PFAFFIAN SYSTEMS OF FROBENIUS TYPE AND SOLVABILITY OF GENERIC OVERDETERMINED PDE SYSTEMS

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We observe that solving an overdetermined PDE system of generic type is equivalent to finding an integral manifold of the associated Pfaffian system of Frobenius type. Then we discuss the existence, regularity and finiteness of solutions of an overdetermined PDE system from this point of view: For an overdetermined PDE systems without side conditions we find compatibility conditions by complete prolongation and by setting the torsions to be zero. Once all the compatibility conditions are found one can determine the existence of general solutions. For the equations arising from geometry it is important to express the conditions for torsions to vanish in terms of the geometric invariants. In §1 and §2 we review basics of the classical prolongation theory in most general setting and then in the cases of Frobenius type. In the subsequent sections we present examples and applications including analyticity of CR mappings, rigidity of isometric and conformal embeddings, finiteness and existence of infinitesimal symmetries of multi-contact structures. All our discussion is local and our proofs need generic hypotheses.

§1. Prolongation of Pfaffian systems.

On a smooth (C^∞) manifold M of dimension n we consider a Pfaffian

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system with independence condition:

$$\begin{aligned} &\theta^\alpha = 0, \quad \alpha = 1, \cdots, s \\ &\Omega = \omega^1 \wedge \ldots \omega^p \neq 0 \quad \text{(independence condition)}, \end{aligned} \tag{1.1}$$

where $\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p$ are linearly independent 1-forms. (1.1) gives rise to subbundles $I \subset J \subset T^*M$. Here $I =< \theta^1, \cdots, \theta^s >$ and $J =< \theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p >$, where $< \cdots >$ denotes the linear span of what are inside. We look for an integral manifold of dimension p that satisfies the independence condition. (1.1) is said to be quasi-linear if $dI \subset J$. (1.1) is said to be of Frobenius type if $s + p = n$, that is, $\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p$ form a coframe of M. Frobenius types are quasi-linear.

Let $N \subset M$ be an integral manifold (1.1) of any dimension $k = 0, 1, \cdots, p$ then $\theta^\alpha|_N = 0$, and therefore, $d\theta^\alpha|_N = 0$. A k-dimensional integral element is a k-dimensional subspace $(x, E) \in T_xM$, for some $x \in M$, on which $\theta^\alpha = 0$ and $d\theta^\alpha = 0$, for all $\alpha = 1, \cdots, s$. By $V(I, J)$ we denote the set of all p-dimensional integral elements (x, E) satisfying $\Omega|_E \neq 0$. The idea of constructing an integral manifold of dimension p is constructing k-dimensional integral manifold N_k with N_{k-1} as initial data, inductively for $k = 1, \cdots, p$, so that we have a nested sequence of integral manifolds

$$N^0 \subset N^1 \subset \cdots \subset N^p.$$

Let

$$\{x\} = E^0 \subset E^1 \subset \cdots \subset E^p = E$$

be the corresponding flag of integral elements. The notion of involutivity is the existence of such a flag for each element of $V(I, J)$ satisfying certain regularity conditions. If the system is analytic (C^ω) one can construct such a nested sequence of integral manifolds by using the Cauchy-Kowalewski theorem. This is the idea of the Cartan-Kähler theorem which asserts that an involutive analytic Pfaffian system has integral manifolds. If (I, J) is not involutive we construct an involutive system which is equivalent to the original system in the following two steps (cf. [GJ] Ch.3):

Step 1. *Reduce (1.1) to a submanifold $M' \subset M$ so that $V'(I, J) \to M'$ is surjective:*

Let M_1 be the image of $V(I, J) \to M$. If $M = M_1$ then we do nothing. If $M_1 \neq M$ then we note that any integral manifold of (I, J) must lie in M_1, and so we set

$$V_1(I, J) = \{(x, E) \in V(I, J) : E \subset T_xM_1\}.$$

Now consider the projection

$$V_1(I,J) \to M_1$$

with image M_2. If $M_2 = M_1$ we stop; otherwise we continue as before. Eventually we arrive either at the empty set, in which case (I,J) has no integral manifolds, or else at M' with $V'(I,J) \to M'$ being surjective and with all $(x,E) \in V'(I,J)$ satisfying $E \subset T_xM'$.

