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GRADUATE SCHOOL

Constraint Preserving Boundary Conditions for Hyperbolic Formulations of Einstein's Equations

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Abstract

Einstein's system of equations in the ADM decomposition involves two subsystems of equations: evolution equations and constraint equations. For numerical relativity, one typically solves the constraint equations only on the initial time slice, and then uses the evolution equations to advance the solution in time. Our interest is in the case when the spatial domain is bounded and appropriate boundary conditions are imposed. A key difficulty, which we address in this thesis, is what boundary conditions to place at the artificial boundary that lead to long time stable numerical solutions. We develop an effective technique for finding well-posed constraint preserving boundary conditions for constrained first order symmetric hyperbolic systems. By using this technique, we study the preservation of constraints by some first order symmetric hyperbolic formulations of Einstein's equations derived from the ADM decomposition linearized around Minkowski spacetime with arbitrary lapse and shift perturbations, and the closely related question of their equivalence with the linearized ADM system. Our main result is the finding of well-posed maximal nonnegative constraint preserving boundary conditions for each of the first order symmetric hyperbolic formulations under investigation, for which the unique solution of the corresponding initial boundary value problem provides a solution to the linearized ADM system on polyhedral domains.

We indicate how to transform first order symmetric hyperbolic systems with constraints into equivalent unconstrained first order symmetric hyperbolic systems (extended systems) by building-in the constraints. We analyze and prove the equivalence between the original and extended systems in both the case of pure Cauchy problem and initial boundary value problem. These results seem to be very useful for transforming constrained numerical simulations into unconstrained ones. As applications, we derive the extended systems corresponding to the very same hyperbolic formulations of Einstein's equations for which boundary conditions consistent with the constraints have been found. Boundary conditions for these extended systems that make them equivalent to the original constrained systems are provided.

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Chapter 1

Introduction

1.1 Background and Motivations

In a nutshell, general relativity says that "matter tells space how to curve, while the curvature of space tells matter how to move," in a now famous phrase that the physicist John Wheeler once said. So, general relativity serves both as a theory of space and time and as a theory of gravitation. Einstein began with two basic but subtle and powerful ideas: gravity and acceleration are indistinguishable and matter in free-fall always takes the shortest possible path in curved spacetime. One of Einstein's most important discoveries was a system of equations which relates spacetime and matter.

While the theory of general relativity has tremendous philosophical implications and has given rise to exotic new physical concepts like black holes and dark matter, it is also crucial in some areas of modern technology such as global positioning systems. All these predictions and applications make general relativity a spectacularly successful theory. However, though a number of its major predictions have been carefully verified by experiments and observations, there are other key predictions, as the existence of gravitational waves, that remain to be fully tested. Einstein's equations possess solutions describing wavelike undulations in the spacetime. These gravitational waves correspond to ripples in spacetime itself, they are not waves of any substance or medium. Like electromagnetic waves, gravitational waves move at the speed of light and carry energy. In spite of carrying enormous amounts of energy from some of the most violent events in the universe, gravitational waves are almost unobservable. For detecting and analyzing them, state–of–the–art detectors have been built in the United States and overseas. The development of these observatories, just coming online now, is one of the grandest scientific undertakings of our time, and the most expensive project ever funded by the National Science Foundation. With a network of gravitational waves detectors, mankind could open up a whole new window on the cosmic space. The study of the universe using gravitational waves would not be just a simple extension of the optical and electromagnetic possibilities, it would be the exploitation of an entirely new spectrum that could unveil parts and aspects of the universe inaccessible so far.

This enormous technological effort to build ultra-sensitive detectors has been followed by an intense quest for developing computer methods to solve Einstein's equations. Having invested so much to detect gravitational waves, it is crucial that we be able to interpret the waveforms detected. Most recent investigations in numerical relativity have been based on first order hyperbolic formulations derived from the Arnowitt, Deser, and Misner, or ADM, decomposition [6] (see also [53]) of Einstein's equations and some results have been obtained in spherical symmetry and axisymmetry. However, in the general three spatial dimensions case, which is needed for the simulation of realistic astrophysical systems, it has not been possible to obtain long term stable and accurate evolutions. One might argue that present day computational resources are still insufficient to carry out high enough resolution three dimensional simulations. However, the difficulty is likely to be more fundamental than that. It seems that there is insufficient understanding of the structure of Einstein's equations and there are too many unsolved questions related to how to approach them numerically.

Einstein's system of equations can be decomposed into two subsystems of equations (ADM decomposition): evolution equations and constraint equations (Hamiltonian constraint and momentum constraints). For numerical relativity, one typically solves the constraint equations only on the initial time slice, and then uses the evolution equations to advance the solution in time. A very difficult task is to derive good boundary conditions, and this problem is crucial if one takes into account that it seems impossible to have in the near future the computational power to put the boundaries far away from sources, far enough that they would not affect the region of numerical spacetime being looked at. Traditionally, most numerical relativity treatments have been careful to impose initial data that satisfies the constraints. However, very rarely boundary conditions that lead to well-posedness are used and much less frequently are they consistent with the constraints. Stewart [47] has addressed this subject within Frittelli-Reula formulation [25] linearized around flat space with unit lapse and zero shift in the quarter plane. Both main system and constraints propagate as first order strongly hyperbolic systems. This implies that vanishing values of the constraints at $t = 0$ will propagate along characteristics. One wants the values of the incoming constraints at the boundary to vanish. However, one can not just impose them to vanish along the boundaries since the constraints involve derivatives of the fields across the boundary, not just the values of the fields themselves. If the Laplace-Fourier transforms are used, the linearity of the differential equations gives algebraic equations for the transforms of the fields. Stewart deduces boundary conditions for the main system in terms of Laplace-Fourier transforms that preserve the constraints by imposing the incoming modes for the system of constraints to vanish and translating these conditions in terms of Laplace-Fourier transforms of the main system variables. The idea of imposing the vanishing of the incoming constraints as boundary conditions is pursued further in [14] within Einstein-Christoffel formulation [7] in the simple case of spherical symmetry. The radial derivative is eliminated in favor of time derivative in the expression of the incoming constraints by using the main evolution system. In [15], these techniques are refined and employed for the linearized generalized Einstein-Christoffel formulation [36] around flat spacetime with vanishing lapse and shift perturbations on a cubic box. By considering well posed boundary conditions for the constraint system and trading normal derivatives for time and tangential ones, face systems are obtained that need to be solved first together with the compatibility conditions at the edges of the faces. The solutions of the face systems are used to impose well posed constraint preserving boundary conditions for the main system. A construction with several points in common with the one just described can be also found in [49]. These two papers, [15] and [49], are the closest to our work. A different approach can be found in [23], [24], where the authors stray away from the general trend of seeking to impose the constraints along the boundary. Their method consists in making the four components of the Einstein tensor projection along the normal to the boundary vanish. In the case of Einstein-Christoffel formulation restricted to spherical symmetry, the same boundary conditions as in [14] are obtained.

Before we end this brief review, it should also be mentioned here the work done on boundary conditions for Einstein's equations in harmonic coordinates [48], [49], when Einstein's equations become a system of second order hyperbolic equations for the metric components. The question of the constraints preservation does not appear here, as it is hidden in the gauge choice (the constraints have to be satisfied only at the initial surface, the harmonic gauge guarantees their preservation in time).

What follows next is a summary of the contents of this dissertation, with emphasis on the ideas that connect the different parts.

1.2 Thesis Organization

This dissertation is divided into three main parts. The first part is represented by Chapter 2 and mainly describes our results concerning first order symmetric hyperbolic (FOSH) systems of partial differential equations. A special attention is being placed on FOSH systems with constraints and their well-posedness with or without boundary conditions. This first part represents a portion of the background theory needed for the rest of the dissertation. The second part, Chapter 3, is focused on the ADM decomposition of Einstein's equations due to Arnowitt, Deser, and Misner [6] and some important first order hyperbolic formulations derived from it. A novelty in this part is represented by the introduction and analysis in Subsection 3.4.4 of a new first order symmetric hyperbolic formulation of the linearized ADM decomposition due to Arnold [2]. The third part, represented by Chapter 4, is the most important part of this thesis. Here, we address a key difficulty in numerical relativity, the derivation of boundary conditions that lead to well posedness and consistent with the constraints. In the beginning of Chapter 4 we introduce and analyze a simpler model problem which gives good insight for the more complex case of Einstein's equations. The core of Chapter 4 consists of the analysis of three important first order symmetric hyperbolic formulations of Einstein's equations for which we provide well-posed constraint-preserving boundary conditions.

In the remainder of this introduction, we will describe the principal results of this dissertation.

1.3 Principal Results

We have developed an effective and general technique for finding well-posed constraint preserving boundary conditions for constrained first order symmetric hyperbolic systems. The key point of this technique is the matching of the general forms of maximal nonnegative boundary conditions for the main system and the system of constraints.

By applying this technique, we study the preservation of constraints by the linearized Einstein-Christoffel system around Minkowski spacetime with arbitrary lapse and shift perturbations, and the closely related question of the equivalence of that system and the linearized ADM system. Our interest is in the case when the spatial domain is bounded and appropriate boundary conditions are imposed. However, we also consider the pure Cauchy problem with the result that the linearized Einstein-Christoffel and ADM systems are equivalent. Our main result is the finding of two distinct sets of well-posed maximal nonnegative constraint preserving boundary conditions for which the unique solution of the corresponding linearized Einstein-Christoffel initial boundary value problem provides a solution to the linearized ADM system on polyhedral domains.

We have also obtained similar results for a very recent symmetric hyperbolic formulation of Einstein's equations introduced by Alekseenko and Arnold in [3]. A new first order symmetric hyperbolic formulation of linearized Einstein equations due to Arnold [2] is analyzed. Again, the main result is the finding of well-posed constraint preserving boundary conditions. In fact, same ideas should be applicable to some other formulations and/or in different contexts, as, for example, linearization about some other backgrounds. However, while the strategy of finding adequate boundary conditions is similar, the technical apparatus employed depends very much on the formulation under investigation.

Returning to the more general framework of constrained first order symmetric hyperbolic systems, we indicate how to transform such systems into equivalent unconstrained first order symmetric hyperbolic systems (extended systems) by building in the constraints. We also analyze and prove the equivalence between the original and extended systems in both the case of pure Cauchy problem and initial boundary value problem. As applications, we derive extended systems corresponding to the (EC), (AA), and (A) formulations respectively and boundary conditions that make them equivalent to the original constrained systems. These results seem to be useful for transforming constrained numerical simulations into unconstrained ones.

Chapter 2

Symmetric Hyperbolic Systems

2.1 Introduction

In this chapter some basic results on first order symmetric hyperbolic (or FOSH) systems of partial differential equations are briefly reviewed, with special attention being given to systems with constraints and boundary conditions. All these results represent background material relevant to the discussions of the hyperbolic formulations of the Einstein equations which follow in the next chapters. Much more information on hyperbolic systems can be found in the books by John [33], Kreiss and Lorenz [35], Gustafsson, Kreiss and Oliger [27], and Evans [17], among many others.

The second section of this chapter is intended to enlist the basic definition of FOSH systems and some relevant existence and uniqueness results. The third section is dedicated to the analysis of constrained initial value problems in a more abstract framework and in the case of FOSH systems of partial differential equations. The emphasis is on the equivalence between a given system subject to constraints and a corresponding extended unconstrained system. The fourth section deals with boundary conditions for FOSH systems and the connections between the initial boundary value problem for a given FOSH system with constraints and that for the extended system. Section 2.3 and a substantial part of Section 2.4 represent our contribution to the subject.

2.2 Initial Value Problems

In this section we will be concerned with a linear first order system of equations for a column vector $u = u(x,t) = u(x_1, \ldots, x_N, t)$ with m components $u_1, \ldots, u_m$. Such a system can be written as

$$Lu = \partial_t u + \sum_{i=1}^{N} A_i(x,t)\partial_i u = f(x,t) \text{ in } \mathbb{R}^N \times (0,T], \tag{2.1}$$

where $T > 0$. Here $A_1, \ldots, A_N$ are given $m \times m$ matrix functions, and f is a given m-vector field. We will further assume that A_i are of class C^2, with bounded derivatives over $\mathbb{R}^N \times [0,T]$, and $f \in H^1(\mathbb{R}^N \times (0,T); \mathbb{R}^m)$.

As initial data we prescribe the values of u on the hyperplane $t = 0$

$$u = u_0 \text{ on } \mathbb{R}^N \times \{t = 0\}, \tag{2.2}$$

with $u_0 \in H^1(\mathbb{R}^N; \mathbb{R}^m)$. For each $w \in \mathbb{R}^N$, define

$$A(w)(x,t) = \sum_{i=1}^{N} w_i A_i(x,t) \quad (x \in \mathbb{R}^N,\ t \geq 0). \tag{2.3}$$

The system (2.1) is called symmetric hyperbolic if $A_i(x,t)$ is a symmetric $m \times m$ matrix for each $x \in \mathbb{R}^N$, $t \geq 0$ $(i = 1, \ldots, N)$. Thus, the $m \times m$ matrix $A(w)(x,t)$ has only real eigenvalues and the corresponding eigenvectors form a basis of $\mathbb{R}^N$ for each $w \in \mathbb{R}^N$, $x \in \mathbb{R}^N$, and $t \geq 0$.

Remark 1. *More general systems having the form*

$$A_0 \partial_t u + \sum_{i=1}^{N} A_i(x,t)\partial_i u + B(x,t)u = f(x,t) \tag{2.4}$$

are also called symmetric*, provided the matrix functions A_i are symmetric for $i = 0, \ldots, N$, and A_0 is positive definite. The results set forth below can be easily extended to such systems.*

As in [17], Section 7.3.2., let us define the bilinear form

$$A[u,v;t] := \int_{\mathbb{R}^N} \sum_{i=1}^{N} (A_i(\,\cdot\,,\,t\,)\partial_i u) \cdot v\,dx$$

for $0 \le t \le T$, u, $v \in H^1(\mathbb{R}^N;\mathbb{R}^m)$.

Definition. We say

$$u \in L^2((0,T);\, H^1(\mathbb{R}^N;\mathbb{R}^m)), \text{ with } u' \in L^2((0,T);\, L^2(\mathbb{R}^N;\mathbb{R}^m)),$$

is a weak solution of the initial value problem (2.1), (2.2) provided

(i) $(u',v) + A[u,v;t] = (f,v)$ for each $v \in H^1(\mathbb{R}^N;\mathbb{R}^m)$ and a.e. $0 \le t \le T$, and

(ii) $u(0) = u_0$.

Here $(\,\cdot\,,\,\cdot\,)$ denotes the inner product in $L^2(\mathbb{R}^N;\mathbb{R}^m)$.

By using energy methods and the vanishing viscosity technique (see [17], Section 7.3.2.), the following existence and uniqueness result can be proven:

Theorem 1. *The initial value problem* (2.1), (2.2) *has a unique weak solution.*

In what follows, we will be more interested in first order symmetric hyperbolic systems with constant coefficients. For such systems, a more general result (including regularity) is valid.

Theorem 2. *[17], Section 7.3.3. (also [50], Section 16.1.) Assume $u_0 \in H^s(\mathbb{R}^N;\mathbb{R}^m)$, with $s > N/2 + m$. Then there is a unique solution $u \in C^1(\mathbb{R}^N \times [0,\infty);\mathbb{R}^m)$ of the initial value problem* (2.1), (2.2).

The main tool used for proving this theorem is the Fourier transform. The unique C^1 solution is given by:

$$u(x,t) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix\cdot w}[e^{itA(w)}\hat{u}_0(w) + \int_0^t e^{i(t-s)A(w)}\hat{f}(w,s)ds]\,dw. \qquad (2.5)$$

2.3 Constrained Initial Value Problems

2.3.1 Abstract Framework

We introduce in this subsection an extended system corresponding to a given constrained system defined on Hilbert spaces and investigate the equivalence of these two systems.

