Numerical Relativity

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Nonlinear PDE
Theory and Approximation

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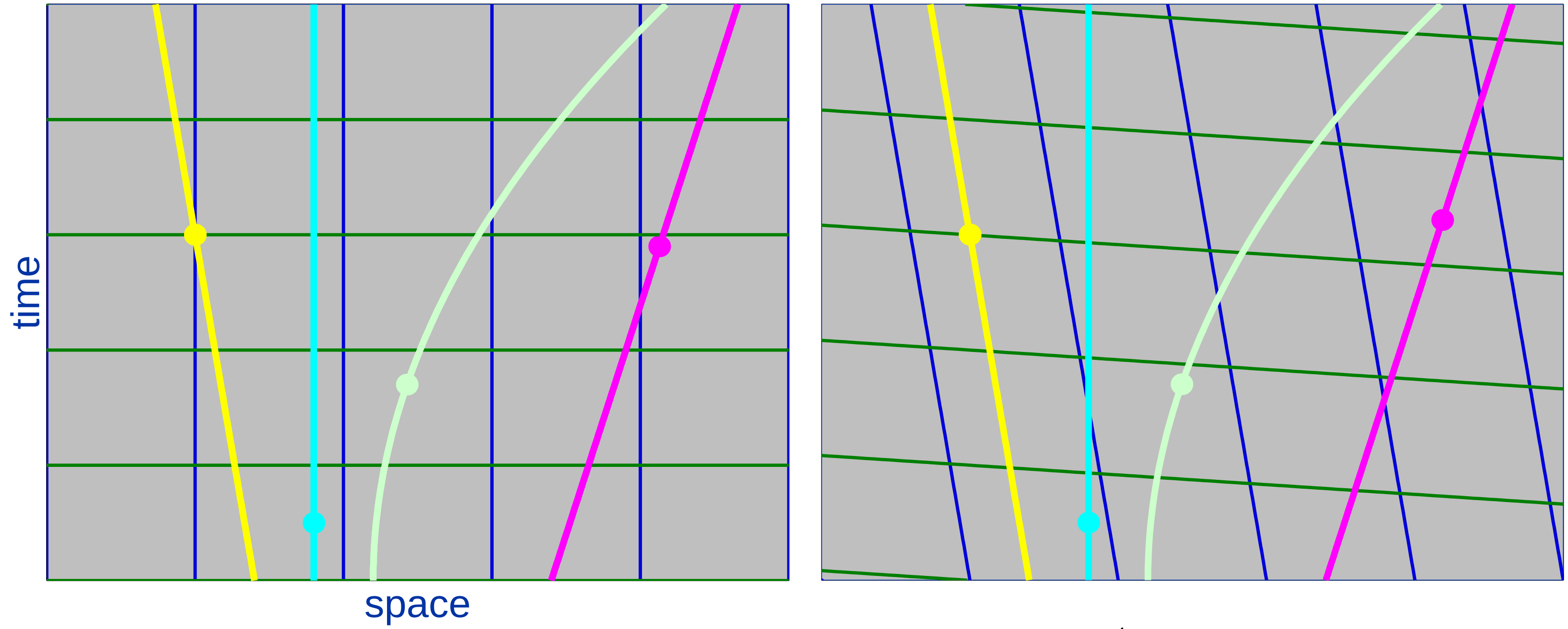
Outline

- Relativity
- The Einstein equations as geometry
- The Einstein equations as PDEs
  - ADM 3+1 decomposition
  - Constraints and initial data
  - Linearization
  - Hyperbolicity
- A new symmetric hyperbolic formulation

Spacetime and special relativity

Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve independence. – H. Minkowski, 1908

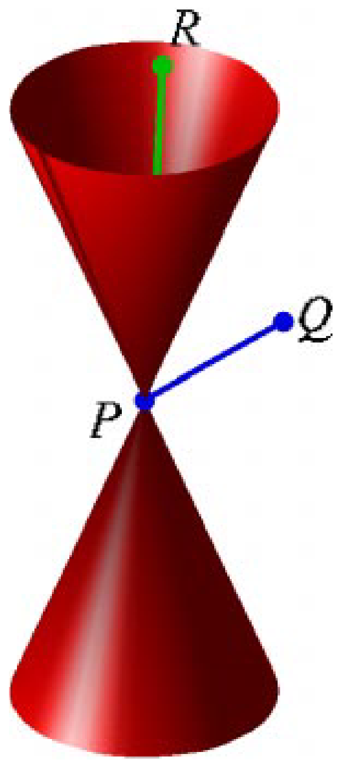


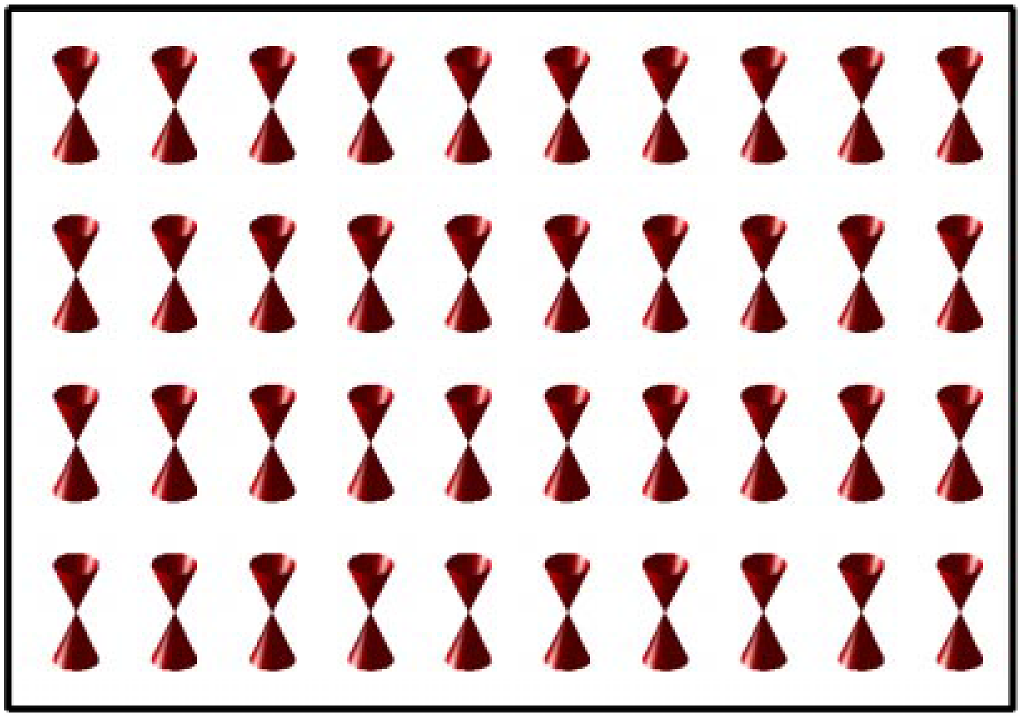
The Minkowski spacetime of special relativity is $\mathbb{R}^4$. There is no preferred coordinate system but there is a method for transforming coordinates between observers in relative motion which leaves the speed of light invariant.

