal differential equations the main difficulty of investigation of the integro-differential equations is related to the existence of the integral term in these equations. Additionally, the classical approach for investigation of symmetry groups of differential equations can not be applied for these equations

2. Symmetry group of point transformations for the integro-differential equations: definition and main property

We consider the following integro-differential equation:

$$F\left(x,u(x),\underset{1}{u}(x),\ldots,\underset{m}{u}(x),T(u)(x)\right)=0 \tag{2.1}$$

in which T is the integro-differential operator given by

$$T(u)(x)=\int_{\Omega} f\left(x,s_{(q)},u\left(s_{(q)},x_{(n-q)}\right),\underset{1}{u}\left(s_{(q)},x_{(n-q)}\right),\ldots,\underset{m}{u}\left(s_{(q)},x_{(n-q)}\right)\right)ds_{(q)}, \tag{2.2}$$

where $u(x)$ is dependent variable with independent variable $x=(x_1,\ldots,x_n)$. Here, the notations $x=(x_{(q)},x_{(n-q)})$ and $u(x)=u(x_{(q)},x_{(n-q)})$ with $x_{(q)}=(x_1,\ldots,x_q)$, $x_{(n-q)}=(x_{q+1},\ldots,x_n)$ and $1\leq q\leq n$, are also used for the variables x and $u(x)$ (if $q=n$, then $x=x_{(n)}$). The symbol $\underset{k}{u}(x)$ or $\underset{k}{u}(x_{(q)},x_{(n-q)})$ denotes the set of all k^{th} order partial derivatives of $u(x)$ with respect to $x_1,\ldots,x_n$, that is,

$$\underset{k}{u}(x)=\underset{k}{u}(x_{(q)},x_{(n-q)})=\left\{\frac{\partial^k u}{\partial x_{i_1}\ldots\partial x_{i_k}}=\partial_{x_{i_1}}\ldots\partial_{x_{i_k}}u=\partial_{i_1}\ldots\partial_{i_k}u\right\} \tag{2.3}$$

and $s_{(q)}=(s_1,\ldots,s_q)$. The functions f and F are sufficiently smooth and Ω is a given region of the Euclidean space R^q. A special case of the equation (2.1) is the equation:

$$F\left(x,u(x),\underset{1}{u}(x),\ldots,\underset{m}{u}(x),\ T_0(u)(x_{(n-q)})\right)=0 \tag{2.4}$$

with the nonlocal variable

$$T_0(u)(x_{(n-q)})=\int_\Omega f\left(s_{(q)},x_{(n-q)},u(s_{(q)},x_{(n-q)}),\underset{1}{u}(s_{(q)},x_{(n-q)}),\ldots,\underset{m}{u}(s_{(q)},x_{(n-q)})\right)ds_{(q)}. \tag{2.5}$$

The nonlocal variable $A_0=T_0(u)$ of the equation (2.4) has also the property that $\partial A_0/\partial x_k=0$ for $k=1,\ldots,q$. Similar local properties of the nonlocal variables permit to find the additional determining equations as the additional information on the symmetry group properties of the corresponding equation.

For the equation (2.1), we will consider the point group (non canonical) given by

$$\begin{aligned}\tilde{x}_i&=e^{aG}(x_i)=x_i+a\xi_i(x,u)+O(a^2),\quad i=1,\ldots,n,\\ \tilde{u}&=e^{aG}(u)=u+ah(x,u)+O(a^2)\end{aligned} \tag{2.6}$$

with the infinitesimal generator

$$G=\sum_{j=1}^{n}\xi_j\frac{\partial}{\partial x_j}+h\frac{\partial}{\partial u}, \tag{2.7}$$

where $\xi_i=\xi_i(x,u)$ and $h=h(x,u)$ are sufficiently smooth functions. The extension (m^{th} order) of the group (2.6) to a jet space of the independent and dependent variables and derivatives of dependent variable is defined by

$$\begin{aligned}\tilde{x}_i&=e^{a\underset{m}{G}}(x_i)=x_i+a\xi_i(x,u)+O(a^2),\quad i=1,\ldots,n,\\ \tilde{u}&=e^{a\underset{m}{G}}(u)=u+ah(x,u)+O(a^2),\end{aligned} \tag{2.8}$$

