

Polynomial Poisson and Associative Algebras for Classical and Quantum Superintegrable Systems with a Third Order Integral of Motion

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We consider a general superintegrable Hamiltonian system in a two-dimensional space with a scalar potential. It allows one quadratic and one cubic integral of motion. We construct the most general cubic Poisson algebra generated by these integrals for the classical case. For the quantum case we construct the associative cubic algebra and we present specific realizations. We use them to calculate the energy spectrum. All classical and quantum superintegrable potentials separable in cartesian coordinates with a third order integral were found. The general formalism is applied to two of these potentials.

This is a joint work with Pavel Winternitz

1 Cubic Poisson algebras

We begin with the classical case. We suppose that we have a superintegrable system with a quadratic Hamiltonian and one second order and one third order integral of motion.

$$\begin{aligned}
 H &= a(q_1, q_2)P_1^2 + 2b(q_1, q_2)P_1P_2 + c(q_1, q_2)P_2^2 + V(q_1, q_2) \\
 A &= A(q_1, q_2, P_1, P_2) = d(q_1, q_2)P_1^2 + 2e(q_1, q_2)P_1P_2 + \\
 &\quad f(q_1, q_2)P_2^2 + g(q_1, q_2)P_1 + h(q_1, q_2)P_2 + Q(q_1, q_2) \quad (1.1) \\
 B &= B(q_1, q_2, P_1, P_2) = u(q_1, q_2)P_1^3 + 3v(q_1, q_2)P_1^2P_2 + 3w(q_1, q_2)P_1P_2^2 \\
 &\quad + x(q_1, q_2)P_2^3 + j(q_1, q_2)P_1^2 + 2k(q_1, q_2)P_1P_2 + l(q_1, q_2)P_2^2 + \\
 &\quad m(q_1, q_2)P_1 + n(q_1, q_2)P_2 + S(q_1, q_2)
 \end{aligned}$$

with

$$\{H, A\} = \{H, B\} = 0 \quad (1.2)$$

where $\{, \}$ is the Poisson bracket

We will try to close the algebra at the lowest order possible, namely 3. We put

$$\begin{aligned}
 \{A, B\} &= C \\
 \{A, C\} &= \alpha A^2 + 2\beta AB + \gamma A + \delta B + \epsilon \quad (1.3) \\
 \{B, C\} &= \mu A^3 + \nu A^2 - \beta B^2 - 2\alpha AB + \xi A - \gamma B + \zeta
 \end{aligned}$$

The coefficients α , β and μ are constants, but the other ones can be polynomials in the Hamiltonian H . The degrees of these polynomials is dictated by the fact that H and A are second order polynomials in the momenta and B is a third order one. Hence C can be a fourth order polynomial. We have

$$\begin{aligned}\alpha &= \alpha_0, \beta = \beta_0, \mu = \mu_0 & (1.4) \\ \gamma &= \gamma_0 + \gamma_1 H, \delta = \delta_0 + \delta_1 H, \epsilon = \epsilon_0 + \epsilon_1 H + \epsilon_2 H^2 \\ \nu &= \nu_0 + \nu_1 H, \xi = \xi_0 + \xi_1 H + \xi_2 H^2 \\ \zeta &= \zeta_0 + \zeta_1 H + \zeta_2 H^2 + \zeta_3 H^3\end{aligned}$$

A Casimir operator K of a polynomial algebra is defined as an operator Poisson commuting with all elements of the algebra. For the algebra(2.5) this means

$$\{K, A\} = \{K, B\} = \{K, C\} = 0 \quad (1.5)$$

and this implies

$$K = C^2 - 2\alpha A^2 B - 2\beta A B^2 - 2\gamma A B - \delta B^2 - 2\epsilon B + \frac{1}{2}\mu A^4 + \frac{2}{3}\nu A^3 + \xi A^2 + 2\zeta A \quad (1.6)$$

thus K is a polynomial of order 8 in the momenta. We can expect K to be a polynomial in H and we write

$$K = k_0 + k_1 H + k_2 H^2 + k_3 H^3 + k_4 H^4 \quad (1.7)$$

where k_0, \dots, k_4 are constants.

