Conflict Analysis in Mixed Integer Programming
Main Idea

- Problem is divided into smaller subproblems
  - branching tree
- Some subproblems are infeasible
- Analyzing the infeasibilities can yield information about the problem
- Information can be used at other subproblems to prune the search tree
Example

\[ x_1 = 1 \]
\[ x_2 = 0 \]
\[ x_3 = 0 \]

\[ \begin{align*}
  c_1 &: x_1 - x_2 - x_3 \leq 0 \\
  c_2 &: x_1 + x_2 - x_3 \leq 1
\end{align*} \]

- a sub problem is infeasible
Example

- A sub problem is infeasible
- Conflict analysis yields a feasible constraint, that cuts off other nodes in the tree

\[
\begin{align*}
    c_1: & \quad x_1 - x_2 - x_3 \leq 0 \\
    c_2: & \quad x_1 + x_2 - x_3 \leq 1 \\
    c_3: & \quad x_1 - x_3 \leq 0
\end{align*}
\]
Outline

- Conflict Analysis in SAT
- Conflict Analysis in MIP
- Computational Results
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Satisfiability Problem (SAT)

- Boolean variables \( x_1, \ldots, x_n \in \{0,1\} \)
- Clauses
  \[
  C_1: \quad l_{11} \lor \cdots \lor l_{1k_1}
  \]
  \[
  \vdots
  \]
  \[
  C_m: \quad l_{n1} \lor \cdots \lor l_{mk_m}
  \]
  with literals
  \[
  l_{ik} = x_j \text{ or } \overline{l_{ik}} = x_j = 1 - x_j
  \]
- Task:
  \[
  \text{find assignment } x^* \in \{0,1\}^n \text{ that satisfies all clauses, or prove that no such assignment exists}
  \]
Binary Constraint Propagation

- clause: $x_6 \lor \overline{x}_7 \lor x_9 \lor x_{10}$
Binary Constraint Propagation

- clause: $x_6 \lor \overline{x}_7 \lor x_9 \lor x_{10}$
- fixings: $x_6 = 0, \overline{x}_7 = 1, x_9 = 0$
Binary Constraint Propagation

- **clause:** $x_6 \lor \overline{x}_7 \lor x_9 \lor x_{10}$
- **fixings:** $x_6 = 0, x_7 = 1, x_9 = 0$
- **deduction:** $\overline{x}_6 \land x_7 \land \overline{x}_9 \rightarrow x_{10}$
Conflict Graph

- decision variables
- deduced variables
- conflict
Conflict Graph

- decision variables
- deduced variables
- conflict

\[ \chi_{13} \lor \overline{\chi}_{14} \lor \overline{\chi}_{15} \]
choose cut that separates decision variables from conflict vertex
Conflicl Graph – Conflict Cuts

- choose cut that separates decision variables from conflict vertex
Conflict Graph - Conflict Cuts

- choose cut that separates decision variables from conflict vertex
- conflict clause: $x_6 \lor \overline{x}_7 \lor x_8 \lor x_{12}$
Conflict Graph - Trivial Cuts

- $\lambda$-cut: $x_{13} \lor \overline{x}_{14} \lor \overline{x}_{15}$
- decision cut: $x_1 \lor \overline{x}_4 \lor x_6 \lor x_8$
- **Unique Implication Point**: lies on all paths from the last decision vertex to the conflict vertex
- **Unique Implication Point**: lies on all paths from the last decision vertex to the conflict vertex
- **First UIP**: the UIP closest to $\lambda$ (except $\lambda$ itself)
\[ x_3 \vee \overline{x}_{11} \]

- Put all vertices fixed after First-UIP to the conflict side, remaining vertices to the reason side

- First-UIP cut: \[ x_3 \vee \overline{x}_{11} \]
Conflict Analysis Algorithm (FUIP)

- BCP detected a conflict
Initialize conflict queue with the variables involved in the conflict
Conflict Analysis Algorithm (FUIP)

- Initialize conflict queue with the variables involved in the conflict
As long as there is more than one variable of the last depth level in the queue, resolve the last deduction.
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Conflict Analysis Algorithm (FUIP)

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If there is only one variable of the last depth level left, stop
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The remaining variables define the conflict set
The conflict clause consists of all (negated) assignments in the conflict set: $x_3 \lor \overline{x}_{11}$
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  - deductions lead to conflict $\rightarrow$ conflict graph
  - cut in conflict graph $\rightarrow$ conflict clause