$$\partial_i \tilde{u} = e^{a\underset{m}{G}}(\partial_i u) = \partial_i u + a h_i^{(1)}\left(x, u, \underset{1}{u}\right) + O\left(a^2\right),$$
$$\vdots$$
$$\partial_{i_1} \ldots \partial_{i_m} \tilde{u} = e^{a\underset{m}{G}}\left(\partial_{i_1} \ldots \partial_{i_m} u\right) = \partial_{i_1} \ldots \partial_{i_m} u + a h_{i_1 \ldots i_m}^{(m)}\left(x, u, \underset{1}{u}, \ldots, \underset{m}{u}\right) + O\left(a^2\right),$$

where $\underset{m}{G}$ is the infinitesimal generator of this extended group:

$$\underset{m}{G} = G + \sum_{i=1}^{n} h_i^{(1)} \frac{\partial}{\partial(\partial_i u)} + \ldots + \sum_{i_1 \ldots i_m} h_{i_1 \ldots i_m}^{(m)} \frac{\partial}{\partial\left(\partial_{i_1 \ldots i_m} u\right)} \tag{2.9}$$

and

$$\begin{aligned}
h^{(0)} &= h(x, u), \\
h_i^{(1)} &= D_i h^{(0)} - \sum_{j=1}^{n} \left(\partial_j u\right) D_i x_j, \\
h_{i_1 i_2}^{(2)} &= D_{i_2} h_{i_1}^{(1)} - \sum_{j=1}^{n} \left(\partial_{i_1} \partial_j u\right) D_{i_2} x_j, \\
&\vdots \\
h_{i_1 \ldots i_m}^{(m)} &= D_{i_m} h_{i_1 \ldots i_{m-1}}^{(m-1)} - \sum_{j=1}^{n} \left(\partial_{i_1} \ldots \partial_{i_{m-1}} \partial_j u\right) D_{i_m} x_j .
\end{aligned} \tag{2.10}$$

Here, D_i is the total derivative operator:

$$D_i = \partial_i + (\partial_i u)\frac{\partial}{\partial u} + \sum_{j=1}^{n} \left(\partial_i \partial_j u\right) \frac{\partial}{\partial\left(\partial_j u\right)} + \ldots + \sum_{i_1 \ldots i_m} \left(\partial_i \partial_{i_1} \ldots \partial_{i_m} u\right) \frac{\partial}{\partial\left(\partial_{i_1} \ldots \partial_{i_m} u\right)} \tag{2.11}$$

We can write the group of transformations (2.6) in the following form:

$$\begin{aligned}
\tilde{x}_{(q)} &= x_{(q)} + a x_{(q)}\left(x_{(q)}, x_{(n-q)}, u\left(x_{(q)}, x_{(n-q)}\right)\right) + O\left(a^2\right), \\
\tilde{x}_{(n-q)} &= x_{(n-q)} + a x_{(n-q)}\left(x_{(q)}, x_{(n-q)}, u\left(x_{(q)}, x_{(n-q)}\right)\right) + O\left(a^2\right), \\
\tilde{u}\left(\tilde{x}_{(q)}, \tilde{x}_{(n-q)}\right) &= u\left(x_{(q)}, x_{(n-q)}\right) + a h\left(x_{(q)}, x_{(n-q)}, u\left(x_{(q)}, x_{(n-q)}\right)\right) + O\left(a^2\right),
\end{aligned} \tag{2.12}$$

where $x_{(q)} = \left(x_1, \ldots, x_q\right)$ and $x_{(n-q)} = \left(x_{q+1}, \ldots, x_n\right)$. The group (2.6) is also a group of the transformations $\left(\left(s_{(q)}, x_{(n-q)}\right), u\left(s_{(q)}, x_{(n-q)}\right)\right) \rightarrow \left(\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right), \tilde{u}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right)\right)$ given by

$$\begin{aligned}
\tilde{s}_{(q)} &= s_{(q)} + a x_{(q)}\left(s_{(q)}, x_{(n-q)}, u\left(s_{(q)}, x_{(n-q)}\right)\right) + O\left(a^2\right), \\
\tilde{x}_{(n-q)} &= x_{(n-q)} + a x_{(n-q)}\left(s_{(q)}, x_{(n-q)}, u\left(s_{(q)}, x_{(n-q)}\right)\right) + O\left(a^2\right), \\
\tilde{u}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right) &= u\left(s_{(q)}, x_{(n-q)}\right) + a h\left(s_{(q)}, x_{(n-q)}, u\left(s_{(q)}, x_{(n-q)}\right)\right) + O\left(a^2\right).
\end{aligned} \tag{2.13}$$

