Rational and Algebraic Invariants of a Group Action

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Synopsis

Propose algebraic constructions of a **generating set of invariants** that comes with a simple **rewriting algorithm**.

Original motivations:

- Differential elimination/completion for symmetric differential systems [Mansfield 2001]
- Avoid the implicit function theorem in [Fels & Olver 1999]

E. Hubert and I. Kogan, *Rational Invariants of a Group Action. Construction and Rewriting.* Journal of Symbolic Computation.

Rational and Algebraic Invariants of a Group Action

Contents

1 Rational Invariants of a Group Action

1.1 Definitions

Algebraic Group $\mathcal{G}$

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

$\mathcal{G} \subset \mathbb{K}^l$ an algebraic variety $\qquad G \subset \mathbb{K}[\lambda_1, \ldots, \lambda_l]$ its ideal

$$m: \begin{array}{ccc} \mathcal{G} \times \mathcal{G} & \to & \mathcal{G} \\ (\lambda, \mu) & \mapsto & \lambda \cdot \mu \end{array} \qquad \text{and} \qquad i: \begin{array}{ccc} \mathcal{G} & \to & \mathcal{G} \\ \lambda & \mapsto & \lambda^{-1} \end{array}$$

$$\lambda \cdot \mu \in \mathbb{K}[\lambda, \mu] \qquad \text{and} \qquad \lambda^{-1} \in \mathbb{K}[\lambda]$$

$$e \in \mathcal{G} \quad e \cdot \lambda = \lambda \cdot e = \lambda$$

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K} \times \{-1, 1\}$	$SO(2)$
G	$(\lambda_1 \lambda_2 - 1)$	$(\lambda_2^2 - 1)$	$(\lambda_1^2 + \lambda_2^2 - 1)$
$\lambda \cdot \mu$	$(\lambda_1\mu_1, \lambda_2\mu_1)$	$(\lambda_1 + \mu_1, \lambda_2\mu_2)$	$(\lambda_1\mu_1 - \lambda_2\mu_2, \lambda_1\mu_2 + \lambda_2\mu_1)$
e	$(1, 1)$	$(0, 1)$	$(1, 0)$
λ^{-1}	(λ_2, λ_1)	$(-\lambda_1, \lambda_2)$	$(\lambda_1, -\lambda_2)$

Rational Action on $\mathcal{Z} = \mathbb{K}^n$

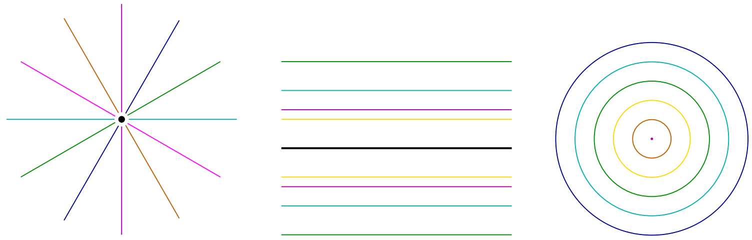
$$g: \begin{array}{ccc} \mathcal{G} \times \mathcal{Z} & \to & \mathcal{Z} \\ (\lambda, z) & \mapsto & \lambda \star z = \left(\dfrac{g_1(\lambda, z)}{h(\lambda, z)}, \ldots, \dfrac{g_n(\lambda, z)}{h(\lambda, z)} \right) \end{array} \qquad (\lambda \cdot \mu) \star z = \lambda \star (\mu \star z)$$

Orbit of $z \in \mathcal{Z}$

$$\mathcal{O}_z = \{\lambda \star z \mid \lambda \in \mathcal{G}\}$$

$$h, g_1, \ldots, g_n \in \mathbb{K}[\lambda_1, \ldots, \lambda_l, z_1, \ldots, z_n]$$

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K} \times \{-1, 1\}$	$SO(2)$
G	$(\lambda_1 \lambda_2 - 1)$	$(\lambda_2^2 - 1)$	$(\lambda_1^2 + \lambda_2^2 - 1)$
$\lambda \star z$	$\begin{pmatrix} \lambda_1 z_1 \\ \lambda_1 z_2 \end{pmatrix}$	$\begin{pmatrix} z_1 + \lambda_1 \\ \lambda_2 z_2 \end{pmatrix}$	$\begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$
	scaling	translation+reflection	rotation



scaling translation+reflection rotation

Field of Rational Invariants $\mathbb{K}(z)^G$

Rational invariant: $\frac{p}{q} \in \mathbb{K}(z)$ $\qquad \frac{p(\lambda \star z)}{q(\lambda \star z)} = \frac{p(z)}{q(z)} \mod G$

