Involutive differential systems and tableaux over Lie algebras

Lorenzo Nicolodi
Università di Parma
`lorenzo.nicolodi@unipr.it`
(based on work with Emilio Musso)

IMA, July 27, 2006

Many classes of surfaces/submanifolds in homogeneous spaces constitute integrable systems. Is there some common structure?

Examples include:

- Pseudospherical surfaces (sine-Gordon eq.), and CMC surfaces (sinh-Gordon eq.)

- Harmonic maps from $\mathbb{R}^{1,1}$ or $\mathbb{R}^2$ into Lie groups and symmetric spaces

- Isometric immersions of space forms into space forms

- Isothermic and Willmore surfaces in Möbius and Laguerre geometry

- The *curved flat* system of Ferus and Pedit and the G/G_0-*system* of Terng

- Demoulin and Godeaux-Rozet surfaces in projective differential geometry and their counterparts in Lie sphere geometry

Properties of integrable systems include:

- Possibility of writing down explicit solutions

- Existence of Bäcklund's transformations

- Soliton solutions

- Hamiltonian formulation

- Formulation in terms of a Lax pair of operators: $dA/dt = [B, A]$

- Zero-curvature formulation: $d\theta_\lambda = -\theta_\lambda \wedge \theta_\lambda$

- A construction of solutions using loop group factorizations (dressing actions).

- An inverse scattering transform to solve the Cauchy problem

Curved flats in symmetric spaces.

$N = G/K$ semisimple symmetric space;

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{symmetric decomposition}$$

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

Def. An immersion $\varphi : M \to G/K$ is a **curved flat** if each $d\varphi(T_pM)$ is an abelian subalgebra of $\mathfrak{g}$ (where $T_{\varphi(p)}N \cong \mathfrak{p} \subset \mathfrak{g}$)

Theorem. Let $F : M \to G$ be a frame of φ (i.e. $\varphi = FK$) with $F^{-1}dF = \theta_{\mathfrak{k}} + \theta_{\mathfrak{p}}$. Then F frames a curved flat $\iff$ $\theta_\lambda := \theta_{\mathfrak{k}} + \lambda\theta_{\mathfrak{p}}$ satisfies

$$d\theta_\lambda + 1/2[\theta_\lambda \wedge \theta_\lambda] = 0, \quad \forall \lambda \in \mathbb{R}$$

Remark. By a canonical choice of frame for a curved flat, the Maurer-Cartan equations for θ reduces to the G/K-**system** of Terng.

Curved flats and isothermic surfaces.

Consider $N := S^3 \times S^3/\Delta$.

$SO(4,1)$ acts transitively on N.

A map $\varphi : M^2 \to N$ amounts to a pair of maps $f, \tilde{f} : M \to S^3$ whose values never coincide. Suppose both f and $\tilde{f}$ are immersions, then

Theorem. $(f, \tilde{f}) : M \to N$ is a curved flat $\iff$ f and $\tilde{f}$ are isothermic and are Darboux transforms of each other.

Basic observation:

- Using the moving frame method, the equations defining the submanifolds arise as compatibility conditions for the linear equations satisfied by frames adapted to the submanifolds

- The (framed) submanifolds can be interpreted as integral manifolds of appropriate **linear** Pfaffian differential systems (PDS), possibly **in involution**.

The purpose is to:

- Find a unifying approach to the different classes of integrable submanifolds in the context of linear PDSs

- Understand the mechanism underlying the involutiveness of the associated linear PDSs

The **common structure** we are looking for is the concept of **tableau over a Lie algebra**

It builds on and generalizes the notion of involutive tableau in the theory of EDS as developed by Y. Matsushima (1950s).

[B] Bryant, *et al.*: *Exterior differential systems*, MSRI Publications, 18, Springer-Verlag, 1991.

[BDPT] Brück-Du-Park-Terng: Submanifold geometry of real Grassmannian systems, *Mem. Amer. Math. Soc.* **155** (2002)

[FP] Ferus-Pedit: Curved flats in symmetric spaces, *Manuscripta Math.* **91** (1996)

[MN1] Musso-Nicolodi: Tableaux over Lie algebras, integrable systems and classical surface theory, *Comm. Anal. Geom.* (to appear)

[MN2] Musso-Nicolodi: `math.DG/0602676` on the arXiv.

