BLACK HOLE SPACETIMES

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Abstract

I describe the properties of an observer horizon and of the event horizon of a black hole spacetime, with emphasis on those aspects important for numerical simulation. The boundary of the spacetime region which can causally effect a given spacetime point \( \mathcal{P} \) constitutes the event horizon of the observer at \( \mathcal{P} \). Thus the horizon is determined by the maximum signal propagation speed, i.e. by the characteristics of the partial differential equations underlying the theory. In relativity, these are the light rays. A black hole event horizon is the boundary of the causal past of the collection of observers at all “distant” spacetime points. In the flat spacetime of special relativity this boundary is empty and there are no black holes. In the curved spacetime of general relativity, black holes are produced by the lensing effect of a body undergoing gravitational collapse to a singularity. The final singularity is an impediment to numerical simulation. Fortunately, theoretical arguments suggest that the singularity lies inside the black hole and does not affect observations by distant astronomers. This allows singularity-avoiding strategies for computing the gravitational radiation emitted in the formation of black holes. This has been achieved with some success for a single black hole. Current efforts concentrate on handling the inspiral and merger of a binary black hole, which has a more complicated horizon structure.

1 Newtonian spacetime

In keeping with the tutorial spirit, the material in this talk is mature enough to have appeared in standard reviews, such as Living Reviews. Therefore, to those that did the original work, please pardon me for concentrating on the ideas and omitting references.

I begin with a brief description of the 4-dimensional space-time geometry underlying Newtonian physics. Let \( x^\alpha = (t, x^i) = (t, x, y, z) \) be spacetime coordinates. In a standard inertial frame in which Newton’s equations are formulated, \( t \) represents absolute time and \( x^i \) are Cartesian coordinates in
which the Euclidean line element for the distance between neighboring points is
\[ dl^2 = \delta_{ij} dx^i dx^j. \]  
(1)

Here the Euclidean metric tensor is the Kronecker \( \delta_{ij} \). The time slices \( t = \text{const} \) are absolute but the time direction is not absolute but changes under a change \( x^i \to x^i + v^i t \) to new inertial coordinates, as illustrated in Fig. ???

The significance of absolute time is that it is possible for signals to propagate instantaneously, i.e. with infinite speed, in Newtonian physics. For instance, at any given time \( t \) a particle of mass \( M \) located at coordinates \( z^i \) creates a gravitational potential \( \Phi(t, x^i) = -Gm/R \), where \( R^2 = \delta_{ij} (x^i - z^i)(x^j - z^j) \). That implies the gravitational signal created by a sudden acceleration of the particle is instantaneously received at the same time by an observer carrying a gravimeter (see Fig. ???).

In principle, an observer can be aware of everything that occurs in his past. In spacetime language, the observer’s horizon, the spacetime boundary of what he can detect, is the present time.

2 Special relativity

This situation changes in special relativity where the line element has the standard Minkowski form
\[ ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = -c^2 dt^2 + dl^2. \]  
(2)

Here \( c \) is the speed of light, i.e. the “physical” speed of the characteristics of Maxwell’s equations for the electromagnetic field when analyzed as a hyperbolic system. Similarly, it is the propagation speed of all fields representing elementary particles. As an example, a massless scalar field \( \Psi \) satisfies the relativistic wave equation
\[ \eta^{\alpha\beta} \partial_\alpha \partial_\beta \Psi = \left( -\frac{1}{c^2} \partial_t^2 + \Delta^2 \right) \Psi = 0, \]  
(3)

where \( \eta^{\alpha\beta} \) is the inverse Minkowski metric and \( \Delta^2 = \delta^{ij} \partial_i \partial_j = \partial_x^2 + \partial_y^2 + \partial_z^2 \) is the Laplace operator.

The Minkowski line element reduces to the change in Newtonian absolute time in the degenerate limit
\[ \lim_{c \to \infty} \frac{1}{c^2} ds^2 = -dt^2. \]  
(4)
Similarly, the Euclidean metric arises from the degenerate limit of the inverse Minkowski metric or equivalently, the relativistic wave equation reduces to the Laplace equation
\[ \lim_{c \to \infty} \eta^{\alpha\beta} \partial_\alpha \partial_\beta \Psi = \Delta^2 \Psi, \]  
so that in the Newtonian limit \( c \to \infty \), the hyperbolic wave equation reduces to the elliptic Laplace equation.