The spacetime interval

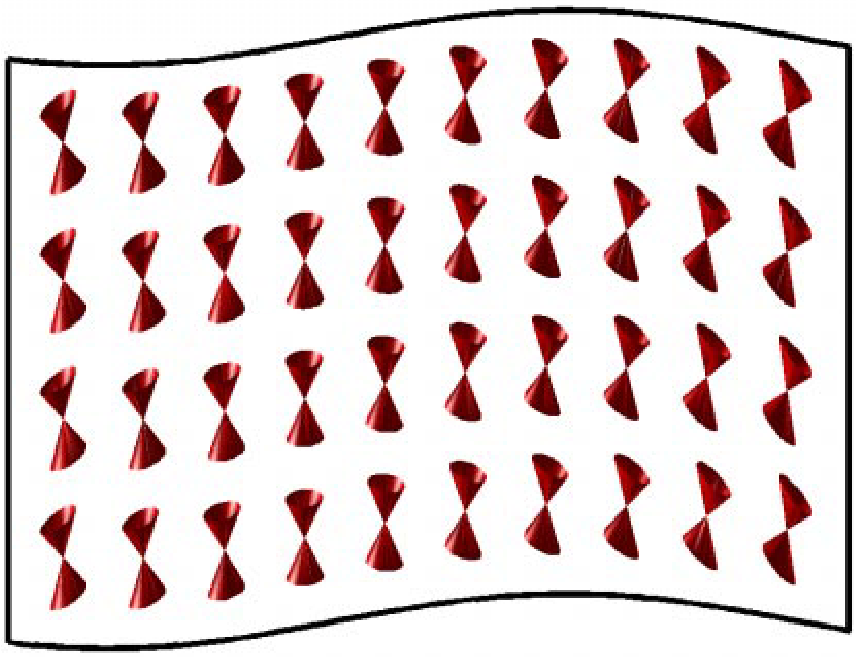
$$I = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - c^2(t_1 - t_2)^2$$

is observer independent. $\sqrt{|I|}$ gives the proper length or proper time between the events.

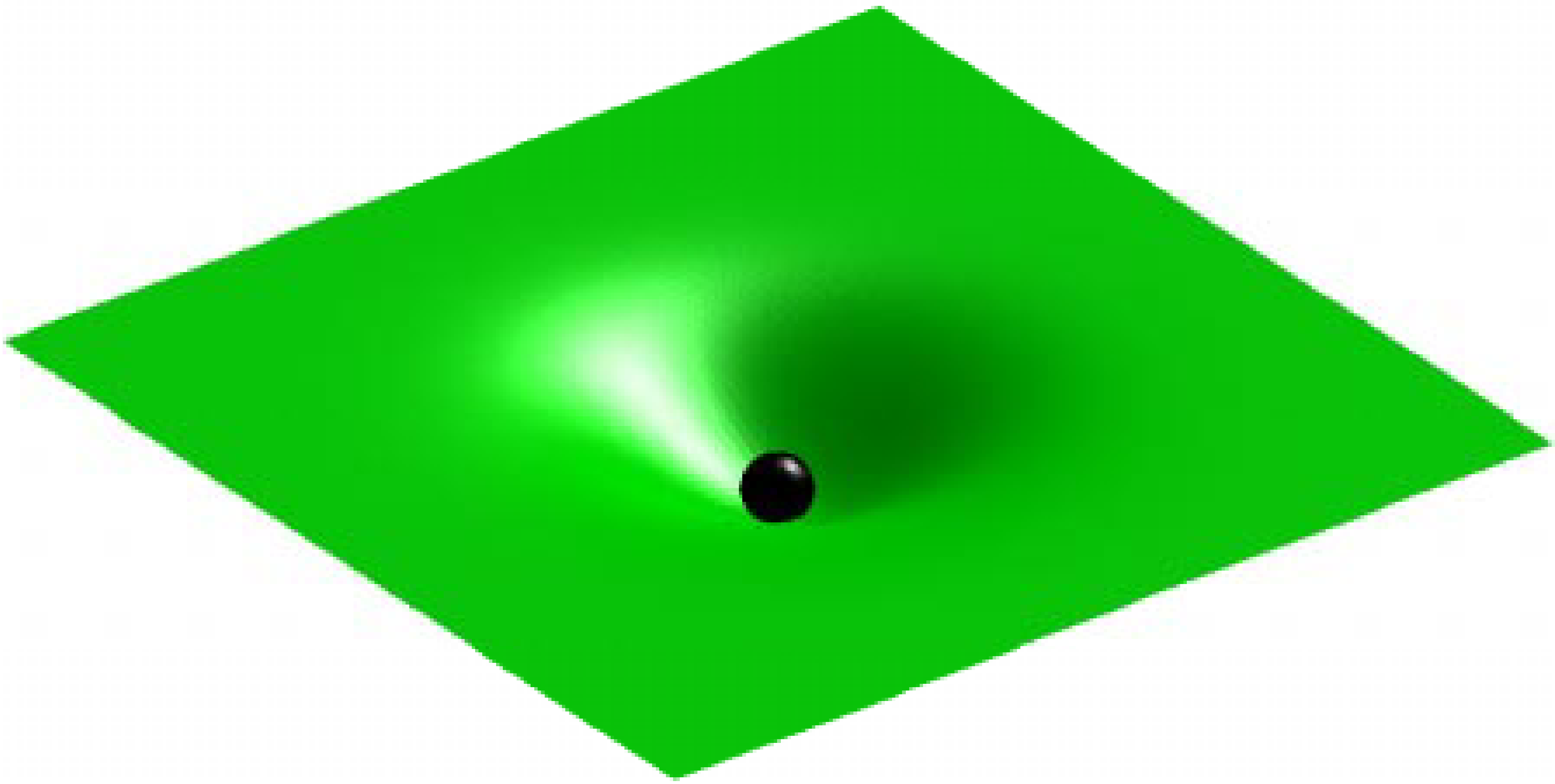




In general relativity, spacetime is a 4-dimensional manifold. Locally it looks like Minkowski space, but it may curve.



According to GR, gravity—rather than being a force-field defined throughout space—is a manifestation of the geometry: freely falling bodies move along geodesics in spacetime, and the "force" of gravity is just the result of the curvature. Einstein's equations relate the curvature at a point of spacetime to the mass-energy there.



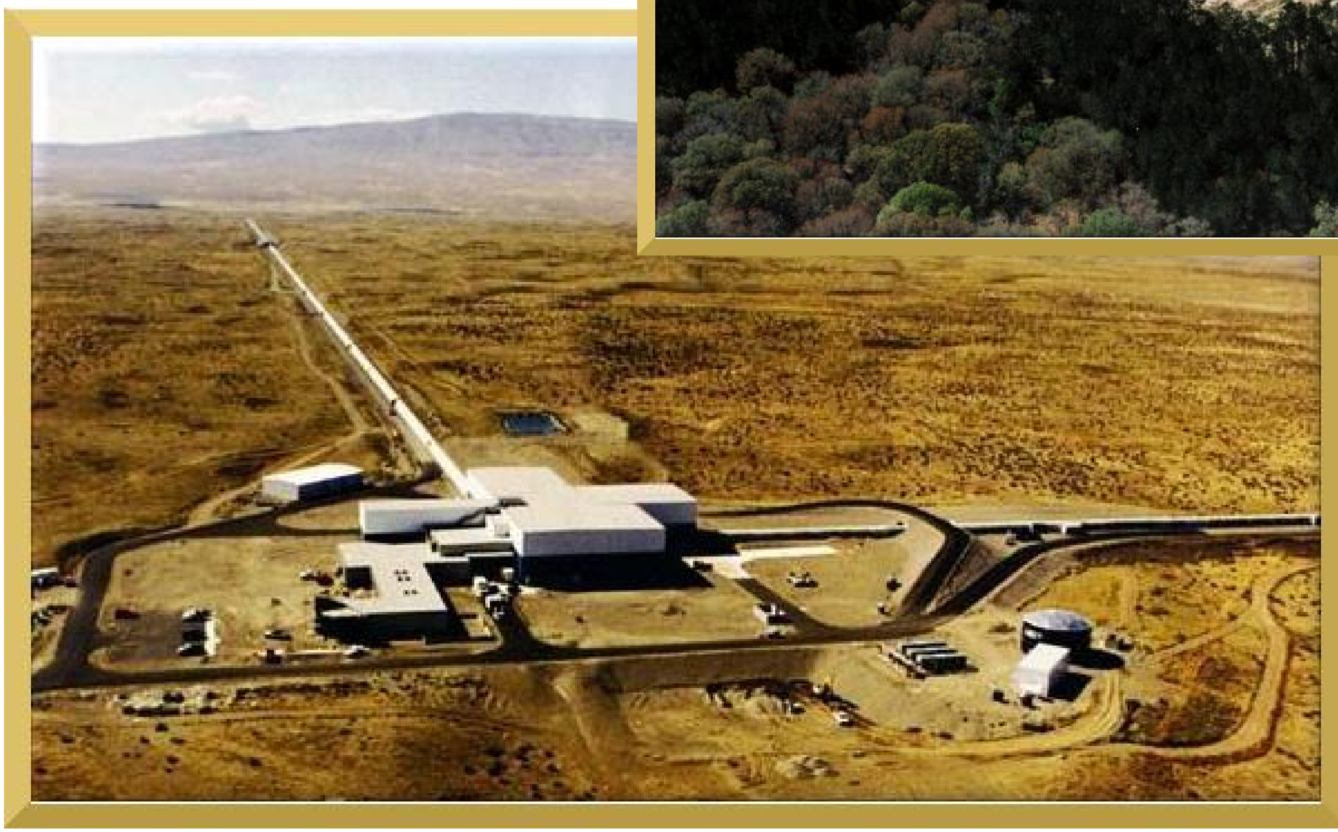
Spacetime grips mass, telling it how to move, and mass grips spacetime, telling it how to curve. – J. A. Wheeler