Clearly that the m^{th} order extension of (2.13) on the elements of the sets $\underset{1}{u}\left(s_{(q)}, x_{(n-q)}\right), \ldots, \underset{m}{u}\left(s_{(q)}, x_{(n-q)}\right)$ are defined by the corresponding expressions of (2.8) with $\tilde{x}_{(q)} = \tilde{s}_{(q)}$ and $x_{(q)} = s_{(q)}$.

Definition 2.1: If the equation (2.1) transforms to the following form invariant equation under the group of point transformations (2.6), then this group is called a point symmetry group for the equation (2.1):

$$F\left(\tilde{x}, \tilde{u}(\tilde{x}), \underset{1}{\tilde{u}}(\tilde{x}), \ldots, \underset{m}{\tilde{u}}(\tilde{x}), \tilde{T}(\tilde{u})(\tilde{x})\right) = 0 \tag{2.14}$$

with

$$\tilde{T}(\tilde{u})(\tilde{x}) = \int_{\tilde{\Omega}} f\left(\tilde{x}, \tilde{s}_{(q)}, \tilde{u}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right), \underset{1}{\tilde{u}}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right), \ldots, \underset{m}{\tilde{u}}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right)\right) d\tilde{s}_{(q)}, \tag{2.15}$$

where $\tilde{\Omega}$ is the image of Ω under the transformation $s_{(q)} \to \tilde{s}_{(q)}$ given by (2.13).

In this definition, the points $\left(\left(\tilde{x}_{(q)}, \tilde{x}_{(n-q)}\right), \tilde{u}\left(\tilde{x}_{(q)}, \tilde{x}_{(n-q)}\right)\right)$ and $\left(\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right), \tilde{u}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right)\right)$ are regarded as transformed forms of the points $\left(\left(x_{(q)}, x_{(n-q)}\right), u\left(x_{(q)}, x_{(n-q)}\right)\right)$ and $\left(\left(s_{(q)}, x_{(n-q)}\right), u\left(s_{(q)}, x_{(n-q)}\right)\right)$ under groups of point transformations (2.12) and (2.13), respectively. This shows that, if $f \not\equiv 0$ and $1 \le q < n$, then the group (2.6) employed in this definition, must have a "separated" form with respect to the variables $x_{(n-q)} = \left(x_{q+1}, \ldots, x_n\right)$ which are not integral variables for the considered equation. In other words, the vector function $x_{(n-q)} = \left(x_{q+1}, \ldots, x_n\right)$ must depend only on $x_{(n-q)}$, that is, has the form of $x_{(n-q)} = x_{(n-q)}\left(x_{(n-q)}\right)$. A similar condition must be satisfied also for the definition given in for a point symmetry group of an integro-differential equation similar to (2.4). Note also that, this condition cannot be eliminated by employing corresponding Lie–Bäcklund operator of the group of transformations (2.6).

3. Determining equations of nonlocal equations: a theoretical approach

Here, we present a direct method for investigation the symmetry group properties of the nonlocal equations. This method uses *Definition 2.1* and the so-called determining equation which is obtained without using the Lie-Bäcklund type operators. To introduce the determining equation, we obtain the extension of (2.6) on the nonlocal variable $T(u)$. For this reason, we will employ both forms (2.12) and (2.13) of the same group (2.6). We first consider the transformation $\tilde{s}_{(q)} \to s_{(q)}$ given by (2.13) for the integration variable $\tilde{s}_{(q)}$ of (2.15). The Jacobean of this transformation can be calculated as

$$J\left(s_{(q)}, x_{(n-q)}\right) = 1 + a\sum_{j=1}^{q} D_j x_j\left(s_{(q)}, x_{(n-q)}, u\left(s_{(q)}, x_{(n-q)}\right)\right) + O\left(a^2\right). \tag{3.1}$$

