①

ADDITIVITY

There are a number of equivalent additivity questions in quantum information theory

We let $S(\rho) = -\mathrm{Tr}\,\rho \log \rho$ be the entropy of a positive trace 1 operator

~~Question A~~

Question a)

Is minimum entropy output additive?

Let Φ be a quantum channel

$$\Phi(\rho) = \sum_k A_k \rho A_k^\dagger, \text{ with } \sum_k A_k^\dagger A_k = I$$

$$S_{min}(\Phi) = \min_{|v\rangle\langle v|} S(\Phi(|v\rangle\langle v|))$$

Is $S_{min}(\Phi_1) + S_{min}(\Phi_2) = S_{min}(\Phi_1 \otimes \Phi_2)$

$\geq$: easy

②

b) Is entanglement of formation additive?

If $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a density matrix on a tensor product space, ~~and~~ then

$$E_F(\rho_{AB}) = \min \sum_i p_i \, S(\mathrm{Tr}_A |v_i\rangle\langle v_i|)$$

where $\rho_{AB} = \sum_i p_i |v_i\rangle\langle v_i|$

Is $E_F(\rho_{AB} \otimes \sigma_{AB}) = E_F(\rho_{AB}) + E_F(\sigma_{AB})$

$\leq$ is easy

Recall $S(\mathrm{Tr}_A |v_i\rangle\langle v_i|)$ is the amount of entanglement in a pure state $|v_i\rangle\langle v_i|$.

③

c) minimum entanglement state in a subspace.

Let Π be a projection matrix in a tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$

$$E_{min}(\Pi) = \min_{|v\rangle\langle v| \leq \Pi} S(Tr_B(|v\rangle\langle v|))$$

Is $E_{min}(\Pi_1 \otimes \Pi_2) = E_{min}(\Pi_1) + E_{min}(\Pi_2)$

$\leq$ easy

d) Strong superadditivity of entanglement of formation.

Additivity

$$E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2)$$

$$\rho_1 \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}$$
$$\rho_2 \in \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2}$$

Generalization

For $\rho \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$

Is $E_F(\rho) \geq E_F(Tr_2\, \rho) + E_F(Tr_1\, \rho)$

For E_F, trace over $A_1 \otimes A_2$, A_1 and A_2 respectively.

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So multiplicativity of ν_p

$\Rightarrow$ additivity of min entropy.

Unfortunately, there is a counterexample

ν_p not multiplicative if $p \geq 5$

In fact, looking at the counterexample for $p \geq 3$, there is a local maximum of ν_p at an entangled input state.

For $1 \leq p \leq 2$, everything looks very nice in counterexample

All we need for additivity of entropy is ν_p multiplicative for $1 < p \leq 1+\epsilon$.

Christopher King
For Δ a depolarizing channel
$\forall$ $\nu_p(\Delta \otimes \Phi) = \nu_p(\Delta)\, \nu_p(\Phi)$
(Φ is arbitrary)

$$\Delta(\rho) = \lambda\rho + (1-\lambda)\, I/d$$

Also have χ additive in same case

Entanglement breaking channels

Suppose we have three Hilbert space

$\mathcal{H}_{ref}$

$\mathcal{H}_{in} \xrightarrow{\Phi} \mathcal{H}_{out}$

$|v\rangle\langle v|$ in $\mathcal{H}_{in} \otimes \mathcal{H}_{ref}$
possibly entangled

$\rho = (\Phi \otimes I)(|v\rangle\langle v|)$ in $\mathcal{H}_{out} \otimes \mathcal{H}_{ref}$
If ρ is always separable (i.e., not entangled),

then Φ is an entanglement breaking channel.

For entanglement breaking channels

S_{min}, χ are additive
ν_p is multiplicative.