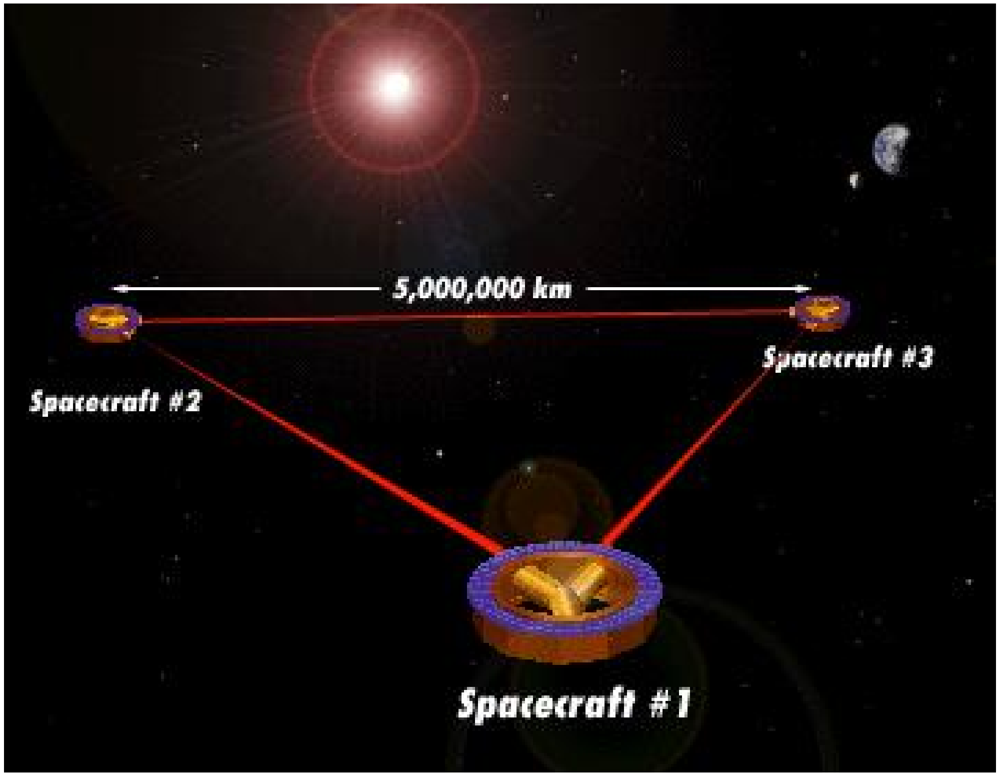
Gravity Waves

A subtle consequence of Einstein's equations is that relatively accelerating masses emit gravitational waves, small perturbations in the spacetime metric tensor, which propagate at the speed of light: *ripples in the rigid fabric of spacetime*.

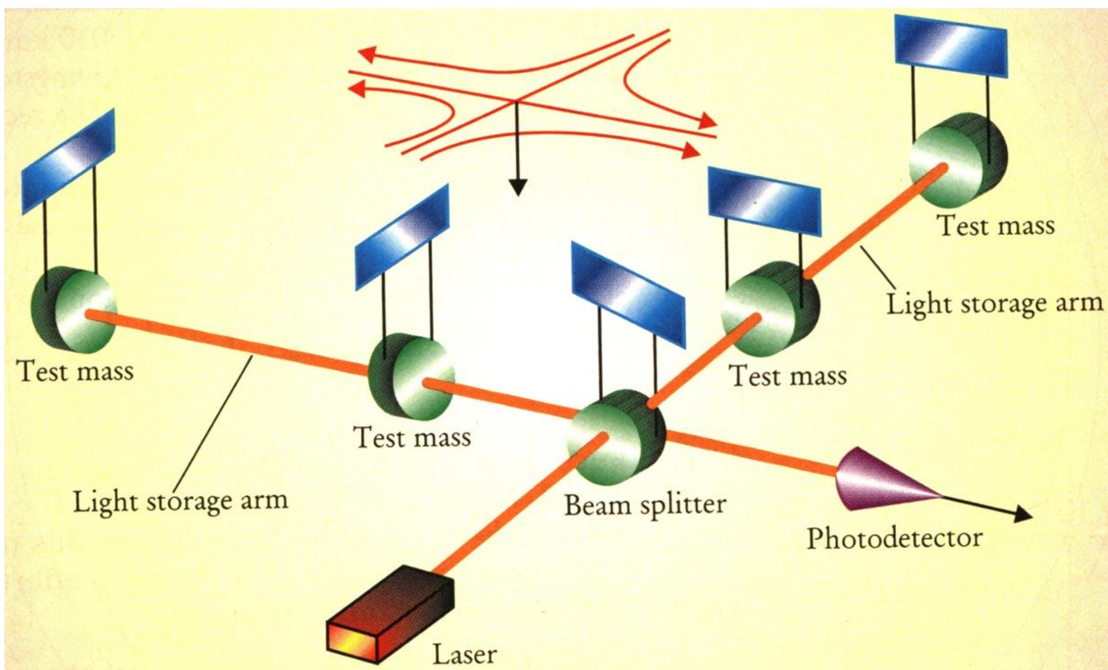
An international network of interferometric detectors is being built to detect them.

LIGO	LIGO	VIRGO	GEO	TAMA	LISA ?
WA	LA	Cascina	Hannover	Mitaka	space


5,000,000 km
Spacecraft #2
Spacecraft #3
Spacecraft #1

A passing gravity wave causes oscillatory decreases in distances between objects along one direction transverse to the wave direction, and increases in the perpendicular direction. The idea behind LIGO is to detect gravity waves by measuring these changes in distance using a sophisticated interferometer as a super-sensitive ruler.