Then, after changing the integral variable in (2.15) and applying (2.12) and (2.13) to (2.15) we obtain

$$\tilde{T}(\tilde{u})(\tilde{x}) = \int_{\Omega} f\left(\tilde{x}, \tilde{s}_{(q)}, \tilde{u}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right), \ldots, \underset{m}{\tilde{u}}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right)\right) J\left(s_{(q)}, x_{(n-q)}\right) ds_{(q)} \tag{3.2}$$

in which $\tilde{s}_{(q)}$, $\tilde{x}_{(q)}$, $\tilde{x}_{(n-q)}$ and $\tilde{u}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right)$ are defined by the corresponding expressions of (2.12) and (2.13) (furthermore, the elements of the sets $\underset{1}{\tilde{u}}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right), \ldots, \underset{m}{\tilde{u}}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right)$ are defined by the corresponding expressions from (2.8) with $\tilde{x}_{(q)} = \tilde{s}_{(q)}$ and $x_{(q)} = s_{(q)}$). Then, by the Taylor expansion, the first multiplier of the integrant of (3.2) can be represented as

$$\begin{aligned} &f\left(\tilde{x}, \tilde{s}_{(q)}, \tilde{u}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right), \ldots, \underset{m}{\tilde{u}}\left(\tilde{s}_{(q)}, \tilde{x}_{(n-q)}\right)\right) = \\ &f\left(x, s_{(q)}, u\left(s_{(q)}, x_{(n-q)}\right), \ldots, \underset{m}{u}\left(s_{(q)}, x_{(n-q)}\right)\right) + a\left(\underset{m}{Q} f\right)\left(x, s_{(q)}\right) + O\left(a^2\right), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \underset{m}{Q} = &\sum_{k=1}^{n} x_k(x, u(x)) \frac{\partial}{\partial x_k} + \sum_{k=1}^{q} x_k(x, u(x))\Big|_{x_{(q)}=s_{(q)}} \frac{\partial}{\partial s_k} + h(x, u(x))\Big|_{x_{(q)}=s_{(q)}} \frac{\partial}{\partial\left(u(x)\big|_{x_{(q)}=s_{(q)}}\right)} + \\ &\sum_{k=1}^{n} h_k^{(1)}\left(x, u(x), \underset{1}{u}(x)\right)\Big|_{x_{(q)}=s_{(q)}} \frac{\partial}{\partial\left(\left(\partial_{x_k} u(x)\right)\big|_{x_{(q)}=s_{(q)}}\right)} + \ldots + \sum_{i_1 \ldots i_m} h_{i_1 \ldots i_m}^{(m)}\left(x, u(x), \ldots, \underset{m}{u}(x)\right)\Big|_{x_{(q)}=s_{(q)}} \times \\ &\frac{\partial}{\partial\left(\left(\partial_{x_{i_1}} \ldots \partial_{x_{i_m}} u(x)\right)\big|_{x_{(q)}=s_{(q)}}\right)} \end{aligned} \tag{3.4}$$

is a differential operator defined on the functions of the form $j\left(x, s_{(q)}, u\left(s_{(q)}, x_{(n-q)}\right), \ldots, \underset{m}{u}\left(s_{(q)}, x_{(n-q)}\right)\right)$. Substituting the expressions (3.1) and (3.3) into (3.2), we obtain

$$\tilde{T}(\tilde{u})(\tilde{x}) = T(u)(x) + aP_T(u)(x) + O\left(a^2\right), \tag{3.5}$$

where P_T is generally nonlinear operator given by

$$P_T(u)(x) = \int_{\Omega}\left[\underset{m}{Q} + \sum_{k=1}^{q} D_k x_k\left(s_{(q)}, x_{(n-q)}, u\left(s_{(q)}, x_{(n-q)}\right)\right)\right] f ds_{(q)} . \tag{3.6}$$

Consequently, the extension of the point group (2.6) on the nonlocal variable $A = T(u)$ is defined by (3.5). Therefore, we can consider the generator given by

$$\underset{m}{G}^T = \underset{m}{G} + P_T(u)\frac{\partial}{\partial(T(u))}. \tag{3.7}$$

