EFFECT OF TOTAL MODELING UNCERTAINTY ON THE ACCURACY OF NUMERICAL SIMULATIONS

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OVERVIEW OF PRESENTATION

- Total Modeling Uncertainty - Definition
- A General Approach Suitable for All Simulations
- Specialization - Linear Structural Dynamics
- Quantification of Total Modeling Uncertainty - Examples
  - Frequency uncertainty
  - Mode shape uncertainty
  - Mass and stiffness uncertainty
  - Uncertainty expressed in terms of generalized “modal” metrics
- Accuracy of Numerical Simulations - Examples
  - Frequency response of a truss beam
  - Airblast response of a reinforced concrete wall
TOTAL MODELING UNCERTAINTY

- **Truth Assumption**
  That which we observe in the physical world

- **Simulation Uncertainty**
  Statistical representation of the difference between simulated and observed phenomena, e.g. a covariance matrix of a discrete time series or frequency response function (FRF).

- **Total Modeling Uncertainty**
  Statistical representation of the difference between simulated and observed phenomena *expressed in terms of generic, dimensionless metrics that may be averaged over a family of generically similar phenomena*, e.g. ratio of measured to predicted natural frequency; normalized analysis-test eigenvector products.
SOURCES OF VARIABILITY / UNCERTAINTY

- Test Article or Product Variability
  - Tolerances
  - Material properties
  - Construction

- Experimental Variability
  - Variability in experimental setup
  - Measurement variability
  - Variability in data processing

- Modeling Variability
  - Variability in modeling assumptions - parameterization
  - Parametric variability
  - Computational variability

- Total Modeling Uncertainty - All of the Above
GENERAL APPROACH

- **Observed Phenomena**

\[
Y(t) = \begin{bmatrix}
y_1(t) \\
y_2(t) \\
\vdots \\
y_n(t)
\end{bmatrix} = \begin{bmatrix}
y_1(t_1), & y_1(t_2), & \cdots & y_1(t_m) \\
y_2(t_1), & y_2(t_2), & \cdots & y_2(t_m) \\
\vdots & \vdots & \ddots & \vdots \\
y_n(t_1), & y_n(t_2), & \cdots & y_n(t_m)
\end{bmatrix}
\]

- **Corresponding Numerical Simulation**

\[
0Y(t) = \begin{bmatrix}
^0y_1(t) \\
^0y_2(t) \\
\vdots \\
^0y_n(t)
\end{bmatrix} = \begin{bmatrix}
^0y_1(t_1), & ^0y_1(t_2), & \cdots & ^0y_1(t_m) \\
^0y_2(t_1), & ^0y_2(t_2), & \cdots & ^0y_2(t_m) \\
\vdots & \vdots & \ddots & \vdots \\
^0y_n(t_1), & ^0y_n(t_2), & \cdots & ^0y_n(t_m)
\end{bmatrix}
\]

- **Singular Value Decomposition**

\[
Y(t) = U\Sigma V^T = \phi D \eta
\]

\[
0Y(t) = ^0U^0\Sigma^0V^T = ^0\phi^0D^0 \eta
\]
PRINCIPAL COMPONENTS ANALYSIS

- Singular Values and Singular (Eigen) Vectors
  - $D_{ii} = \text{Singular value (from the diagonal matrix, } D)$
  - $\phi_i = \text{Left (column) eigenvector}$
  - $\eta_i = \text{Right (row) eigenvector}$

- Orthonormal properties of Eigenvectors
  \[
  \phi_i^T \phi_j = \delta_{ij} \quad \eta_i \eta_j^T = \delta_{ij} \\
  0 \phi_i^T 0 \phi_j = \delta_{ij} \quad 0 \eta_i 0 \eta_j^T = \delta_{ij}
  \]

- Principal Components: Energy-ordered “Modes”
  \[
  YY^T = (\phi D \eta)(\eta^T D \phi^T) = \phi D^2 \phi^T \\
  \sum_i \sum_j y_i^2(t_j) = Tr(YY^T) = Tr(\phi D^2 \phi^T) = \sum_i D_{ii}^2
  \]
PC METRICS FOR QUANTIFYING TOTAL MODELING UNCERTAINTY

- Express $\phi_i$’s and $\eta_i$’s as linear combinations of $^0\phi_i$’s and $^0\eta_i$’s
  
