OPTIMIZATION AND ROBUST POWER GRIDS

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Large-scale blackouts

- August 2003: North America. 50 million people affected during two days; New York City loses power

- September 2003: Switzerland-France-Italy. 57 million people affected during one day; Italy loses power

- 1999 Brazil: 75 million people affected during four hours

Why?

Too much consumption? **No.** Inadequate generation? **No.**

→ The transmission network failed
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**U.S.-Canada joint commission:** The leading cause of the blackout was “inadequate system understanding”
A power grid has 3 components:

- GENERATION
- TRANSMISSION
- DISTRIBUTION
AC power flows

• Voltage at a node \( k \) is of the form \( U_k e^{j \theta_k} \), where \( j = \sqrt{-1} \)

• Power flowing on edge \( \{k, m\} \) equals \( p_{km} + j q_{km} \), where

\[
(A)\quad p_{km} = U_k^2 g_{km} - U_k U_m g_{km} \cos \theta_{km} - U_k U_m b_{km} \sin \theta_{km}
\]

\[
(B)\quad q_{km} = -U_k^2 (b_{km} + b_{km}^{sh}) + U_k U_m b_{km} \cos \theta_{km} - U_k U_m g_{km} \sin \theta_{km}
\]

Here, \( \theta_{km} \doteq \theta_k - \theta_m \)

\( g_{km}, \; b_{km}, \; b_{km}^{sh} \) are known parameters (series conductance, series reactance, shunt susceptance)

• Net power injected at node \( k \) equals \( P_k + j Q_k \), where

\[
(C)\quad P_k = \Sigma_{\{k,m\}} p_{km} \quad \text{(active power)}
\]

\[
(D)\quad Q_k = \Sigma_{\{k,m\}} q_{km} \quad \text{(reactive power)}
\]

• Node \( k \) is a generator \( \Rightarrow P_k > 0 \); if \( k \) is a load \( \Rightarrow P_k < 0 \). Similarly with the reactive powers.

• **Power flow problem:** given known loads, find values for all the variables \( U_k \) and \( \theta_k \) so that equations \( (A) - (D) \) hold

Boundary condition: the angles \( \theta_k \) can be fixed at some nodes
Linearization – DC power flows

Power flowing on edge \( \{k, m\} \) equals \( p_{km} + jq_{km} \), where

\[
(A) \quad p_{km} = U_k^2 g_{km} - U_k U_m g_{km} \cos \theta_{km} - U_k U_m b_{km} \sin \theta_{km}, \quad (\theta_{km} = \theta_k - \theta_m)
\]

Approximation:

- \( \theta_{km} \) is small, and hence \( \sin \theta_{km} \approx \theta_{km} \), \( \cos \theta_{km} \approx 1 \).

- \( U_k \approx 1 \), for each node \( k \).

\[
(A') \quad x_{km} p_{km} - \theta_k + \theta_m = 0 \quad (x_{km} = 1/b_{km})
\]

As before, for any node \( k \),

\[
(C') \quad \Sigma_{\{k,m\}} p_{km} = \text{net supply of power at } k \quad \text{(power) flow conservation}
\]

Fact: Suppose the network is connected, and supply = demand

then there is a unique set of power flows \( p_{km} \) satisfying system \( (A')-(C') \)
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\]

**Fact:** Suppose the network is connected, and supply = demand

then there is a unique set of power flows \( p_{km} \) satisfying system \((A')-(C')\)

But ... we also need \( |p_{km}| \leq u_{km} \) for every edge \( \{k, m\} \)
How a cascade occurs

- An initial fault (= edge deletion) or set of faults occurs
- We now have a new network (the residual network)
- In the new network, some of the power flows may exceed the edge “capacities”
- This may trigger a new set of faults, or equipment shut-downs
What can be done

• **Static model.** Model the possible initial faults as a set of potential contingencies, or scenarios. Example: any single edge could fail, or any pair of edges. Then we want to invest in a network so as to make it resilient under any scenario.

