SYMMETRIES OF Δ^2

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question of Ernie Kalnins

<u>Notation</u>: work on Euclidean space $\mathbb{R}^n$ with standard metric g_{ab}, write $\nabla_a \equiv \partial/\partial x^a$, and Laplacian $\Delta = g^{ab}\nabla_a\nabla_b = \nabla^a\nabla_a$.

Definitions

- <u>symmetry of Δ^2</u> $\quad \mathcal{D}$ such that $\Delta^2\mathcal{D} = \underbrace{\delta}_{\text{some linear differential operator}}\Delta^2$

- <u>equivalence</u> $\quad \mathcal{D}_1 \equiv \mathcal{D}_2$ means $\mathcal{D}_1 - \mathcal{D}_2 = \underbrace{\mathcal{P}}_{\text{some linear differential operator}}\Delta^2$

- <u>conformal Killing tensor</u> $\quad \nabla^{(a}\underbrace{V}_{\text{symmetric trace-free}}{}^{bcd\cdots e)} = g^{(ab}\underbrace{\phi}_{\text{some tensor}}{}^{cd\cdots e)}$

  equivalently $\nabla^{(a}V^{\overbrace{bcd\cdots e)}^{s \text{ indices}}} = \frac{s}{n+2s-2}\, g^{(ab}\nabla_f V^{cd\cdots e)f}$ $\quad$ (A)

- $\nabla^{(a}\nabla^b\nabla^c \underbrace{W}_{\text{symmetric trace-free}}{}^{d\cdots e)} = g^{(ab}\phi^{cd\cdots e)}$ $\quad$ (B) $\quad$ (generalised conformal Killing tensor (Nikitin, Smirnov, ...))

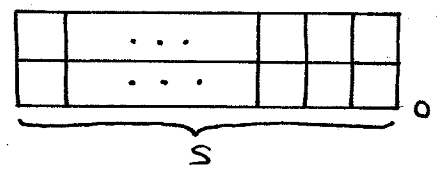
Theorem 1 $\quad \mathcal{D}$ symmetry of $\Delta^2 \Rightarrow \mathcal{D}$ is canonically equivalent to one of the form

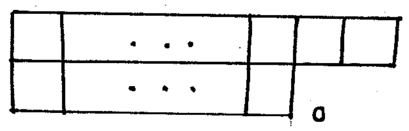
$$\underline{V^{bcd\cdots e}\nabla_b\nabla_c\nabla_d\cdots\nabla_e + W^{d\cdots e}\Delta\nabla_d\cdots\nabla_e + \text{LOTs}}$$

where $V^{bcd\cdots e}$ satisfies (A)

$W^{d\cdots e}$ satisfies (B)

Facts (from BGG-machinery) Equations (A) and (B) are $\mathfrak{so}(n+1,1)$-invariant and

{Sol$^{\underline{n}}$s of (A)} $\cong$ 

{Sol$^{\underline{n}}$s of (B)} $\cong$ 

finite-dimensional irreducible representations of $\mathfrak{so}(n+1,1)$
(cf. Nikitin-Prilipko dimension formulae)

Theorem 2 Suppose $V^{bcd\cdots e}$ and $W^{d\cdots e}$ satisfy (A) and (B). Then there are canonically defined symmetries

$$\mathcal{D}_V = V^{bcd\cdots e}\nabla_b\nabla_c\nabla_d\cdots\nabla_e + \mathrm{LOTs}$$

$$\mathcal{D}_W = W^{d\cdots e}\Delta\nabla_d\cdots\nabla_e + \mathrm{LOTs}$$

Proof: ambient metric! $\square$

Examples 1$^{\text{st}}$ order $\quad \mathcal{D}_V f = V^a\nabla_a f + \frac{n-4}{2n}(\nabla_a V^a)f$

2$^{\text{nd}}$ order $\quad \mathcal{D}_V f = V^{ab}\nabla_a\nabla_b f + \frac{n-2}{n+2}(\nabla_a V^{ab})\nabla_b f + \frac{(n-2)(n-4)}{4(n+1)(n+2)}(\nabla_a\nabla_b V^{ab})f$

$$\mathcal{D}_W f = W\Delta f - (\nabla^a W)\nabla_a f - \frac{n-4}{2(n+2)}(\Delta W)f$$

Composition $\ydiagram{1,1} \ni U, V$ give first order symmetries

$$\boxed{\mathcal{D}_U \mathcal{D}_V = \mathcal{D}_{U \circledcirc V} - \frac{1}{n}\mathcal{D}_{U \bullet V} + \frac{1}{2}\mathcal{D}_{[U,V]} - \frac{1}{4}\frac{(n-2)(n-4)}{n^2(n+1)}\mathcal{D}_{\langle U,V\rangle}}$$

$U \circledcirc V \in \ydiagram{2,2}_{\,\circ}$ is Cartan Product

$U \bullet V = \frac{1}{2}(UV + VU) - \text{trace}$ as matrices $\in \ydiagram{2}_{\,\circ}$

$[U,V] = UV - VU$ as matrices $\in \ydiagram{1,1}$ is Lie Bracket

$\langle U,V\rangle = n\,\text{trace}\, UV$ as matrices $\in \mathbb{R}$ is Killing Form

Corollary The symmetry algebra of Δ^2 is isomorphic to

$$\boxed{\bigotimes \ydiagram{1,1} \Big/ \left\langle U \otimes V - U \circledcirc V + \frac{1}{n} U \bullet V - \frac{1}{2}[U,V] + \frac{1}{4}\frac{(n-2)(n-4)}{n^2(n+1)}\langle U,V\rangle \right\rangle}$$

$\left(\ydiagram{1,1} \otimes \ydiagram{1,1} = \ydiagram{2,2}_{\,\circ} \oplus \ydiagram{2}_{\,\circ} \oplus \mathbb{R} \oplus \ydiagram{1,1} \oplus \ydiagram{2,1,1}_{\,\circ} \oplus \ydiagram{1,1,1,1} \text{ under } so(n+1,1)\right)$

Kostant $\Rightarrow$ there are no further relations

Cf. Identities of Dan Fox