[T] Terng: Soliton equations and differential geometry, *J. Differential Geom.* **45** (1997)

[TW] Terng-Wang: Curved flats, exterior differential systems and conservation laws; Curvature in Geometry, Lecce 2003, Birkhäuser, 2005.

Plan of the talk:

Part I: Tableaux, Involutive Tableaux

Part II: Tableaux over Lie algebras; examples: Cartan's tableaux (i.e., tableaux associated with semisimple Lie algebras)

Part III: The linear Pfaffian system associated with a tableau over a Lie algebra: involutiveness

Part IV: Interpretation of the G/G_0-system in terms of the Pfaffian system defined by a Cartan's tableau

Further developments

Part I: Tableaux

A **tableau** is a linear subspace $\mathbf{A} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$, where $\mathfrak{a}$, $\mathfrak{b}$ are (real or complex) finite dimensional vector spaces.

An h-dimensional subspace $\mathfrak{a}_h \subset \mathfrak{a}$ is called *generic* w.r.t. $\mathbf{A}$ if the dimension of $\mathrm{Ker}\,(\mathbf{A}, \mathfrak{a}_h)$ $:= \{Q \in \mathbf{A} \,|\, Q_{|\mathfrak{a}_h} = 0\}$ is a minimum

A flag $(0) \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_n = \mathfrak{a}$ of $\mathfrak{a}$ is said *generic* if $\mathfrak{a}_h$ is generic, for all $h = 1, \ldots, n$.

The **characters** of $\mathbf{A}$ are the non-negative integers $s_j(\mathbf{A})$, $j = 1, \ldots, n$, defined inductively by

$$s_1(\mathbf{A}) + \cdots + s_j(\mathbf{A}) = \mathrm{codim}\,\mathrm{Ker}\,(\mathbf{A}, \mathfrak{a}_j),$$

for any generic flag $(0) \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_n = \mathfrak{a}$.

From the definition, it is clear that

$$\dim \mathfrak{b} \geq s_1 \geq s_2 \geq \cdots \geq s_n, \quad \dim \mathbf{A} = s_1 + \cdots + s_n.$$

If $s_\nu \neq 0$, but $s_{\nu+1} = 0$, we say that $\mathbf{A}$ has *principal character* s_ν and call ν the *Cartan integer* of $\mathbf{A}$.

The **first prolongation** $\mathbf{A}^{(1)}$ of $\mathbf{A}$ is the kernel of the linear map

$$\delta^{1,1} : \mathrm{Hom}(\mathfrak{a}, \mathbf{A}) \cong \mathbf{A} \otimes \mathfrak{a}^* \to \mathfrak{b} \otimes \Lambda^2(\mathfrak{a}^*)$$

$$\delta^{1,1}(F)(A_1, A_2) := \frac{1}{2}\left(F(A_1)(A_2) - F(A_2)(A_1)\right),$$

for $F \in \mathrm{Hom}(\mathfrak{a}, \mathbf{A})$, and $A_1, A_2 \in \mathfrak{a}$.

The h-**th prolongation** of $\mathbf{A}$ is defined inductively by setting $\mathbf{A}^{(h)} = \mathbf{A}^{(h-1)^{(1)}}$, for $h \geq 1$ (by convention $\mathbf{A}^{(0)} = \mathbf{A}$ and $\mathbf{A}^{(-1)} = \mathfrak{b}$). $\mathbf{A}^{(h)}$ identifies with

$$\mathbf{A}^{(h)} = \left(\mathbf{A} \otimes S^h(\mathfrak{a}^*)\right) \cap \left(\mathfrak{b} \otimes S^{h+1}(\mathfrak{a}^*)\right).$$

An element $Q_{(h)} \in \mathfrak{b} \otimes S^{h+1}(\mathfrak{a}^*)$ belongs to $\mathbf{A}^{(h)}$ if and only if $i(X)Q_{(h)} \in \mathbf{A}^{(h-1)}$, for all $X \in \mathfrak{a}$.

Theorem. $\dim \mathbf{A}^{(1)} \leq s_1 + 2s_2 + \cdots + ns_n$.

$\mathbf{A}$ is said **involutive** if equality holds.

Theorem. For any tableau $\mathbf{A}$ there exists an integer h_0 such that $\mathbf{A}^{(h)}$ is involutive, for all $h \geq h_0$.

Theorem. If $\mathbf{A}$ is involutive, then $\mathbf{A}^{(1)}$ is involutive and

$$s_j^{(1)} := s_j(\mathbf{A}^{(1)}) = s_n(\mathbf{A}) + \cdots + s_j(\mathbf{A}),$$

$j = 1, \ldots, n$.