• **Online model.** What should be done when a cascade is developing? Both centralized and distributed models are appropriate.
Given a network, and a list of fault scenarios,

- **Add a minimum cost set of edges**

- So that in **every** fault scenario, **all** of the resulting power flows are within tolerance:

\[ |p_{km}| \leq u_{km} \] for each scenario \( s \), and every \( \{k, m\} \) not faulty in \( s \)

In other words, no cascade ever occurs

**Is this a reasonable model?**
Given a network, and a list of fault scenarios,

- **Add a minimum cost set of edges**

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for each scenario \( s \), and every \( \{k, m\} \) not faulty in \( s \)

In other words, no cascade ever occurs

**Is this a reasonable model?**

- It is very conservative, but within scope of traditional integer programming
First model

Given a network, and a list of fault scenarios,

• **Add a minimum cost set of edges**

• So that in **every** fault scenario, **all** of the resulting power flows are within tolerance:

\[
|p_{km}| \leq u_{km} \text{ for each scenario } s, \text{ and every } \{k, m\} \text{ not faulty in } s
\]

In other words, no cascade ever occurs

**Is this a reasonable model?**

→ real-life constraint: very difficult to add anything but parallel edges
Previous work

Fito, Pilotto, Martins, Carvalho, Bianco (2000),

Oliveira, Binato, Bahiense, Thomé, Pereira (2001)

Approach used:

1. Formulate the problem as a single mixed-integer model (big M)

2. Solve the model using commercial MIP solver.

Impractical unless the network is not large and the number of scenarios is small
New formulation

• 0/1 variable: for an edge $\{k, m\}$,

$$y_{km} = 1$$ if we add a parallel edge to $\{k, m\}$, 0 otherwise

• continuous variable: for an edge $\{k, m\}$,

$$\hat{p}_{km} = \text{power flow on parallel edge.}$$

Constraints:

$$|\hat{p}_{km}| \leq u_{km}y_{km}$$

$$|p_{km} - \hat{p}_{km}| \leq u_{km}(1 - y_{km}) \quad \rightarrow \text{not a big-M formulation}$$

Plus: for every fault scenario, flow conservation and voltage-laws on the residual network

$$\Sigma_{\{k,m\}} P_{km}^s + \hat{p}_{km}^s = P_k,$$

$$x_{km}P_{km}^s - \theta_k^s + \theta_m^s = 0$$

$$x_{km}\hat{p}_{km}^s - \theta_k^s + \theta_m^s = 0$$

($s = \text{scenario}$)
Projection approach

Motivated by previous work on network loading problem (Alevras, Grötschel, Stoer; Chopra, Tsai; Barahona; Avella, Mattia, Sassano)

**Key idea:** only work with $y$ variables; always maintain a “working formulation” $Ay \geq b$ of valid inequalities

### Algorithm:

**Step 1.** Let $y^*$ be an optimal solution to $\{ \min c^Ty : Ay \geq b, \ y \text{ binary} \}$

**Step 2.** If there is a (flow, angle) vector $(p^*, \hat{p}^*, \theta^*)$, such that $(y^*, p^*, \hat{p}^*, \theta^*)$ are feasible for the complete problem, **STOP**.

**Step 3.** Else, find a valid inequality that is violated by $y^*$, add it to $Ay \geq b$, and go to **Step 1**.

### Better Algorithm:

**Step 1.** Let $y^*$ be an optimal solution to $\{ \min c^Ty : Ay \geq b, \ 0 \leq y \leq 1 \}$

**Step 2.** If we can find a valid inequality that is violated by $y^*$, add it to $Ay \geq b$ and go to **Step 1**.

**Step 3.** Otherwise, if $y^*$ is binary **STOP** – we have solved the problem.

**Step 4.** Otherwise, *branch* on some fractional variable $y_j^*$.

→ How do we implement **Step 2**?
Farkas’ Lemma

\( y^* \) is infeasible for scenario \( s \) if:

\[
1 > \max \alpha \\
\text{s.t.}
\]

\[
\sum_{\{k,m\}} p^s_{km} + \hat{p}^s_{km} = \alpha P_k,
\]

\[
x_{km} p^s_{km} - \theta^s_k + \theta^s_m = 0
\]

\[
x_{km} \hat{p}^s_{km} - \theta^s_k + \theta^s_m = 0
\]

\[
|\hat{p}^s_{km}| \leq u_{km} y^*_{km}
\]

\[
|p^s_{km} - \hat{p}^s_{km}| \leq u_{km}(1 - y^*_{km})
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\( y^* \) is infeasible for scenario \( s \) if:

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|p^s_{km} - \hat{p}^s_{km}| \leq u_{km}(1 - y^*_{km})
\]

If so, duals give a violated inequality

\[
\Sigma_{km} \pi_{km} y_{km} \geq \beta
\]
Farkas’ Lemma

$y^*$ is infeasible for scenario $s$ if:

$$1 > \max \alpha$$

s.t.