Let us consider the following system subject to constraints:

$$\dot{y} = Ay + f, \tag{2.6}$$

$$By = 0, \tag{2.7}$$

$$y(0) = y_0, \tag{2.8}$$

where $A : D(A) \subseteq Y \to Y,\ B : D(B) \subseteq Y \to X$ are densely defined closed linear operators on the Hilbert spaces $(X, \langle \cdot\,, \cdot \rangle_X)$ and $(Y, \langle \cdot\,, \cdot \rangle_Y)$, and $f : [0, \infty) \to Y$. Moreover, suppose A is skew-symmetric and

$$A(\operatorname{Ker} B) \subseteq \operatorname{Ker} B. \tag{2.9}$$

Of course, we assume that the compatibility condition $By_0 = 0$ is satisfied. Moreover, another more subtle compatibility condition must hold: $Bf = 0$. This is because, for any fixed $\bar{t} \in [0, \infty)$, $B([y(t) - y(\bar{t})]/(t - \bar{t})) = 0$, for all $t > 0$, and passing to the limit as $t \to \bar{t}$, it turns out that $B(\dot{y}(\bar{t})) = 0,\ \forall \bar{t} \in [0, \infty)$ (since B is a closed operator). By operating on (2.6) with B, it follows that

$$0 = B(\dot{y}(\bar{t})) = BAy(\bar{t}) + Bf(\bar{t}) = Bf(\bar{t}),\ \forall \bar{t} \in [0, \infty),$$

where the last equality comes from (2.9).

Remark 2. *If $f = 0$, then the energy of the solution is preserved:*

$$E(t) = \frac{1}{2}\|y(t)\|^2 = \frac{1}{2}\|y_0\|^2,\ \forall t \geq 0.$$

Theorem 3. *The pair* $(y,z)^T$ *is a solution of the associated unconstrained system*

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & -B^* \\ B & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{2.10}$$

$$y(0) = y_0, \;\; z(0) = 0, \tag{2.11}$$

if and only if $z \equiv 0$ *and* y *satisfies the constrained system* (2.6)–(2.8).

Proof. If $z \equiv 0$ and y satisfies (2.6)–(2.8), then it is clear that $(y, 0)$ satisfies (2.10)–(2.11).

Now suppose (y, z) satisfies (2.10)–(2.11). Observe that we can split Y as a direct sum as following

$$Y = \operatorname{Ker} B \oplus (\operatorname{Ker} B)^{\perp} = \operatorname{Ker} B \oplus \overline{\operatorname{Im} B^*}. \tag{2.12}$$

According to this decomposition of Y,

$$y(t) = y_1(t) + y_2(t), \tag{2.13}$$

where $y_1(t) \in \operatorname{Ker} B$, and $y_2(t) \in \overline{\operatorname{Im} B^*}$.

Since both $\operatorname{Ker} B$ and $\overline{\operatorname{Im} B^*}$ are closed and the corresponding projections are continuous,

$$\dot{y}(t) = \dot{y}_1(t) + \dot{y}_2(t), \tag{2.14}$$

with $\dot{y}_1(t) \in \operatorname{Ker} B$, and $\dot{y}_2(t) \in \overline{\operatorname{Im} B^*}$.

From (2.9), (2.10), and (2.11), we obtain that

$$\dot{y}_1 = Ay_1 + f, \tag{2.15}$$

$$y_1(0) = y_0, \tag{2.16}$$

and

$$\dot{y}_2 = Ay_2 - B^*z, \tag{2.17}$$

$$\dot{z} = By_2, \tag{2.18}$$

$$y_2(0) = 0, \ z(0) = 0. \tag{2.19}$$

Observe that

$$\frac{1}{2}(\|z\|^2)^{\cdot} = \langle \dot{z}, z\rangle_X = \langle By_2, z\rangle_X = \langle y_2, B^*z\rangle_Y = \langle y_2, Ay_2 - \dot{y}_2\rangle_Y = -\langle y_2, \dot{y}_2\rangle_Y = -\frac{1}{2}(\|y_2\|^2)^{\cdot}.$$

Therefore,

$$\|z(t)\|^2 + \|y_2(t)\|^2 = \|z(0)\|^2 + \|y_2(0)\|^2 = 0,$$

which implies $z \equiv 0$ and $y_2 \equiv 0$. Thus, $y = y_1$ and (2.6)–(2.8) are satisfied. □

2.3.2 Constrained First Order Symmetric Hyperbolic Problems

In this subsection, we will prove a result similar to Theorem 3 for the initial value problem

$$\partial_t u = Au + f, \tag{2.20}$$

$$Bu = 0, \tag{2.21}$$

$$u(x, 0) = u_0, \tag{2.22}$$

where $A = \sum_{i=1}^N A_i \partial_i$, with $A_i \in \mathbb{R}^{m \times m}$ constant symmetric matrices, and $B = \sum_{i=1}^N B_i \partial_i$, with $B_i \in \mathbb{R}^{p \times m}$ constant matrices. Of course, we assume that (2.9) and the compatibility conditions $Bu_0 = 0$, $Bf(\cdot, t) = 0$, $\forall t \geq 0$, hold.

Theorem 4. *(equivalence for classical solutions) If $u_0 \in H^s(\mathbb{R}^N; \mathbb{R}^m)$ and $f(\cdot, t) \in H^s(\mathbb{R}^N; \mathbb{R}^m)$, $\forall t \geq 0$, for $s > N/2 + m$, then the pair $(u, z)^T \in C^1(\mathbb{R}^N \times [0, \infty); \mathbb{R}^{m+p})$ is a solution of* *the associated unconstrained system*

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} A & -B^* \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ z \end{pmatrix} + \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{2.23}$$

$$u(x,0) = u_0, \ \ z(x,0) = 0, \tag{2.24}$$

if and only if $z \equiv 0$*, and* $u \in C^1(\mathbb{R}^N \times [0,\infty); \mathbb{R}^m)$ *satisfies the constrained system* (2.20)–(2.22)*.*

Proof. If $z \equiv 0$ and u satisfies (2.20)–(2.22), then it is clear that $(y,0)^T$ satisfies (2.23)–(2.24).

Now, let us prove the converse. Denote by

$$\overline{A} = \begin{pmatrix} A & -B^* \\ B & 0 \end{pmatrix} = \sum_{j=1}^{N} \overline{A}^j \frac{\partial}{\partial x_j}.$$

From (2.5), we know that the solution of (2.23)–(2.24) has the following expression

$$\begin{aligned} \begin{pmatrix} u \\ z \end{pmatrix}(x,t) \ &= \tfrac{1}{(2\pi)^{N/2}} \textstyle\int_{\mathbb{R}^N} e^{ix\cdot y}[e^{-it\overline{A}(y)} \begin{pmatrix} \hat{u}_0 \\ 0 \end{pmatrix}(y) \\ &+ \textstyle\int_0^t e^{-i(t-s)\overline{A}(y)} \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix}(y,s)ds]\, dy. \end{aligned} \tag{2.25}$$

The next step in the proof is to show that

$$\overline{A}^n(y)\begin{pmatrix} \hat{u}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^n(y)\hat{u}_0 \\ 0 \end{pmatrix} \tag{2.26}$$

for all positive integer n.

We are going to prove (2.26) by induction.

For $n = 1$, we have

$$\overline{A}(y) \begin{pmatrix} \hat{u}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} A(y)\hat{u}_0 \\ B(y)\hat{u}_0 \end{pmatrix}.$$

But, since $Bu_0 = 0$, it follows that $B(y)\hat{u}_0 = 0$ by taking the Fourier transform. So

$$\overline{A}(y) \begin{pmatrix} \hat{u}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} A(y)\hat{u}_0 \\ 0 \end{pmatrix}.$$

Assume that (2.26) is true for $n = k - 1$ and let us prove it for $n = k$.

$$\overline{A}^k(y) \begin{pmatrix} \hat{u}_0 \\ 0 \end{pmatrix} = \overline{A}(y) \begin{pmatrix} A^{k-1}(y)\hat{u}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^k(y)\hat{u}_0 \\ B(y)A^{k-1}(y)\hat{u}_0 \end{pmatrix}.$$

Since $u_0 \in \operatorname{Ker} B$, from (2.9), we can see that

$$BA^{k-1}u_0 = 0. \tag{2.27}$$

Applying the Fourier transform to (2.27), we get

$$B(y)A^{k-1}(y)\hat{u}_0 = 0.$$

Thus,

$$\overline{A}^k(y) \begin{pmatrix} \hat{u}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} A^k(y)\hat{u}_0 \\ 0 \end{pmatrix}$$

and the proof of (2.26) is complete.

From (2.26), observe that

$$e^{-it\overline{A}(y)} \begin{pmatrix} \hat{u}_0 \\ 0 \end{pmatrix}(y) = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \overline{A}^n(y) \begin{pmatrix} \hat{u}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-itA(y)}\hat{u}_0 \\ 0 \end{pmatrix}. \tag{2.28}$$

Same arguments show that

$$e^{-i(t-s)\overline{A}(y)} \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix} (y) = \sum_{n=0}^{\infty} \frac{[-i(t-s)]^n}{n!} \overline{A}^n(y) \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i(t-s)A(y)}\hat{f} \\ 0 \end{pmatrix}. \qquad (2.29)$$

From (2.28) and (2.29), it follows that

$$\begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix},$$

with

$$u(x,\, t) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix\cdot y}[e^{-itA(y)}\hat{u}_0(y) + \int_0^t e^{-i(t-s)A(y)}\hat{f}(y,s)ds]\, dy$$

the unique C^1 solution of (2.20)–(2.22). □

2.4 Boundary Conditions

In general, one has to be careful when choosing boundary conditions for a hyperbolic equation (or system). This can be seen even in the simple case of a first order equation in one space dimension (the transport equation). It seems that any acceptable boundary conditions should give the incoming modes into the spatial domain, but they must not try to change the behavior of the outgoing modes. In several dimensions the situation is much more complicated since there is not easy to identify the incoming and outgoing modes. Worse, there may also be waves moving tangent to the boundary, and it is not very clear how these modes could be casted into the boundary conditions.

A few approaches to these questions have been proposed. Some answers have been given by Friedrichs [18] via the "energy method" (see also the work done by Courant and Hilbert). This method provides criteria which are sufficient for constructing boundary conditions that lead to a well-posed problem. Other sufficient conditions have been pointed out by Lax and Phillips in their very interesting work [37]. A necessary and sufficient condition for having a well-posed initial boundary condition has been proved by Hersh [29], but his result was only for systems with constant coefficients and defined on a half-space with non-characteristic boundary conditions. Using Fourier and Laplace transforms, he constructed solutions and derived a necessary and sufficient condition for well-posedness. Later on (in the 1970s and 1980s), more technical approaches came up. Kreiss [34], Majda and Osher [39], among others, proved pretty complicated algebraic results concerning boundary conditions. Remarkably, Kreiss [34] gave a criteria that determine whether a boundary condition is admissible or not. The main point of his approach was the possibility to solve for incoming modes in terms of outgoing modes and boundary conditions. Majda and Osher [39] generalized Kreiss' theory to the case of uniformly characteristic boundary. Other significant contributions to this subject have been made by Rauch [40], Higdon [30], Secchi [42]–[46], among many others.

2.4.1 Maximal Non-Negative Boundary Conditions

In this subsection we prove a well-posedness result for first order symmetric hyperbolic initial boundary value problems that closely follows the ideas of [37], [40], and [18]. Moreover, we give an algebraic characterization of maximal non-negative boundary conditions which will be used later for determining constraint preserving boundary conditions for some constrained first order symmetric hyperbolic systems.

Consider the symmetric hyperbolic system of equations (2.1) on $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, and $f \in L^2(\Omega_T, \mathbb{R}^m)$.

Set $n(x) = (n_1, \ldots, n_N)$ be the outer normal to Ω at $x \in \partial\Omega$, and denote by $A_n(x,t)$ the boundary matrix

$$A_n(x,t) = \sum_{i=1}^{N} n_i A^i. \tag{2.30}$$

We supplement (2.1) with the initial condition (2.2), with $u_0 \in H^1(\Omega)$, and with linear boundary conditions of the following form

$$E(x,t)u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,T). \tag{2.31}$$

Of course, we suppose that the compatibility condition $E(x,0)u_0(x) = 0$ holds on $\partial\Omega$.

In fact, by choosing a function of $H^1(\Omega_T, \mathbb{R}^m)$ satisfying (2.2) and (2.31), and changing the variable, we may assume that $u_0 = 0$; hereafter, we stick with this choice of u_0.

Also, the boundary condition (2.31) may be regarded as

$$u(x,t) \in N(x,t) = \operatorname{Ker} E(x,t), \ \forall (x,t) \in \partial\Omega \times (0,T). \tag{2.32}$$

Denote the formal adjoint of L by L^*

$$L^* u = -\partial_t u - \sum_{i=1}^{N} \partial_i (A^i u).$$

Associated to (2.1), (2.2), and (2.31) (or (2.32)), we consider the adjoint problem

$$L^* v = f \quad \text{in } \Omega_T, \tag{2.33}$$

$$v(x,t) \in (A_n(x,t)N(x,t))^{\perp}, \ \forall (x,t) \in \partial\Omega \times (0,T), \tag{2.34}$$

$$v(x,T) = 0 \quad \text{in } \Omega. \tag{2.35}$$

Next, define the admissible spaces of solutions for both the original problem and the adjoint problem:

$$\mathcal{H} = \{u \in H^1(\Omega_T, \mathbb{R}^m) : \ u \text{ satisfies (2.2) and (2.31) (or (2.32))}\}$$

and

$$\mathcal{H}_* = \{v \in H^1(\Omega_T, \mathbb{R}^m) : \ v \text{ satisfies (2.34) and (2.35) }\}.$$

Observe that, if $u \in \mathcal{H}$ and $v \in \mathcal{H}_*$, then from Green's formula

$$(v, Lu) - (u, L^* v) = \int_{\partial\Omega \times (0,T)} v^T A_n u \, d\sigma,$$

it follows that

$$(v, Lu) = (u, L^*v).$$

Definition. The function $u \in L^2(\Omega_T, \mathbb{R}^m)$ is said to be a weak solution of (2.1), (2.2), and (2.31) if

$$(v, f) - (L^*v, u) = 0, \ \forall v \in \mathcal{H}_*.$$

Definition. The function $u \in L^2(\Omega_T, \mathbb{R}^m)$ is said to be a strong solution of (2.1), (2.2), and (2.31), if the pair (u, f) belongs to the closure of the graph of L; in other words, if u is the limit in the L^2 norm of a sequence of functions $\{u_k\} \subset \mathcal{H}$ such that $f_k = Lu_k \to f$ in the L^2 norm.

Remark 3. *If u solves* (2.1), (2.2), *and* (2.31) *in the strong sense, then it also solves the problem in the weak sense.*

Definition. The boundary condition (2.31) (or (2.32)) is called non-negative if the matrix $A_n(x,t)$ is non-negative over $N(x,t)$

$$u^T A_n u \geq 0, \ \forall u \in N(x,t), \ (x,t) \in \partial\Omega \times (0,T). \tag{2.36}$$

Theorem 5. *([37], [40]) If the boundary condition* (2.31) *is maximal non-negative at each point $(x,t) \in \partial\Omega \times (0,T)$ (meaning that* (2.36) *holds and there is no other larger subspace containing $N(x,t)$ and having the same property), then* (2.1), (2.2), *and* (2.31) *has a unique strong solution for any given integrable function f.*

In order to prove this theorem, we need a couple of intermediate results.