Thus *every prolongation of an involutive tableau is involutive*. Moreover, *the principal character and the Cartan integer are invariant under prolongation of an involutive tableau*.

Remark. $\mathbf{A}$ is involutive iff $H^{q,p}(\mathbf{A}) = (0)$, $\forall\ q \geq 1,\ p \geq 0$ (Guillemin-Sternberg, Serre, 1964).

If $H^{q,2}(\mathbf{A}) = (0)$, for all $q \geq 1$, the tableau $\mathbf{A}$ is called 2-**acyclic**.

This notion plays an essential role when one needs to prolong a non involutive linear Pfaffian system (cf. Kuranishi, Goldschmidt)

Example. Let V and W be vector spaces with coordinates $x^1, \ldots, x^n$ and $y^1, \ldots, y^s$ dual to bases $v_1, \ldots, v_n$ for V and $w_1, \ldots, w_s$ for W. Consider the first-order constant coefficient system of PDEs for maps $f : V \to W$ given in coordinates by

$$(*) \qquad B_a^{\lambda i} \frac{\partial y^a}{\partial x^i}(x) = 0 \quad (\lambda = 1, \ldots, r).$$

The linear solutions $y^a(x) = A^a_j x^j$ to this system give rise to a tableau $\mathbf{A} \subset \mathrm{Hom}(V, W)$.

Conversely, any tableau $\mathbf{A} \subset \mathrm{Hom}(V, W)$ determines a PDE system of this type.

NB: $\mathbf{A}^{(q)}$ is the set of homogeneous polynomial solutions of degree $q + 1$ to $(*)$.

The annihilator of $\mathbf{A}$, denoted $\mathbf{B} := \mathbf{A}^{\perp} \subset V \otimes W^*$ is called the *symbol* of the system $(*)$.

Theorem. The PDE system associated to $\mathbf{A}$ is involutive $\iff$ $\mathbf{A}$ is involutive.

Example. Let $(M, \mathcal{I}, \omega)$ be a Pfaffian system,

$\mathcal{I} = \{\theta^1, \dots, \theta^s, d\theta^1, \dots, d\theta^s\}$ (algebraic ideal),

$\omega = \omega^1 \wedge \cdots \wedge \omega^n$ (independence condition)

Let $\pi^1, \dots, \pi^t$ be 1-forms such that

$$\theta^1, \dots, \theta^s;\ \omega^1, \dots, \omega^n;\ \pi^1, \dots, \pi^t$$

be a local adapted coframe of M.

$(\mathcal{I}, \omega)$ is called *linear* (or *quasi-linear*) $\iff$

$d\theta^a \equiv 0 \mod \{\theta^1, \dots, \theta^s,\ \omega^1, \dots, \omega^n\}$, $0 \le a \le s$,

($\iff$ the variety of admissible integral elements $V_n(\mathcal{I}, \omega) \subset G_n(TM, \omega)$ is an affine bundle over M)

A linear Pfaffian system is described locally by

$$\begin{cases} \theta^a = 0 \\ d\theta^a \equiv A^a_{\epsilon i} \pi^\epsilon \wedge \omega^i + \frac{1}{2} c^a_{ij} \omega^i \wedge \omega^j \quad \mod \{\theta^1, \dots, \theta^s\} \\ \omega = \omega^1 \wedge \cdots \wedge \omega^n \neq 0, \end{cases}$$

$c^a_{ij} = -c^a_{ji}$, $1 \le a \le s$; $1 \le i, j \le n$; $1 \le \epsilon \le t$.

Invariants of a linear PDS: torsion and tableau

At a point $x \in M$, the following are equivalent:

i) $V_n(\mathcal{I}, \omega)_{|x} \neq \emptyset$;
ii) $c^a_{ij}(x) = 0$, $\forall a, i, j$; (“torsion vanishes at x.”)

At $x \in M$, let $V^* = \text{span}\,\{\omega^i\}$ and $\frac{\partial}{\partial \omega^i}$ the basis of its dual V. Further, let $W^* = \text{span}\,\{\theta^a\}$, and $\frac{\partial}{\partial \theta^a}$ the basis of its dual W.