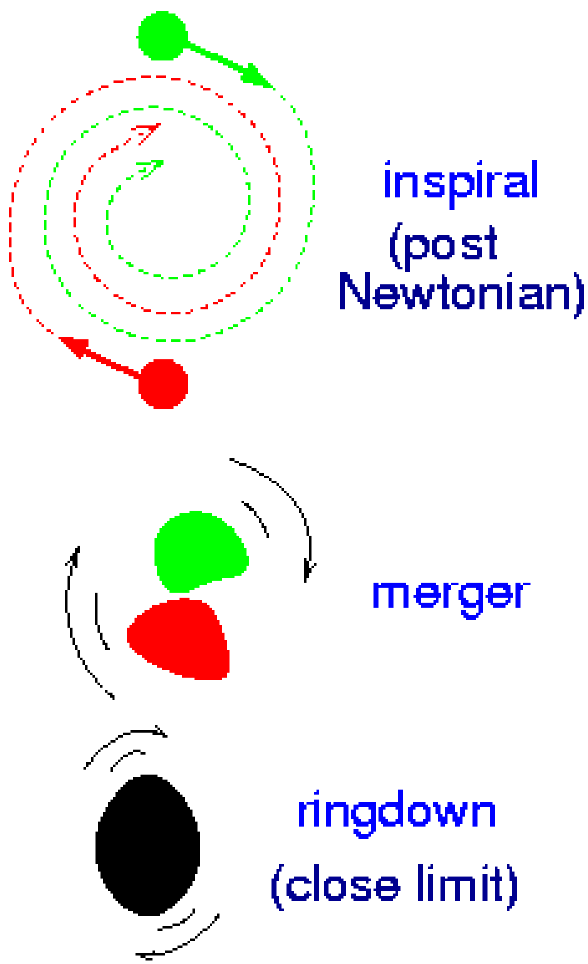
Black hole collisions are expected to be a leading source of detectable gravity waves. Success of the observatories depends on both *detection* and *simulation* of such events.

To detect black holes of a few solar masses colliding in nearby galaxies (10^{23} meters), LIGO will have to be able to detect distance changes of about 10^{-18} meters, one hundred-millionth of the diameter of a hydrogen atom.

But simulation is really hard!

The simulation of black hole mergers requires the numerical solution of the *Einstein equations* with appropriate initial and boundary data. This is a massive computational problem, currently beyond our abilities, and sure to be a great source of problems for many years to come.

BH collisions



322 GRAVITATIONAL COLLAPSE [9.2

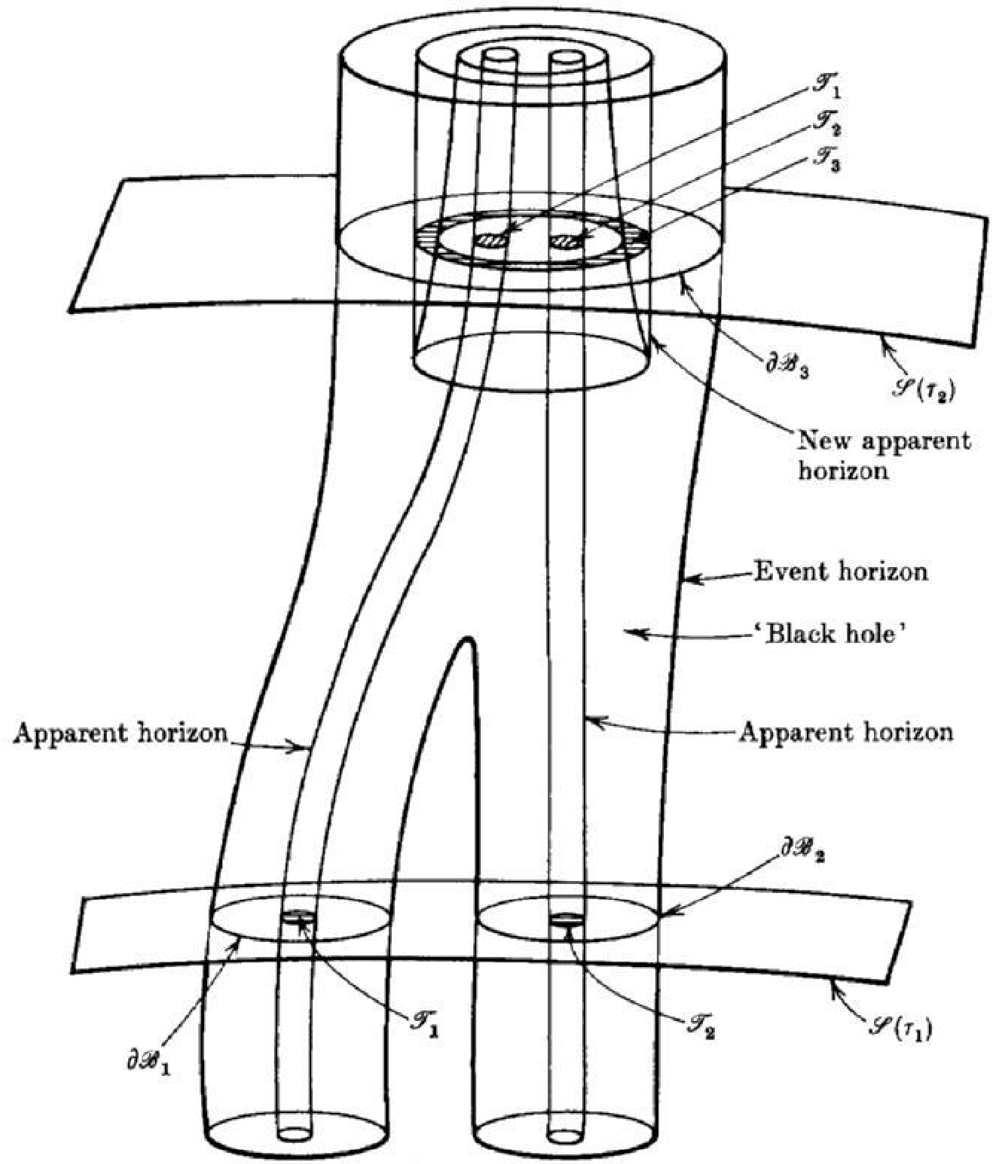


FIGURE 60. The collision and merging of two black holes. At time τ_1, there are apparent horizons $\partial\mathscr{T}_1$, $\partial\mathscr{T}_2$ inside the event horizons $\partial\mathscr{B}_1$, $\partial\mathscr{B}_2$ respectively. By time τ_2, the event horizons have merged to form a single event horizon; a third apparent horizon has now formed surrounding both the previous apparent horizons.

The Einstein equations are simple!

$$\boxed{\boldsymbol{G} = 8\pi\boldsymbol{T}}$$

$\boldsymbol{T}$ is the *energy-momentum tensor*, which describes the mass and energy present, and is given by a *matter model*. E.g., for a perfect fluid $\boldsymbol{T} = (\rho + p)\, U \otimes U + \rho\boldsymbol{g}$. For a vacuum, $\boldsymbol{T} = 0$.