Now, substituting the expressions (2.8) and (3.5) into (2.14), we have

$$\begin{aligned} &F\Big(x + a x(x,u(x)) + O(a^2), u(x) + a h(x,u(x)) + O(a^2), \underset{1}{u}(x) + \\ &a h^{(1)}\Big(x,u(x),\underset{1}{u}(x)\Big) + O(a^2),\ldots,\underset{m}{u}(x) + a h^{(m)}\Big(x,u(x),\underset{1}{u}(x),\ldots,\underset{m}{u}(x)\Big) + \\ &O(a^2), T(u)(x) + aP_T(u)(x) + O(a^2)\Big) = 0 \end{aligned} \tag{3.8}$$

in which $h^{(k)}$ is the set of all coefficients $h^{(k)}_{i_1 \ldots i_k}$ with a given k and $x = (x_1,\ldots,x_n)$. By Definition 2.1, the equations (2.1) and (3.8) are equivalent, if (2.6) is a point symmetry group of (2.1). Then, by the Taylor expansion from (3.8), we obtain the criterion of the invariance of the integro-differential equation (2.1) under the point transformations (2.6) in the following form:

$$\underset{m}{G}^T F\Big(x,u(x),\underset{1}{u}(x),\ldots,\underset{m}{u}(x),T(u)(x)\Big) = 0, \tag{3.9}$$

in which $u(x)$ is an arbitrary solution of (2.1). The equation (3.9) is regarded as the determining equation of (2.1). This result can also be generalized for other classes of the equations with nonlocal structure. We consider, for example, the following loaded equation:

$$F\Big(x,u(x),\underset{1}{u}(x),\ldots,\underset{m}{u}(x),T(u)(x),T_1(u)(x)\Big) = 0. \tag{3.10}$$

Here, T is defined by (2.2) and T_1 is the loaded operator defined by

$$T_1(u)(x) = f_1(x,u(z)) \tag{3.11}$$

with a sufficiently smooth function $f_1(x,u)$ and a fixed point $z \in \Omega$. For the equation (3.10), we will employ the following definition. If the equation (3.10) is form invariant, that is, it becomes as

$$F\Big(\tilde{x},\tilde{u}(\tilde{x}),\underset{1}{\tilde{u}}(\tilde{x}),\ldots,\underset{m}{\tilde{u}}(\tilde{x}),\tilde{T}(\tilde{u})(\tilde{x}),\tilde{T}_1(\tilde{u})(\tilde{x})\Big) = 0 \tag{3.12}$$

under (2.6), then we call (2.6) as a point symmetry group for the equation (3.10). Here, $\tilde{T}_1(\tilde{u})(\tilde{x}) = f_1(\tilde{x},\tilde{u}(\tilde{z}))$ and $\tilde{z} \in \tilde{\Omega}$ is the point obtained from z by (2.6). In this case, the extension of (2.6) on $A_1 = T_1(u)$ has the simple form

$$\tilde{T}_1(\tilde{u})(\tilde{x}) = T_1(u)(x) + aP_{T_1}(u)(x) + O(a^2), \tag{3.13}$$

where

$$P_{T_1}(u)(x)=\sum_{k=1}^{n}x_k(x,u(x))\frac{\partial f_1(x,u(z))}{\partial x_k}+h(z,u(z))\frac{\partial f_1(x,u(z))}{\partial u(z)}. \tag{3.14}$$

Therefore, the extended generator becomes as

$$\underset{m}{G}^{(T,T_1)}=\underset{m}{G}+P_T(u)\frac{\partial}{\partial(T(u))}+P_{T_1}(u)\frac{\partial}{\partial(T_1(u))}. \tag{3.15}$$

Then, the determining equation of (3.10) can be obtained in the following natural form:

$$\underset{m}{G}^{(T,T_1)}F\left(x,u(x),\underset{1}{u}(x),...,\underset{m}{u}(x),T(u)(x),T_1(u)(x)\right)=0, \tag{3.16}$$

which must be satisfied for all solutions $u(x)$ of (3.10).

It can be proven that the determining equation of the equations having finite number nonlocal terms has an analogy form to (3.16). A similar result is true also for a system of equations with finite number dependent variables $u_1(x),...,u_N(x)$. In this case, a determining equation similar to (3.16) must be written for each equation of this system.

The structure of the generator $\underset{m}{G}^{T}$ (or $\underset{m}{G}^{(T,T_1)}$) differs from the structure of the corresponding Lie-Bäcklund generator including the nonlocal variables as the additional dependent variables. Here, a principle point is that the problem of solving of the determining equations given by such generators requires essentially little calculations than the determining equations given by Lie-Bäcklund type generators.

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