Step 2. *Assuming $V(I,J) \to M$ is surjective we do prolongation.*

To recall the definitions, let $G_p(M)$ be the Grassmann bundle of p-planes in TM. Let $\pi^1, \cdots, \pi^r$ be a set of 1-forms so that

$$\theta^1, \cdots, \theta^s, \omega^1, \cdots, \omega^p, \pi^1, \cdots, \pi^r$$

form a basis of T^*M. Let $(x,E) \in V(I,J)$. Since $\Omega|_E \neq 0$, on a neighborhood of $(x,E) \in G_p(M)$ we have $\theta^\alpha = m^\alpha_\rho \omega^\rho$, $\pi^\epsilon = \ell^\epsilon_\rho \omega^\rho$ and $\Omega \neq 0$. Thus $\{m^\alpha_\rho, \ell^\epsilon_\rho\}$ are local fibre coordinates in $G_p(M)$. The canonical Pfaffian system on $G_p(M)$ is the tautological Pfaffian system given by

$$\begin{aligned} &\theta^\alpha - m^\alpha_\rho \omega^\rho = 0, \quad \pi^\epsilon - \ell^\epsilon_\rho \omega^\rho = 0, \\ &\Omega \neq 0, \end{aligned} \tag{1.2}$$

where the summation convention is used for $\alpha = 1, \cdots, s$, $\rho = 1, \cdots, p$, $\epsilon = 1, \cdots, r$. The first prolongation $(I^{(1)}, J^{(1)})$ is the restriction to $V(I,J) \subset G_p(M)$ of the canonical Pfaffian system. Since $m^\alpha_\rho = 0$ on $V(I,J)$ the first prolongation of (1,1) is

$$\begin{aligned} &\theta^\alpha = 0, \quad \pi^\epsilon - \ell^\epsilon_\rho \omega^\rho = 0 \\ &\Omega \neq 0. \end{aligned} \tag{1.3}$$

Integral manifolds of (I,J) and those of $(I^{(1)}, J^{(1)})$ are in a one-to-one correspondence. Under a certain generic hypothesis we have

Theorem (Cartan-Kuranishi). *Given a Pfaffian system (I,J) there is q_0 such that prolongations $(I^{(q)}, J^{(q)})$ are involutive for $q \geq q_0$.*

§2 Pfaffian systems of Frobenius type.

Now we focus our interest to the systems of Frobenius type, that is, (1.1) with $s+p=n$. In this case no further equations are obtained by prolongation and the existence of general integral manifolds is determined only by Step 1. The notion of involutivity is very subtle as we see in E. Mansfield's lecture [M]. However, for $V(I,J)$ of Frobenius type the following are equivalent:

i) $V(I,J)\to M$ is surjective.
ii) (I,J) is integrable in the sense of the Frobenius theorem.
iii) (I,J) is invoultive.

It is easy to prove the following

Lemma 2.1. *Let M be a smooth manifold of dimension n. Let $\theta := (\theta^1,\cdots,\theta^s)$ be a set of independent 1-forms on M and $\mathcal{D} :=<\theta>^{\perp}$ be the $(n-s)$, $r\le s$, dimensional plane field annihilated by θ. Suppose that N is a submanifold of M of dimension $n-r$ defined by $T_1=\cdots=T_r=0$, where T_i are smooth real-valued functions of M such that $dT_1\wedge\cdots\wedge dT_r\neq 0$ on N. Then the following are equivalent :*
(i) $\mathcal{D}$ is tangent to N.
(ii) $dT_j\equiv 0, \quad \mathrm{mod}\ \ \theta^1,\ldots,\theta^s$ on N, for each $j=1,\ldots,r$.