  $\phi \equiv (^0\phi)\psi : ^0\phi^T\phi = \psi$
  
  $\eta \equiv (\nu)^0\eta : ^0\eta\eta^T = \nu$

- Evaluate differences between observation and simulation
  
  $\Delta\phi = \phi - ^0\phi \equiv ^0\phi(\psi - I) = ^0\phi\Delta\psi$
  
  $\Delta D = D - ^0D$;  $\Delta D^0D^{-1} = \Delta\tilde{D}$
  
  $\Delta\eta = \eta - ^0\eta \equiv (\nu - I)^0\eta = \Delta\nu^0\eta$

- Vectorize $\Delta\psi, \Delta\tilde{D}, \Delta\nu$
  
  $\Delta \tilde{r} = \begin{Bmatrix} vec(\Delta\psi) \\ vec(\Delta\tilde{D}) \\ vec(\Delta\nu) \end{Bmatrix}$
EVALUATION OF TOTAL MODELING UNCERTAINTY

- Using Replicate Test Data for a Particular Test Article

\[ S_{rr} = E[(\Delta \tilde{r} - \mu_{\Delta r})(\Delta \tilde{r} - \mu_{\Delta r})^T] = \frac{1}{N-1} \sum_{i=1}^{N} (\Delta \tilde{r}_i - \mu_{\Delta r})(\Delta \tilde{r}_i - \mu_{\Delta r})^T \]

\[ \mu_{\Delta r} = E[\Delta \tilde{r}] = \frac{1}{N} \sum_{i=1}^{N} \Delta \tilde{r}_i \] (Systematic Error)

- Using Test Data from Generically Similar Test Articles

\[ S_{PP} = E[(\Delta \tilde{r} \Delta \tilde{r}^T)] = \frac{1}{N} \sum_{i=1}^{N} \Delta \tilde{r}_i \Delta \tilde{r}_i^T \]

\[ \mu_{\Delta \tilde{r}} \] is assumed to be zero
FAST RUNNING SIMULATION

- Recall Singular Value Decomposition

\[ Y(t) = \phi \  D \ \eta \]
\[ y_i(t_j) = \sum_{k=1}^{p} \phi_{ik} \ D_{kk} \ \eta_k(t_j) = \sum_{k=1}^{p} \phi_{ik} \ D_{kk} \ \eta_{kj} \]

- Recall Difference Analysis

\[ \phi = ^0\phi + \Delta\phi \cong ^0\phi(I + \Delta\psi) \]
\[ D = ^0D + \Delta D = ^0D(I + \Delta\tilde{D}) \]
\[ \eta = ^0\eta + \Delta\eta \cong ^0\eta(I + \Delta\nu) \]

- Fast Running Simulation

\[ \hat{Y}_{ij} = ^0\phi(I + \Delta\psi) \ ^0D(I + \Delta\tilde{D}) \ ^0\eta(I + \Delta\nu) \]
EVALUATING THE ACCURACY OF NUMERICAL SIMULATIONS

- **Linear Covariance Propagation**
  \[ S_{yy} \cong S_{\hat{y} \hat{y}} = T_{\hat{y}r} S_{rr} T_{\hat{y}r}^T \]
  \[ T_{\hat{y}r} = \partial \hat{Y} / \partial \tilde{r} \]

- **Interval Propagation - Vertex Method**
  \[ \Delta Y = \left\{ \max_{i,j} \{ Y(c_i), Y(e_j) \} - \min_{i,j} \{ Y(c_i), Y(e_j) \} \right\} \]
  \[ c_i = \text{coordinates of vertices of rectangular hyperspace} \]
  \[ e_j = \text{coordinates of extrema within rectangular hyperspace} \]

- **Monte Carlo Simulation**
  - Diagonalize \( S_{rr} : \Delta \tilde{r} = \Theta \Delta s \)
  - Execute PC Model for randomly selected values of \( \Delta s \)
SPECIALIZATION TO LINEAR STRUCTURAL DYNAMICS

- Classical Normal Mode Analysis
  \[(0^0K - 0^0\lambda_j 0^0M)0^0\phi_j = 0\]
  \(0^0\lambda_j = \text{Analytical eigenvalues}\)
  \(0^0\phi_j = \text{Analytical Eigenvectors}\)
  \(0^0M = \text{Analytical mass matrix}\)
  \(0^0k = \text{Analytical stiffness matrix}\)

- Modal Mass and Stiffness Matrices
  \(0^0m = 0^0\phi^T 0^0M 0^0\phi = I\)
  \(0^0k = 0^0\phi^T 0^0K 0^0\phi = 0^0\lambda\)

- Assumed “True” Modal Mass and Stiffness Matrices
  \(m = 0^0m + \Delta m = I + \Delta m\)
  \(k = 0^0k + \Delta k = 0^0\lambda + \Delta k\)
MODAL METRICS FOR QUANTIFYING
MODELING UNCERTAINTY

- Normalization of Test Modes, $\phi$
  \[ \phi_j^T M \phi_j = 1 \]

- Cross-orthogonality of Analysis and Test Modes
  \[ \phi_j^T M \phi_j = \phi_j^T M \phi = \psi \]

- Difference Between Analysis and Test Modes
  \[ \Delta \lambda = \lambda - \lambda \]
  \[ \Delta \phi = \phi - \phi = \phi (\psi - I) = \phi \Delta \psi \]