I.e., Φ entanglement breaking, Ψ arbitrary

$\nu_p(\Phi \otimes \Psi) = \nu_p(\Phi)\nu_p(\Psi) \quad \chi(\Phi \otimes \Psi) = \chi(\Phi) + \chi(\Psi)$

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Proof of additivity for e.b.c, for S_{min}

This uses strong subadditivity of entropy

Take $\Psi \otimes \Phi$, Ψ arbitrary
Φ entanglement breaking

$I \otimes \Phi(\rho_{AB})$ is separable

So $I \otimes \Phi(\rho_{AB}) = \sum_j q_j |a_j\rangle\langle a_j| \otimes |b_j\rangle\langle b_j|$

$\sigma_{AB} = \Psi \otimes \Phi(\rho_{AB}) = \sum_j q_j \Psi(|a_j\rangle\langle a_j|) \otimes |b_j\rangle\langle b_j|$

system A   system B

Look at state

$\sigma_{ABC} = \sum_j q_j \Psi(|a_j\rangle\langle a_j|) \otimes |b_j\rangle\langle b_j| \otimes |j\rangle\langle j|$

system C

We use strong subadditivity on σ_{ABC} to show there is a ~~tensor product~~ output state for Φ and Ψ whose combined entropy at most $S(\sigma_{AB})$

(a) $S(\sigma_{AB}) \geq S(\sigma_{ABC}) - S(\sigma_{BC}) + S(\sigma_B)$

But $\sigma_B = \sum q_i |b_i\rangle\langle b_i|$

$= Tr_A \sum q_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|$

$= Tr_A (I \otimes \Phi(\rho_{AB}))$

$= \Phi(Tr_A \rho_{AB})$

So $S(\sigma_B) \geq S_{min}(\Phi)$

Need to show
$S(\sigma_{ABC}) - S(\sigma_{BC}) \geq S_{min}(\Psi)$

$\sigma_{ABC} = \sum q_j \Psi(|a_j\rangle\langle a_j|) \otimes |b_j\rangle\langle b_j| \otimes |j\rangle\langle j|$

$S(\sigma_{ABC}) = S(\sigma_{AC})$ since $|b_j\rangle$ is uniquely determined by $|j\rangle$

Similarly $S(\sigma_{BC}) = S(\sigma_C)$

$S(\sigma_{AC}) = \sum q_i S(\Psi |a_j\rangle\langle a_j|) + \underbrace{\sum_i q_i \log q_i}_{= S(\sigma_C)}$

So $S(\sigma_{AC}) - S(\sigma_C) \geq S_{min}(\Psi)$

QED.

The key ingredient in the proof of multiplicativity for the depolarizing channel is the Lieb-Thirring inequality

$$\mathrm{Tr}\left[\left(A^{\frac{1}{2}}BA^{\frac{1}{2}}\right)^p\right] \leq \mathrm{Tr}\left(A^{p/2}B^pA^{p/2}\right) = \mathrm{Tr}\left(A^pB^p\right)$$
for $p \geq 1$

Recall the depolarizing channel

$$\Delta_\lambda(\rho) = \lambda\rho + (1-\lambda)\, I/d$$

We also have the dephasing channel

Suppose $E_i = |e_i\rangle\langle e_i|$, $|e_i\rangle$ orthonormal basis

Dephasing channel Multiplies off-diag-terms by λ.

$$\Phi_\lambda(\rho) = \lambda\rho + (1-\lambda)\sum_{i=1}^{d} E_i\rho E_i$$

We can express the depolarizing channel Δ_λ as a convex combination of dephasing channels

In addition, if our channel is acting on a ρ with eigenvectors $|v_j\rangle$, we can express Δ_λ as a convex combination of dephasing channels, each of which has the E_i complementary to $|v_j\rangle$.

That is, $\langle v_j | E_i | v_j \rangle = \frac{1}{d}$

This will let us work on the dephasing channel.

Lemma. For a phase damping channel,

$$\|(\Phi_\lambda \otimes I)\rho_{12}\|_p \le d^{(1-1/p)} \nu_p(\Delta_\lambda) \left[\sum_{i=1}^{d} \mathrm{Tr}(\rho_2^{i})^p\right]^{\frac{1}{p}}$$

$$\rho_2^{(i)} = \mathrm{Tr}_1 [(E_i \otimes I)\rho_{12}]$$

Lieb Thirring inequality

$$\mathrm{Tr}((A^{1/2} B A^{1/2})^p) \le \mathrm{Tr}(A^p B^p)$$

Clever step: Factor $(\Phi_\lambda \otimes I)\rho_{12}$ into $A^{1/2} B A^{1/2}$ so this inequality can be applied.