Our basic observation is the following algorithm for Step 1: For each $\alpha=1,\cdots,s$, set

$$d\theta^\alpha = T^\alpha_{ij}\omega^i\wedge\omega^j, \quad \mathrm{mod}\theta,$$

where $\theta=(\theta^1,\cdots,\theta^s)$ and T^α_{ij} is skew symmetric in (ij). Let $\mathcal{T}_1$ be the set of functions $\{T^\alpha_{ij}\}$. If $\mathcal{T}_1$ are identically zero then $V(I,J)\to M$ is surjective, which is the Frobenius integrability condition for θ, and by Frobenius theorem we have $(n-p)$-parameter family of integral manifolds. Otherwise, let M_1 be the common zero set of $\mathcal{T}_1$ and set

$$dT^\alpha_{ij} = T^\alpha_{ij,k\ell}\omega^k\wedge\omega^\ell,\ \mathrm{mod}\ \theta,$$

where $T^\alpha_{ij,k\ell}$ is skew symmetric in (ij) and in $(k\ell)$. Let $\mathcal{T}_2$ be the set of functions $T^\alpha_{ij,k\ell}$. If $\mathcal{T}_2$ are identically zero on M_1 then $V_1(I,J)\to M_1$ is surjective, and by Frobenius theorem there exist $(\mathrm{dim}M_1-p)$-parameter family of solutions. If $\mathcal{T}_2$ are not identically zero, let M_2 be the submanifold of M_1 defined by $\mathcal{T}_2=0$ and continue as before. Eventually we arrive either at an empty set, in which case there is no integral manifolds, or at an integrable Pfaffian system on a submanifold $M' \subset M$, in which case there exist $(\dim M' - p)$-parameter family of integral manifolds.

§3. Complete prolongation of overdetermined PDE systems.

Let $u = (u^1, \cdots, u^q)$ be a system of real-valued functions of independent variables $x = (x^1, \cdots, x^p)$. Consider a system of partial differential equations of order m

$$\Delta_\lambda(x, u^{(m)}) = 0, \quad \lambda = 1, \cdots \ell, \tag{3.1}$$

where $u^{(m)}$ denotes all the partial derivatives of u of order up to m. We assume that (3.1) is over-determined, that is, $\ell > q$.

As we differentialte (3.1) μ times we have partial derivatives of u up to order $m + \mu$. Since

$$\frac{\text{number of equations}}{\text{number of partial derivatives}} \to \frac{\ell}{q}, \qquad \text{as } \mu \to \infty,$$

generically it is possible to solve for all the partial derivatives of u of a sufficiently high order, say k, as functions of derivatives of lower order, by the implicit function theorem, namely,

$$u^\alpha_K = H^\alpha_K(x, u^{(k-1)}), \tag{3.2}$$

for all multi-index K with $|K| = k$, and for all $\alpha = 1, \cdots, q$. (3.2) is called a complete system of order k and we say (3.1) admits prolongation to a complete system of order k. (3.2) can be regarded as a Pfaffian system of Frobenius type on submanifold M of the $(k-1)$th jet space $J^{(k-1)}(X, U)$, where X is the set of independent variables x and U is the space of dependent variables u. K. Yamaguchi and T. Yatsui studied in [Y],[YY1] and [YY2] the equivalence problem by point transformations and formulated the geometry of $J^{(k-1)}$ with the distribution of p planes $\mathcal{D}$ given by (3.2). Solving (3.1) is finding an integral manifold of $\mathcal{D}$ that is contained in the submanifold M that is defined by (3.1) and its prolongation. We do this by Step 1 as in §2. For examples, we consider a PDE systems of first order for one unknown function $u(x, y)$

$$\begin{cases} u_x = A(x, y, u) \\ u_y = B(x, y, u). \end{cases} \tag{3.3}$$

This is a complete system of first order. In this case M is the whole 0-th order jet space $\mathbb{R}^3 = \{(x, y, u)\}$ we consider the Pfaffian system

$$\theta := du - Adx - Bdy = 0 \tag{3.4}$$

with the independence condition

$$dx \wedge dy \neq 0. \tag{3.5}$$

Then $d\theta = Tdx \wedge dy, \quad \mathrm{mod}\theta, \quad$ where

$$T = A_y + A_u b - B_x - B_u A. \tag{3.6}$$

If $T = 0$ on $\mathbb{R}^3$, then for any initial condition $u(x_0, y_0) = u_0$ there exists a unique solution satisfying the initial condition and thus there exists a 1-parameter family of solutions. If T depends only on (x, y) then and not identically equal to zero then $dx \wedge dy = 0$ on $T = 0$ and therefore, there is no solution.