The *Einstein tensor* $\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{g})$ is a second order tensor built from the metric $\boldsymbol{g}$ in three steps:

1. construct the Riemann curvature tensor
2. take its trace to get the Ricci tensor
3. trace-reverse the Ricci tensor to get the Einstein tensor

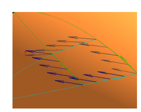
The metric

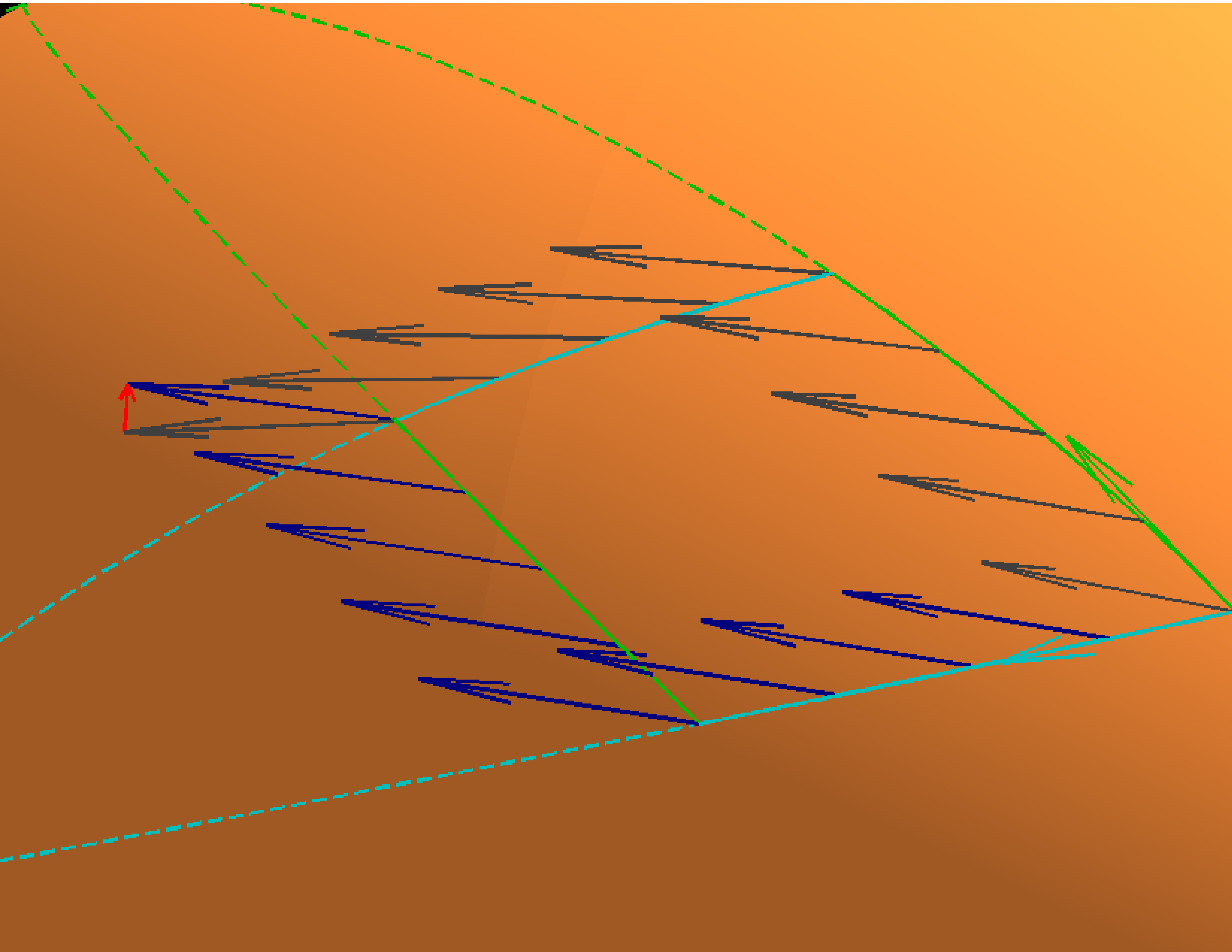
A *metric tensor* $\boldsymbol{g}$ on a manifold: given a point m in the manifold and two tangent vectors $\boldsymbol{X}$, $\boldsymbol{Y}$ at m, it computes a number $\boldsymbol{g}_m(\boldsymbol{X}, \boldsymbol{Y})$, linear and symmetric in $\boldsymbol{X}$ and $\boldsymbol{Y}$, and smoothly varying in m.

On a *pseudo-Riemannian manifold*, such as spacetime, the metric is not positive definite. It determines a lightcone of vectors for which $g_m(\boldsymbol{X}, \boldsymbol{X}) = 0$, separating the spacelike and timelike vectors.

The metric defines the length of vectors and angles between them. It determines a notion of *parallel transport* of a tangent vector from one point to another along a curve, and therefore a notion of directional differentiation of vectorfields.

The Riemann curvature tensor measures the failure of two directional derivatives to commute.





The Riemann tensor maps three vectorfields trilinearly to a fourth vectorfield: it is a fourth order tensor. It depends nonlinearly on the metric $\boldsymbol{g}$.

Taking a trace gives a scalar-valued bilinear map on vectorfields, the *Ricci tensor* $\boldsymbol{R}$.

$$\boldsymbol{G} = \boldsymbol{R} - \tfrac{1}{2}(\operatorname{tr}\boldsymbol{R})\boldsymbol{g}$$

If $\phi : M \to N$ is any diffeomorphism of manifolds and we have a metric g on M, then we can push forward to get a metric $\phi_* g$ on N. With this choice of metric ϕ is an isometry. It is obvious that the Riemann/Ricci/Einstein curvature tensors associated with $\phi_* g$ on N are just the push-forwards of the those associated with g on M. So if g satisfies the vacuum Einstein equations, so does $\phi_* g$.

In particular we can map a manifold to itself diffeomorphically, leaving it unchanged in all but a small region. This shows that the Einstein equations plus boundary conditions can never determine a unique metric on a manifold.

Uniqueness can never be for more than an equivalence class of metrics under diffeomorphism.

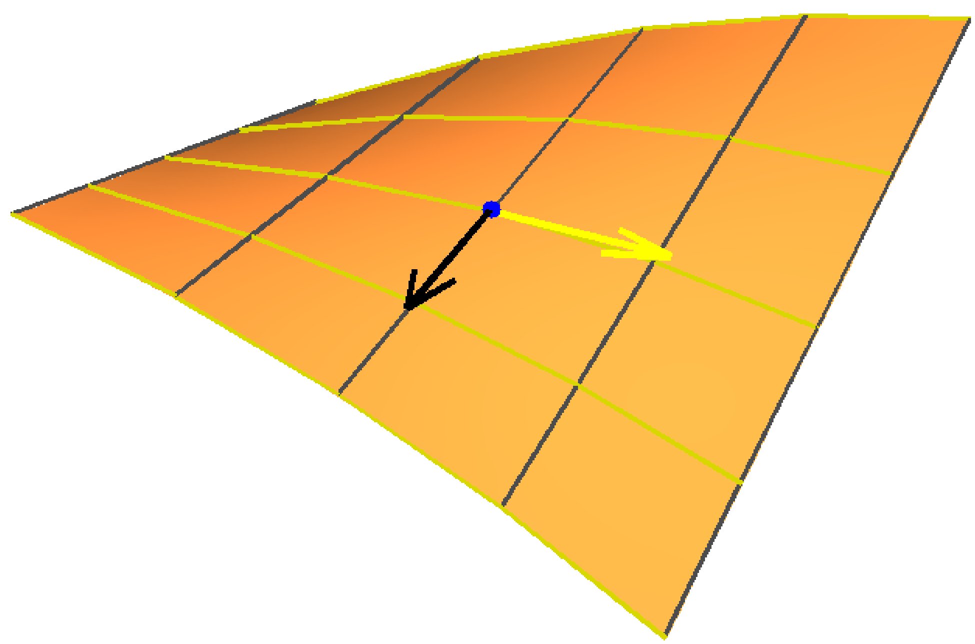
$$\boldsymbol{G} = \frac{\mathsf{G}}{c^4}\, 8\pi \boldsymbol{T}$$

$\frac{\mathsf{G}}{c^4} \approx 8 \times 10^{-50}$ sec^2/g cm

The Einstein equations: PDE viewpoint

Although they represent relatively simple geometry, the Einstein equations are among the most complicated PDEs in mathematical physics.