We define the tableau $\mathbf{A} \subset W \otimes V^*$ by

$$\mathbf{A} := \text{span}\,\{A^a_{\epsilon i} \frac{\partial}{\partial \theta^a} \otimes \omega^i \,:\, \epsilon = 1, \ldots, t\}$$

Theorem. The linear PDS $(\mathcal{I}, \omega)$ is involutive at $x \iff$ 1) the torsion at x is zero and 2) $\mathbf{A}$ is involutive.

Part II: Tableaux over Lie algebras [MN1]

Let $(\mathfrak{g}, [\,,])$ be a finite dimensional Lie algebra, $\mathfrak{a}, \mathfrak{b}$ vector subspaces of $\mathfrak{g}$ such that $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{g}$, and $\mathbf{A} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$ a tableau.

Define the polynomial map $\tau : \mathbf{A} \to \mathfrak{b} \otimes \Lambda^2(\mathfrak{a}^*)$ defined by

$$\begin{aligned}\tau(Q)(A_1, A_2) := {} & [A_1 + Q(A_1), A_2 + Q(A_2)]_{\mathfrak{b}} \\ & - Q([A_1 + Q(A_1), A_2 + Q(A_2)]_{\mathfrak{a}}),\end{aligned}$$

where $X_{\mathfrak{a}}$ (resp. $X_{\mathfrak{b}}$) denotes the $\mathfrak{a}$ (resp. $\mathfrak{b}$) component of X.

By a **tableau over** $\mathfrak{g}$ is meant a tableau $\mathbf{A} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$ such that:

1) $\mathbf{A}$ is involutive (or, more generally, 2-acyclic);
2) $\tau(Q) \in \mathrm{Im}\,\delta^{1,1} \subset \mathfrak{b} \otimes \Lambda^2(\mathfrak{a}^*)$, for each $Q \in \mathbf{A}$.

Example. If $\mathbf{A} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$ is involutive and $\mathfrak{g}$ is the abelian Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, then $\tau(Q) = 0$, for each $Q \in \mathbf{A}$. $\mathbf{A}$ can be considered as a tableau over $\mathfrak{g}$. Therefore, the concept of tableau over a Lie algebra is a natural (non-abelian) generalization of the classical notion of involutive (or 2-acyclic) tableau.

Example. Let $\mathfrak{g}$ be a semisimple Lie algebra with Killing form $\langle\, , \rangle$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Cartan decomposition. Then

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, \quad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0.$$

Assume that $\mathrm{rank}\, \mathfrak{g}/\mathfrak{g}_0 = k$ and $\mathfrak{a}$ be a maximal (k-dimensional) abelian subspace of $\mathfrak{g}_1$. Then $\mathfrak{g}_1 = \mathfrak{a} \oplus \mathfrak{m}$, where

$$\mathfrak{m} = \mathfrak{a}^{\perp} \cap \mathfrak{g}_1$$

Further, let
$(\mathfrak{g}_0)_{\mathfrak{a}} = \{X \in \mathfrak{g}_0 \,:\, [X, \mathfrak{a}] = 0\}$
$(\mathfrak{g}_0)_{\mathfrak{a}}^{\perp} = \{X \in \mathfrak{g} \,:\, \langle X, Y\rangle = 0, \forall\, Y \in (\mathfrak{g}_0)_{\mathfrak{a}}\}$,
$\mathfrak{b} := \mathfrak{g}_0 \cap (\mathfrak{g}_0)_{\mathfrak{a}}^{\perp}$.

Then, for any regular element $A \in \mathfrak{a}$, the maps

$$\mathrm{ad}_A : \mathfrak{m} \to \mathfrak{b}, \quad \mathrm{ad}_A : \mathfrak{b} \to \mathfrak{m}$$

are vector space isomorphisms and

$$X \in \mathfrak{m} \mapsto -\mathrm{ad}_X \in \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$$

is injective, hence $\mathfrak{m}$ *can be identified with a linear subspace of* $\mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$.

Proposition [MN1]. If $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{m}$ are defined as above, then $\mathfrak{m}$, regarded as a subspace of $\mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$, is a tableau over $\mathfrak{g}$.

Def. $\mathfrak{m}$ is called a **Cartan tableau** over $\mathfrak{g}$.

Remark. The notion of tableau over a Lie algebra originates from the method of moving frames and reflects the existence of canonical adapted frames. (e.g., the Wilczynski–Cartan frame in projective differential geometry)

Part III: The linear Pfaffian differential system associated with a tableau over a Lie algebra

Let $\mathbf{A} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$ be a tableau over $\mathfrak{g}$ and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We set $Y := G \times \mathbf{A}$ and refer to it as the *configuration space*.