Example 3.1.

$$\begin{cases} u_x = a(x, y) + u \\ u_y = b(x, y). \end{cases} \tag{3.7}$$

By (3.6) $T = a_y + b - b_x$. If the functions a and b satisfy $a_y + b - b_x = 0$, then there exists 1-parameter family of solutions. Otherwise, $T = 0$ gives a relation between x and y, which violates the condition $dx \wedge dy \neq 0$, hence there is no solution.

Example 3.2.

$$\begin{cases} u_x = a(x, y) + u^2 \\ u_y = b(x, y), \quad b \neq 0. \end{cases} \tag{3.8}$$

In this case $\theta = du - (a + u^2)dx - bdy$ and $T = a_y + 2ub - b_x$. T cannot be identically zero for all (x, y, u). Thus $T = 0$ gives

$$u = \frac{1}{2b}(-a_y + b_x). \tag{3.9}$$

Thus, if a solution exists then (3.9) is the solution and (3.9) is indeed a solution if it satisfies (3.8), namely,

$$\begin{cases} \{\frac{1}{2b}(-a_y + b_x)\}_x = a + \{\frac{1}{2b}(-a_y + b_x)\}^2 \\ \{\frac{1}{2b}(-a_y + b_x)\}_y = b. \end{cases} \tag{3.10}$$

However, we derive (3.10) as follows: dT modulo θ on the submanifold $\{T = 0\}$ is

$$\begin{aligned}\{a_{xy} + \frac{b_x}{b}(-a_y + b_x) - b_{xx} + 2b[a + (\frac{-a_y + b_x}{2b})^2]\}dx \\ + \{a_{yy} + \frac{b_y}{b}(-a_y + b_x) - b_{xy} + 2b^2\}dy.\end{aligned}$$

By setting the coefficient to dx and the coefficient to dy to be zero, we obtain (3.10).

Now we consider systems of second order

$$\begin{cases} u_x + u_y = b(x, y) \\ u_{yy} = a(x, y, u, u_x, u_y). \end{cases} \tag{3.11}$$

Differentiate the first equation of (3.11) with respect to x and y, respectively, and solving for all the second order derivatives of u , we obtain

$$\begin{cases} u_{xx} = b_x - b_y + a \\ u_{xy} = b_y - a \\ u_{yy} = a. \end{cases} \tag{3.12}$$

Thus (3.11) admits prolongation to a complete system of second order. In the first jet space $\mathbb{R}^5 = \{(x, y.u, p, q)\}$, where p and q stands for u_x and u_y, respectively. Let M be a real submanifold of dimension 4 defined by the first equation of (3.11), that is, $p + q = b(x, y)$. We consider 1-forms

$$\begin{cases} \theta^0 = du - pdx - (b - p)dy \\ \theta^1 = dp - (b_x - b_y + a)dx - (b_y - a)dy \\ \theta^2 = dq - (b_y - a)dx - ady. \end{cases}$$

Observe that on M we have $\theta^2 = -\theta^1$ and that $\{\theta^0, \theta^1\}$ defines a 2-dimensional plane field $\mathcal{D}$ on M, whose integral manifolds are the first jet graph of solutions. To check the Frobenius integrability conditions we see that on M

$$\begin{cases} d\theta^0 = 0, \quad \mathrm{mod} \quad \{\theta^0, \theta^1\} \\ d\theta^1 = Tdx \wedge dy, \quad \mathrm{mod} \quad \{\theta^0, \theta^1\}, \end{cases}$$

where

$$T = -b_{yy} + a_y + a_u b + a_x + a_p b_x + a_q b_y. \tag{3.13}$$

If $T = 0$ on M, then there exists 2-parameter family of solutions. Otherwise, we restrict $\{\theta^0, \theta^1\}$ on the submanifold M_1 of M given by $T = 0$.