Lemma 6. *For all $u \in \mathcal{H}$ satisfying the condition* (2.36)*, there exists a positive constant $C > 0$ that does not depend on u such that*

$$\|u\|_{L^2(\Omega_T, \mathbb{R}^m)} \leq C \|Lu\|_{L^2(\Omega_T, \mathbb{R}^m)}. \tag{2.37}$$

Proof. Symmetric operators with smooth A^i satisfy the following identity

$$u^T L u = \frac{1}{2}\frac{\partial}{\partial t}(u^T u) + \frac{1}{2}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}(u^T A^i u) + u^T K u, \tag{2.38}$$

where

$$K = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial A^i}{\partial x_i}. \tag{2.39}$$

If we integrate (2.38) over $\Omega_s = \Omega \times (0, s)$, we get

$$\int_{\Omega_s} u^T L u = \frac{1}{2}\int_{\Omega\times\{s\}} u^T u + \frac{1}{2}\int_{\partial\Omega\times(0,s)} u^T A_n u + \int_{\Omega_s} u^T K u. \tag{2.40}$$

From (2.36) and (2.40), it is easy to see that

$$\int_{\Omega\times\{s\}} u^T u \le 2\int_{\Omega_s} u^T L u + 2\|K\|\int_{\Omega_s} u^T u \le (1+2\|K\|)\int_{\Omega_s} u^T u + \int_{\Omega_T}(Lu)^T(Lu). \tag{2.41}$$

Denote by

$$\phi(s) = \int_{\Omega_s} u^T u.$$

Then (2.41) recasts into

$$\phi'(s) \le (1+2\|K\|)\phi(s) + \|Lu\|^2_{L^2(\Omega_T,\mathbb{R}^m)}. \tag{2.42}$$

Applying the Gronwall's lemma to (2.42), it follows that

$$\phi(s) \le \frac{e^{s(1+2\|K\|)}}{1+2\|K\|}\|Lu\|^2_{L^2(\Omega_T,\mathbb{R}^m)}.$$

Hence,

$$\phi(T) = \|u\|^2_{L^2(\Omega_T,\mathbb{R}^m)} \le \frac{e^{T(1+2\|K\|)}}{1+2\|K\|}\|Lu\|^2_{L^2(\Omega_T,\mathbb{R}^m)}.$$

Then, the inequality (2.37) holds for

$$C = \sqrt{\frac{e^{T(1+2\|K\|)}}{1+2\|K\|}}.$$

□

A simple consequence of this lemma is stated next.

Corollary 7. *For any square integrable function f there is at most one solution $u \in \mathcal{H}$ satisfying non-negative boundary conditions to the problem* (2.1), (2.2), *and* (2.31).

Lemma 8. *If the boundary conditions* (2.32) *are maximal non-negative for L, then the adjoint boundary conditions* (2.34) *are non-negative for L^*.*

Proof. According to the expression of L^*, the boundary matrix for L^* is equal to $-A_n$. Hence it suffices to show that $v^T A_n v$ is non-positive for each v satisfying the adjoint boundary condition.

Arguing by contradiction, suppose that there is a v such that $v^T A_n v > 0$.

Consider the linear space $N(x,t) \oplus v$. The elements in this space have the form $u + av$, where $u \in N(x,t)$ and a is a real number.

Observe that

$$(u+av)^T A_n (u+av) = u^T A_n u + 2av^T A_n u + a^2 v^T A_n v = u^T A_n u + a^2 v^T A_n v \geq 0,$$

which is in contradiction with the maximality of $N(x,t)$. □

Proof. (of Theorem 5) *Existence.* We claim that $L(\mathcal{H})$ is dense in $L^2(\Omega_T, \mathbb{R}^m)$. Arguing by contradiction, let us suppose the contrary. Then there exists a non-trivial function v orthogonal to $L(\mathcal{H})$, i.e.

$$(Lu, v) = 0, \ \forall u \in \mathcal{H}.$$

Therefore, v is a weak solution to the adjoint problem corresponding to $f = 0$.

Now, let us prove that $v = 0$. From [37], Theorem 1.1, we get that v is a strong solution; therefore there exists a sequence $v_n \to v$ in L^2, such that $f_n = Lv_n \to 0$ in L^2 and v_n satisfies the (non-negative) adjoint boundary conditions, which implies

$$\|v_n\| \leq C\|f_n\| \to 0.$$

So, $v = 0$. In conclusion, if f is any element of $L^2(\Omega_T, \mathbb{R}^m)$, then we can find a sequence $\{u_n\} \subset \mathcal{H}$ so that $Lu_n \to f,$ as $n \to \infty,$ in L^2.

From Lemma 6, it follows that $\{u_n\}$ is a Cauchy sequence in L^2. Thus, there is $u \in L^2(\Omega_T, \mathbb{R}^m$ such that $u_n \to u$ and $Lu_n \to f$ in L^2.

This implies that u is a strong solution of (2.1), (2.2), and(2.31) for the given $f \in L^2(\Omega_T, \mathbb{R}^m)$.

Uniqueness and Continuous Dependence. Follow from Lemma 6. □

There are results concerning the regularity of the solution of (2.1), (2.2), and (2.31). If $f \in L^1((0,T); L^2(\Omega))$ and $u_0 \in L^2(\Omega)$ then the solution u belongs to $C((0,T); L^2(\Omega))$ (see [40]). In fact, this regularity result can be improved by imposing more regularity for f and u_0 and, in addition, compatibility conditions at the corner $\{t = 0\} \times \partial\Omega$. These compatibility conditions are computed in the following fashion. Denote by $\pi(x)$ the orthogonal projection onto $N(x)^{\perp}$. The compatibility condition of order j comes from expressing $\partial_t^j(\pi u)$ at $\{t = 0\} \times \partial\Omega$ in terms of u_0 and f and requiring that the resulting expression vanishes. For example, for $j = 0$, we find $\pi u_0 = 0$ on $\partial\Omega$, or $u_0 \in N(x)$ for all $x \in \partial\Omega$. For $j = 1$, we have to impose: $\pi u_0 = 0$ and $\pi(f(0,\cdot) - \sum_{i=1}^N A_i\partial_i u_0) = 0$ on $\partial\Omega$.

Definition. ([40]) A smooth vector field $\underline{\gamma}$ on $\overline{\Omega}$ is called *tangential* if and only if, for every $x \in \partial\Omega$, $\langle\underline{\gamma}(x), \underline{n}(x)\rangle = 0$. For s a positive integer, the space $H^s_{\tan}(\Omega)$ consists of those $u \in L^2(\Omega)$ with the property that for any $l \leq s$ and tangential fields $\{\underline{\gamma}_i\}_{i=1}^l$, $\underline{\gamma}_1\underline{\gamma}_2 \cdots \underline{\gamma}_l u \in L^2(\Omega)$ (see [8] for more on $H^s_{\tan}$ spaces).

Theorem 9. *([40]) Suppose $s \geq 1$ is an integer and A_i, N, $\partial\Omega$ are of class $C^{s,1}$. Suppose* *the data* $u_0 \in H^s$ *and* $\partial_t^j f \in L^1((0,T); H_{\tan}^{s-j}(\Omega))$ *for* $0 \le j \le s$ *and in addition there is a* $0 < T' \le T$ *such that* $\partial_t^j f \in L^1((0,T'); H^{s-j}(\Omega))$, $0 \le j \le s$. *If the data satisfy the compatibility conditions up to order* $s-1$, *then the solution* u *of* (2.1), (2.2), *and* (2.31) *lies in* $C^j([0,T]; H_{\tan}^{s-j}(\Omega))$ *for* $0 \le j \le s$.

We close this subsection by giving an algebraic characterization of maximal non-negative boundary conditions.

Suppose that the boundary matrix $A_n(x,t)$ has l_0 0–eigenvalues $\lambda_1(x,t), \ldots ,\lambda_{l_0}(x,t)$, l_- negative eigenvalues $\lambda_{l_0+1}(x,t), \ldots ,\lambda_{l_0+l_-}(x,t)$, and l_+ positive eigenvalues $\lambda_{l_0+l_-+1}(x,t)$, $\ldots$, $\lambda_m(x,t)$. Let $\underline{e}_1(x,t), \ldots, \underline{e}_{l_0}, \underline{e}_{l_0+1}, \ldots, \underline{e}_{l_0+l_-}, \underline{e}_{l_0+l_-+1}, \ldots, \underline{e}_m(x,t)$ be the corresponding eigenvectors. Naturally, at $(x,t) \in \partial\Omega \times (0,T)$, any vector of $N(x,t)$ has the form $v = \sum_{i=1}^m \alpha_i \underline{e}_i(x,t)$. The non-negative condition (2.36) implies:

$$v^T A_n(x,t) v = \sum_{i=1}^m \lambda_i \alpha_i^2 \ge 0,$$

or

$$\sum_{\lambda_i > 0} \lambda_i \alpha_i^2 \ge -\sum_{\lambda_j < 0} \lambda_j \alpha_j^2. \tag{2.43}$$

Observe that the dimension of $N(x,t)$ must be equal to the number of positive and null eigenvalues counted with their multiplicities.

Now, any $v \in N(x,t)$ can be written as $v = Q\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_m)^T$, and $Q = (\underline{e}_1(x,t), \ldots, \underline{e}_m(x,t))$.

Since $N(x,t)$ is a subspace of codimension l_-, there exists a $l_- \times m$ matrix $E(x,t)$ such that

$$N(x,t) = \{v \, : \, E(x,t)v = 0\}.$$

So, $EQ\alpha = 0$, or

$$S_0\alpha_0 + S_-\alpha_- - S_+\alpha_+ = 0, \tag{2.44}$$

where $\alpha_0 = (\alpha_1, \ldots, \alpha_{l_0})$, $\alpha_- = (\alpha_{l_0+1}, \ldots, \alpha_{l_0+l_-})$, $\alpha_+ = (\alpha_{l_0+l_-+1}, \ldots, \alpha_m)$, and $S_0 = EQ_0$, $S_- = EQ_-$, $S_+ = -EQ_+$, with $Q_0 = (\underline{e}_1(x,t), \ldots, \underline{e}_{l_0}(x,t))$, $Q_- = (\underline{e}_{l_0+1}(x,t), \ldots,$

$\underline{e}_{l_0+l_-}(x,t))$, and $Q_+ = (\underline{e}_{l_0+l_-+1}(x,t),\dots,\underline{e}_m(x,t))$.

Since $N(x,t)$ is maximal non-negative, it follows that $\operatorname{Ker} A_n(x,t) \subseteq N(x,t)$, and so,

$$S_0 = EQ_0 = 0_{l_- \times l_0}.$$

Therefore, (2.44) reads

$$S_-\alpha_- - S_+\alpha_+ = 0. \tag{2.45}$$

Observe that the $l_- \times l_-$ matrix S_- is invertible. If not, there exists $\alpha_- \neq 0$ so that $S_-\alpha_- = 0$, and so,

$$v = \sum_{i=l_0+1}^{l_0+l_-} \alpha_i \underline{e}_i(x,t)$$

belongs to $N(x,t)$. But this is in contradiction with (2.43). Hence, S_- is invertible.

So, (2.45) recasts into

$$\alpha_- = S_-^{-1} S_+ \alpha_+.$$

Now, we must have (2.43) satisfied, which leads to

$$\| \begin{pmatrix} \sqrt{|\lambda_{l_0+1}|} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \sqrt{|\lambda_{l_0+l_-}|} \end{pmatrix} S_-^{-1} S_+ \begin{pmatrix} 1/\sqrt{\lambda_{l_0+l_-+1}} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1/\sqrt{\lambda_m} \end{pmatrix} \| \leq 1.$$

Let us state what we have just proved into the following proposition.

Proposition 10. *The boundary condition* (2.31) *is maximal non-negative if and only if there exists a $l_- \times l_+$ matrix $M(x,t)$ such that*

$$E(x,t) = \begin{pmatrix} \underline{e}^T_{l_0+1}(x,t) \\ \vdots \\ \underline{e}^T_{l_0+l_-}(x,t) \end{pmatrix} - M(x,t) \begin{pmatrix} \underline{e}^T_{l_0+l_-+1}(x,t) \\ \vdots \\ \underline{e}^T_{m}(x,t) \end{pmatrix}, \tag{2.46}$$

with

$$\| \begin{pmatrix} \sqrt{|\lambda_{l_0+1}|} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \sqrt{|\lambda_{l_0+l_-}|} \end{pmatrix} M \begin{pmatrix} 1/\sqrt{\lambda_{l_0+l_-+1}} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1/\sqrt{\lambda_m} \end{pmatrix} \| \leq 1.$$

The following corollary is an immediate consequence of Proposition 10.

Corollary 11. *A sufficient condition for having a maximal non-negative boundary condition of the form* (2.31)*, with* $E(x,t)$ *given by* (2.46)*, is*

$$\|M\| \leq \frac{\min_{\lambda_j>0} \sqrt{\lambda_j}}{\max_{\lambda_j<0} \sqrt{|\lambda_j|}}.$$

2.4.2 Boundary Conditions for Constrained Systems

In this subsection we establish some connections between the problems (2.20)–(2.22) and (2.23)–(2.24) restricted to a bounded smooth domain Ω. More precisely, we analyze the links between the boundary subspaces and boundary matrices associated to the two problems.

First of all, it is easy to see that the boundary matrix associated to (2.23)–(2.24) is given by

$$\overline{A}_n(x) = \begin{pmatrix} A_n(x) & B_n^T(x) \\ B_n(x) & 0 \end{pmatrix},$$

where, as in Subsection 2.4.1, $A_n(x) = -\sum_{j=1}^N n_j A^j$, and $B_n(x) = -\sum_{j=1}^N n_j B^j$, with $n(x) = (n_1, \dots, n_N)$ the outer normal to Ω at $x \in \partial\Omega$.

Lemma 12.

$$A_n(x)(\mathrm{Ker}\, B_n(x)) \subset \mathrm{Ker}\, B_n(x),$$

for all $x \in \partial\Omega$.

Proof. Let $v \in \mathrm{Ker}\, B_n(x)$ and $u \in \mathrm{Ker}\, B$ such that $v = \hat{u}(n)$. Applying the Fourier transform to (2.9), it follows that $B_n(x)A_n(x)\hat{u}(n) = 0$, or $B_n(x)A_n(x)v = 0$. So, $A_n(x)(\mathrm{Ker}\, B_n(x)) \subset$

$\operatorname{Ker} B_n(x)$, for all $x \in \partial\Omega$. □

Lemma 13.

$$\operatorname{Ker} \overline{A}_n(x) = (\operatorname{Ker} A_n(x) \cap \operatorname{Ker} B_n(x)) \times \operatorname{Ker} B_n^T(x), \tag{2.47}$$

for all $x \in \partial\Omega$*.*

Proof. Clearly,

$$(\operatorname{Ker} A_n(x) \cap \operatorname{Ker} B_n(x)) \times \operatorname{Ker} B_n^T(x) \subset \operatorname{Ker} \overline{A}_n(x), \tag{2.48}$$

for all $x \in \partial\Omega$.

Let $\begin{pmatrix} u \\ z \end{pmatrix} \in \operatorname{Ker} \overline{A}_n(x)$. Then

$$A_n u + B_n^T z = 0, \tag{2.49}$$

$$B_n u = 0. \tag{2.50}$$

From (2.50), $u \in \operatorname{Ker} B_n$, and so, $A_n u \in \operatorname{Ker} B_n$, from the previous lemma. Applying B_n to (2.49), it follows that $B_n B_n^T z = 0$, which implies $\|B_n^T z\| = 0$, by multiplying it with z^T. Thus $z \in \operatorname{Ker} B_n^T$. Returning to (2.49), observe that $u \in \operatorname{Ker} A_n$. Putting together all this information, we have that

$$\begin{pmatrix} u \\ z \end{pmatrix} \in (\operatorname{Ker} A_n \cap \operatorname{Ker} B_n) \times \operatorname{Ker} B_n^T.$$

Hence,

$$\operatorname{Ker} \overline{A}_n(x) \subset (\operatorname{Ker} A_n(x) \cap \operatorname{Ker} B_n(x)) \times \operatorname{Ker} B_n^T(x), \tag{2.51}$$

for all $x \in \partial\Omega$.

From (2.48) and (2.51), we get (2.47). □

Lemma 14. *If* u *is a non-negative vector for* A_n *and* $z \perp B_n u$*, then* $\begin{pmatrix} u \\ z \end{pmatrix}$ *is non-negative for* $\overline{A}_n$*.*

Proof. Under the given hypotheses, the conclusion is a simple consequence of the following equalities

$$(u^T, z^T)\overline{A}_n \begin{pmatrix} u \\ z \end{pmatrix} = u^T A_n u + 2z^T B_n u = u^T A_n u.$$

□

An immediate corollary of this lemma is

Corollary 15. *The subspace* $\overline{N} = N \times [B_n(N)]^{\perp}$ *is non-negative for* $\overline{A}_n$ *if and only if* N *is non-negative for* A_n.