To get PDEs we choose coordinates x^α, $0 \le \alpha \le 3$, on the manifold. These determine a basis $\boldsymbol{a}_\alpha$ in the tangent space at each point, and so the metric is given by a symmetric 4×4 matrix $g_{\alpha\beta} = \boldsymbol{g}(\boldsymbol{a}_\alpha, \boldsymbol{a}_\beta)$ with inverse $g^{\alpha\beta}$.



If a vectorfield has coordinates v^α, then its derivative (defined via parallel transport, so dependent on the metric, but not on the choice of coordinate system) has coordinates

$$\nabla_\beta v^\alpha = \frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\delta} v^\delta$$

where the *Christoffel symbols* are given by

$$\Gamma^\alpha_{\beta\delta} = \frac{1}{2} g^{\alpha\lambda} \left(\frac{\partial g_{\beta\lambda}}{\partial x^\delta} + \frac{\partial g_{\lambda\delta}}{\partial x^\beta} - \frac{\partial g_{\beta\delta}}{\partial x^\lambda} \right).$$

The *Riemann curvature tensor* then has coordinates ${R_{\alpha\beta\gamma}}^{\delta}$ such that $(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)V^\delta = {R_{\alpha\beta\gamma}}^{\delta}V^\gamma$.

$${R_{\alpha\beta\gamma}}^{\delta} = \frac{\partial\Gamma^\delta_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial\Gamma^\delta_{\alpha\gamma}}{\partial x^\beta} + \Gamma^\epsilon_{\beta\gamma}\Gamma^\delta_{\epsilon\alpha} - \Gamma^\epsilon_{\alpha\gamma}\Gamma^\delta_{\epsilon\beta}$$

Ricci tensor: $R_{\alpha\beta} = {R_{\alpha\delta\beta}}^{\delta}$

Einstein tensor: $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}(g^{\gamma\delta}R_{\gamma\delta})\, g_{\alpha\beta}$

The Einstein equations as PDEs

- Ten quasilinear 2nd order PDEs in 4 independent variables and 10 unknowns. (Each expands out to over 1000 terms.)
- The equations are not independent (the Bianchi identities imply $\nabla_\alpha G^{\alpha\beta} \equiv 0$).
- Gauge freedom: if $g_{\alpha\beta}(x)$ is a solution of the vacuum Einstein equations and $x' = \psi(x)$ any diffeomorphism, then $g'_{\alpha\beta}(x')$ is another solution, where

$$g_{\alpha\beta}(x) = \frac{\partial \psi^\gamma}{\partial x^\alpha}(x)\frac{\partial \psi^\delta}{\partial x^\beta}(x) g'_{\gamma\delta}(x').$$

- System is not elliptic, parabolic, or hyperbolic in any usual sense.

Given a Riemannian 3-manifold S with metric $\boldsymbol{\gamma}$ and another symmetric 2-tensor $\boldsymbol{\kappa}$, find a *Cauchy development*: a Lorentzian 4-manifold M and an imbedding $S \hookrightarrow M$ so that $\boldsymbol{\gamma}$ is the induced metric on S and $\boldsymbol{\kappa}$ is its second fundamental form.

Local existence and uniqueness (Choquet-Bruhat, '52): If the data $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ satisfy the necessary constraints, there exists a maximal Cauchy development, unique up to isometry.

The ADM $3+1$ decomposition

Choose $x^0 = t$ timelike, x^i spacelike $(i = 1, 2, 3)$ and express the 4-metric $g_{\alpha\beta}$ in terms of a time-dependent spatial 3-metric h_{ij}, shift b_i, and lapse a:

$$\left(\begin{array}{c|ccc} g_{00} & g_{01} & g_{02} & g_{03} \\ \hline g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{array}\right) = \left(\begin{array}{c|ccc} |b|^2 - \alpha^2 & b_1 & b_2 & b_3 \\ \hline b_1 & h_{11} & h_{12} & h_{13} \\ b_2 & h_{21} & h_{22} & h_{23} \\ b_3 & h_{31} & h_{32} & h_{33} \end{array}\right)$$

The corresponding partition of the Einstein tensor gives

Hamiltonian constraint (time-indep.) $\longrightarrow$

$$\left(\begin{array}{c|ccc} G_{00} & G_{01} & G_{02} & G_{03} \\ \hline G_{10} & G_{11} & G_{12} & G_{13} \\ G_{20} & G_{21} & G_{22} & G_{23} \\ G_{30} & G_{31} & G_{32} & G_{33} \end{array}\right)$$

$\longleftarrow$ Momentum constraint (1st order)

evolution eqs.

$$\frac{\partial h_{ij}}{\partial t} = -2ak_{ij} + \nabla_i b_j + \nabla_j b_i$$

$$\frac{\partial k_{ij}}{\partial t} = a[R_{ij} + (\operatorname{tr} k)k_{ij} - 2k_{il}k^l_j] + b^l \nabla_l k_{ij} - k_{il}\nabla_j b^l + k_{lj}\nabla_i b^l - \nabla_i \nabla_j a$$

$$\operatorname{tr} R - (\operatorname{tr} k)^2 - k_{ij}k^{ij} = 0$$

$$\nabla^j k_{ij} - \nabla_i \operatorname{tr} k = 0$$

The ADM solution procedure

- Choose lapse α and shift β_j in advance or as the computation progresses (could solve other PDEs).
- Determine initial data γ_{ij}, K_{ij} satisfying the four constraint equations. (initial data problem)
- Evolve the initial data using the evolution equations.

Theorem. If the constraints are satisfied for $t = 0$ and the evolution equations are satisfied, the constraints are satisfied for all time (Bianchi identities).

(For numerical work it may be useful to reimpose the constraints from time to time.)