Example 3.3 (Linear Case).

$$\begin{cases} u_x + u_y = b(x,y) \\ u_{yy} = \alpha(x,y) + c_1 u + c_2 u_x + c_3 u_y. \end{cases}$$

Then

$$T(b^{(2)}, \alpha^{(1)}, c_1, c_2, c_3) = -b_{yy} + \alpha_x + \alpha_y + c_1 b + c_2 b_x + c_3 b_y,$$

which depends only on (x, y). If the functions α, b and the constants c_1, c_2, c_3 satisfy $T = 0$ then there are 2-parameter family of solutions. If T is not identically zero, $T = 0$ gives a relation between x and y, which violates $dx \wedge dy \neq 0$, therefore, no solutions exist.

Example 3.4(Nonlinear case).

$$\begin{cases} u_x + u_y = b(x), \quad (b' \neq 0) \\ u_{yy} = \alpha(x,y) + u_x^2. \end{cases}$$

In this example we assume b depends only on x, for simplicity. Then on $M := \{p + q = b(x)\}$ we have independent 1-forms

$$\begin{cases} \theta^0 = du - pdx - (b - p)dy \\ \theta^1 = dp - (b' + \alpha + p^2)dx + (\alpha + p^2)dy. \end{cases}$$

Then on M

$$d\theta^1 \equiv T dx \wedge dy, \quad \text{mod} \quad \{\theta^0, \theta^1\}$$

where $T = \alpha_y + \alpha_x + 2pb'$. Hence $T = 0$ implies $p = -\frac{1}{2b'}(\alpha_x + \alpha_y)$, which defines a 3-dimensional submanifold M_1 of M. If $dT \equiv 0, \text{mod}\{\theta^0, \theta^1\}$ on M_1 then there is a 1-parameter family of solutions. In fact, if $u(x, y)$ is a solution, so is $u +$ constant. On M_1

$$dT = (\alpha_x + \alpha_y)_x + (\alpha_x + \alpha_y)_y dy + 2b' dp + 2pb^{''} dx,$$

substituting $p = -\frac{\alpha_x + \alpha_y}{2b'}$ and $dp = (b' + \alpha + p^2)dx + (\alpha + p^2)dy$ and setting each of the coefficients to dx and to dy to be zero we obtain

$$\begin{cases} (\alpha_x + \alpha_y)_x + 2(b')^2 + 2b'\alpha + \frac{(\alpha_x + \alpha_y)^2}{2b'} - \frac{b^{''}}{b'}(\alpha_x + \alpha_y) = 0 \\ (a_x + a_y)_y - 2b'\alpha - \frac{(\alpha_x + \alpha_y)^2}{2b'} = 0. \end{cases} \tag{3.14}$$

If the functions α and b satisfy (3.14) there is a 1-parameter family of solutions. Otherwise, no solutions exist.

§4. Applications.

By complete prolongation to a complete system of order k solving (3.1) becomes problem of finding integral manifolds of a Pfaffian system of Frobenius type on a submanifold $M \subset J^{(k-1)}(X,U)$, which is essentially an ODE problem. By the fundamental theorem of ODE a generic overdetermined PDE system (3.1) has the following properties:

i) Existence and finiteness of solutions: A solution is determined by its $(k-1)$th jet at a point, therefore, there are at most finitely many parameter family of solutions. In fact, there are $(\dim M' - p)$-parameter family of solutions. If M' is empty there are no solutions.

ii) Regularity of solutions: If each Δ of (3.1) is C^∞ in its argument then the solutions are C^∞. If Δ is C^ω then the solutions are C^ω.