Inhomogeneous Boundary Conditions

In this part, we consider the problem of finding $u(x,t) \in \mathbb{R}^m$, $x \in \Omega \subset \mathbb{R}^N$, $t > 0$, for (2.20), subject to initial condition (2.22), constraints (2.21), and linear inhomogeneous boundary conditions

$$E(x,t)u(x,t) = g(x,t) \text{ on } \partial\Omega \times (0,T). \tag{2.52}$$

Assume g is a vector function defined on $\partial\Omega$ for all time $t > 0$ such that there exists $\tilde{g}$ satisfying the constraint equation (2.21) and $g = E\tilde{g}$ on the boundary $\partial\Omega$ for all time $t > 0$. Then, by substituting $\tilde{u} = u - \tilde{g}$, we arrive to the constrained initial *homogeneous* boundary value problem

$$\dot{\tilde{u}} = A\tilde{u} + A\tilde{g} - \dot{\tilde{g}} + f, \tag{2.53}$$

$$\tilde{u}(x,0) = u_0(x) - \tilde{g}, \tag{2.54}$$

$$B\tilde{u} = 0, \tag{2.55}$$

$$E\tilde{u} = 0. \tag{2.56}$$

It is easy to see that the compatibility conditions for this last problem are satisfied and, in fact, (2.53)–(2.56) is equivalent to the original constrained initial *inhomogeneous* boundary value problem (2.20)–(2.22) and (2.52). Therefore, in this way and for a restricted set of boundary inhomogeneities, the treatment of inhomogeneous boundary conditions reduces to the treatment of the corresponding homogeneous ones.

Chapter 3

Hyperbolic Formulations of Einstein Equations

3.1 Introduction

Einstein's equations can be viewed as equations for geometries, that is, its solutions are equivalent classes under spacetime diffeomorphisms of metric tensors. To break this diffeomorphism invariance, Einstein's equations must be first transformed into a system having a well-posed Cauchy problem. The initial method to solve this problem has been by "fixing the gauge" [26], [12], or in other words, by imposing some conditions on the metric components which would select only one representative from each equivalent class of Einstein's solutions. By using an ingenious choice of gauge fixing, the so called "harmonic gauge," Einstein's equations can be converted to a set of coupled wave equations, one for each metric component. Thus, by prescribing at an initial hypersurface values for the variables and their normal derivatives, one gets unique solutions to this system of wave equations. In Appendix B, we explain this approach, restricted to the linearized case for the sake of simplicity. For a more detailed discussion on this topic, see [41], [19], and references therein.

Another way to deal with the gauge freedom of Einstein's equations is by prescribing the time foliation along evolution, that is, by prescribing a lapse-shift pair along evolution [6]. Einstein's equations are then decomposed into evolution equations and constraint equations on the foliation hypersurfaces. An analogous decomposition occurs for Maxwell's equations, which are canonically split into constraint (divergence) equations and evolution (curl) equations. Both the constraints and the evolution equations are not uniquely determined by this procedure: by taking combinations of the constraints or/and by adding any combination of constraints to any of the evolution equations one gets an equivalent decomposition.

There have been numerical schemes to solve Einstein's equations based on the ADM decomposition [6], but they have had only a very limited success. The instabilities observed in the ADM based schemes might be at least in part caused by the weakly hyperbolicity of the first order differential reduction of the ADM evolution equations, as argued in [36]. In fact, it is known that some of the stability problems of numerical schemes are due to properties of the equations themselves. By rewriting the equations in a different form, one can obtain better stability of computations for the very same numerical methods.

There have been a large number of first order hyperbolic formulations derived from the ADM decomposition in recent years [4], [5], [25], [7], [9], [10], [36], [3], among others. To give a survey of all of them appears to be a very difficult and extensive task, which is out of the scope of this dissertation (see [28] for a more comprehensive review). All such formulations must be equivalent since they describe the same physical phenomenon. However, they can admit different kinds of unphysical solutions which can grow rapidly in time and overwhelm the physical solution in numerical computations. This is one reason (signaled in [36]) why some formulations of Einstein's equations behave numerically better than others. From this point of view, first order symmetric hyperbolic formulations of Einstein's equations present a special attraction for a couple of reasons. First of all, there is a large body of experience and numerical codes that are stable for numerical simulations of FOSH systems derived from various applications (transport equations, wave equations, electromagnetism, etc.). Also, the symmetric hyperbolicity of the system ensures well–posedness and gives bounds on the solution growth. In fact, hyperbolicity refers to algebraic conditions on the principal part of the equations which imply well–posedness, that is, if appropriate initial data is given on an appropriate hypersurface, then a unique solution can be found in a neighborhood of that hypersurface, and that solution depends continuously, with respect to an appropriate norm, on the initial data.

In this chapter, we discuss some FOSH formulations of Einstein's equations introduced in recent years. Moreover, a new formulation due to Arnold [2] is presented and analyzed.

3.2 Einstein Equations

In general relativity, spacetime is a 4-dimensional manifold of events endowed with a pseudo-Riemannian metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta. \tag{3.1}$$

This metric determines curvature on the manifold, and Einstein's equations relate the curvature at a point of spacetime to the mass-energy there:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{3.2}$$

where $G_{\alpha\beta}$ is the *Einstein tensor* and $T_{\alpha\beta}$ is the *energy-momentum tensor.*

The Einstein tensor G is a second order tensor built from the given metric $g_{\alpha\beta}$ as follows:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}.$$

Here $R_{\alpha\beta}$ is the *Ricci tensor:*

$$R_{\alpha\beta} = R^\delta_{\alpha\delta\beta},$$

where

$$R^\delta_{\alpha\beta\gamma} = \partial_\alpha \Gamma^\delta_{\beta\gamma} - \partial_\beta \Gamma^\delta_{\alpha\gamma} + \Gamma^\epsilon_{\beta\gamma}\Gamma^\delta_{\epsilon\alpha} - \Gamma^\epsilon_{\alpha\gamma}\Gamma^\delta_{\epsilon\beta}$$

is the *Riemann curvature tensor.*

The *Christoffel symbols* are defined as

$$\Gamma^{\alpha}_{\beta\delta} = \frac{1}{2} g^{\alpha\lambda} (\partial_{\delta} g_{\beta\lambda} + \partial_{\beta} g_{\lambda\delta} - \partial_{\lambda} g_{\beta\delta}).$$

By R we denote the *scalar curvature*

$$R = g^{\alpha\beta} R_{\alpha\beta}.$$

The energy-momentum tensor $T_{\alpha\beta}$ can be better understood by looking at two of the simplest energy-momentum tensors in general relativity, namely, the energy-momentum tensors for incoherent matter or dust and for a perfect fluid.

a) *Incoherent matter (non-interacting incoherent matter or dust)*

Such a field may be characterized by two quantities, the *4-velocity* vector field of flow

$$u^a = \frac{dx^a}{d\tau},$$

where τ is the proper time along the world-line of a dust particle and a scalar field

$$\rho_0 = \rho_0(x)$$

describing the *proper density* of the flow, that is, the density which would be measured by an observer moving with the flow (a co-moving observer).

The simplest second-rank tensor we can construct from these two quantities is

$$T^{ab} = \rho_0 u^a u^b$$

and this turn out to be the *energy-momentum tensor* for the matter field.

Now, let us investigate this tensor in special relativity in Minkowski coordinates. In this case

$$u^a = \frac{dx^a}{d\tau} = \gamma(1, u_x, u_y, u_z),$$

where $\gamma = (1 - u^2)^{-1/2}$ and τ is the proper time defined by

$$d\tau^2 = ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = dt^2(1 - u^2) = \gamma^{-2} dt^2.$$

Then, the T^{00} component of T^{ab} is

$$T^{00} = \rho_0 \frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = \rho_0 \frac{dt^2}{d\tau^2} = \gamma^2 \rho_0.$$

This quantity has a simple physical interpretation. First of all, in special relativity, the mass of a body in motion is greater than its rest mass by a factor γ ($m = \gamma m_0$). In addition, if we consider a moving three-dimensional volume element, then its volume decreases by a factor γ through the Lorentz contraction. Thus, from the point of view of a fixed as opposed to a comoving observer, the density increases by a factor γ^2. Hence, if a field of material of proper density ρ_0 flows past a fixed observer with velocity $\underline{u}$, then the observer will measure a density $\rho = \gamma^2 \rho_0$.

The component T^{00} may therefore be interpreted as the *relativistic energy density* of the matter field since the only contribution to the energy of the field is from the motion of the matter.

The components of T^{ab} are

$$T^{ab} = \rho \begin{pmatrix} 1 & u_x & u_y & u_z \\ u_x & u_x^2 & u_x u_y & u_x u_z \\ u_y & u_x u_y & u_y^2 & u_y u_z \\ u_z & u_x u_z & u_y u_z & u_z^2 \end{pmatrix}.$$

Next, we will show that the equations governing the force-free motion of a matter field of dust can be written in the following very succinct way

$$\partial_b T^{ab} = 0.$$

When $a = 0$, this equation becomes exactly the classical *equation of continuity*

$$\frac{\partial \rho}{\partial t} + div\,(\rho \underline{u}) = 0,$$

which expresses the conservation of matter with density ρ moving with velocity $\underline{u}$. Since matter is the same as energy, it follows that the conservation of energy equation for dust is

$$\partial_b T^{0b} = 0.$$

Writing the equations corresponding to $a \in \{1, 2, 3\}$, we get

$$\frac{\partial}{\partial t}(\rho \underline{u}) + \frac{\partial}{\partial x}(\rho u_x \underline{u}) + \frac{\partial}{\partial y}(\rho u_y \underline{u}) + \frac{\partial}{\partial z}(\rho u_z \underline{u}) = 0.$$

Combining this with the equation of continuity, we obtain

$$\rho[\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u}] = 0,$$

which is the Euler equation of motion for a perfect fluid in classical fluid dynamics (in the absence of pressure and external forces).

We have seen that the requirement that the energy-momentum tensor has zero divergence in special relativity is equivalent to demanding conservation of energy and conservation of momentum in the matter field (hence the name *energy-momentum tensor*).

If we use a non-flat metric, then the *conservation law*

$$\partial_b T^{ab} = 0$$

is replaced by its covariant counterpart

$$\nabla_b T^{ab} = 0.$$

b) *Perfect Fluid*

A *perfect fluid* is characterized by three quantities

1. a *4-velocity* $u^a = dx^a/d\tau$,
2. a *proper density* $\rho_0 = \rho_0(x)$,
3. a *scalar pressure* $p = p(x)$.

Observe that, if p vanishes, a perfect fluid reduces to incoherent matter. This suggests that we take the energy-momentum tensor for a perfect fluid to be of the form

$$T^{ab} = \rho_0 u^a u^b + pS^{ab},$$

for some symmetric tensor S^{ab}. Since this tensor depends on the velocity and the metric, the simplest assumption we can make is

$$S^{ab} = \lambda u^a u^b + \mu g^{ab},$$

where λ and μ are constants.

Considering the conservation law

$$\partial_b T^{ab} = 0$$

in special relativity in Minkowski coordinates and demanding that it reduces in an appropriate limit to the continuity equation and the Euler equation in the absence of body forces, we obtain that $\lambda = 1$ and $\mu = -1$. Therefore, the energy-momentum tensor of a perfect fluid is

$$T^{ab} = (\rho_0 + p)u^a u^b - pg^{ab}.$$

In the full theory, we again take the covariant form $\nabla_b T^{ab} = 0$ for the conservation law. In addition, p and ρ are related by an *equation of state* which, in general, is an equation of the form $p = p(\rho, T)$, where T is the absolute temperature. Usually, T is constant and so, that equation of state reduces to $p = p(\rho)$.

The Einstein equations can be viewed in three different ways:

1. The field equations are differential equations for determining the metric tensor g_{ab} from a *given* energy-momentum tensor T_{ab}. An important case of the equations is when $T_{ab} = 0$, in which case we are concerned with finding *vacuum* solutions.

2. The field equations are equations from which the energy-momentum tensor can be read off corresponding to a *given* metric tensor g_{ab}. In fact, this rarely turns out to be very useful in practice because the resulting T_{ab} are usually physically unrealistic. In particular, it frequently turns out that the energy density goes negative in some region, which is rejected as unphysical.

3. The field equations consist of *ten equations* connecting *twenty quantities* (the ten components of g_{ab} and the ten components of T_{ab}). In this way, the field equations are viewed as *constraints* on the simultaneous choice of g_{ab} and T_{ab}. This point of view is useful when one can partly specify the geometry and the energy-momentum tensor from physical considerations and then the equations are used to determine both quantities completely.

Remark 4. *1. Each of the 10 equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ is a 2nd order PDE in 4 independent variables and 10 unknowns. These 10 equations involve lots of terms.*

2. The equations are not independent, due to the Bianchi identities $\nabla_\alpha G^{\alpha\beta} = 0$. The energy-momentum tensor also must satisfy these identities.

3. Solutions are not unique, because of gauge freedom (any diffeomorphism of the manifold gives a reparametrization, and hence another solution).

A subtle consequence of Einstein's equations is that relatively accelerating bodies emit gravitational waves. These gravitational waves are very slight variations in the spacetime

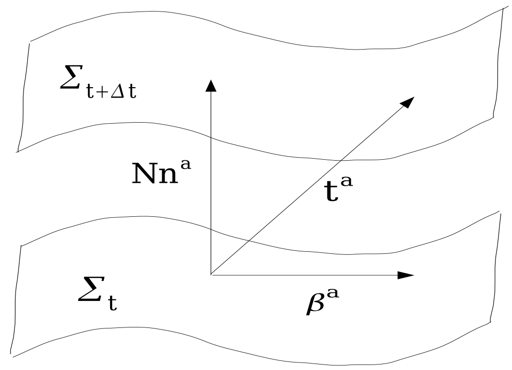

Figure 3.1: A spacetime diagram for the ADM decomposition illustrating the definition of the lapse function, N, and shift vector β^a.

metric tensor, which propagate at the speed of light. It is generally believed that extremely violent movements of huge masses, such as collisions of black holes, should generate detectable gravitational waves. However, because of their tiny amplitude, gravitational waves have eluded detection until now. When gravitational waves will be detected, we will must determine the cosmological event that could have caused them. This is in fact an inverse problem and, as usual, we need the solution of the direct problem, which in this case is the numerical solution of Einstein's equations.

3.3 ADM (3+1) Decomposition

In numerical relativity, the Einstein equations are usually solved as an initial-boundary value problem. In other words, the spacetime is foliated and each slice Σ_t is characterized by its intrinsic geometry γ_{ij} and extrinsic curvature K_{ij}. Subsequent slices are connected via the lapse function N and shift vector β^i (see Figure 3.1). The ADM decomposition [6] (also [53]) of the line element

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \; i, j = 1, 2, 3,$$

allows one to express six of the ten components of Einstein's equations in vacuum as a system of evolution equations for the metric γ_{ij} and the extrinsic curvature K_{ij}:

$$\dot{\gamma}_{ij} = -2NK_{ij} + 2\nabla_{(i}\beta_{j)}, \tag{3.3}$$

$$\dot{K}_{ij} = N[R_{ij} + (K^l_l)K_{ij} - 2K_{il}K^l_j] + \beta^l\nabla_l K_{ij} + K_{il}\nabla_j\beta^l + K_{lj}\nabla_i\beta^l - \nabla_i\nabla_j N, \tag{3.4}$$

$$R^i_i + (K^i_i)^2 - K_{ij}K^{ij} = 0, \tag{3.5}$$

$$\nabla^j K_{ij} - \nabla_i K^j_j = 0, \tag{3.6}$$

where we use a dot to denote time differentiation. The spatial Ricci tensor R has components given by second order spatial partial differential operators applied to the spatial metric components γ_{ij}. Indices are raised and traces taken with respect to the spatial metric, and parenthesized indices are used to denote the symmetric part of a tensor.

The system of equations for γ_{ij} and K_{ij} is first order in time and second order in space. It is not hyperbolic in any usual sense, and direct numerical approaches have been unsuccessful. Therefore, many authors have considered reformulations of (3.3), (3.4) into more standard first order hyperbolic systems. Typically, these approaches involve introducing other variables, like the first spatial derivatives of the spatial metric components γ_{ij} (or quantities closely related to them). In the rest of this chapter we present some of the most important first order hyperbolic formulations derived from the ADM decomposition, as well as their linearizations around the Minkowski spacetime. In the last section of this chapter we introduce a new first order symmetric hyperbolic formulation of the linearized ADM equations around Minkowski's spacetime that has surprising resemblances with Maxwell's equations. This motivates the introduction of the linearized ADM equations around the flat spacetime in the following subsection.

3.3.1 Linearized ADM Decomposition

In this subsection, we derive the linearized ADM decomposition around the Minkowski spacetime.