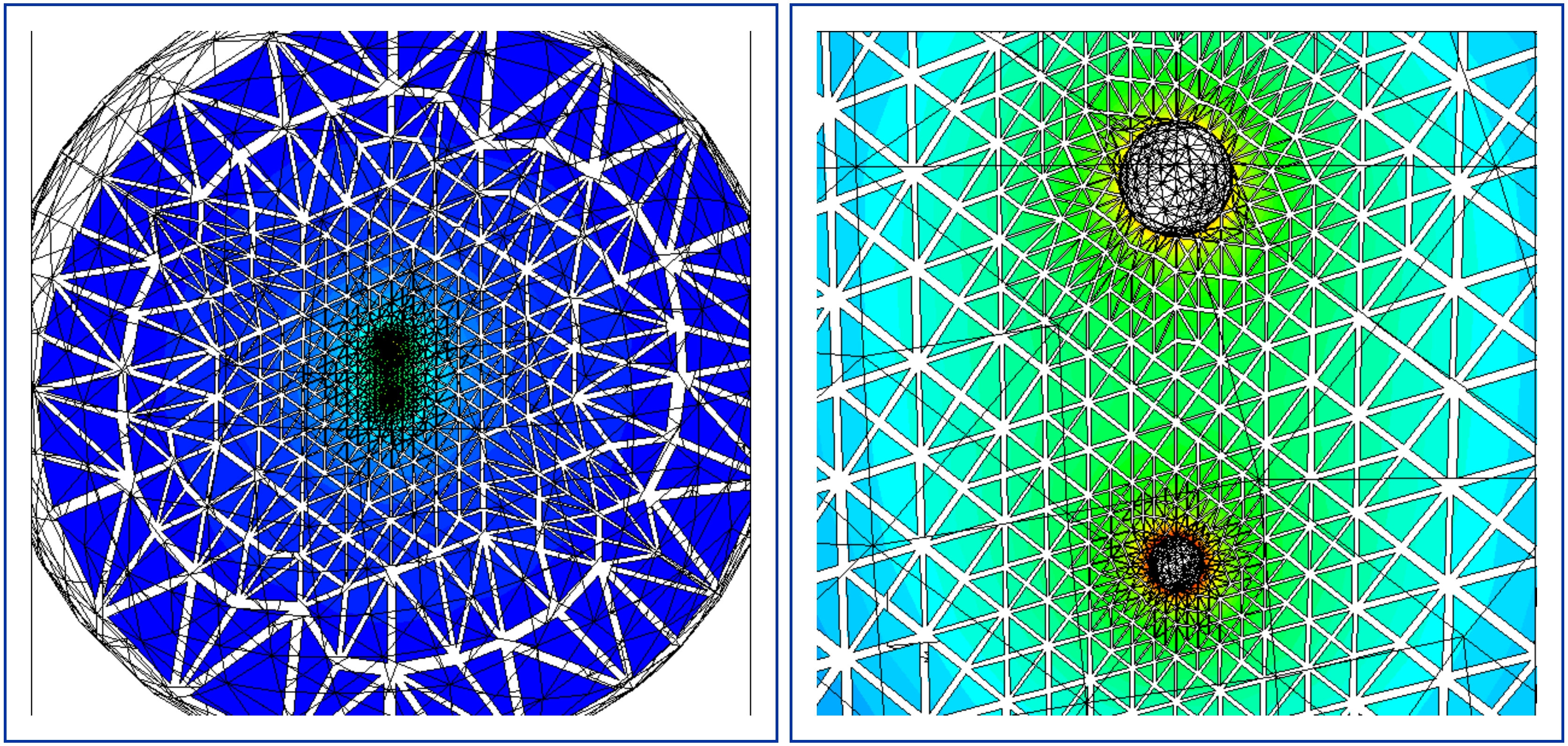
The initial data problem is 4 equations in 12 unknowns. The York–Lichnerowicz conformal decomposition provides a way to divide the unknowns into 8 freely-specifiable quantity and 4 quantities satisfying 4 elliptic equations.

- The spatial metric $\boldsymbol{\gamma}$ is decomposed as $\psi^4\hat{\boldsymbol{\gamma}}$ with $\hat{\boldsymbol{\gamma}}$ the normalized (e.g., $\det = 1$) *background metric* to be specified, and ψ the *conformal factor*, to be computed.
- The extrinsic curvature $\boldsymbol{K}$ is decomposed into its trace-free part and its trace, with the latter to be specified: $\boldsymbol{K} = \psi^{-2}\boldsymbol{A} + \frac{1}{3}\mathrm{tr}(K)\boldsymbol{\gamma}$.
- $\boldsymbol{A}$ is decomposed as a divergence-free trace-free tensor $\boldsymbol{A}^*$, to be specified, and the symmetric trace-free gradient of a potential vector $\boldsymbol{W}$.

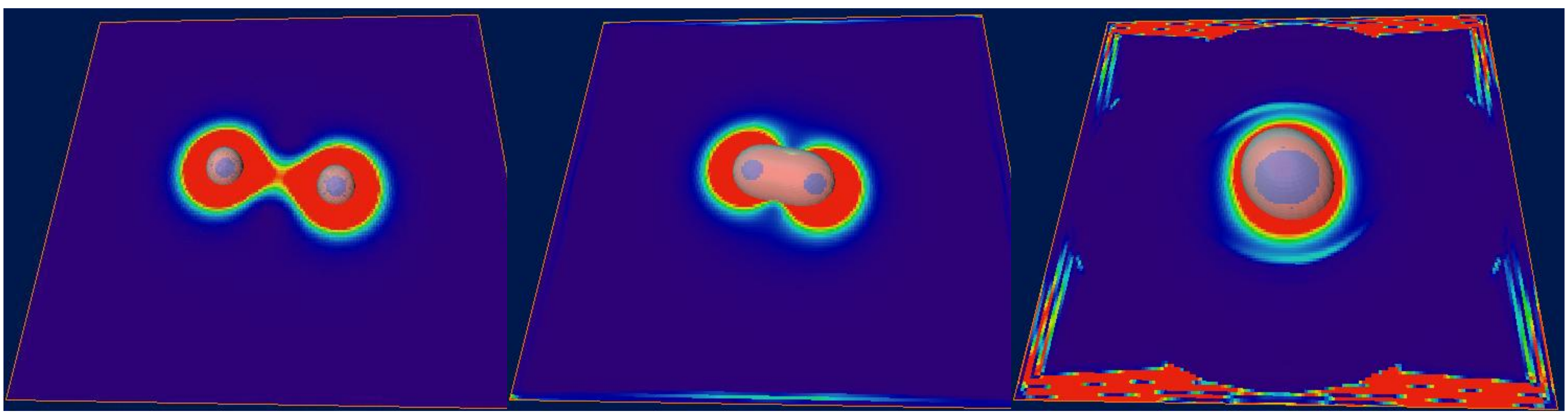
We want to find initial data which represents two black holes which, when evolved, eventually collide and merge into one black hole, spewing forth gravity waves.

There is a great deal of freedom in developing initial data compatible with the constraints, but it is not so clear how to find data which is physically relevant to black hole collisions.

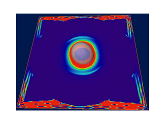
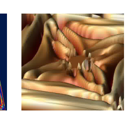
One approach (which may not be the best) is to choose the free quantities $\mathrm{tr}(\boldsymbol{K})$, $\boldsymbol{A}^*$, and $\hat{\gamma}$ by linear superposition of single blackholes moving with constant velocity and spin (boosted Kerr black holes in Kerr-Schild coordinates). We then solve an elliptic system for the conformal factor ψ and the vector potential $\boldsymbol{W}$ so the constraints are satisfied.



Arnold–Mukherjee '96



Brandt, Correll, Gomez, Huq, Laguna, Lehner, Marronetti, Matzner,
Neilsen, Pullin, Schnetter, Shoemaker, Winicour '2000

Computational difficulties

- Not clear how to find a well-posed formulation appropriate for computation. Hyperbolicity. . .
- Stable evolution scheme.
- Treatment of constraints.
- Gauge freedom; choice of lapse and shift.
- Outer boundary conditions.
- Black hole singularities.
- Horizon identification.
- Excision, inner boundary conditions.

Eventually:

- Incorporation of matter models.
- Extraction of far-field wave forms.
- Solution of the inverse problem.

The ADM evolution equations are not hyperbolic in any usual sense. Many authors have considered first-order hyperbolic formulations (N.B.: second order formulations deserve more attention): Fritelli & Reula; Baumgarte & Shapiro; Shibata & Nakamura; Wilson, Mathews & Maronetti; Kidder, Scheel, & Teukolsky, . . .

These are generally derived from ADM by introducing all the first spatial derivatives of the metric (or extrinsic curvature) or quantities closely related to them (18 new variables); combining constraint equations with the evolution; and playing with the lapse and shift.

The systems are quite big and complicated, and it is not clear to what extent numerical methods on them perform better.

It appears that there is a more canonical way. . .