Field of rational invariants: $\mathbb{K}(z)^G$

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K} \times \{-1,1\}$	$SO(2)$
$\mathbb{K}(z)^G$	$\mathbb{K}\left(\frac{z_1}{z_2}\right)$	$\mathbb{K}\left(z_2^2\right)$	$\mathbb{K}\left(z_1^2+z_2^2\right)$

1.2 Results

Rational Invariants

Algorithm

In : $G, \quad (g_1(\lambda,z),\dots,g_n(\lambda,z),h(\lambda,z)) \in \mathbb{K}[\lambda,z]$

Out : $\{r_1,\dots,r_\kappa\} \subset \mathbb{K}(z)^G$

$$\begin{array}{rccl} \longrightarrow_Q: & \mathbb{K}(z)^G & \rightarrow & \mathbb{K}(y_1,\dots,y_\kappa) \\ & r & \mapsto & R \end{array} \qquad r = R(r_1,\dots,r_\kappa)$$

So : $\mathbb{K}(z)^G = \mathbb{K}(r_1,\dots,r_\kappa)$

I : Gröbner basis of an unmixed dimensional ideal of dimension s

II : Gröbner basis of a zero dimensional ideal
Relies on the choice of a generic linear space of codimension s

Code: `www.inria.fr/cafe/Evelyne.Hubert/Publi/RationalInvariants`

Algebraic Moving Frame

We introduce replacement invariants

$$\xi = (\xi_1,\dots,\xi_n), \quad \xi_i \in \overline{\mathbb{K}(z)}^G$$

with property

$$r(z_1,\dots,z_n) = r(\xi_1,\dots,\xi_n) \quad \forall r \in \mathbb{K}(z)^G$$

ξ is the algebraic counterpart of the Cartan normalized invariants.

2 Intermezzo

2.1 Gröbner bases

Gröbner bases in $\mathbb{K}[z_1,\dots,z_n]$

I a (radical) ideal in $\mathbb{K}[z] = \mathbb{K}[z_1,\dots,z_n]$

Hilbert: $I = (q_1,\dots,q_l)$

Reduc^o: $z^\alpha = z_1^{\alpha_1}\dots z_n^{\alpha_n}$
$q = z^\alpha - \sum_{\beta<\alpha} c_\beta\, z^\beta \qquad z^{\alpha+\gamma} \quad \longrightarrow_q z^\gamma \quad \sum_{\beta<\alpha} c_\beta\, z^\beta$

Gröbner: $\{q_1, \dots, q_l\}$ a Gröbner basis if $\quad p \in I \quad \Leftrightarrow \quad p \longrightarrow_Q^* 0$

Prop: Reduced Gröbner bases are canonical representative for ideals

Algo: INPUT: $p_1, \dots, p_m$ a generating set of I
OUTPUT: $Q = \{q_1, \dots, q_l\}$ a reduced Gröbner basis of I

$\rightsquigarrow$ $\mathbb{K}$-basis for $\mathbb{K}[z]/I$, its Hilbert polynomial, resolution.

Observe: $p_1, \dots, p_m \in k[z] \ \Rightarrow \ Q \subset k[Z] \qquad k \subset \mathbb{K}$

$\rightsquigarrow$ The coefficients of Q give the field of definition of I.

3 Construction and rewriting of rational invariants

Synopsis

3.1 Graph ideal $\rightsquigarrow$ Rosenlicht (1956),..

Graph of the action & its ideal O

- Graph of the action

$$\mathcal{O} = \{(z, z') \in \mathcal{Z} \times \mathcal{Z} \mid \exists \lambda \in \mathcal{G} \text{ s.t. } z' = \lambda \star z\}$$

- Its ideal: $O = (G + (\, Z - \lambda \star z \,)) \cap \mathbb{K}[z, Z]$

$Z = (Z_1, \dots, Z_n)$ new set of variables

$$(Z - \lambda \star z) = (\, h\, Z_i - g_i \mid 1 \leq i \leq n \,) : h^\infty$$

- O^e the extension of O to $\mathbb{K}(z)[Z]$.
  $\rightsquigarrow$ the ideal of a generic orbit