2 Cubic Associative algebras and their algebraic realizations

Let us first consider a general quantum superintegrable two-dimensional system of the form (1.1) with

$$P_1 = -i\hbar\partial_1, P_2 = -i\hbar\partial_2 \quad (2.1)$$

$$[H, A] = [H, B] = 0 \quad (2.2)$$

We assume that our integrals close in a polynomial algebra.

$$\begin{aligned} [A, B] &= C \\ [A, C] &= \alpha A^2 + \beta\{A, B\} + \gamma A + \delta B + \epsilon \\ [B, C] &= \mu A^3 + \nu A^2 - \beta B^2 - \alpha\{A, B\} + \xi A - \gamma B + \zeta \end{aligned} \quad (2.3)$$

The Casimir operator of a polynomial algebra is an operator that commutes with all elements of the associative algebra. The Casimir operator satisfies :

$$[K, A] = [K, B] = [K, C] = 0 \quad (2.4)$$

$$\begin{aligned} K &= C^2 - \alpha\{A^2, B\} - \beta\{A, B^2\} + (\alpha\beta - \gamma)\{A, B\} + (\beta^2 - \delta)B^2 \\ &+ (\beta\gamma - 2\epsilon)B + \frac{\mu}{2}A^4 + \frac{2}{3}(\nu + \mu\beta)A^3 + \left(-\frac{1}{6}\mu\beta^2 + \frac{\beta\nu}{3} + \frac{\delta\mu}{2} + \alpha^2 + \xi\right)A^2 \\ &+ \left(-\frac{1}{6}\mu\beta\delta + \frac{\delta\nu}{3} + \alpha\gamma + 2\zeta\right)A \end{aligned} \quad (2.5)$$

We construct a realization of the cubic associative algebra by the means of the deformed oscillator technique. We use a deformed oscillator algebra $\{b^t, b, N\}$ which satisfies the relation

$$[N, b^t] = b^t, [N, b] = -b, b^t b = \Phi(N), b b^t = \Phi(N + 1) \quad (2.6)$$

We want that $\Phi(x)$ is a real function that satisfies the boundary condition $\Phi(0) = 0$, $\Phi(x) > 0$ for $x > 0$. These constraints impose the existence of a Fock type representation of the deformed oscillator algebra [4,5]. There is a Fock basis $|n\rangle$, $n=0,1,2,\dots$. With the relation

$$N|n\rangle = n|n\rangle, b^t|n\rangle = \sqrt{\Phi(N+1)}|n+1\rangle \quad (2.7)$$

$$b|0\rangle = 0, b|n\rangle = \sqrt{\Phi(N)}|n-1\rangle \quad (2.8)$$

We consider the case of a nilpotent deformed oscillator algebra, i.e., there should be an integer p such that,

$$b^{p+1} = 0, (b^t)^{p+1} = 0 \quad (2.9)$$

These relations imply that we have

$$\Phi(p+1) = 0 \quad (2.10)$$

In this case we have a finite-dimensional representation of dimension $p+1$.

Let us show that there is a realization of the form :

$$A = A(N), B = b(N) + b^t \rho(N) + \rho(N) b \quad (2.11)$$

The functions $A(x)$, $b(x)$ et $\rho(x)$ will be determined by the algebra.

We shall distinguish two cases..

Case 1 $\beta \neq 0$

$$\begin{aligned}
 A(N) &= \frac{\beta}{2} \left((N+u)^2 - \frac{1}{4} - \frac{\delta}{\beta^2} \right) \\
 b(N) &= \frac{\alpha}{4} \left((N+u)^2 - \frac{1}{4} \right) + \frac{\alpha\delta - \gamma\beta}{2\beta^2} \\
 &\quad - \frac{\alpha\delta^2 - 2\gamma\delta\beta + 4\beta^2\epsilon}{4\beta^4} \frac{1}{(N+u)^2 - \frac{1}{4}}
 \end{aligned} \tag{2.12}$$

Thus the structure function depends only on the function ρ . This function is arbitrary. In Case 1 we choose.

$$\rho(N) = \frac{1}{3 * 2^{12} \beta^8 (N+u)(1+N+u)(1+2(N+u))^2} \tag{2.13}$$

From our expression for $A(x)$, $b(x)$ and $\rho(x)$, the third relation of the cubic associative algebra and the expression of the Casimir operator we find the structure function $\Phi(N)$. For this case the structure function is a polynomial of order 10 in N . The coefficients of this polynomial are function of $\alpha, \beta, \mu, \gamma, \delta, \epsilon, \nu, \xi$ and ζ .