The finite propagation speed \( c \) leads to a new feature of horizons in special relativity. An observer at the spacetime point \( \mathcal{O} \) with coordinates \( x^\alpha = 0 \) cannot detect events outside his past light cone defined by
\[ x^\alpha x_\alpha = -c^2 t^2 + r^2 = 0, \quad t \leq 0, \]  
where \( r^2 = x^2 + y^2 + z^2 \). For that reason, the past light cone is called the event horizon of the observer at \( \mathcal{O} \). We make the standard convention choice of units in which \( c = 1 \) so that light rays follow 45° straight lines (geodesics), as depicted in Fig. ???

The full light cone at \( \mathcal{O} \), consisting of the past and future portions, is the chief invariant feature of the spacetime geometry of special relativity, in the same way that absolute time is an invariant feature of Newtonian spacetime. In fact the lightcone collapses to an absolute time slice as \( c \to \infty \).

The light cones, also called null cones, underly the global structure of Minkowski spacetime. Consider retarded time coordinates \((u, r, \theta, \phi)\), where \( u = ct - r \) is retarded time and \((r, \theta, \phi)\) are standard spherical coordinates. The retarded time hypersurfaces \( u = \text{const} \) foliate spacetime with future light cones. This foliation can be used to formulate a characteristic evolution scheme in the analogous way that the standard time foliation is used for Cauchy evolution. However, major differences in the evolution scheme arise because the the normal \( \ell_\alpha = -\partial_\alpha u \) to the null cone corresponds to the tangent direction \( \ell^\alpha = \eta^{\alpha\beta} \ell_\beta \) along the null rays. As a result, the standard Cauchy "3+1" formulation is degenerate when extended to null hypersurfaces.

In retarded time coordinates, the Minkowski metric becomes
\[ ds^2 = -du^2 - 2dudr + r^2 d\Omega^2 = r^2 [-\frac{du^2}{r^2} + 2dud\left(\frac{1}{r}\right) + d\Omega^2], \]
where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the unit sphere metric. Note that the intrinsic metric of the null cone \( u = \text{const} \) has the degenerate form
\[ ds^2 = r^2 d\Omega^2, \]
which reflects the fact that all directions tangent to the light cone are space-like except for the null direction. Setting \( \ell = 1/r \), the conformally related metric
\[
ds^2 = \ell^2 ds^2 = -\ell^2 du^2 + 2dud\ell + d\Omega^2,
\]
can be smoothly extended to include the points \( \ell = 1/r = 0 \) lying at infinity along the null rays. This 3-dimensional set of boundary points having topology \( S^2 \times R \) are called future null infinity, denoted by \( \mathcal{I}^+ \). Setting \( \ell = 0 \) in Eq. (9), we obtain the rank 2 metric \( ds^2 = d\Omega^2 \), showing that \( \mathcal{I}^+ \) is itself a null hypersurface. Similarly, \( \mathcal{I}^- \) represents the conformal compactification of points at past null infinity. This gives rise to the Penrose picture Fig. ??? of compactified Minkowski space, where null directions are drawn at 45 degree angles. In this picture, \( i^0 \) is the point at spatial infinity, analogous to the one point compactification of Euclidean 3-space into a 3-sphere. Similarly, \( i^+ \) and \( i^- \) are points at future and past time infinity, respectively.

Outgoing radiation fields from an isolated source in Minkowski spacetime propagate to \( \mathcal{I}^+ \). As an example, a scalar wave from a compact source has asymptotic dependence \( \Phi \sim A(u, \theta, \phi)/r \) and the radiation amplitude is given by the limit \( A = r\Phi|_{\mathcal{I}^+} \). Thus \( \mathcal{I}^+ \) models the location of telescopes observing distant objects in the universe. In the compactified picture, matter which travels slower than the velocity of light enters spacetime at \( i^- \) and leaves at \( i^+ \), whereas fields propagating with the speed of light, such as electromagnetic waves, enter at \( \mathcal{I}^- \) and leave at \( \mathcal{I}^+ \).