We present here some applications of the complete prolongation:

Proposition 4.1 (Analyticity of CR mappings). *Let M_1 and M_2 be C^ω real hypersurfaces in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, with $m \le n$. Let $f = (f^1, \cdots, f^n) : M_1 \to M_2$ be a CR mapping which is C^k, for sufficiently large k then f is C^ω.*

A CR mapping f extends to a holomorphic mapping of a neighborhood of M_1 into a neighborhood of M_2 if and only if f is C^ω (cf. [CM]). Let $L_1, \cdots, L_{m-1}$ be the tangential Cauchy-Riemann operators on M_1 and let M_2 be defined as the zero set of a real valued C^ω function $r(z,\bar z)$ with $dr \neq 0$, where $z = (z_1, \cdots, z_n)$, then f is a CR mapping iff

$$\begin{aligned} &L_j f^\lambda = 0, \quad \text{(tangential Cauchy-Riemann equations)} \\ &r(f,\bar f) = 0, \end{aligned} \tag{4.1}$$

for all $j = 1, \cdots, m-1$ and for all $\lambda = 1, \cdots, n$.
D. Burns first observed that the ananyticity of CR mappings can be shown by constructing a complete system of finite order, see [B] and [BS]. When $m = n$ and the Leviforms of M_1 and M_2 are finitely degenerated then (4.1) admits prolongation to a complete system of a finite order that depends on the degeneracy of the Levi forms ([H1]), under a certain additional generic assumption. In particular, if the Levi forms are nondegenrate then (4.1) admits prolongation to a complete system of third order. The cases $m \le n$ have been studied in [H3].

Proposition 4.2 (Infinitesimal rigidity of embeddings [CH2]). *Let* (M^n, g) *be a Riemannian manifold and* $f : M \to \mathbb{R}^{(n+1)}$ *be a smooth mapping. Then*
i) If f *is an isometric embedding then the equations for infinitesimal isometric deformation of* f *generically admit prolongation to a complete system of second order. Therefore, the dimension of the space of infinitesimal deformations* $\rho(f) = \frac{1}{2}(n+1)(n+2)$, *which is the dimension of the euclidean group of* $\mathbb{R}^{n+1}$.
ii) If f *is a conformal embedding then the equations for infinitesimal conformal deformation of* f *generically admit prolongation to a complete system of third order. Therefore, the dimension of the space of infinitesimal deformations* $\rho^c(f) = \frac{1}{2}(n+2)(n+3)$, *which is the dimension of the group of conformal transformations of* $\mathbb{R}^{n+1}$.

Let $v : M \to \mathbb{R}^{n+1}$. v is an infinitesimal deformation of an isometric (conformal, resp.) embedding f iff

$$< df, dv >= 0. \tag{4.2}$$

(4.2) is a system of linear equations for $v = (v^1, \cdots, v^{n+1})$ of first order, which is the linearization of the isometric (conformal, resp.) embedding equations at f. We differentiate (4.2) two times (three times, resp.) and eliminate all the third order(fourth order, resp.) derivatives of v using the symmetry to obtain a complete system of second order(third order, resp.).

Proposition 4.3 (Symmetry algebra for $2n$ vector fields in $\mathbb{R}^{2n+1}$). *Let* M *be a* C^∞ *manifold of dimension* $2n+1, n \geq 1$, *and* $X_j, j = 1, \cdots, 2n$ *be linearly independent* C^∞ *vector fields on* M. *Let* θ *be a* C^∞ 1*-form that annihilates all* X_j*'s. Suppose that* $(d\theta)^n \wedge \theta \neq 0$. *Then the set of vector fields* V *satisfying*

$$[V, X_j] = \lambda_j X_j, \tag{4.3}$$

for some scalar function λ_j, *forms a finite dimensional Lie algebra. In the case* $n = 2$, *the dimension is either* 8 *or* ≤ 4.

V is an infinitesimal symmetry preserving each of the integral curves of $X_j, j = 1, \cdots, 2n$, which is a special case of the multi-contact structure as defined in [CDKR]. In [HO] we constructed a complete system of third order from (4.3). At the conference on CR geometry, Levico, September 2004, A. Koranyi raised the problem of determining the dimension of symmetry algebra of multi-contact structure. After many discussions among M. Eastwood, G. Schmalz, J. Landsberg, K. Yamaguchi, Jongwon Oh and the author it turned out that in the case of two vector fields in $\mathbb{R}^3$ the symmetry is the same as the point symmetry of the second order ODE studied by E. Cartan. More generally, for the second order PDE system of finite type (3.2) for one unknown function of n independent variables the symmetry algebra is $sl(n+2)$ if the associate geometry is flat.

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