Use $\rho_{12} = (I \otimes \Psi)(\sigma_{12})$ to get

~~addi~~ multiplicativity

E_F and χ

Stinespring
Any completely positive trace-preserving Φ can be realized

$$\Phi : \rho \to U(\rho \otimes |0\rangle\langle 0|)U^\dagger$$
$$\downarrow$$
$$Tr_2 (\rho \otimes |0\rangle\langle 0|)U^\dagger$$

Let us call $U(\rho \otimes |0\rangle\langle 0| U^\dagger$
$\mathcal{U}(\rho)$

Φ realized as an
1) embedding in a larger Hilbert space
2) unitary
3) partial trace

$\mathcal{U}$ is steps 1 and 2

$$S(\Phi(|\psi\rangle\langle\psi|))$$
$$= S(Tr_2\, \mathcal{U}(|\psi\rangle\langle\psi|))$$
$$= E_P(\mathcal{U}(|\psi\rangle\langle\psi|)$$

pure state entanglement

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entanglement of formation

$$E_F(\mathcal{U}(\rho)) = \min_{\rho=\sum p_i\rho_i} \sum p_i E_P(\mathcal{U}(\rho_i))$$

$$= \min_{\rho=\sum p_i\rho_i} \sum p_i S(Tr_2\, \mathcal{U}(\rho_i))$$

$$= \min_{\rho=\sum p_i\rho_i} \sum p_i S(\Phi(\rho_i))$$

where ρ_i are pure states.

This is the second term in the Holevo capacity.

This shows E_F is equivalent to constrained Holevo capacity, where we maximize over states with fixed $\rho = \sum p_i\rho_i$

To go from constrained to unconstrained Holevo capacity, we use technique called Channel Extension

Equivalence of additivity questions

3 ingredients in proofs

Linear programming
Stinespring dilation theorem
Channel extension.

Linear programming
Need to compute

$$\min_{\rho = \sum_i p_i |v_i\rangle\langle v_i|} \sum p_i S(\Phi(|v_i\rangle\langle v_i|))$$

Assume all $|v_i\rangle\langle v_i|$, $S(\Phi(|v_i\rangle\langle v_i|))$ precomputed in big table

$$\min_{p_i \geq 0} \sum p_i S(\Phi(|v_i\rangle\langle v_i|))$$
subject to
$$\sum_i p_i |v_i\rangle\langle v_i| = \rho$$

Dual LP

$$\max_{\tau} \operatorname{Tr} \tau \rho$$

subject to $\langle v|\tau|v\rangle \leq S(\Phi(|v\rangle\langle v|))$ $\forall |v\rangle$

τ is maximized over Hermitian matrices.

Dual LP has same optomum as primal

Proof one direction is easy

$$\min \sum_i p_i S(\Phi(|v_i\rangle\langle v_i|)) \geq \sum p_i \langle v_i|\tau|v_i\rangle$$

$$= \mathrm{Tr} \sum_i p_i \tau |v_i\rangle\langle v_i|$$

$$= \mathrm{Tr}\, \tau \rho$$

Other direction needs to invoke duality theorem.

Additivity of min entropy $\Rightarrow$ additivity of constrained Holevo capacity (fix χ)

$$\chi_\rho(\Phi) = \max_{\sum p_i \rho_i = \rho} S(\Phi(\rho)) - \sum_i p_i S(\Phi(\rho_i))$$

By LP, there is τ so.

$$\mathrm{Tr}\, \rho_i \tau \leq S(\Phi(\rho_i))$$

with equality for signal states ρ_i

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Find another channel $\tilde{\Phi}$

so $$S(\tilde{\Phi}(\rho)) = \underbrace{S(\Phi(\rho)) + C - \mathrm{Tr}\,\rho\tau}$$

This side $= C$ for signal states ρ_i

$\geq C$ for other states

So signal states for this channel are also minimum entropy output states

If we have additivity of min entropy output for this channel,
gives additivity of χ for (constrained) this channel
gives additivity of χ (constrained) for the original channel.

Cannot find such a channel.
Can find a sequence of ~~such~~ channels so it works "in the limit."

(17) Additivity of χ also proved for a number of other channels

unital qubit channels

Unital means $\Phi(I/2) = I/2$.