The concept of a horizon applies to any collection of observers occupying a set \( S \) of spacetime points. The causal past of \( S \), denoted by \( J^-(S) \), consists of the points reachable from \( S \) by continuous curves comprised of past-directed segments of light rays or past directed timelike curves. The horizon for the collection of observers located on \( S \) is \( \partial J^-(S) \), the boundary of its causal past. It follows that any horizon is traced out (generated) by light rays. For a sphere of observers \( S_0 \) located at a constant retarded time slice \( u = u_0 \) of \( \mathcal{I}^+ \), the horizon is just the future light cone \( N_0 \) extending to \( S_0 \).

Note that if we continue one of the null geodesics past the vertex of \( N_0 \) it leaves the horizon and enters the interior of the past \( S_0 \). This is a general feature: any horizon terminates (in the past) whenever two of its generating light rays intersect, as is evident from Fig. ???i. Similarly, any horizon terminates when neighboring geodesics focus, as in the case of the point caustic on the light cone.
The horizon determined by a generic slice of $\mathcal{I}^+$ is a generalized null cone (null hypersurface) that does not have a point vertex. A case that is important for understanding the collision of black holes is the horizon formed by tracing the light rays orthogonal to the surface of an ellipsoid embedded at constant time in Minkowski space. When the rays are traced into the future the ellipsoidal wavefront expands to intersect $\mathcal{I}^+$ in a spherical slice $\mathcal{S}$, but one which is not located at a constant retarded time. The horizon of $\mathcal{S}$ is formed by tracing these null rays back into the past of the ellipsoid. In the generic case that the ellipsoid is not a surface of revolution (cigar), the horizon pinches off at a 2-dimensional set $X$ of cross-over points which are bounded by a curve of caustics, as shown in Fig. ???. The cross-over set arises from the intersection of two null hypersurfaces with opposite null directions $\ell_1$ and $\ell_2$ which emanate from opposing sides of the ellipsoid. $X$ must be spacelike because, just as in the case of the null cone, all directions in a general null hypersurface are spacelike, except for the null direction. The caustics arise from rays emanating from the curve where the opposite sides of the ellipsoid join. At this curve, $\ell_1 \to \ell_2$ and the cross-over becomes a caustic between neighboring rays. $X$ becomes asymptotically null as it approaches the caustic curve $C$. The caustics are of the structurally stable cusp type, as classified by catastrophe optics. This global picture of the horizon, and its formation at $X$ and $C$ is structurally stable, i.e. its qualitative structure is invariant under perturbations of the ellipsoid or by perturbations introduced by intermediate lenses along the rays. That is why this same horizon structure shows up in the study of black holes where gravitational lensing is at play.

A black hole horizon $\mathcal{H}^+$ is the boundary of the set of points that are visible to the set of all observers $\mathcal{I}^+$, i.e. $\mathcal{H}^+ = \partial \mathcal{J}^- (\mathcal{I}^+)$. In Minkowski spacetime, $\mathcal{H}^+$ is empty.

3 The light cone in general relativity

Light cones in general relativity are locally the same as in special relativity. The spacetime metric now has the general form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

where $g_{\alpha\beta}$ is a function of the coordinates with the property that coordinates can be chosen at any point to transform it into the Minkowski form. Whereas
the local properties of the light cone traced out by null geodesics from a point and of an observer’s event horizon are unchanged, the curvature of spacetime can lead to drastic global effects. In Newtonian language, these effects arise from the bending of light by matter (gravitational lensing) and by the delay time in the propagation of light. Such language is useful for treating post-Newtonian corrections but are confusing for describing fully nonlinear effects such as the formation of black holes.

The curved spacetime of an isolated source which does not collapse to form a black hole or singularity has a Penrose picture analogous to the Minkowski space case, except the asymptotic properties at the corner points $i^-, i^0$ and $i^+$ are more complicated. Again, the absence of black holes is expressed by $\mathcal{H}^+ = \partial J^- (\mathcal{I}^+) = 0$.