A trivial solution to the ADM system (3.3)–(3.6) is the Minkowski spacetime in Cartesian coordinates, given by $\gamma_{ij} = \delta_{ij}$, $K_{ij} = 0$, $\beta^i = 0$, $N = 1$. To derive the linearization of (3.3)–(3.6) about this solution, we write $\gamma_{ij} = \delta_{ij} + \bar{g}_{ij}$, $K_{ij} = \bar{K}_{ij}$, $\beta^i = \bar{\beta}^i$, $N = 1 + \bar{N}$, where the bars indicate perturbations, assumed to be small. If we substitute these expressions into (3.3)–(3.6) and ignore terms which are at least quadratic in the perturbations and their derivatives, then we obtain a linear system for the perturbations. Dropping the bars, the system is

$$\dot{g}_{ij} = -2K_{ij} + 2\partial_{(i}\beta_{j)}, \tag{3.7}$$

$$\dot{K}_{ij} = \partial^l \partial_{(j} g_{i)l} - \frac{1}{2}\partial^l \partial_l g_{ij} - \frac{1}{2}\partial_i \partial_j g_l^l - \partial_i \partial_j N, \tag{3.8}$$

$$C := \partial^j (\partial^l g_{lj} - \partial_j g_l^l) = 0, \tag{3.9}$$

$$C_j := \partial^l K_{lj} - \partial_j K_l^l = 0. \tag{3.10}$$

The usual approach to solving the system (3.7)–(3.10) is to begin with initial data $g_{ij}(0)$ and $K_{ij}(0)$ defined on $\mathbb{R}^3$ and satisfying the constraint equations (3.9), (3.10), and to define g_{ij} and K_{ij} for $t > 0$ via the Cauchy problem for the evolution equations (3.7), (3.8). It can be easily shown that the constraints are then satisfied for all times. Indeed, if we apply the Hamiltonian constraint operator defined in (3.9) to the evolution equation (3.7) and apply the momentum constraint operator defined in (3.10) to the evolution equation (3.8), we obtain the first order symmetric hyperbolic system

$$\dot{C} = -2\partial^j C_j, \quad \dot{C}_j = -\frac{1}{2}\partial_j C.$$

Thus if C and C_j vanish at $t = 0$, they vanish for all time.

3.4 Hyperbolic Formulations

In this section, we present a number of popular hyperbolic formulations of Einstein's equations derived from the ADM decomposition. In the last section, we present and analyze a new first order symmetric hyperbolic formulation of Einstein's equations in the linearized case due to Arnold [2].

3.4.1 Kidder–Scheel–Teukolsky (KST) Family

In this subsection, we present a many-parameter family of hyperbolic representations of Einstein's equations intoduced by Kidder, Scheel, and Teukolsky [36].

In order to write the evolution equations (3.3) and (3.4) in first-order form, we have to eliminate the second order derivatives of the spatial metric. For this purpose, we introduce new variables

$$d_{kij} = \partial_k \gamma_{ij}. \tag{3.11}$$

Since

$$R_{ij} = \frac{1}{2}\gamma^{ab}(\partial_{(i}\partial_a\gamma_{bj)} + \partial_a\partial_{(i}\gamma_{j)b} - \partial_a\partial_b\gamma_{ij} - \partial_{(i}\partial_{j)}\gamma_{ab}) + \text{lower order},$$

the evolution system (3.3), (3.4), together with (3.11) give

$$\dot{\gamma}_{ij} = -2NK_{ij} + \nabla_i\beta_j + \nabla_j\beta_i, \tag{3.12}$$

$$\dot{K}_{ij} = \frac{1}{2}N\gamma^{ab}(\partial_{(i}d_{abj)} + \partial_a d_{(ij)b} - \partial_a d_{bij} - \partial_{(i}d_{j)ab}) - \partial_i\partial_j N + \dots, \tag{3.13}$$

$$\dot{d}_{kij} = -2N\partial_k K_{ij} - 2K_{ij}\partial_k N + \partial_k(\nabla_i\beta_j + \nabla_j\beta_i). \tag{3.14}$$

Since we have introduced a new variable that we will evolve independently of the metric, we get additional constraints,

$$C_{kij} := d_{kij} - \partial_k\gamma_{ij} = 0, \tag{3.15}$$

$$C_{klij} := \partial_{[k}d_{l]ij} = 0. \tag{3.16}$$

The system (3.12)–(3.14) has been proven to be only weakly hyperbolic. Its characteristic matrix has eigenvalues $\{0, \pm 1\}$, but does not have a complete set of eigenvectors. Fortunately, the hyperbolicity of the system can be changed by *densitizing* the lapse and adding constraints to the evolution equations.

We densitize the lapse by writing: $N = \gamma^{\sigma} e^{Q}$, where $\gamma = \det(\gamma_{ij})$, σ is the densitization parameter, which is an arbitrary constant, and Q is the lapse density, which will be chosen independent of the dynamical fields.

By adding terms proportional to the constraints, we can modify the evolution system (3.12)–(3.14) without affecting the physical solution:

$$\dot{K}_{ij} = (\ldots) + \theta N \gamma_{ij} C + \zeta N \gamma^{ab} C_{a(ij)b}, \tag{3.17}$$

$$\dot{d}_{kij} = (\ldots) + \eta N \gamma_{k(i} C_{j)} + \chi N \gamma_{ij} C_k, \tag{3.18}$$

where $(\ldots)$ represents the same thing that was before, C is the Hamiltonian constraint, C_i are the momentum constraints, and $\{\theta, \zeta, \eta, \chi\}$ are arbitrary parameters.

By carrying out the computations, the evolution system, up to the principal part, is now given by (**System 1** in [36]):

$$\begin{aligned} \dot{\gamma}_{ij} &= 0, \\ \dot{K}_{ij} &= \frac{1}{2} N \gamma^{ab} [\partial_a d_{bij} - (1+\zeta)\partial_a d_{(ij)b} - (1-\zeta)\partial_{(i} d_{abj)} + \\ &\quad (1+2\sigma)\partial_{(i} d_{j)ab} - \theta \gamma_{ij} \gamma^{cd} \partial_a d_{cdb} + \theta \gamma_{ij} \gamma^{cd} \partial_a d_{bcd}], \\ \dot{d}_{kij} &= -2N \partial_k K_{ij} + N \gamma^{ab} (\eta \gamma_{k(i} \partial_a K_{bj)} + \chi \gamma_{ij} \partial_a K_{bk} - \eta \gamma_{k(i} \partial_{j)} K_{ab} - \\ &\quad -\chi \gamma_{ij} \partial_k K_{ab}). \end{aligned} \tag{3.19}$$

The eigenvalues of the characteristic matrix of this system are $\{0, \pm 1, \pm c_1, \pm c_2, \pm c_3\}$, where $c_1 = \sqrt{2\sigma}$, $c_2 = 2^{-3/2}\sqrt{\eta - 4\eta\sigma - 2\chi - 12\sigma\chi - 3\eta\zeta}$, and $c_3 = 2^{-1/2}\sqrt{2 + 4\theta - \eta - 2\theta\eta + 2\chi + 4\theta\chi - \eta\zeta}$.

If all c_i are real, then it can be proven that the system is strongly hyperbolic unless one of the following conditions occurs:

$$c_i = 0, \tag{3.20}$$

$$c_1 = c_3 \neq 1, \tag{3.21}$$

$$c_1 = c_3 = 1 \neq c_2. \tag{3.22}$$

By differentiating the constraints in time, we obtain the following system for the evolution of the constraints (up to the principal part)

$$\dot{C} = -\frac{1}{2}(2 - \eta + 2\chi)N\gamma^{pq}\partial_p C_q, \tag{3.23}$$

$$\dot{C}_i = -(1+2\theta)N\partial_i C + \frac{1}{2}N\gamma^{pq}\gamma^{rs}[(1-\zeta)\partial_q C_{prsi} + (1+\zeta)\partial_p C_{siqr} - (1+2\sigma)\partial_p C_{qirs}], \tag{3.24}$$

$$\dot{C}_{kij} = 0, \tag{3.25}$$

$$\dot{C}_{klij} = \frac{1}{2}\eta N(\gamma_{j[l}\partial_{k]}C_i + \gamma_{i[l}\partial_{k]}C_j) + \chi N\gamma_{ij}\partial_{[k}C_{l]}. \tag{3.26}$$

The eigenvalues for the characteristic matrix of this last system is a subset of the eigenvalues of the evolution equations $\{0, \pm c_2, \pm c_3\}$. Moreover, the constraint evolution system is strongly hyperbolic whenever the regular evolution system is strongly hyperbolic.

We define two new variables: the generalized extrinsic curvature P_{ij} and the generalized derivative of the metric M_{kij}

$$P_{ij} = K_{ij} + \hat{z}\gamma_{ij}K, \text{ so } K_{ij} = P_{ij} - \frac{\hat{z}}{1+3\hat{z}}\gamma_{ij}P, \tag{3.27}$$

$$M_{kij} = \frac{1}{2}[\hat{k}d_{kij} + \hat{e}d_{(ij)k} + \gamma_{ij}(\hat{a}d_k + \hat{b}b_k) + \gamma_{k(i}(\hat{c}d_{j)} + \hat{d}b_{j)})], \tag{3.28}$$

where we introduce seven additional parameters $\{\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, \hat{k}, \hat{z}\}$, and $d_j = \gamma^{ab}d_{jab}$, $b_j = \gamma^{ab}d_{abj}$.

The System 1, (3.19), rewritten for these new variables, up to the principal parts, becomes

(**System 2** in [36]):

$$
\begin{aligned}
\dot{\gamma}_{ij} &= 0, \\
\dot{P}_{ij} &= -N\gamma^{ab}(\mu_1\partial_a M_{bij} + \mu_2\partial_a M_{(ij)b} + \mu_3\partial_{(i}M_{abj)} + \mu_4\partial_{(i}M_{j)ab} + \\
&\quad +\mu_5\gamma_{ij}\gamma^{cd}\partial_a M_{cdb} + \mu_6\gamma_{ij}\gamma^{cd}\partial_a M_{bcd}), \\
\dot{M}_{kij} &= -N(\nu_1\partial_k P_{ij} + \nu_2\partial_{(i}P_{j)k} + \nu_3\gamma^{ab}g_{k(i}\partial_a P_{bj)} + \nu_4\gamma_{ij}\gamma^{ab}\partial_a P_{bk} + \\
&\quad +\nu_5\gamma^{ab}\gamma_{k(i}\partial_{j)}P_{ab} + \nu_6\gamma_{ij}\gamma^{ab}\partial_k P_{ab}),
\end{aligned}
\tag{3.29}
$$

where $\mu_i = \mu_i(\sigma,\, \theta,\, \zeta,\, \eta,\, \chi,\, \hat{a},\, \hat{b},\, \hat{c},\, \hat{d},\, \hat{e},\, \hat{k},\, \hat{z})$ and $\nu_i = \nu_i(\sigma,\, \theta,\, \zeta,\, \eta,\, \chi,\, \hat{a},\, \hat{b},\, \hat{c},\, \hat{d},\, \hat{e},\, \hat{k},\, \hat{z})$.

The System 2, (3.29), is also strongly hyperbolic. Moreover, it has the same eigenvalues as System 1, (3.19), but the eigenvectors are different.

Conclusion: By densitizing the lapse, adding constraints to the evolution equations, and changing variables, Kidder, Scheel, and Teukolsky [36] got a twelve-parameter family of strongly hyperbolic formulations of Einstein's equations. They have observed that the choice of parameters can have a huge impact on the amount of time and accuracy of numerical simulations. Unfortunately, nobody knows why one particular parameter choice behaves much better than others.

Some well-known formulations can be recovered by making appropriate choices for parameters. For example, we can recover the Fritteli–Reula (FR) system [25] if the following choice of parameters is taken in (3.29):

$$\{\sigma,\, \theta,\, \zeta,\, \eta,\, \chi,\, \hat{z},\, \hat{k},\, \hat{a},\, \hat{b},\, \hat{c},\, \hat{d},\, \hat{e}\} = \{1/2,\, -1,\, -1,\, 4,\, -2,\, -1,\, 1,\, -1,\, 0,\, 0,\, 0,\, 0\}.$$

In the next subsection, we present another popular hyperbolic formulation which can be recovered from (3.29).

3.4.2 Einstein–Christoffel (EC) formulation

The EC formulation was originally derived directly from the ADM system (3.3)–(3.6) by Anderson and York [7] in 1999. It can be also recovered if we make the following choice of parameters in (3.29):

$$\{\sigma,\, \theta,\, \zeta,\, \eta,\, \chi,\, \hat{z},\, \hat{k},\, \hat{a},\, \hat{b},\, \hat{c},\, \hat{d},\, \hat{e}\} = \{1/2,\, 0,\, -1,\, 4,\, 0,\, 0,\, 1,\, 0,\, 0,\, 2,\, -2,\, 0\}.$$

In this case, the system is (up to the principal parts):

$$\begin{aligned} \partial_t \gamma_{ij} &= 0, \\ \partial_t P_{ij} &= -N\gamma^{ab}\partial_a M_{bij}, \\ \partial_t M_{kij} &= -N\partial_k P_{ij}. \end{aligned} \tag{3.30}$$

Essentially the coupled part of this system, i.e. the last two equations, is a set of six coupled quasilinear scalar wave equations with nonlinear source terms.

Linearized Einstein–Christoffel Formulation

First, we replace the lapse N in (3.3)–(3.6) with $\alpha\sqrt{\gamma}$ where α denotes the lapse density. A trivial solution to this system is Minkowski spacetime in Cartesian coordinates, given by $\gamma_{ij} = \delta_{ij}$, $K_{ij} = 0$, $\beta^i = 0$, $\alpha = 1$. To derive the linearization, we write $\gamma_{ij} = \delta_{ij} + \bar{g}_{ij}$, $K_{ij} = \bar{K}_{ij}$, $\beta^i = \bar{\beta}^i$, $\alpha = 1 + \bar{\alpha}$, where the bars indicate perturbations, assumed to be small. If we substitute these expressions into (3.3)–(3.6) (with $N = \alpha\sqrt{\gamma}$), and ignore terms which are at least quadratic in the perturbations and their derivatives, then we obtain a linear system for the perturbations. Dropping the bars, the system is

$$\dot{g}_{ij} = -2K_{ij} + 2\partial_{(i}\beta_{j)}, \tag{3.31}$$

$$\dot{K}_{ij} = \partial^l \partial_{(j} g_{i)l} - \frac{1}{2}\partial^l \partial_l g_{ij} - \partial_i \partial_j g^l_l - \partial_i \partial_j \alpha, \tag{3.32}$$

$$C := \partial^j(\partial^l g_{lj} - \partial_j g^l_l) = 0, \tag{3.33}$$

$$C_j := \partial^l K_{lj} - \partial_j K^l_l = 0, \tag{3.34}$$

where we use a dot to denote time differentiation.

Remark 5. *For the linear system the effect of densitizing the lapse is to change the coefficient of the term $\partial_i \partial_j g^l_l$ in* (3.32)*. Had we not densitized, the coefficient would have been $-1/2$ instead of -1, and the derivation of the linearized EC formulation below would not be possible.*

The usual approach to solving the system (3.31)–(3.34) is to begin with initial data $g_{ij}(0)$ and $K_{ij}(0)$ defined on $\mathbb{R}^3$ and satisfying the constraint equations (3.33), (3.34), and to define g_{ij} and K_{ij} for $t > 0$ via the Cauchy problem for the evolution equations (3.31), (3.32). It can be easily shown that the constraints are then satisfied for all times. Indeed, if we apply the Hamiltonian constraint operator defined in (3.33) to the evolution equation (3.31) and apply the momentum constraint operator defined in (3.34) to the evolution equation (3.32), we obtain the first order symmetric hyperbolic system

$$\dot{C} = -2\partial^j C_j, \quad \dot{C}_j = -\frac{1}{2}\partial_j C.$$

Thus if C and C_j vanish at $t = 0$, they vanish for all time.