A basis $(A_1, \dots, A_k, B_1, \dots, B_h, C_1, \dots, C_s)$ of $\mathfrak{g}$ is said *adapted* to $\mathbf{A}$ if

1) $\mathfrak{a} = \mathrm{span}\,\{A_1, \dots, A_k\}$,
2) $\mathrm{Im}\,\mathbf{A} := \sum_{Q \in \mathbf{A}} \mathrm{Im}\,Q = \mathrm{span}\,\{B_1, \dots, B_h\}$,
3) $\mathfrak{b} = \mathrm{span}\,\{B_1, \dots, B_h, C_1, \dots, C_s\}$,

An adapted basis is a *generic* if the flag

$$(0) \subset \mathrm{span}\,\{A_1\} \subset \cdots \subset \mathrm{span}\,\{A_1, \dots, A_k\} = \mathfrak{a}$$

is generic with respect to $\mathbf{A}$.

For a generic adapted basis, let

$$(\alpha^1, \dots, \alpha^k, \beta^1, \dots, \beta^h, \gamma^1 \dots, \gamma^s)$$

denote the dual coframe on G. Given a basis

$$Q_\epsilon = Q^j_{\epsilon i} B_j \otimes \alpha^i \quad (\epsilon = 1, \dots m)$$

of the tableau $\mathbf{A}$, Y identifies with $G \times \mathbb{R}^m$ by

$$(g, p^\epsilon Q_\epsilon) \in Y \mapsto (g; p^1, \dots, p^m) \in G \times \mathbb{R}^m.$$

Def. [MN1] The EDS **associated with** $\mathbf{A}$ is the Pfaffian system $(\mathcal{I}, \omega)$ on Y generated (as a differential ideal) by the linearly independent 1-forms

$$\begin{cases} \eta^j := \beta^j - p^\epsilon Q^j_{\epsilon i} \alpha^i, \quad (j = 1, \dots, h), \\ \gamma^1, \dots, \gamma^s, \end{cases}$$

with indepenent condition $\omega = \alpha^1 \wedge \dots \wedge \alpha^k \neq 0$.

Integral manifolds. Consider an immersed submanifold

$$\Phi = (g; q^1, \dots, q^m) : N^k \to G \times \mathbf{A} \cong G \times \mathbb{R}^m.$$

(N^k, Φ) is an *integral manifold* of $(\mathcal{I}, \omega)$ iff

1) $(\alpha^1 \wedge \cdots \wedge \alpha^k)_{|N} \neq 0$;
2) $\beta^j = q^\epsilon Q^j_{\epsilon i} \alpha^i$, $j = 1, \dots, h$;
3) $\gamma^1 = \cdots = \gamma^s = 0$.

Theorem [MN1]. Let $\mathbf{A}$ be a tableau over a Lie algebra $\mathfrak{g}$. Then, the EDS $(\mathcal{I}, \omega)$ on Y associated with $\mathbf{A}$ is a linear Pfaffian system in involution. In particular, the characters of $\mathbf{A}$ coincide with the Cartan characters of $(\mathcal{I}, \omega)$.

Remark. This provides a general scheme for producing *linear Pfaffian systems in involution* starting from tableaux over Lie algebras

Developments.

- Identify the geometry associated to a given tableau/system, i.e., find submanifolds in some homogeneous space whose integrability conditions are given by the PDS associated with the tableau

- Study the algebraic structure of tableaux to understand when a tableaux generate integrable geometries

- Study the Cauchy problem

- Analyze characteristic cohomology ...

Part IV: The G/G_0-system and the PDS associated with a Cartan tableaux

Example (abelian case). Let $\mathbf{A} \subset \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^h)$ be an m-dimensional involutive tableau over the (abelian) Lie algebra $\mathfrak{g} = \mathbb{R}^k \oplus \mathbb{R}^h$, spanned by the lin. indep. $h \times k$ matrices $Q_\epsilon = (Q^j_{\epsilon i})$.

The $\mathbf{A}^{(1)}$-system: Consider the linear, homogeneous, constant coefficient PDE system for maps $P = (p^1, \dots, p^m) : \mathbb{R}^k \to \mathbf{A} \cong \mathbb{R}^m$ defined by the differential inclusion $dP_{|x} \in \mathbf{A}^{(1)}$, for all $x \in \mathbb{R}^k$ ($\mathbf{A}^{(1)}$ is the first prolongation of $\mathbf{A}$).