4 Oppenheimer-Snyder collapse

The simplest model of black hole formation was developed by Oppenheimer and Snyder in the 1930’s by pasting together two famous spherically symmetric solutions of Einstein’s equations. The inner region is a spherical portion of the Friedmann solution describing a universe with a homogeneous distribution of dust (pressureless fluid), except in the time reversed scenario so that the big bang singularity occurs in the future. The exterior region is the Schwarzschild solution, the unique spherically symmetric vacuum solution. As the dust undergoes gravitational collapse, the model proceeds from the regime of Newtonian physics to the highly nonlinear regime where the singularity forms, as illustrated in Fig. ?? which shows a 2-dimensional $(t, z)$ spacetime slice, suppressing the $(x, y)$ plane of spatial dimensions.

At early times, the null cones emanating from the center expand out to $\mathcal{I}^+$, as in Minkowski space. However, at late times, instead of expanding to $\mathcal{I}^+$ the null cones are refocused to a point (the singularity) due to the strong effects of gravity. Intermediating this behavior, there is one null cone which hovers forever at a constant surface area, so its light rays are just invisible from $\mathcal{I}^+$. This null cone is the black hole horizon $\mathcal{H}^+$. The spatial slices of $\mathcal{H}^+$ are the spherical surface of the black hole whose null worldtube traces out the horizon. The radius related to the surface area $4\pi R^2$ of the black hole determines the black hole mass according to $R = 2M$.

The final singularity is an impediment to numerical evolution. Fortu-
nately, distant astronomers at $I^+$ can only observe the region of spacetime outside the horizon which is causally unaffected by the region inside the black hole. It is generally believed, even in the absence of spherical symmetry, that the singularities produced in the gravitational collapse of an astrophysical object will be hidden inside a black hole. That means the singularity can in principle be harmlessly excised from the evolution without losing any astronomically relevant data. If the location of the horizon were known, this could be achieved by using it as the inner boundary of the computational domain. However, because the horizon is determined by the global construction $\mathcal{H}^+ = \partial J^- (I^+)$, it cannot be located during the course of a Cauchy evolution. Instead the excision strategy is based on the so-called apparent horizon inside the black hole.

The apparent horizon, if it has formed, is located at a given time in side the black hole where the outgoing null cone momentarily hovers. This allows it to be located “on the fly” during a numerical evolution. If the Cauchy time slicing is judiciously chosen, the apparent horizon will asymptote to the actual horizon at late times, so an excision strategy based on the apparent horizon accomplishes the right thing. But it is not easy computationally because the points on the apparent horizon move faster than the speed of light (in a spacelike direction), as they must in order to catch up to the real event horizon.

5 Compactified picture of spherically symmetric collapse

Figure ??? is a compactified Penrose diagram for the 2-dimensional spacetime picture depicted in Fig. ????. It shows typical time slices used in present evolution codes. Standard spacelike slices used for Cauchy evolution require an outer artificial boundary. Alternatively, “hyperbolic” spacelike slices used for Cauchy evolution and the null slices used for characteristic evolution extend to $I^+$, the physical boundary of the spacetime where radiation fields can be calculated. At late times, the Cauchy slices penetrate the horizon and some technique must be applied to avoid the singularity.

One of the hidden but unavoidable features of this global picture is the enormous ratio of time scales between the inner and outer regions. For a black
hole of mass $M$ the time between horizon formation and the final singularity is of order $M$ (in gravitational units) whereas the evolution time measured in the exterior region by observers at $I^+$ is infinite. This infinite redshift produced by black holes places a severe burden on the computation.

6 Vacuum Schwarzschild black hole - wormhole

The exterior Schwarzschild solution can be analytically continued to form a purely vacuum spacetime. The spacetime picture is confusing because the global topology is no longer $R^4$ but $R^2 \times S^2$. In Fig. ??, we have factored out the $S^2$, coordinatized by standard angular coordinates $(\theta, \phi)$, to depict the the timelike $R^2$, coordinatized by $(T, r)$ where $T$ is a global time coordinate and $4\pi r^2$ is the area of the spheres of symmetry. There is a left-right reflection symmetry which results in two disjoint boundaries at infinity. The spacetime also has a time reflection symmetry which results in a singularity in the past as well as in the future. The $r = 2M$ Schwarzschild surface bifurcates into two branches: the black hole event horizon and a white hole branch which is directly visible by external observers.