Look for solution as a perturbation of flat space, unit lapse, zero shift:

$$
\begin{aligned}
h_{ij} &= \delta_{ij} + \gamma_{ij} & \qquad a &= 1 + \alpha \\
k_{ij} &= 0 + \kappa_{ij} & \qquad b_i &= 0 + \beta_i
\end{aligned}
$$

To first order, γ, κ, α, β satisfy

$$
\begin{aligned}
\dot\gamma &= -2\kappa + 2\epsilon\beta \\
\dot\kappa &= P\gamma - \nabla\nabla\alpha
\end{aligned}
\qquad\qquad
\begin{aligned}
\operatorname{div} M\gamma &= 0 \\
M\kappa &= 0
\end{aligned}
$$

$$
P\gamma := \epsilon \operatorname{div}\gamma - \tfrac{1}{2}\Delta\gamma - \tfrac{1}{2}\nabla\nabla \operatorname{tr}\gamma, \qquad M\kappa := \operatorname{div}\kappa - \nabla \operatorname{tr}\kappa
$$

$$
\operatorname{div} M \equiv \operatorname{tr} P, \quad M \operatorname{curl}_{\mathsf{r}} \equiv M\nabla\nabla \equiv 0, \quad \operatorname{div} M\epsilon \equiv 0,
$$

$$
MP \equiv -\tfrac{1}{2}\nabla \operatorname{div} M, \qquad \operatorname{curl}_{\mathsf{r}}\tau - \operatorname{curl}_{\mathsf{c}}\tau = \operatorname{Skw} M\tau
$$

Constraint preservation

$$\dot{\gamma} = -2\kappa + 2\epsilon\beta$$
$$\dot{\kappa} = P\gamma - \nabla\nabla\alpha$$

$$p := -\frac{1}{2}\operatorname{div} M\gamma = 0$$
$$q := M\kappa = 0$$

$$\dot{p} = \operatorname{div} q$$
$$\dot{q} = \nabla p$$

$$p(0) = q(0) = 0 \implies p \equiv 0,\ q \equiv 0$$

$$\dot{\gamma} = -2\kappa + 2\epsilon\beta, \qquad \dot{\kappa} = P\gamma - \nabla\nabla\alpha$$

$$\ddot{\gamma} = L\gamma + 2\epsilon\dot{\beta} + 2\nabla\nabla\alpha, \qquad \ddot{\kappa} = L\kappa - \nabla\nabla\dot{\alpha}$$

$$L\gamma := -2P\gamma = \Delta\gamma + \nabla\nabla\,\mathrm{tr}\,\gamma - 2\epsilon\,\mathrm{div}\,\gamma$$

L is *not* self-adjoint. Its symbol is diagonalizable with 3 positive eigenvalues and 3 zero eigenvalues.

One idea is to choose the lapse and shift related to γ. If, e.g., $\dot{\beta} = \mathrm{div}\,\gamma$ and $\alpha = -\,\mathrm{tr}\,\gamma/2$, then $\ddot{\gamma} = \Delta\gamma$. This is nice theoretically (linear analogue of *harmonic gauge*). But for numerical work it creates additional equations for lapse and shift and, more importantly, takes away valuable gauge freedom. $\alpha = -\,\mathrm{tr}\,\gamma/2 + \tilde{\alpha}$ is the analogue of *densitizing the lapse*.

A new symmetric hyperbolic formulation

$$\dot\gamma = -2\kappa + 2\epsilon\beta$$
$$\dot\kappa = P\gamma - \nabla\nabla\alpha$$

$$\operatorname{div} M\gamma = 0$$
$$M\kappa = 0$$

$$\ddot\kappa = -2P\kappa - \nabla\nabla\dot\alpha \qquad (P\epsilon \equiv 0)$$

$$-2P\kappa = -\operatorname{curl}_s\operatorname{curl}_s\kappa + \cancel{\tfrac{1}{2}\epsilon M}\kappa + \tfrac{1}{2}(\cancel{\operatorname{div} M}\kappa)$$

Define $\lambda = \dot\kappa$, $\mu = \operatorname{curl}_s\kappa$
(so $\kappa = \kappa(0) + \int_0^t \lambda$, $\gamma = \gamma(0) + \int_0^t (-2\kappa + 2\epsilon\beta)$).

$$\boxed{\begin{aligned}\dot\lambda &= -\operatorname{curl}_s\mu - \nabla\nabla\dot\alpha\\ \dot\mu &= \operatorname{curl}_s\lambda\end{aligned}}$$

FOSH

$\lambda(0) = P\gamma(0) - \nabla\nabla\alpha(0), \qquad \mu(0) = \operatorname{curl}_s\kappa(0)$

$$\dot{\lambda} = -\operatorname{curl}_{\mathsf{s}} \mu - \nabla\nabla\dot{\alpha}, \quad \dot{\mu} = \operatorname{curl}_{\mathsf{s}} \lambda$$

$$\lambda(0) = P\gamma(0) - \nabla\nabla\alpha(0), \quad \mu(0) = \operatorname{curl}_{\mathsf{s}} \kappa(0)$$

$$M\gamma(0) = 0 \quad \Longrightarrow \quad M\lambda(0) = MP\gamma(0) = -\tfrac{1}{2}\nabla \operatorname{div} M\gamma(0) = 0$$
$$M\kappa(0) = 0 \quad \Longrightarrow \quad M\mu(0) = M \operatorname{curl}_{\mathsf{s}} \kappa(0) = M \operatorname{curl}_{\mathsf{r}} \kappa(0) = 0$$

If the initial data $\lambda(0)$, $\mu(0)$ is derived from ADM initial data which satisfies the Hamiltonian and momentum constraints, then they satisfy the constraints $M\lambda(0) = M\mu(0) = 0$.

$$p := M\lambda,\ q := M\mu = 0 \quad \Longrightarrow \quad \dot{p} = -\tfrac{1}{2}\operatorname{curl} q, \quad \dot{q} = \tfrac{1}{2}\operatorname{curl} p$$

If the initial data satisfy the constraints $M\lambda(0) = M\mu(0) = 0$, and λ and μ satisfy the evolution, then $M\lambda = M\mu = 0$ for all time.

If we seek plane wave solutions

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} l \\ u \end{pmatrix} f(ct + n \cdot x)$$

we find the characteristic speeds

0, 0, 0, 0

$1,\ 1,\ -1,\ -1$

$1/2,\ 1/2,\ -1/2,\ -1/2$

An alternative to unconstrained evolution

$$\dot{\lambda} = -\operatorname{curl}_{\mathsf{s}} \mu - \nabla\nabla\dot{\alpha}$$
$$\dot{\mu} = \operatorname{curl}_{\mathsf{s}} \lambda$$

$$\dot{\lambda} = -\operatorname{curl}_{\mathsf{s}} \mu - M^*\nu - \nabla\nabla\dot{\alpha}$$
$$\dot{\mu} = \operatorname{curl}_{\mathsf{s}} \lambda - M^*\xi$$
$$\dot{\nu} = M\lambda \; - \; k^2\nu$$
$$\dot{\xi} = M\mu \; - \; k^2\xi$$

This is a FOSH system in 18 variables.

$M\lambda(0) = M\mu(0) = 0$, $\nu(0) = \xi(0) = 0 \implies \nu \equiv \xi \equiv 0.$

An analogous procedure can be carried out for the full nonlinear system. It is considerably more complicated. In particular, the evolution equations for γ and κ do not decouple from the FOSH system for λ and μ.

It remains to be seen whether this system is better suited to computation.

They're just a set of PDEs, Larry. — Bryce DeWitt

The numerical solution of the Einstein equations, and in particular the numerical simulation of gravitational wave emission from the collision of black holes, presents tremendous challenges.

The huge effort to construct observatories based on gravitational radiation depends on our meeting this challenge.

Success will almost surely require the collaboration of mathematicians expert in the theory and approximation of nonlinear PDE.

http://ima.umn.edu/nr