This system can be written

$$\delta^{1,1}(dP) = 0,$$

where $\delta^{1,1}$ is the Spencer coboundary of the tableau $\mathbf{A}$ (Rk: $\delta^{1,1} : C^{1,1} = \mathbf{A} \otimes (\mathbb{R}^k)^* \to C^{0,2}$)

Lemma. A map $P : \mathbb{R}^k \to \mathbf{A} \cong \mathbb{R}^m$ is a solution to the $\mathbf{A}^{(1)}$-system iff the $\mathbb{R}^h$-valued 1-form

$$\theta = (\theta^1, \ldots, \theta^h) \in \Omega^1(\mathbb{R}^k) \otimes \mathbb{R}^h, \theta^j = p^\epsilon Q^j_{\epsilon a} dx^a$$

is closed.

As a conseguence, we have

Corollary. Let $P : \mathbb{R}^k \to \mathbf{A} \cong \mathbb{R}^m$ be a solution to the $\mathbf{A}^{(1)}$-system and let $y = (y^1, \ldots, y^h)$ be a primitive of θ (i.e., $\theta = dy$). Then

$$\mathbb{R}^k \ni x \mapsto (x, y(x), P(x)) \in \mathbb{R}^k \oplus \mathbb{R}^h \oplus \mathbb{R}^m$$

is an integral manifold of the PDS $(\mathcal{I}, \omega)$ associated with $\mathbf{A}$. Moreover, every integral manifold of $(\mathcal{I}, \omega)$ arises in this way.

Conclusion: The integral manifolds of $(\mathcal{I}, \omega)$ correspond to the solutions of the $\mathbf{A}^{(1)}$-system. Moreover, $(\mathcal{I}, \omega)$ is in involution (as differential system) and the Cartan characters with those of the tableau $\mathbf{A}$ (cf. Matsushima).

Example (the G/G_0-system). Let G/G_0 be a semisimple symmetric space of rank k and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ a Cartan decomposition of G/G_0. Let $(A_1, \ldots, A_k)$ be a regular basis for the maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}_1$. According to Terng [T], the $\underline{G/G_0\text{-system}}$ is the system of PDEs for maps $V : U \subset \mathfrak{a} \to \mathfrak{m} := \mathfrak{g}_1 \cap \mathfrak{a}^\perp$ defined by

$$\left[A_i, \frac{\partial V}{\partial x^j}\right] - \left[A_j, \frac{\partial V}{\partial x^i}\right] = \Big[[A_i, V], [A_j, V]\Big],$$

$1 \le i < j \le k$, where x^i are the coordinates with respect to $(A_1, \ldots, A_k)$.

Lemma. $V : \mathfrak{a} \to \mathfrak{m}$ is a solution of the G/G_0-system iff the $\mathfrak{g}$-valued 1-form

$$\theta = \alpha + [\alpha, V] \in \Omega^1(\mathfrak{a}) \otimes \mathfrak{g},$$

satisfies the Maurer–Cartan equation, where $\alpha = \alpha^i \otimes A_i$ is the tautological 1-form on $\mathfrak{a}$.

Corollary. Let $V : \mathfrak{a} \to \mathfrak{m}$ be a solution of the G/G_0-system and let $g : \mathfrak{a} \to G$ be a *primitive* of θ (i.e. a solution to $g^{-1}dg = \theta$). Then

$$\mathfrak{a} \ni x \mapsto (g(x), V(x)) \in G \times \mathfrak{m}$$

is an integral manifold of the PDS $(\mathcal{I}, \omega)$ on $Y = G \times \mathfrak{m}$ associated with the Cartan tableau $\mathfrak{m} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$. Conversely, any integral manifold of $(\mathcal{I}, \omega)$ arises in this way.

Conclusion: The integral manifolds of the PDS $(\mathcal{I}, \omega)$ associated to the Cartan tableau $\mathfrak{m} \subset \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$ are given by the solutions of the corresponding G/G_0-system. Moreover, $(\mathcal{I}, \omega)$ is in involution and its Cartan characters coincide with those of the tableau $\mathfrak{m}$ (i.e., $s_1 = n$, $s_j = 0$, $j = 2, \dots, n$). In particular, the general solution depends on n functions in one variabe.