$$\sum_{k,m} p_{km}^s + \hat{p}_{km}^s = \alpha P_k,$$

$$x_{km}p_{km}^s - \theta_k^s + \theta_m^s = 0$$

$$x_{km}\hat{p}_{km}^s - \theta_k^s + \theta_m^s = 0$$

$$|\hat{p}_{km}^s| \leq u_{km}y_{km}^*$$

$$|p_{km}^s - \hat{p}_{km}^s| \leq u_{km}(1 - y_{km}^*)$$

Round up:

$$\sum_{km} [\pi_{km}] y_{km} \geq [\beta]$$
Balas; Balas, Ceria and Cornuéjols

( → drop the “s” from the notation)

Suppose \( 0 < y_{ij}^* < 1 \) for some particular edge \( \{i, j\} \)

→ Are there vectors \((y^0, p^0, \hat{p}^0, \theta^0)\) and \((y^1, p^1, \hat{p}^1, \theta^1)\) such that

(a) both vectors are feasible for:

\[
\begin{align*}
\sum_{k,m} p_{km} + \hat{p}_{km}^s &= P_k \\
x_{km} p_{km} - \theta_k^s + \theta_m^s &= 0 \\
x_{km} \hat{p}_{km}^s - \theta_k^s + \theta_m^s &= 0 \\
|\hat{p}_{km}^s| &\leq u_{km} y_{km} \\
|p_{km}^s - \hat{p}_{km}^s| &\leq u_{km} (1 - y_{km})
\end{align*}
\]

(b) \( y_{ij}^0 = 0, \ y_{ij}^1 = 1 \), and \( y^* = \lambda y^1 + (1 - \lambda) y^0 \) (necessarily \( \lambda = y_{ij}^* \))

If not, Farkas’ lemma again gives a violated inequality

\[
\sum_{km} \pi_{km} y_{km} \geq \beta
\]

(and again round coeffs.)

System is larger, but we get stronger inequalities
Combinatorial inequalities

• We would expect both cutting schemes to “discover” combinatorial inequalities.

• However, that is not the case.

• That is not surprising, because the polyhedron may not have *any* combinatorial facets.
Combinatorial inequalities

• We would expect both cutting schemes to “discover” combinatorial inequalities

• However, that is not the case

• That is not surprising, because the polyhedron may not have any combinatorial facets

→ Use heuristics to separate cover inequalities $\sum_{e \in C} y_e \geq 1$, and

“circulant systems”: $\sum_{e \in C \setminus f} y_e \geq 1, \ \forall f \in C \ \Rightarrow \sum_{e \in C} y_e \geq 2$
Numerical stability – back to Farkas’ lemma

$y^*$ is infeasible for scenario $s$ if:

1. $1 > \max \alpha$

s.t.

\[ \sum_{k,m} p_{km}^s + \hat{p}_{km}^s = \alpha P_k, \]
\[ x_{km} p_{km}^s - \theta_k^s + \theta_m^s = 0 \]
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Numerical stability – back to Farkas’ lemma

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s.t.

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$$x_{km} p_{km}^s - \theta_k^s + \theta_m^s = 0$$

$$x_{km} \hat{p}_{km}^s - \theta_k^s + \theta_m^s = 0$$

$$|\hat{p}_{km}^s| \leq u_{km} y_{km}^* \rightarrow \text{dual } \gamma \text{ corrected to } \gamma(1 - \epsilon), \quad \epsilon = 10^{-6}$$

$$|p_{km}^s - \hat{p}_{km}^s| \leq u_{km} (1 - y_{km}^*) \rightarrow \text{dual } \gamma \text{ corrected to } \gamma(1 + \epsilon)$$

Use corrected duals to generated a (possibly) violated inequality

$$\Sigma_{km} \pi_{km} y_{km} \geq \beta$$
Sample computational results

- network data derived from one of IEEE test cases

- recent workstation; Cplex 9

<table>
<thead>
<tr>
<th>nodes</th>
<th>edges</th>
<th>scenarios</th>
<th>time (sec.)</th>
<th>cuts</th>
<th>B &amp; B nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>409</td>
<td>28</td>
<td>7962</td>
<td>11619</td>
<td>275</td>
</tr>
<tr>
<td>300</td>
<td>409</td>
<td>29</td>
<td>23492</td>
<td>20143</td>
<td>866</td>
</tr>
<tr>
<td>300</td>
<td>409</td>
<td>37</td>
<td>11511</td>
<td>10651</td>
<td>133</td>
</tr>
<tr>
<td>300</td>
<td>409</td>
<td>32</td>
<td>67193</td>
<td>36523</td>
<td>811</td>
</tr>
</tbody>
</table>
Second model

• In the first model, we were strengthening a network so that in the event of a fault, the residual network will operate without any overloads, so cascading is stopped immediately

• But in real-life, a small or slow cascade or a small blackout are tolerable

• We need a model of a developing cascade that better accounts for the dynamics

• Fact: an overloaded line does not fail immediately
Cascade model

Adapted from Carreras, Lynch, Sachtjen, Dobson and Newman (2001).