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Mixed Integer Program

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \in \mathbb{R}^p \times \mathbb{Z}^q
\end{align*}
\]

- linear objective function \( c \)
- linear constraints \( Ax \leq b \)
- real or integer valued variables \( x \)
Conflict Analysis for MI P

Two main differences to SAT:

- non-binary variables
  - conflict graph: bound changes instead of fixings
  - conflict clause $\rightarrow$ conflict constraint
Conflict Analysis for MIP

Two main differences to SAT:

- non-binary variables
  - conflict graph: changes instead of fixings
  - conflict clause $\rightarrow$ conflict constraint
  - technical issue
Conflict Analysis for MIP

Two main differences to SAT:

- non-binary variables
  - conflict graph: changes instead of fixings
  - conflict clause $\rightarrow$ conflict constraint

- main reason for infeasibility: LP relaxation
  - conflict graph has no link from the decision and deduction vertices to the conflict vertex
Conflict Analysis for MIP

Two main differences to SAT:

- non-binary variables
  - conflict graph: changes instead of fixings
  - conflict clause $\rightarrow$ conflict constraint

- main reason for infeasibility: LP relaxation
  - conflict graph has no link from the decision and deduction vertices to the conflict vertex

analyze LP
Back to SAT: Conflict Graph

- One clause detected the conflict
- Only a few variables linked to the conflict vertex

\[ x_{13} \lor \overline{x}_{14} \lor \overline{x}_{15} \]
Infeasible LP: Conflict Graph

- The LP as a whole is responsible for the conflict
- All local bound changes are linked to the conflict vertex
Infeasible LP: Conflict Graph

- LP analysis selects some of these local bounds
Infeasible LP: Conflict Graph

- LP analysis selects some of these bounds
- cut yields conflict constraint

\[(x_3 = 1) \lor (x_5 = 1) \lor (x_{11} \leq 3)\]
Conflict Analysis for infeasible LPs

- if the LP relaxation is infeasible, the whole relaxation is involved in the conflict
  - all constraints
  - all global bounds
  - all local bounds

- try to find a small subset of the local bounds that still leads to an infeasible LP relaxation
  - variant of minimal infeasible subsystem problem (see Amaldi, Pfetsch, Trotter)
  - heuristic: use dual ray to relax local bounds
Infeasible LP: Dual Ray Heuristic

- LP relaxation: \( \max \ c^T x \)
  s.t. \( Ax \leq b \)
  \( 0 \leq x \leq \mu \leq u \)

local bounds
Infeasible LP: Dual Ray Heuristic

- LP relaxation: \[ \max \ c^T x \]
  \[ \text{s.t.} \quad A x \leq b \]
  \[ 0 \leq x \leq \mu \leq u \]

- dual LP: \[ \min \ b^T y + \mu^T r \]
  \[ \text{s.t.} \quad A^T y + r \geq c \]
  \[ y, \ r \geq 0 \]
Infeasible LP: Dual Ray Heuristic

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  \[ \text{s.t.} \quad Ax \leq b \]
  \[ 0 \leq x \leq \mu \leq u \]

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  \[ \text{s.t.} \quad A^T y + r \geq c \]
  \[ y, r \geq 0 \]

- Dual ray: \((y^*, r^*) \geq 0, A^T y^* + r^* = 0, b^T y^* + \mu^T r^* < 0\)
Infeasible LP: Dual Ray Heuristic

- LP relaxation: \( \max c^T x \)
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  \[ \text{s.t.} \quad Ax \leq b \]
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  \[ y, r \geq 0 \]

- dual ray: \[ (y^*, r^*) \geq 0, A^T y^* + r^* = 0, b^T y^* + \mu^T r^* < 0 \]

- relax bounds: \[ \mu_i := u_i, \text{ s.t. still } b^T y^* + \mu^T r^* < 0 \]
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- Conflict Analysis in MIP
  - deductions lead to infeasible LP
  - LP analysis links conflict vertex
  - cut in conflict graph $\rightarrow$ conflict constraint

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    - LP analysis links conflict vertex
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- Computational Results
## Computational Results

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<tr>
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<th>Infeas. ALU</th>
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# Computational Results

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## Computational Results

### ALU Prop #7

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# Computational Results (new)

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<td>Nodes</td>
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<td>Node Winner</td>
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