Case 2 $\beta = 0$ et $\delta \neq 0$

$$A(N) = \sqrt{\delta}(N+u), b(N) = -\alpha(N+u)^2 - \frac{\gamma}{\sqrt{\delta}}(N+u) - \frac{\epsilon}{\delta} \tag{2.14}$$

In Case 2 we choose a trivial expression $\rho(N) = 1$. I will give the explicit expression of the sutructure function for this case.

$$\Phi(N) = \left(\frac{K}{-4\delta} - \frac{\gamma\epsilon}{4\delta^{\frac{3}{2}}} - \frac{\zeta}{4\sqrt{\delta}} + \frac{\epsilon^2}{4\delta^2} \right) \tag{2.15}$$

$$\begin{aligned}
& + \left(\frac{-\alpha\epsilon}{2\delta} - \frac{d}{4} - \frac{\gamma^2}{4\delta} + \frac{\gamma\epsilon}{2\delta^{\frac{3}{2}}} + \frac{\alpha\gamma}{4\sqrt{\delta}} + \frac{\zeta}{2\sqrt{\delta}} + \frac{\nu\sqrt{\delta}}{12} \right) (N+u) \\
& + \left(\frac{-\nu\sqrt{\delta}}{4} - \frac{3\alpha\gamma}{4\sqrt{\delta}} + \frac{\gamma^2}{4\delta} + \frac{\epsilon\alpha}{2\delta} + \frac{\alpha^2}{4} + \frac{\xi}{4} + \frac{\mu\delta}{8} \right) (N+u)^2 \\
& + \left(\frac{-\alpha^2}{2} + \frac{\gamma\alpha}{2\delta^{\frac{1}{2}}} + \frac{\nu\sqrt{\delta}}{6} - \frac{\mu\delta}{4} \right) (N+u)^3 + \left(\frac{\alpha^2}{4} + \frac{\mu\delta}{8} \right) (N+u)^4
\end{aligned}$$

We will consider a representation of the cubic associative algebra in which the generator A and the Casimir operator K are diagonal. We use parafermionic realization which the parafermionic number operator N and the Casimir operator K are diagonal. The basis of this representation is the Fock basis for the parafermionic oscillator. The vector $|k, n\rangle, n = 0, 1, 2, \dots$ satisfies the following relation :

$$N|k, n\rangle = n|k, n\rangle, K|k, n\rangle = k|k, n\rangle \quad (2.16)$$

The vectors $|k, n\rangle$ are also eigenvectors of the generator A .

$$A|k, n\rangle = A(k, n)|k, n\rangle$$

We have the following constraints for the structure function,

$$\Phi(0, u, k) = 0, \Phi(p+1, u, k) = 0 \quad (2.17)$$

With these two relations we can find the energy spectrum. Many solutions for the system exist. Unitary representations of the deformed parafermionic oscillator obey the following constraint $\Phi(x) > 0$ for $x=1, 2, \dots, p$ [4,5].

3 Examples

Simon Gravel [23] founded 8 classical potentials separable in cartesian coordinates with a third order integral.

Case C6

$$H = \frac{P_1^2}{2} + \frac{P_2^2}{2} + \frac{\omega^2}{2}y^2 + V(x), A = \frac{P_1^2}{2} - \frac{P_2^2}{2} - \frac{\omega^2}{2}y^2 + V(x) \quad (3.1)$$

$$B = LP_1^2 + \left(\frac{\omega^2}{2}x^2y - 3y\right)V(x)P_1 - \frac{1}{\omega^2}\left(\frac{\omega^2}{2}x^2 - 3V(x)\right)V(x)_xP_2 \quad (3.2)$$

where V satisfies a quartic equation

$$\begin{aligned} -9V(x)^4 + 14\omega^2x^2V(x)^3 + (6d - 3\frac{\omega^4}{4}x^4)V(x)^2 + (\frac{3\omega^6}{2}x^6 - 2\omega^2x^2)V(x) \\ + (cx^2 - d - d\frac{\omega^4}{2}x^4 - \frac{\omega^8}{16}x^8) = 0 \end{aligned} \quad (3.3)$$

In the quantum case V satisfies a fourth order differential equation[23]

$$\hbar^2V^{(4)} = 12\omega^2xV' + 6(V^2)'' - 2\omega^2x^2V'' + 2\omega^4x^2 \quad (3.4)$$

That can be solved in terms of the painlevé transcendent P_{IV} . (3.3) is the solution of (3.4) for $\hbar \rightarrow 0$ and c and d are integration constants. In general, eq. (3.3) has 4 roots and the expressions for them are quite complicated. A special case occurs if ω, c and d satisfy.

$$c = \frac{2^3\omega^8b^3}{3^6}, d = \frac{\omega^4b^2}{3^3} \quad (3.5)$$

where b is an arbitrary constant. Then eq. (3.3) has a double root and

we obtain

$$V(x) = \frac{\omega^2}{18}(2b + 5x^2 \pm 4x\sqrt{b + x^2}), V(x) = (-\frac{\omega^2 b}{3^3} + \frac{\omega^2}{2}x^2) \quad (3.6)$$

For $V(x)$ satisfying (4.6) the cubic algebra is

$$\begin{aligned} \{A, B\} &= C \\ \{A, C\} &= -4\omega^2 B \end{aligned} \quad (3.7)$$

$$\begin{aligned} \{B, C\} &= 8A^3 + 12HA^2 - 4H^3 - 4\frac{4b^2\omega^4}{27}A + \frac{4b^3\omega^6}{729} \\ K &= 4H^4 - \frac{4}{27}b^2\omega^4H^2 + \frac{8b^3\omega^6}{729}H \end{aligned} \quad (3.8)$$

The trajectories were obtained numerically directly from the equations of motion. We have closed trajectories and are shown on Fig 1. Simon Gravel [23] founded 21 quantum potentials separable in cartesian coordinates with a third order integral. We will consider this interesting case.