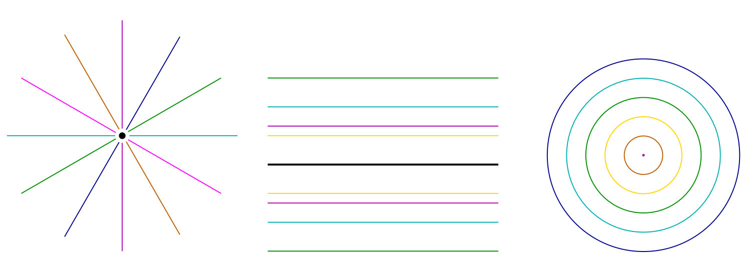
Construction of rational invariants

Invariance: $\quad (z, z') \in \mathcal{O} \ \Rightarrow \ (\lambda \star z, z') \in \mathcal{O}$

Thm: The reduced Gröbner basis of O^e is contained in $\mathbb{K}(z)^G[Z]$.

Examples

$\mathcal{G}$ $\qquad \mathbb{K}^* \qquad \mathbb{K} \times \{-1, 1\} \qquad SO(2)$



Q $\qquad Z_2 - \frac{z_2}{z_1} Z_1 \qquad Z_2^2 - z_2^2 \qquad Z_2^2 + Z_1^2 - (z_1^2 + z_2^2)$

Rewriting & Generation

Q reduced Gröbner basis of O^e

$\{r_1,\dots,r_\kappa\}$ the coefficients of Q

Theorem:

$$\mathbb{K}(z)^G = \mathbb{K}(r_1,\dots,r_\kappa)$$

¡2-¿
Rewriting $\quad \frac{p}{q} \in \mathbb{K}(z)^G$

- $y_1,\dots,y_\kappa$ a new indeterminates
- $Q_y := Q(r_i \leftarrow y_i)$
- $p(Z) \longrightarrow^*_{Q_y} \sum_\alpha a_\alpha(y)\, Z^\alpha$
- $q(Z) \longrightarrow^*_{Q_y} \sum_\alpha b_\alpha(y)\, Z^\alpha$
- $\frac{p(z)}{q(z)} = \frac{a_\alpha(r)}{b_\alpha(r)}$

Example of rewriting for the scaling

$$Q = \{Z_2 - \frac{z_2}{z_1} Z_1\} \qquad r = \frac{z_2}{z_1} \qquad Q_y = \{Z_2 - yZ_1\}$$

$$\frac{p}{q} = \frac{z_1^2 + 4z_1z_2 + z_2^2}{z_1^2 - 3z_2^2}$$

$$p(Z) = Z_1^2 + 4Z_1Z_2 + Z_2^2 \quad \longrightarrow_{Q_y} \quad (y^2+4y+1)Z_2^2$$

$$q(Z) = Z_1^2 - 3Z_2^2 \quad \longrightarrow_{Q_y} \quad (y^2-3)Z_2^2$$

$$q(z)p(Z) \equiv p(z)q(Z) \,\mathrm{mod} O^e \;\Rightarrow\; q(z)(r^2-3)Z_2^2 = p(z)(r^2+4r+1)Z_2^2$$

$$\frac{z_1^2 + 4z_1z_2 + z_2^2}{z_1^2 - 3z_2^2} = \frac{r^2+4r+1}{r^2-3} \text{ where } r = \frac{z_1}{z_2}$$

Previously

Müller-Quade & Beth 99 • Case of linear group actions.

- Proof: $(Q) = (Z - z) \cap \mathbb{K}(z)^G[Z]$

Vinberg & Popov 89 • There exists a generating set Q of O^e the coefficients $\{r_1,\dots,r_\kappa\}$ of which are in $\mathbb{K}(z)^G$

- $\{r_1, \ldots, r_\kappa\}$ separate orbits
- A set of rational invariant that separate orbits is a generating set for $\mathbb{K}(z)^G$

Rosenlicht 56
- The coefficients of the Chow form of O^e are rational invariants and separate orbits
- A set of rational invariant that separate orbits is a generating set for $\mathbb{K}(z)^G$

3.2 Graph-section ideal $\rightsquigarrow$ Fels & Olver (1999)

Cross-section of degree d

A variety $\mathcal{P}$ that intersects generic orbits in d simple points.