0. Initial fault contingency occurs – network is modified.

1. For $k = 1, 2, \ldots$, perform Round $k$:

(a) Compute power flows in current residual network

(b) Compute, and remove from network, those lines taken out of service now

→ How does the process stop?
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**Definition:** The network **fails** if at some point less than a given fraction \( \alpha \) of the demand can be served. Example: \( \alpha = 90\% \).
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**Definition:** The network **fails** if at some point less than a given fraction \( \alpha \) of the demand can be served. Example: \( \alpha = 90\% \).

**Definition:** If, after a given number \( N \) of rounds the network has not failed, then it **survives**. Example: \( N = 50 \).
Example

<table>
<thead>
<tr>
<th>Round</th>
<th>Faults</th>
<th>Connected components</th>
<th>Demand served (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
<td>100.0</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>6</td>
<td>95.53</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td>8</td>
<td>94.20</td>
</tr>
<tr>
<td>4</td>
<td>34</td>
<td>16</td>
<td>78.81</td>
</tr>
</tbody>
</table>
What can be done:

• This talk: **passive model.** We want to strengthen a network, at minimum cost, so as to make it survivable with respect to a set of initial fault contingencies

• i.e., the network should be able to “ride out” each potential cascade we consider, with no action taken online

• Example: for each \{i, j\}, if we pay a cost \( c_{ij} \), we increase its capacity from the original amount \( u_{ij} \) to a higher value \( \hat{u}_{ij} \).
Prototype Algorithm

Benders’-like approach; maintain a “working formulation” \( Ay \geq b \) of valid inequalities

**Step 1.** Let \( y^* \) be an optimal solution to \( \{ \min c^T y : Ay \geq b, \ y \text{ binary} \} \)

**Step 2.** Strengthened network as per \( y^* \)

**Step 3.** Evaluate \( y^* \): simulate cascade.

**Step 4.** If network fails, then find a valid inequality that is violated by \( y^* \).

Add the inequality to \( Ay \geq b \) and go to **Step 1.**

**Step 5.** Otherwise, \( y^* \) is optimal. Stop.

Step 4?
Example:

Suppose we want to satisfy a fraction $\alpha = 0.65$ of the demand

Round 1: edges 0, 1, 2, 3, 4 fail; 5 components

We satisfy 80% of the demand, so not a failure
Example:

Suppose we want to satisfy a fraction $\alpha = .65$ of the demand

**Round 2:** edges $5, 6, 7, 8, 9, 10$ fail; 10 components

We satisfy $61\%$ of the demand, so **failure**!!
Example:

Suppose we want to satisfy a fraction $\alpha = .65$ of the demand

**Round 2**: edges $5, 6, 7, 8, 9, 10$ fail; 10 components

Assuming round 1 has occurred, $y_5 + y_6 + y_7 \geq 1$ is valid
Recap

Suppose we want to satisfy a fraction \( \alpha = .65 \) of the demand

**Round 1:** edges: 0, 1, 2, 3, 4 fail; 5 components

**Round 2:** edges: 5, 6, 7, 8, 9, 10 fail; 10 components

Valid inequality: \( y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 \geq 1 \)
More

• In general, need to add terms of the form \((1 - y_j)\) to make the inequality valid

  ("Braess’ paradox")

• Can be tightened

• When does it define a face?
Computational experience, so far

• 300 nodes, 409 edges, 49 generator nodes, 172 demand nodes
  (≈ 2 copies of IEEE test network)

• Several hundred initial fault scenarios

• A few hours running time to prove optimality in the most difficult cases;

• Several thousand iterations

• Integer programs not hard – evaluation step is the expensive part

<table>
<thead>
<tr>
<th>Iteration</th>
<th>1 118 122 159 700 1919</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gap (%)</td>
<td>+∞ 73 55 33 17 0</td>
</tr>
</tbody>
</table>