The linearized EC formulation provides an alternate approach to obtaining a solution of (3.31)–(3.34) with the given initial data, based on solving a system with better hyperbolicity properties. If g_{ij}, K_{ij} solve (3.31)–(3.34), define

$$f_{kij} = \frac{1}{2}[\partial_k g_{ij} - (\partial^l g_{li} - \partial_i g^l_l)\delta_{jk} - (\partial^l g_{lj} - \partial_j g^l_l)\delta_{ik}]. \tag{3.35}$$

Then $-\partial^k f_{kij}$ coincides with the first three terms of the right-hand side of (3.32), so

$$\dot{K}_{ij} = -\partial^k f_{kij} - \partial_i \partial_j \alpha. \tag{3.36}$$

Differentiating (3.35) in time, substituting (3.31), and using the constraint equation (3.34), we obtain

$$\dot{f}_{kij} = -\partial_k K_{ij} + L_{kij}, \tag{3.37}$$

where

$$L_{kij} = \partial_k \partial_{(i}\beta_{j)} - \partial^l \partial_{[l}\beta_{i]}\delta_{jk} - \partial^l \partial_{[l}\beta_{j]}\delta_{ik}. \tag{3.38}$$

The evolution equations (3.36) and (3.37) for K_{ij} and f_{kij}, together with the evolution equation (3.31) for g_{ij}, form the linearized EC system. As initial data for this system we use the given initial values of g_{ij} and K_{ij} and derive the initial values for f_{kij} from those of g_{ij} based on (3.35):

$$f_{kij}(0) = \frac{1}{2}\{\partial_k g_{ij}(0) - [\partial^l g_{li}(0) - \partial_i g_l^l(0)]\delta_{jk} - [\partial^l g_{lj}(0) - \partial_j g_l^l(0)]\delta_{ik}\}. \tag{3.39}$$

A purpose of this dissertation is to study the preservation of constraints by the linearized EC system and the closely related question of the equivalence of that system and the linearized ADM system. Our interest is in the case when the spatial domain is bounded and appropriate boundary conditions are imposed, but first we consider the result for the pure Cauchy problem in the remainder of this subsection.

Suppose that K_{ij} and f_{kij} satisfy the evolution equations (3.36) and (3.37) (which decouple from (3.31)). If K_{ij} satisfies the momentum constraint (3.34) for all time, then from (3.36) we obtain a constraint which must be satisfied by f_{kij}:

$$\partial^k(\partial^l f_{klj} - \partial_j f_{kl}{}^l) = 0. \tag{3.40}$$

The following theorem shows that the pair of constraints (3.34), (3.40) is preserved by the linearized EC evolution.

Theorem 16. *Let initial data $K_{ij}(0)$, $f_{kij}(0)$ be given satisfying the constraints* (3.34) *and* (3.40). *Then the unique solution of the evolution equations* (3.36), (3.37) *satisfy* (3.34) *and* (3.40) *for all time.*

Proof. It is immediate from the evolution equations that each component K_{ij} satisfies the inhomogeneous wave equation

$$\ddot{K}_{ij} = \partial^k \partial_k K_{ij} - \partial^k L_{kij} - \partial_i \partial_j \dot{\alpha}.$$

Applying the momentum constraint operator defined in (3.34), we see that each component C_j satisfies the homogeneous wave equation

$$\ddot{C}_j = \partial^k \partial_k C_j. \tag{3.41}$$

Now $C_j = 0$ at the initial time by assumption, so if we can show that $\dot{C}_j = 0$ at the initial time, we can conclude that C_j vanishes for all time. But, from (3.36) and the definition of C_j,

$$\dot{C}_j = -\partial^k (\partial^l f_{klj} - \partial_j f_{kl}{}^l), \tag{3.42}$$

which vanishes at the initial time by assumption. Thus we have shown C_j vanishes for all time, i.e., (3.34) holds. In view of (3.42), (3.40) holds as well. □

In view of this theorem it is straightforward to establish the key result that for given initial data satisfying the constraints, the unique solution of the linearized EC evolution equations satisfies the linearized ADM system, and so the linearized ADM system and the linearized EC system are equivalent.

Theorem 17. *Suppose that initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian constraint* (3.33) *and momentum constraint* (3.34)*, respectively, and that initial data $f_{kij}(0)$ is defined by* (3.39). *Then the unique solution of the linearized EC evolution equations* (3.31), (3.36), (3.37) *satisfies the linearized ADM system* (3.31)–(3.34).

Proof. First we show that the initial data $f_{kij}(0)$ defined in (3.35) satisfies the constraint (3.40). Applying the constraint operator in (3.40) to (3.35) we find

$$\partial^k(\partial^l f_{klj} - \partial_j f_{kl}{}^l) = \frac{1}{2}\partial_j(\partial^l\partial^k g_{kl} - \partial^k\partial_k g_l^l) = \frac{1}{2}\partial_j C,$$

which vanishes at time 0 by (3.33). From Theorem 16, we conclude that $C_j = 0$ for all time, i.e., (3.34) holds. Then from (3.31) and (3.34) we see that $\dot{C} = -2\partial^j C_j = 0$, and, since C vanishes at initial time by assumption, C vanishes for all time, i.e., (3.33) holds as well.

It remains to verify (3.32). From (3.37) and (3.31) we have

$$\dot{f}_{kij} = \frac{1}{2}\partial_k\dot{g}_{ij} - \partial^l\partial_{[l}\beta_{i]}\delta_{jk} - \partial^l\partial_{[l}\beta_{j]}\delta_{ik}.$$

Applying the momentum constraint operator to (3.31) and using (3.34), it follows that

$$\frac{1}{2}(\partial^l\dot{g}_{li} - \partial_i\dot{g}_l^l) = \partial^l\partial_{[l}\beta_{i]},$$

so $f_{kij} - [\partial_k g_{ij} - (\partial^l g_{li} - \partial_i g_l^l)\delta_{kj} - (\partial^l g_{lj} - \partial_j g_l^l)\delta_{ki}]/2$ does not depend on time. In view of (3.39), we have (3.35).

Substituting (3.35) in (3.36) gives (3.32), as desired. □

3.4.3 Alekseenko–Arnold (AA) Formulation

In this subsection we present a first order symmetric hyperbolic formulation of the full nonlinear ADM system (3.3)–(3.6) which involves fewer unknowns than other hyperbolic formulations and does not require any arbitrary parameters. The hyperbolic systems involves 14 unknowns, namely the components of the extrinsic curvature and eight particular combinations of the first derivatives of the spatial metric. In order to derive the AA formulation, we introduce the notations $\mathbb{S}$ for the 6-dimensional space of symmetric matrices and $\mathbb{T}$ for the 8-dimensional space of triply-indexed arrays (w_{ijk}) which are skew-symmetric in the first two indices and satisfy the cyclic property

$$w_{ijk} + w_{jki} + w_{kij} = 0. \tag{3.43}$$

Define the operators $M : C^\infty(\mathbb{R}^3, \mathbb{S}) \to C^\infty(\mathbb{R}^3, \mathbb{R}^3)$, $L : C^\infty(\mathbb{R}^3, \mathbb{S}) \to C^\infty(\mathbb{R}^3, \mathbb{T})$, and $L^* : C^\infty(\mathbb{R}^3, \mathbb{T}) \to C^\infty(\mathbb{R}^3, \mathbb{S})$ by

$$(Mu)_i = 2\gamma^{pq}\partial_{[p}u_{i]q}, \tag{3.44}$$

$$(Lu)_{ijk} = \partial_{[i}u_{j]k}, \tag{3.45}$$

$$(L^*v)_{ij} = -\gamma_{qi}\gamma_{rj}\partial_p v^{p(qr)}, \tag{3.46}$$

respectively. Observe that the operators L and L^* are formal adjoints to each other with respect to the scalar products $\langle u, w \rangle := \int u_{ij}w^{ij}dx$ and $\langle v, z \rangle := \int v_{ijk}z^{ijk}dx$ on the spaces $C^\infty(\mathbb{R}^3, \mathbb{S})$ and $C^\infty(\mathbb{R}^3, \mathbb{T})$ respectively.

Next, we introduce new variables

$$f_{ijk} = -\frac{1}{\sqrt{2}}[(L\gamma)_{ijk} + (M\gamma)_{[i}\gamma_{j]k}]. \tag{3.47}$$

Observe that (f_{ijk}) belongs to $\mathbb{T}$ and so, it has eight independent components.

Now we are able to write down the new formulation derived in [3] for the ADM system (3.3)–(3.6).

$$\begin{aligned} \partial_0\gamma_{ij} &= -2NK_{ij} + 2\gamma_{k(i}\partial_{j)}\beta^k, \\ \frac{1}{\sqrt{2}}\partial_0 K_{ij} &= -N(L^*f)_{ij} + \ldots = N\gamma_{mi}\gamma_{nj}\partial_l f^{l(mn)} + \ldots, \\ \frac{1}{\sqrt{2}}\partial_0 f_{ijk} &= [L(NK)]_{ijk} + \ldots = \partial_{[i}(NK)_{j]k} + \ldots. \end{aligned} \tag{3.48}$$

Here $\partial_0 := \partial_t - \beta^i\partial_i$ is the convective derivative and the omitted terms are algebraic expressions involving γ_{ij}, their spatial derivatives $\partial_k\gamma_{ij}$, K_{ij}, the lapse N, and the shift β.

Linearized AA Formulation

The linearized AA formulation provides an alternate approach to obtaining a solution of (3.7)–(3.10) with the given initial data, based on solving a system with better hyperbolicity properties. If $g_{ij} := \gamma_{ij}$ and $K_{ij} := \kappa_{ij}$ solve (3.7)–(3.10), define

$$\lambda_{kij} = -\frac{1}{\sqrt{2}}[\partial_{[k}\gamma_{i]j} + (M\gamma)_{[k}\delta_{i]j}], \tag{3.49}$$

where $(M\gamma)_i = \partial^l \gamma_{il} - \partial_i \gamma^l_l$.

Then, proceeding as in [3], we obtain an evolution system for κ_{ij} and λ_{kij}

$$\frac{1}{\sqrt{2}}\dot{\kappa}_{ij} = \partial^k \lambda_{k(ij)} - \frac{1}{\sqrt{2}}\partial_i\partial_j\alpha, \tag{3.50}$$

$$\frac{1}{\sqrt{2}}\dot{\lambda}_{kij} = \partial_{[k}\kappa_{i]j} + \eta_{kij}, \tag{3.51}$$

where

$$\eta_{kij} = -\frac{1}{2}(\partial_j\partial_{[k}\beta_{i]} + \partial^m\partial_{[m}\beta_{k]}\delta_{ij} - \partial^m\partial_{[m}\beta_{i]}\delta_{kj}). \tag{3.52}$$

The evolution equations (3.50) and (3.51) for κ_{ij} and λ_{kij} form the linearized AA system. As initial data for this system we use the given initial values of γ_{ij} and κ_{ij} and derive the initial values for λ_{kij} from those of γ_{ij} based on (3.49):

$$\lambda_{kij}(0) = -\frac{1}{\sqrt{2}}[\partial_{[k}\gamma(0)_{i]j} + (M\gamma(0))_{[k}\delta_{i]j}]. \tag{3.53}$$

As shown in [3], if γ and κ satisfy the ADM system and λ is defined by (3.49), then κ and λ satisfy the symmetric hyperbolic system (3.50), (3.51). Conversely, to recover the solution to the ADM system from (3.50), (3.51), the same κ should be taken at time 0, and λ should be given by (3.53). Having κ determined, the metric perturbation γ is defined as follows from (3.7)

$$\gamma_{ij} = \gamma_{ij}(0) - 2\int_0^t (\kappa_{ij} - \partial_{(i}\beta_{j)})(s)\, ds. \tag{3.54}$$

3.4.4 Arnold (A) Formulation

In this subsection, we present and analyze the linearized case of a new first order symmetric hyperbolic formulation of Einstein's equations due to Arnold [2].

Notations and Identities

We begin by introducing a number of notations and listing a couple of useful identities. Let $\underline{\underline{\gamma}}$ be a symmetric 3×3–matrix function and $\underline{v}$ a vector field (we use underbars for 3-vectors, double underbars for 3×3 matrices).

$\underline{\underline{\mathrm{skw}}}\,\underline{\underline{\gamma}} = (\underline{\underline{\gamma}} - \underline{\underline{\gamma}}^T)/2$

$\underline{\underline{\mathrm{Skw}}}\,\underline{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$

$\underline{\mathrm{curl}}\,\underline{v} = \Big(\partial u_2/\partial z - \partial u_3/\partial y, \quad \partial u_3/\partial x - \partial u_1/\partial z, \quad \partial u_1/\partial y - \partial u_2/\partial x\Big)$

$\underline{\underline{\mathrm{curl}}}_{\mathrm{r}}\,\underline{\underline{\gamma}}$ is the matrix whose rows are the curls of the rows of $\underline{\underline{\gamma}}$

$\underline{\underline{\mathrm{curl}}}_{\mathrm{c}}\,\underline{\underline{\gamma}}$ is the matrix whose columns are the curls of the columns of $\underline{\underline{\gamma}}$

$\underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\,\underline{\underline{\gamma}} = (\underline{\underline{\mathrm{curl}}}_{\mathrm{r}}\,\underline{\underline{\gamma}} + \underline{\underline{\mathrm{curl}}}_{\mathrm{c}}\,\underline{\underline{\gamma}})/2$

$\underline{\underline{\nabla}}\,\underline{v}$ is the matrix whose rows are the gradients of the entries of $\underline{v}$

$\underline{\mathrm{div}}\,\underline{\underline{\gamma}}$ is the vector whose components are the divergences of the rows of $\underline{\underline{\gamma}}$.

$\underline{\underline{\epsilon}}\,\underline{v} = [\underline{\underline{\nabla}}\,\underline{v} + (\underline{\underline{\nabla}}\,\underline{v})^T]/2$

$\underline{\underline{R}}\underline{\underline{\gamma}} = \underline{\underline{\epsilon}}\,\underline{\mathrm{div}}\,\underline{\underline{\gamma}} - 1/2\,\underline{\underline{\Delta}}\,\underline{\underline{\gamma}} - 1/2\,\underline{\underline{\nabla}}\,\underline{\nabla}\,\mathrm{tr}\,\underline{\underline{\gamma}}$

$\underline{M}\underline{\underline{\gamma}} = \underline{\mathrm{div}}\,\underline{\underline{\gamma}} - \underline{\nabla}\,\mathrm{tr}\,\underline{\underline{\gamma}}$

$\underline{\underline{M}}^*\underline{v} = -\,\underline{\underline{\epsilon}}\,\underline{v} + (\mathrm{div}\,\underline{v})\underline{\underline{\delta}}$ ($\underline{\underline{\delta}}$ being the 3×3 unit matrix)

For two vector fields $\underline{a}$ and $\underline{b}$, denote by $\underline{a} \odot \underline{b}$ the following symmetric matrix function $\underline{a} \odot \underline{b} = (\underline{a}\, \underline{b}^T + \underline{b}\, \underline{a}^T)/2$.