$$O^e = (G + (Z - \lambda \star z)) \cap \mathbb{K}(z)[Z].$$

$$s = \text{dimension of } O^e = \text{dimension of generic orbits}$$

The ideal P defines a cross-section $\mathcal{P}$ of degree d:

- $P \subset \mathbb{K}[Z]$ prime ideal of codimension s
- $I^e = O^e + P$ radical and zero-dimensional
- $\dim_{\mathbb{K}(z)} \mathbb{K}(z)[Z]/I^e = d$

$$P = (a_{i1}Z_1 + \ldots + a_{in}Z_n - b_i,\ 1 \le i \le s)$$

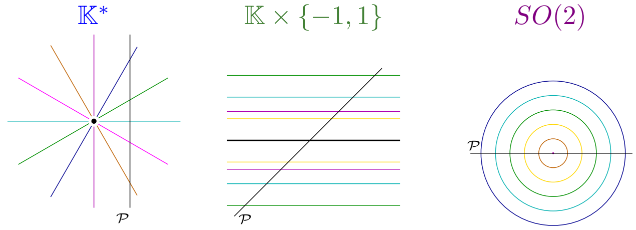
Rational Invariants 2

$$I^e = P + O^e = (\ P + G + (Z - \lambda \star z)\ )\ \cap\ \mathbb{K}(z)[Z]$$

Q a reduced Gröbner basis of I^e $\qquad$ $\{r_1, \ldots, r_\kappa\}$ its coefficients

Theorem: $\mathbb{K}(z)^G = \mathbb{K}(r_1, \ldots, r_\kappa)$ + rewriting

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K} \times \{-1, 1\}$	$SO(2)$



Q	$Z_1 - 1,\ Z_2 - \frac{z_2}{z_1}$	$Z_1 - Z_2,\ Z_2^2 - z_2^2$	$Z_2,\ Z_1^2 - (z_1^2 + z_2^2)$

4 Algebraic Moving frame

Replacement Invariant ξ

- $\mathcal{P}$ a cross-section of degree $d = 1$
  $I^e = (\ Z_1 - r_1(z), \ldots, Z_n - r_n(z))$

$$r(z_1, \ldots, z_n) = r(r_1, \ldots, r_n) \qquad \forall r \in \mathbb{K}(z)^G$$

- $\mathcal{P}$ a cross-section of degree $d > 1$
  $I^G = I^e \cap \mathbb{K}(z)^G[Z] = (Q)$ has d distinct $\overline{\mathbb{K}(z)}^G$-zeros

Thm: $\xi = (\xi_1, \ldots, \xi_n)$ a $\overline{\mathbb{K}(z)}^G$-zero of I^G.

$$r(z) = r(\xi), \quad \forall r \in \mathbb{K}(z)^G$$

Replacement Invariant ξ. Examples

$\mathcal{P}$ a cross-section of degree d

$I^G = I^e \cap \mathbb{K}(z)^G[Z] = (Q)$ has d distinct $\overline{\mathbb{K}(z)}^G$-zeros

Thm: $\xi = (\xi_1, \ldots, \xi_n)$ a $\overline{\mathbb{K}(z)}^G$-zero of I^G. $r(z) = r(\xi),\ r \in \mathbb{K}(z)^G$

$\mathcal{G}$	$\mathbb{K}^*$	$\mathbb{K} \times \{-1, 1\}$	$SO(2)$
Q	$Z_1 - 1,\ Z_2 - \frac{z_2}{z_1}$	$Z_1 - Z_2,\ Z_2^2 - z_2^2$	$Z_2,\ Z_1^2 - (z_1^2 + z_2^2)$
ξ	$(1,\ \frac{z_2}{z_1})$	$(\pm z_2,\ \pm z_2)$	$\left(\pm\sqrt{z_1^2 + z_2^2},\ 0\right)$

Algebraic moving frame

$\mathcal{P}$ an algebraic cross-section $\Rightarrow$ $\mathcal{P} \cap \mathcal{U}$ a local cross-section.

Replacement invariant : $\overline{\mathbb{K}(z)}^G$-zero of I^G

Thm: The normalized invariants $(\bar{\iota}z_1, \ldots, \bar{\iota}z_n)$ form the smooth zero of I^G that agrees with the coordinate functions on $\mathcal{P} \cap \mathcal{U}$.

E. Hubert and I. Kogan. *Smooth and Algebraic Invariants of a Group Action. Local and Global Constructions.* (Preprint).

Merci.

Thanks.