- Modal Metrics for Quantifying Total Modeling Uncertainty
  \[ \Delta m = - (\Delta \psi + \Delta \psi^T) \]
  \[ \Delta \kappa = \lambda^{-1/2} (\Delta \lambda - \lambda \Delta \psi - \Delta \psi^T \lambda) \lambda^{-1/2} \]
  \[ \Delta \tilde{\kappa} = \lambda^{-1/2} (\Delta \lambda - \lambda \Delta \psi - \Delta \psi^T \lambda) \lambda^{-1/2} \]
COMPARISON OF FAST-RUNNING SIMULATIONS

- **General Approach**

\[ y_i(t_j) = \sum_{k=1}^{p} \phi_{ik} D_{kk} \eta_k(t_j) \]

where \( \phi \), \( D \) and \( \eta \) are the *principal component metrics* corresponding to the *energy*-ordered modes of a SVD.

- **Linear Structural Dynamics Approach**

\[ y_i(t_j) = \sum_{k=1}^{p} \phi_{ik} \Gamma_{kk} \eta_k(t_j) \]

where \( \phi \), \( \Gamma \) and \( \eta \) are the *classical modal metrics* from a real symmetric eigenvalue/eigenvector analysis.

- Classical normal mode
- \( \phi_k \)
- Modal participation factor
- \( \Gamma_{kk} \)
- Modal response time history
- \( \eta_k(t) \)
NASA / LaRC EIGHT BAY TRUSS

Force input at Nodes 4 and 7
Acceleration response measured at Nodes 1 through 32
EXAMPLE OF TOTAL MODELING UNCERTAINTY
CLASSICAL MODAL ANALYSIS

Variations of an 8-bay truss beam

Generic family of space structures

Eigenvector Cross-orthogonality

Eigenvector Cross-orthogonality
TOTAL MODELING UNCERTAINTY IN TERMS OF MODAL MASS AND STIFFNESS VARIABILITY

Variations of an 8-bay truss beam

Generic family of space structures

Normalized Modal Stiffness
FRF SIMULATION ACCURACY BASED ON STRUCTURE-SPECIFIC MODELING UNCERTAINTY
COMPARISON OF SIMULATION ACCURACY FOR SPECIFIC AND GENERIC MODELING UNCERTAINTY

Node 4 X-Acceleration / 4 X-Force

Node 20 X-Acceleration / 4 X-Force

Node 4 Z-Acceleration / 4 X-Force

Node 20 Z-Acceleration / 4 X-Force
BLAST RESPONSE OF A BURIED R/C STRUCTURE

Finite Element Model
- ~ 80,000 continuum elements (concrete and soil)
- ~ 20,000 rebar elements

Test Measurements
- 12 Pressure gages (10 on interior walls)
- 12 Accelerometers on interior walls
- 5 Locations each wall
EXAMPLE OF TOTAL MODELING UNCERTAINTY
PRINCIPAL COMPONENTS ANALYSIS

Typical Singular Values

Normalized Singular Values

Left Eigenvector Cross-Orthogonality

Right Eigenvector Cross-Orthogonality
DISPLACEMENT SIMULATION UNCERTAINTY FOR R/C WALL
PRE-UPDATE MODEL OF STRUCTURE 1

Left Horizontal Quarter Point

Upper Vertical Quarter Point

Center Point

Lower Vertical Quarter Point

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DISPLACEMENT SIMULATION UNCERTAINTY FOR R/C WALL
POST-UPDATE MODEL OF STRUCTURE 1

Left Horizontal Quarter Point

Center Point

Upper Vertical Quarter Point

Lower Vertical Quarter Point

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DISPLACEMENT SIMULATION UNCERTAINTY FOR R/C WALL
PRE-UPDATE MODEL OF STRUCTURE 2

Gage 1

Gage 2

Gage 3

Gage 4
SUMMARY

- **Why total modeling uncertainty?**
  The predictive accuracy of numerical simulations can be reliably estimated only to the extent that total modeling uncertainty is quantified.

- **How does one obtain the necessary data?**
  By decomposing corresponding sets of simulated and measured response into their principal components, and statistically processing the differences between corresponding PC (modal) metrics.

- **How is the predictive accuracy of numerical simulations derived from total modeling uncertainty?**
  - By first order covariance propagation,
  - By interval propagation, and/or
  - By Monte Carlo simulation
CONCLUSIONS

- Numerical simulations are increasingly relied upon for making critical decisions.

- The reliability of these simulations depends on uncertainty inherent in the underlying mathematical models.

- A true representation of this uncertainty must include all possible sources of uncertainty, i.e. total modeling uncertainty.

- Decision making under uncertainty demands that total modeling uncertainty be quantified realistically.