The proofs of the following identities imply only elementary computations and we leave them to the reader:

$$\underline{\underline{\mathrm{curl}}}_{\mathrm{r}}{}^* = \underline{\underline{\mathrm{curl}}}_{\mathrm{r}} \tag{3.55}$$

$$\underline{\underline{\mathrm{curl}}}_{\mathrm{c}}{}^* = \underline{\underline{\mathrm{curl}}}_{\mathrm{c}} \tag{3.56}$$

$$\underline{\underline{\mathrm{curl}}}_{\mathrm{s}}{}^* = \underline{\underline{\mathrm{curl}}}_{\mathrm{s}} \tag{3.57}$$

$$\underline{\underline{\mathrm{curl}}}_{\mathrm{c}}\, \underline{\underline{\gamma}} = (\underline{\underline{\mathrm{curl}}}_{\mathrm{r}}\, \underline{\underline{\gamma}})^T \text{ (therefore } \underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\, \underline{\underline{\gamma}} \text{ is symmetric)} \tag{3.58}$$

$$\mathrm{tr}\, \underline{\underline{\mathrm{curl}}}_{\mathrm{r}}\, \underline{\underline{\gamma}} = 0 \tag{3.59}$$

$$\underline{M}\, \underline{\underline{\mathrm{curl}}}_{\mathrm{r}}\, \underline{\underline{\gamma}} = 0 \tag{3.60}$$

$$\underline{M}\, \underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\, \gamma = 1/2\, \underline{\mathrm{curl}}\, \underline{M} \underline{\underline{\gamma}} = -\frac{1}{2} \underline{M}\, \underline{\underline{\mathrm{Skw}}}(\underline{M} \underline{\underline{\gamma}}) \tag{3.61}$$

$$\underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\, \underline{\underline{\gamma}} = \underline{\underline{\mathrm{curl}}}_{\mathrm{r}}\, \underline{\underline{\gamma}} - \frac{1}{2}\, \underline{\underline{\mathrm{Skw}}}(\underline{M} \underline{\underline{\gamma}}) = \underline{\underline{\mathrm{curl}}}_{\mathrm{c}}\, \underline{\underline{\gamma}} + \frac{1}{2}\, \underline{\underline{\mathrm{Skw}}}(\underline{M} \underline{\underline{\gamma}}) \tag{3.62}$$

$$\underline{\underline{\mathrm{skw}}}\, \underline{\underline{\mathrm{curl}}}_{\mathrm{r}}\, \underline{\underline{\gamma}} = -\frac{1}{2}\, \underline{\underline{\mathrm{Skw}}}(\underline{M} \underline{\underline{\gamma}}) \tag{3.63}$$

$$\underline{\underline{R}} \underline{\underline{\gamma}} = \frac{1}{2}\, \underline{\underline{\mathrm{curl}}}_{\mathrm{c}}\, \underline{\underline{\mathrm{curl}}}_{\mathrm{r}}\, \underline{\underline{\gamma}} + \frac{1}{2} (\mathrm{div}\, \underline{M} \underline{\underline{\gamma}}) \underline{\underline{\delta}} \tag{3.64}$$

$$\mathrm{div}\, \underline{M}\, \underline{\underline{\epsilon}}\, \underline{v} = 0 \tag{3.65}$$

$$\underline{M}\, \underline{\underline{\nabla}}\, \underline{\nabla}\, N = 0 \text{ for any function } N \tag{3.66}$$

$$\underline{M}\, \underline{\underline{M}}^* \underline{v} = -\, \underline{\mathrm{div}}\, \underline{\underline{\epsilon}}\, \underline{v} - \underline{\nabla}\, \mathrm{div}\, \underline{v} \tag{3.67}$$

A New FOSH Formulation of Linearized ADM

If we choose lapse N, shift $\underline{\beta}$, and linearize the ADM system about the Minkowski's flat space, the perturbations $\underline{\underline{\gamma}}$, $\underline{\underline{\kappa}}$ of the metric and the extrinsic curvature, respectively, are symmetric matrix fields satisfying

$$\dot{\underline{\underline{\gamma}}} = -2\underline{\underline{\kappa}} + 2\,\underline{\underline{\epsilon}}\,\underline{\beta}, \tag{3.68}$$

$$\dot{\underline{\underline{\kappa}}} = \underline{\underline{R}}\underline{\underline{\gamma}} - \underline{\underline{\nabla}}\,\underline{\nabla}\,N, \tag{3.69}$$

$$\operatorname{div}\underline{M}\underline{\underline{\gamma}} = 0, \tag{3.70}$$

$$\underline{M}\underline{\underline{\kappa}} = 0. \tag{3.71}$$

Of course, (3.68)–(3.71) is exactly the linearized ADM system (3.7)–(3.10) written in vector–matrix notation.

This system should be supplemented with initial conditions

$$\underline{\underline{\gamma}}(0) = \underline{\underline{\gamma}}_0, \quad \underline{\underline{\kappa}}(0) = \underline{\underline{\kappa}}_0, \tag{3.72}$$

(that satisfy the constraints (3.70) and (3.71)) and boundary conditions if the domain has frontier.

Lemma 18.

$$\underline{M}\dot{\underline{\underline{\kappa}}} = -\frac{1}{2}\,\underline{\nabla}(\operatorname{div}\underline{M}\underline{\underline{\gamma}}). \tag{3.73}$$

Proof. It follows from (3.69), (3.70), and the identity:

$$\underline{\operatorname{div}}\,\underline{\underline{\operatorname{curl}}}_{\mathrm{c}}\,\underline{\underline{\gamma}} = \underline{\operatorname{curl}}\,\underline{\operatorname{div}}\,\underline{\underline{\gamma}}.$$

□

Theorem 19. *If the initial conditions* $\underline{\underline{\gamma}}_0$ *and* $\underline{\underline{\kappa}}_0$ *satisfy the Hamiltonian constraint* (3.70) *and the momentum constraint* (3.71) *respectively, then the constraints are automatically satisfied for all time by any solution of the pure Cauchy problem* (3.68), (3.69), *and* (3.72).

Proof. From (3.65), (3.68), and (3.73), observe that

$$\partial_t^2(\operatorname{div}\underline{M}\underline{\underline{\gamma}}) = -2\operatorname{div}\underline{M}\dot{\underline{\underline{\kappa}}} = \Delta(\operatorname{div}\underline{M}\underline{\underline{\gamma}}).$$

Denote by $\phi = \operatorname{div} \underline{M}\underline{\underline{\gamma}}$. Then $\ddot{\phi} = \Delta\phi,\ \phi(0) = 0$, and $\dot{\phi}(0) = 0$. Therefore,

$$\phi = \operatorname{div} \underline{M}\underline{\underline{\gamma}} \equiv 0.$$

Moreover, from Lemma 18,

$$\underline{M}\dot{\underline{\underline{\kappa}}} = -\frac{1}{2}\,\underline{\nabla}(\operatorname{div} \underline{M}\underline{\underline{\gamma}}) = 0,$$

and from $\underline{M}\underline{\underline{\kappa}}_0 = 0$, we get $\underline{M}\underline{\underline{\kappa}} = 0$ for all time. □

From (3.68) and (3.71), it follows that

$$\ddot{\underline{\underline{\kappa}}} = -2\underline{\underline{R}}\underline{\underline{\kappa}} - \underline{\underline{\nabla}}\,\underline{\nabla}\,\dot{N}, \tag{3.74}$$

$$\underline{M}\underline{\underline{\kappa}} = 0. \tag{3.75}$$

Taking into account (3.62), (3.64), and (3.70), the equation (3.74) transforms into

$$\ddot{\underline{\underline{\kappa}}} = -\operatorname{\underline{\underline{curl}}_s}\operatorname{\underline{\underline{curl}}_s}\underline{\underline{\kappa}} - \underline{\underline{\nabla}}\,\underline{\nabla}\,\dot{N}. \tag{3.76}$$

Introduce $\underline{\underline{\nu}} = \dot{\underline{\underline{\kappa}}}$, and $\underline{\underline{\mu}} = \operatorname{\underline{\underline{curl}}_s}\underline{\underline{\kappa}}$. Then the equations (3.76), (3.75) induce the following first order symmetric hyperbolic system with constraints

$$\dot{\underline{\underline{\nu}}} = -\operatorname{\underline{\underline{curl}}_s}\underline{\underline{\mu}} - \underline{\underline{\nabla}}\,\underline{\nabla}\,\dot{N}, \tag{3.77}$$

$$\dot{\underline{\underline{\mu}}} = \operatorname{\underline{\underline{curl}}_s}\underline{\underline{\nu}}, \tag{3.78}$$

$$\underline{M}\underline{\underline{\nu}} = 0, \tag{3.79}$$

$$\underline{M}\underline{\underline{\mu}} = 0. \tag{3.80}$$

Here, the initial data is

$$\underline{\underline{\nu}}(0) = \underline{\underline{R}}\underline{\underline{\gamma}}(0) - \underline{\underline{\nabla}}\,\underline{\nabla}\,N(0),\ \ \underline{\underline{\mu}}(0) = \operatorname{\underline{\underline{curl}}_s}\underline{\underline{\kappa}}(0). \tag{3.81}$$

If $\underline{\underline{\gamma}}_0$ and $\underline{\underline{\kappa}}_0$ satisfy the Hamiltonian constraint (3.70) and the momentum constraint (3.71) respectively, then it is not hard to see that the compatibility conditions $\underline{M}\underline{\underline{\nu}}(0) = 0$ and $\underline{M}\underline{\underline{\mu}}(0) = 0$ are satisfied.

Proposition 20. *Suppose that $\underline{\underline{\gamma}}_0$ and $\underline{\underline{\kappa}}_0$ satisfy the Hamiltonian constraint* (3.70) *and the momentum constraint* (3.71) *respectively. Then, the problems* (3.68)–(3.72) *and* (3.77)–(3.81) *are equivalent.*

Proof. From construction, it is obvious that once we have a solution for (3.68)–(3.72), then (3.77)–(3.81) has solution.

Let us prove that the converse is also valid. Assume we can solve (3.77)–(3.81) and define

$$\underline{\underline{\kappa}} = \underline{\underline{\kappa}}_0 + \int_0^t \underline{\underline{\nu}}(s)\, ds, \tag{3.82}$$

$$\underline{\underline{\gamma}} = \underline{\underline{\gamma}}_0 + \int_0^t [-2\kappa(s) + 2\underline{\underline{\epsilon}}\underline{\beta}(s)]\, ds. \tag{3.83}$$

Obviously, $\underline{\underline{\gamma}}(0) = \underline{\underline{\gamma}}_0$ and $\underline{\underline{\kappa}}(0) = \underline{\underline{\kappa}}_0$ and so, (3.72) is verified. Now, we prove that (3.70) is satisfied. From (3.61) it follows that

$$\partial_t(\underline{M}\underline{\underline{\nu}}) = -\frac{1}{2}\, \underline{\mathrm{curl}}\, \underline{M}\underline{\underline{\mu}}, \tag{3.84}$$

and from here

$$\partial_t(\mathrm{div}\, \underline{M}\underline{\underline{\nu}}) = 0. \tag{3.85}$$

Since $\mathrm{div}\, \underline{M}\underline{\underline{\nu}}_0 = 0$, we get $\mathrm{div}\, \underline{M}\underline{\underline{\nu}} = 0$ for all time.

Next, we show that $\underline{\underline{\kappa}}$ satisfies (3.71). From (3.78), (3.81), and (3.82) it follows that

$$\underline{\underline{\mu}} = \underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\, \underline{\underline{\kappa}} \tag{3.86}$$

for all time. By using (3.77)–(3.78), (3.82), and (3.86) we can see that $\underline{M}\underline{\underline{\kappa}}$ satisfies the initial value problem

$$\partial_t^2(\underline{M}\,\underline{\underline{\kappa}}) = \frac{1}{4}\Delta\underline{M}\,\underline{\underline{\kappa}}, \tag{3.87}$$

$$\underline{M}\,\underline{\underline{\kappa}}(0) = 0,\ \partial_t(\underline{M}\,\underline{\underline{\kappa}})(0) = 0, \tag{3.88}$$

whose only solution is the trivial solution. So, $\underline{M}\,\underline{\underline{\kappa}} = 0$ for all time.

Finally, let us prove that (3.68) and (3.69) are verified. Observe that (3.68) is trivially satisfied by taking the derivative of (3.83) with respect to time. The identity (3.69) is a little bit more delicate, but it follows from (3.64), (3.77), (3.82), and (3.86)

$$\dot{\underline{\underline{\nu}}} = \ddot{\underline{\underline{\kappa}}} = -\underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\,\underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\,\underline{\underline{\kappa}} - \underline{\underline{\nabla}}\,\underline{\nabla}\dot{N} = -2\underline{\underline{R}}\,\underline{\underline{\kappa}} - \underline{\underline{\nabla}}\,\underline{\nabla}\dot{N} = \partial_t(\underline{\underline{R}}\,\underline{\underline{\gamma}} - \underline{\underline{\nabla}}\,\underline{\nabla}N), \tag{3.89}$$

and from (3.81), we get (3.69). □

Equivalent Unconstrained Initial Value Problem

Observe that the constrained problem (3.77)–(3.81) can be put into the form (2.10)–(2.11) for

$$A = \begin{pmatrix} 0 & -\underline{\underline{\mathrm{curl}}}_{\mathrm{s}} \\ \underline{\underline{\mathrm{curl}}}_{\mathrm{s}} & 0 \end{pmatrix},\ B = \begin{pmatrix} \underline{M} & 0 \\ 0 & \underline{M} \end{pmatrix},\ f = \begin{pmatrix} -\underline{\underline{\nabla}}\,\underline{\nabla}\dot{N} \\ 0 \end{pmatrix},\ \text{and } u_0 = \begin{pmatrix} \underline{\underline{\nu}}(0) \\ \underline{\underline{\mu}}(0) \end{pmatrix}.$$

Moreover, the condition (2.9) is satisfied since

$$Bx = 0 \Leftrightarrow \underline{M}\,\underline{\underline{\mu}} = \underline{M}\,\underline{\underline{\nu}} = 0 \Leftrightarrow \underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\,\underline{\underline{\mu}} = \underline{\underline{\mathrm{curl}}}_{\mathrm{r}}\,\underline{\underline{\mu}} \text{ and } \underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\,\underline{\underline{\nu}} = \underline{\underline{\mathrm{curl}}}_{\mathrm{r}}\,\underline{\underline{\nu}} \Leftrightarrow$$

$$\underline{M}\,\underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\,\underline{\underline{\mu}} = \underline{M}\,\underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\,\underline{\underline{\nu}} = 0 \Leftrightarrow BAx = 0.$$

Therefore, we can transform our system into a 18×18 symmetric hyperbolic (unconstrained) system

$$\begin{pmatrix} \dot{\underline{\underline{\nu}}} \\ \dot{\underline{\underline{\mu}}} \\ \dot{\underline{p}} \\ \dot{\underline{q}} \end{pmatrix} = \begin{pmatrix} 0 & -\underline{\underline{\mathrm{curl}}}_{\mathrm{s}} & -\underline{\underline{M}}^* & 0 \\ \underline{\underline{\mathrm{curl}}}_{\mathrm{s}} & 0 & 0 & -\underline{\underline{M}}^* \\ \underline{M} & 0 & 0 & 0 \\ 0 & \underline{M} & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \\ \underline{p} \\ \underline{q} \end{pmatrix} + \begin{pmatrix} -\underline{\underline{\nabla}}\,\underline{\nabla}\dot{N} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.90}$$

The initial data for the above system reads

$$\begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \\ \underline{p} \\ \underline{q} \end{pmatrix}(0) = \begin{pmatrix} \underline{\underline{R}}\,\underline{\underline{\gamma}}_0 - \underline{\underline{\nabla}}\,\underline{\nabla}N(0) \\ \underline{\underline{\mathrm{curl}}}_{\mathrm{s}}\,\underline{\underline{\kappa}}_0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.91}$$

Chapter 4

Boundary Conditions for Einstein's Equations

4.1 Introduction

A very difficult situation is encountered if initial *boundary* value problems are considered for Einstein's equations. It has become clear in the numerical relativity community that in order for constraints to be preserved during evolution, the boundary conditions have to be chosen in an appropriate way. This problem is widely open in its full generality for Einstein's equations. The main difficulty comes from the necessity to deal with unphysical instabilities, which will always be triggered in numerical simulations by round-off errors and lead to constraint violating numerical solutions. In other words, the numerical solution of the initial boundary value problem with constrained initial data fails to solve the constraints soon after the initial time. This serious drawback motivates the quest for boundary conditions that preserve the constraints. Only in the last few years work along this line has been pursued; see [47], [14], [15], [23], [24], [31], among others. In view of our work, of particular interest is the recent paper by Calabrese et al. [15] on the generalized [36] Einstein–Christoffel formulation [7] linearized around Minkowski spacetime, which employs techniques with some points in common with the ones used in our approach of the problem in this chapter.

It is here where the differential equations satisfied by the constraints are very important. In particular, if they form a hyperbolic system, then the study of its well–posedness is proven to be a very good starting point for what boundary conditions we must force upon the evolution system for the dynamical variables. By exploiting this idea, we have been able to develop a technique that provides maximal nonnegative constraint preserving boundary conditions for various hyperbolic formulations of Einstein's equations in the linearized case. This entire chapter represents our original contribution to the subject.

4.2 Model Problem

In this section we consider a constrained first order symmetric hyperbolic system, which, as we shall prove, admits well-posed constraint preserving boundary conditions. The analysis of this model problem gives a good deal of insight for finding same type of boundary conditions for various hyperbolic formulations of Einstein's equations. Although the problem is simple, the techniques used here reveal our basic strategy to tackle the more complex case of Einstein's equations.