Case Q5

$$H = \frac{P_x^2}{2} + \frac{P_y^2}{2} + \hbar^2\left(\frac{x^2 + y^2}{8a^4} + \frac{1}{(x - a)^2} + \frac{1}{(x + a)^2}\right) \quad (3.9)$$

$$A = \frac{P_x^2}{2} - \frac{P_y^2}{2} + \hbar^2\left(\frac{x^2 - y^2}{8a^4} + \frac{1}{(x - a)^2} + \frac{1}{(x + a)^2}\right) \quad (3.10)$$

$$\begin{aligned} B = X_2 &= \{L, P_x^2\} + \hbar^2\left\{y\left(\frac{4a^2 - x^2}{4a^4} - \frac{6(x^2 + a^2)}{(x^2 - a^2)^2}\right), P_x\right\} \\ &+ \hbar^2\left\{x\left(\frac{x^2 - 4a^2}{4a^4} - \frac{2}{x^2 - a^2} + \frac{4(x^2 + a^2)}{(x^2 - a^2)^2}\right), P_y\right\} \end{aligned} \quad (3.11)$$

$$\begin{aligned}
[A, B] &= C \\
[A, C] &= \frac{\hbar^4}{a^4} B
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
[B, C] &= -32\hbar^2 A^3 - 48\hbar^2 A^2 H + 16\hbar^2 H^3 + 48\frac{\hbar^4}{a^2} A^2 + 32\frac{\hbar^4}{a^2} H A - 16\frac{\hbar^4}{a^2} H^2 \\
&\quad + 8\frac{\hbar^6}{a^4} A - 4\frac{\hbar^6}{a^4} H - 12\frac{\hbar^8}{a^6} \\
K &= -16\hbar^2 H^4 + 32\frac{\hbar^4}{a^2} H^3 + 16\frac{\hbar^6}{a^4} H^2 - 40\frac{\hbar^8}{a^6} H - 3\frac{\hbar^{10}}{a^8}
\end{aligned} \tag{3.13}$$

$$u = \frac{-a^2 E}{\hbar^2} + \frac{5}{2}, \quad E = \frac{\hbar^2(p+3)}{2a^2} \tag{3.14}$$

$$\Phi(x) = \left(\frac{4\hbar^8}{a^4}\right) x(p+1-x)(x+1)(x+3) \tag{3.15}$$

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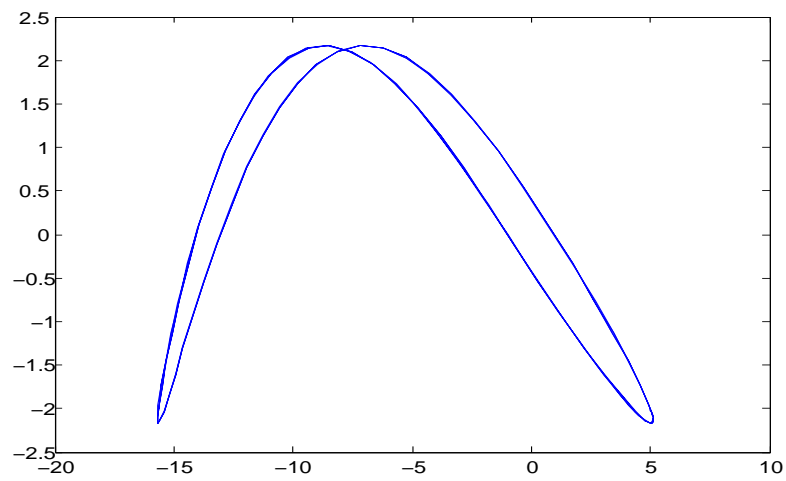


FIG. 1 – A trajectory for $\frac{\omega^2}{18}(2b + 5x^2 + 4x\sqrt{b + x^2})$. Parameter $\omega^2 = 1$ and $d=3$, $v_{x_0} = -1.5$, $x_0 = 5$, $v_{y_0} = -1.2$, $y_0 = -2$, $t=[0,400]$