It is this purely vacuum black hole problem which many of the groups are attempting to simulate in order to avoid dealing with hydrodynamics. The trade-off is a more complicated global problem. One approach takes advantage of the global structure to introduce a hyperboloidal time slicing with wormhole topology which extends to infinity at each end and requires no inner boundary (see Fig. ??).

Spherically symmetric black holes are very special. However, all theoretical and computational evidence supports the “no-hair” hypothesis that any non-rotating black hole behaves like a Schwarzschild black hole at late times. It is further believed that any astrophysical gravitational collapse asymptotes at late times to a Kerr black hole, which is the most general stationary black hole spacetime and is uniquely determined by its mass and angular momentum.
7 Focusing properties of horizons

At late times black holes always come to a simple final equilibrium as Kerr or Schwarzschild black hole but their generic structure at formation is much more complicated. In order to appreciate this consider some elementary focusing properties of null hypersurfaces. A null hypersurface can be described from a stand-alone point of view as a 3-dimensional manifold, similar to the way you describe a timelike Cauchy hypersurface. The difference is that the intrinsic 3-metric is degenerate.

The intrinsic geometry consists of the tangent null vector $\ell^a$ with affine parameterization $\lambda$, so that $\ell^a \partial_a = \partial_\lambda$, and a degenerate rank-2 3-metric satisfying $\gamma_{ab} \ell^a = 0$. The three metric can be factored in the form $\gamma_{ab} = R^2 h_{ab}$ where $R$ is determined by the local surface area (size) and $h_{ab}$ describes the rank-2 conformal geometry of the cross-sections (shape). In the characteristic initial value problem, the shape and affine parameter are unconstrained data that can be given freely. The shape then determines the size through the null component of Einstein’s equations (the Raychauduri equation)

$$\partial_\lambda R = - \frac{R}{4} \Sigma^2 - \frac{4\pi}{R} (\rho + p) (u_a \ell^a)^2 \leq 0,$$

(11)

where

$$\Sigma^2 = \frac{1}{2} \gamma^{ab} \gamma^{cd} (\partial_\lambda \gamma_{ac}) (\partial_\lambda \gamma_{bd})$$

is the square of the optical shear tensor and $\rho$, $p$ and $u_a$ are the density, pressure and 4-velocity of the fluid source, respectively.

The important thing about this equation is that the right hand side is non-positive. That means that a null hypersurface whose surface area hovers, i.e. $\partial_\lambda R = 0$, is structurally unstable to perturbations. In particular, since a black holes hovers asymptotically in the future, structural stability of the horizon requires that the surface area traced back in time along each ray must go to 0 at a caustic point. However, as already explained in the flat space example, most light rays on the horizon actually pinch off in the past at a crossover point before they reach the caustic.
8 Binary black holes

Except for the deformations introduced by the lensing effects of gravity, all black hole horizons found in simulations are qualitatively similar to the structurally stable flat space ellipsoidal horizon. The two chief differences are that gravitational lensing stops the outward expansion in Fig. ??? so that the horizon asymptotes to a stationary Kerr black hole and that the stronger gravitational lensing in the central region causes the cross-over points to form quickest there (tracing backward in time along the central rays). This changes the affine parameter slicing of the horizon and leads to an upward bulge in the central region which creates the pair-of-pants shaped horizon found in simulations of colliding black holes. Although the pair-of-pants horizon in Fig. ??? doesn’t look like the flat space ellipsoidal horizon they have the same intrinsic conformal geometry. The difference is their affine parameterization and how their surface area changes with time.

There is an ambiguity in the pair-of-pants picture because one spatial dimension has been suppressed. A time slice intersecting the pants legs in two circles can correspond either to two black holes (the circles representing individual spheres) or a single toroidal black hole (the two circles resulting from a slice through the torus). In a structurally stable black hole collision, both these facets are present. Figures ??? show a sequence of snapshots depicting two black holes which merge into a single toroidal black hole, which then grows to from a single spherical black hole as the hole in the torus closes up.

These simulations were made using a characteristic code which computes the spacetime near the horizon. What we don’t know and want to know is the gravitational radiation produced in this collision process.