For $t > 0$ and x in $\mathbb{R}^3$, we are interested in finding a solution (w_i, v_i, u_{ij}) for the following first order symmetric hyperbolic system

$$\dot{w}_i = v_i, \quad \dot{v}_i = \partial^j u_{ij} + f_i, \quad \dot{u}_{ij} = \partial_j v_i, \tag{4.1}$$

with the initial data

$$w_i(0) = w_i^0, \quad v_i(0) = v_i^0, \quad u_{ij}(0) = u_{ij}^0, \tag{4.2}$$

and subject to the constraint

$$C := \partial^i v_i = 0. \tag{4.3}$$

Assume that the initial conditions (4.2) are compatible with the constraint (4.3)

$$\partial^i v_i^0 = 0, \quad \partial^i u_{ij}^0 = 0. \tag{4.4}$$

Also, assume that the forcing terms satisfy the compatibility condition

$$\partial^i f_i = 0, \qquad \text{for all time } t \geq 0. \tag{4.5}$$

Then, for the pure Cauchy problem (4.1), (4.2), it can be easily shown that the constraint (4.3) is satisfied for all time. Indeed, if we differentiate twice in time the constraint (4.3) and use the main system (4.1), it follows that

$$\ddot{C} = \partial^i \ddot{v}_i = \partial_t[\partial^i(\partial^j u_{ij} + f_i)] = \partial^i \partial^j \dot{u}_{ij} + \partial_t(\partial^i f_i) = \partial^j \partial_j \partial^i v_i + \partial_t(\partial^i f_i) = \partial^j \partial_j C + \partial_t(\partial^i f_i). \tag{4.6}$$

Since $\partial^i f_i = 0$ for all time, it follows that C satisfies the wave equation on $\mathbb{R}_+ \times \mathbb{R}^3$.

$$\ddot{C} = \Delta C. \tag{4.7}$$

Moreover, from the compatibility conditions (4.4)

$$\partial^i v_i(0) = \partial^i v_i^0 = 0. \tag{4.8}$$

So,

$$C(0) = 0. \tag{4.9}$$

Also,

$$\partial^i \dot{v}_i(0) = \partial^j \partial^i u_{ij}(0) + \partial^i \dot{f}_i(0). \tag{4.10}$$

The first term on the right side vanishes from the second compatibility condition in (4.4). The second term vanishes because of (4.5). Thus,

$$\dot{C}(0) = 0. \tag{4.11}$$

From (4.7), (4.9), and (4.11), we conclude that $C = 0$ for all time.

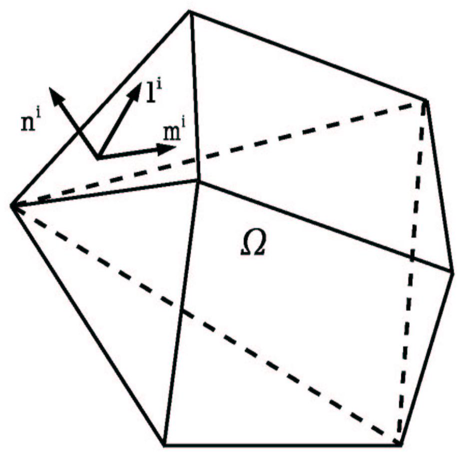

Figure 4.1: A polyhedral domain.

4.2.1 Constraint-Preserving Boundary Conditions for the Model Problem

In this subsection, we are interested in finding suitable boundary conditions for (4.1) on the boundary $\partial\Omega$ of a *bounded* spatial domain Ω such that the solution of the resulting initial-*boundary* value problem satisfies the constraint (4.3) for all time.

On Polyhedral Domains

First, we investigate the existence of constraint-preserving boundary conditions on polyhedral domains, with the result that there exists a set of such boundary conditions. Moreover, these boundary conditions are also maximal nonnegative, and so, the corresponding initial-boundary value problem is well-posed.

Let Ω be a polyhedral domain in $\mathbb{R}^3$. Consider an arbitrary face of $\partial\Omega$ and let n^i denote its exterior unit normal. Denote by m^i and l^i two additional vectors which together n^i form an orthonormal basis (see Figure 4.1). The projection operator orthogonal to n^i is then given by $\tau_i^j := m_i m^j + l_i l^j$ (and does not depend on the particular choice of these tangential vectors). Note that

$$\delta_i^j = n_i n^j + \tau_i^j, \quad \tau_i^j \tau_j^k = \tau_i^k. \tag{4.12}$$

Consequently,

$$a_l b^l = n^j a_j n_i b^i + \tau_l^j a_j \tau_i^l b^i \quad \text{for all vectors } a_l,\ b^l. \tag{4.13}$$

We shall prove that the following set of boundary conditions

$$n^i n^j u_{ij} = 0, \quad m^i v_i = 0, \quad l^i v_i = 0, \tag{4.14}$$

is constraint-preserving and, together with (4.1) and (4.2), leads to a well-posed initial-boundary value problem. Observe that these boundary conditions can be written as well:

$$n^i n^j u_{ij} = 0, \quad \tau_j^i v_i = 0. \tag{4.15}$$

Therefore, they do not depend on the particular choice of the tangential vectors m^i and l^i.

Theorem 21. *Given initial conditions $w_i(0)$, $v_i(0)$, $u_{ij}(0)$ and forcing terms f_i satisfying the compatibility conditions* (4.4) *and* (4.5) *respectively, define w_i, v_i, and u_{ij} for positive time by the evolution equations* (4.1) *and the boundary conditions* (4.14)*, or* (4.15)*. Then, the constraint* (4.3) *is satisfied for all time.*

Proof. First we prove that $C = 0$ on any boundary face of Ω. From the first identity of (4.12), the constraint C can be decomposed as follows

$$C = \partial^i v_i = n^i n^j \partial_j v_i + \tau^{ij} \partial_j v_i. \tag{4.16}$$

By the third equation in (4.1) and the second identity of (4.12), (4.16) reads

$$C = n^i n^j \dot{u}_{ij} + \tau^{ik} \tau_k^j \partial_j v_i. \tag{4.17}$$

From the boundary conditions (4.15), we know that $n^i n^j u_{ij} = 0$ for all time, and so the first term on the right-hand side vanishes. Similarly, we know that $\tau^{ik} v_i = 0$ on the boundary face, and so the second term vanishes as well (since the differential operator $\tau_k^j \partial_j$ is purely tangential). We have established that $C = 0$ holds on any boundary face of Ω.

Finally, since C also satisfies the wave equation (4.7), together with the trivial initial conditions (4.9) and (4.11), we conclude that the constraint C vanishes for all time. □

Theorem 22. *The differential system* (4.1)*, together with initial data* (4.2) *and boundary conditions* (4.15) *is well-posed. Furthermore, if the initial conditions* $w_i(0)$*,* $v_i(0)$*,* $u_{ij}(0)$ *and the forcing terms* f_i *are given satisfying the compatibility conditions* (4.4) *and* (4.5) *respectively, then the solution satisfies the constraint* (4.3) *for all time.*

Proof. We begin by showing that the boundary conditions (4.15) are maximal nonnegative for the hyperbolic system (4.1), and so, according to the classical theory of [18] and [37] (see also [40]), the initial-boundary value problem is well-posed. We recall the definition of maximal nonnegative boundary conditions. Let $V := \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$. Obviously, $\dim V = 15$. The boundary operator A_n associated to the hyperbolic system (4.1) is the symmetric linear operator $V \to V$ which assigns to any element $(w_i,\, v_i,\, u_{ij})$ of V the element $(\tilde{w}_i,\, \tilde{v}_i,\, \tilde{u}_{ij})$ defined by

$$\tilde{w}_i = 0, \quad \tilde{v}_i = -n^j u_{ij}, \quad \tilde{u}_{ij} = -n_j v_i. \tag{4.18}$$

A subspace N of V is called nonnegative for A_n if

$$w^i \tilde{w}_i + v^i \tilde{v}_i + u^{ij} \tilde{u}_{ij} \geq 0, \tag{4.19}$$

whenever $(w_i,\, v_i,\, u_{ij}) \in N$ and $(\tilde{w}_i,\, \tilde{v}_i,\, \tilde{u}_{ij})$ defined by (4.18). We claim that the subspace N defined by the boundary conditions (4.15) is maximal nonnegative. The dimension of N is clearly 12. Since A_n has three positive, nine zero, and three negative eigenvalues, a nonnegative subspace is *maximal* nonnegative if and only if it has dimension 12, and so, the proof reduces to the verification of nonnegativity condition (4.19). Since

$$w^i \tilde{w}_i + v^i \tilde{v}_i + u^{ij} \tilde{u}_{ij} = -2 v^i n^j u_{ij}, \tag{4.20}$$

the verification of (4.19) reduces to showing that $v^i n^j u_{ij} \leq 0$ whenever (4.15) holds. In fact, we prove that $v^i n^j u_{ij} = 0$ if the boundary conditions (4.15) are verified. For this, we use the orthogonal decomposition of $v^i = n^k v_k n^i + \tau^{ki} v_k$. From the second boundary condition

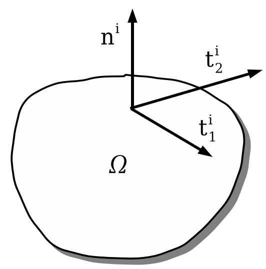

Figure 4.2: A 3D domain.

in (4.15), the tangential part $\tau^{ki}v_k$ of v^i vanishes. Thus, $v^i n^j u_{ij} = n^i n^j u_{ij} n^k v_k$. From the first boundary condition in (4.15), the right-hand side vanishes, and so, $v^i n^j u_{ij} = 0$. This ends the proof of the maximal nonnegativity of the boundary conditions (4.15).

The fact that the solution satisfies the constraint for all time follows from Theorem 21. □

On Piecewise Regular Domains

Here we investigate the existence of constraint-preserving boundary conditions on more general domains, namely on piecewise regular domains. We shall provide a set of such boundary conditions which generalize the boundary conditions (4.15). Although these boundary conditions are not maximal nonnegative in general, and so, the classical theory of [18] and [37] can not be employed, we have been able to prove an energy inequality which is a key ingredient in proving well-posedness.

Let Ω be a bounded domain in $\mathbb{R}^3$. Assume that the boundary of Ω, denoted by $\partial\Omega$ is an almost everywhere regular surface in $\mathbb{R}^3$, that is, for almost every point $p \in \partial\Omega$, there exists an open set V in $\mathbb{R}^3$ and a difeomorphism $\phi : U \to V \cap \partial\Omega$, $(x^1, x^2) \to \phi(x^1, x^2)$, of an open set $U \subset \mathbb{R}^2$ onto $V \cap \partial\Omega \subset \mathbb{R}^3$. The mapping ϕ is called a *parameterization* or a system of (local) coordinates in a neighborhood of p. From [38], Corollary 2, p. 183, we know that for all $p \in \partial\Omega$ there exists a parameterization ϕ in a neighborhood V of p

such that the coordinate curves intersect orthogonally for each $q \in V$ (such a ϕ is called *orthogonal* parameterization). Since working in an orthogonal parameterization turns out to simplify our computations, we assume that ϕ is an orthogonal parameterization. At each point p of $V \cap \partial\Omega$ define the tangent vectors $\underline{t}_i = \partial\phi/\partial x^i$, $i = 1, 2$, and the unit normal to the boundary (see Figure 4.2)

$$\underline{n} = \frac{\underline{t}_1 \wedge \underline{t}_2}{|\underline{t}_1 \wedge \underline{t}_2|}. \tag{4.21}$$

For the given parameterization ϕ, the derivatives of $\underline{n}$ belong to the tangential plane and are given by

$$\begin{aligned} \frac{\partial \underline{n}}{\partial x^1} &= a_{11}\underline{t}_1 + a_{21}\underline{t}_2, \\ \frac{\partial \underline{n}}{\partial x^2} &= a_{12}\underline{t}_1 + a_{22}\underline{t}_2. \end{aligned} \tag{4.22}$$

The trace of the matrix $(a_{ij})_{1\le i,\,j\le 2}$ does not depend on the choice of parameterization. Hence, the *mean curvature* H, which is one half of the trace of $(a_{ij})_{1\le i,\,j\le 2}$, is parameterization independent. Since the principal curvatures k_1 and k_2 are the roots of the quadratic equation

$$\det \begin{pmatrix} a_{11} - k & a_{12} \\ a_{21} & a_{22} - k \end{pmatrix} = 0,$$

we can write

$$H = \frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2}(k_1 + k_2). \tag{4.23}$$

We shall prove that the following set of boundary conditions

$$n^i n^j u_{ij} + 2H n^i w_i = 0, \quad \tau^{ij} v_i = 0, \tag{4.24}$$

is constraint-preserving and, together with (4.1) and (4.2), leads to well-posedness.

First we prove the equivalent of Theorem 21 under the new circumstances.

Theorem 23. *Given initial conditions $w_i(0)$, $v_i(0)$, $u_{ij}(0)$ and forcing terms f_i satisfying the compatibility conditions* (4.4) *and* (4.5) *respectively, define w_i, v_i, and u_{ij} for positive time by the evolution equations* (4.1) *and the boundary conditions* (4.24)*. Then, the constraint* (4.3) *is satisfied for all time.*

Proof. Let $E = \langle\, \underline{t}_1,\, \underline{t}_1 \,\rangle$, $F = \langle\, \underline{t}_1,\, \underline{t}_2 \,\rangle = 0$, and $G = \langle\, \underline{t}_2,\, \underline{t}_2 \,\rangle$ be the coefficients of the first fundamental form in the basis $\{\, \underline{t}_1,\, \underline{t}_2 \,\}$. The derivatives of the vectors $\underline{t}_1$ and $\underline{t}_2$ in the basis $\{\, \underline{t}_1,\, \underline{t}_2,\, \underline{n} \,\}$ are (see [38], p. 232):

$$\begin{aligned} \frac{\partial \underline{t}_1}{\partial x^1} &= \Gamma^1_{11}\underline{t}_1 + \Gamma^2_{11}\underline{t}_2 + L_1\underline{n}, \\ \frac{\partial \underline{t}_1}{\partial x^2} = \frac{\partial \underline{t}_2}{\partial x^1} &= \Gamma^1_{12}\underline{t}_1 + \Gamma^2_{12}\underline{t}_2 + L_2\underline{n}, \\ \frac{\partial \underline{t}_2}{\partial x^2} &= \Gamma^1_{22}\underline{t}_1 + \Gamma^2_{22}\underline{t}_2 + L_3\underline{n}, \end{aligned} \tag{4.25}$$

where Γ^k_{ij} are the Christoffel symbols and $L_1 = -a_{11}E$, $L_2 = -a_{12}E = -a_{21}G$, and $L_3 = -a_{22}G$.

Since the constraint C satisfies the wave equation (4.7) with zero initial conditions (4.9), (4.11), the proof of this theorem reduces to prove that $C = 0$ *a.e.* on the boundary $\partial\Omega$ for all time. From the first identity of (4.12), observe that on the boundary the constraint C is $\partial^i v_i = \delta^{ij}\partial_j v_i = n^i n^j \partial_j v_i + \tau^{ij}\partial_j v_i$. From the main system (4.1), the first term of the right-hand side is equal to $n^i n^j \dot{u}_{ij}$. Since the parameterization is orthogonal, the tangential projection of $\partial_j v_i$ is

$$\tau^{ij}\partial_j v_i = \frac{1}{E} t^i_1 t^j_1 \partial_j v_i + \frac{1}{G} t^i_1 t^j_1 \partial_j v_i,$$

where t^j_i, $1 \le j \le 3$, represent the components of the tangent vector $\underline{t}_i$, $i = 1, 2$. By the Leibnitz rule of differentiation, the right-hand side of this identity can be written as follows

$$\tau^{ij}\partial_j v_i = \frac{1}{E} t^j_1 \partial_j (t^i_1 v_i) + \frac{1}{G} t^j_2 \partial_j (t^i_2 v_i) - \frac{1}{E} v_i t^j_1 \partial_j t^i_1 - \frac{1}{G} v_i t^j_2 \partial_j t^i_2. \tag{4.26}$$

The first two terms of the right-hand side vanish due to the second boundary condition in (4.24) combined with the tangentiality of the differential operators $t^j_i\partial_j$, $i = 1, 2$. Furthermore, by the chain rule, $t^j_k \partial_j t^i_k = \partial t^i_k / \partial x^k$, $k = 1, 2$. Thus,

$$\tau^{ij}\partial_j v_i = -\frac{1}{E} v_i \frac{\partial t^i_1}{\partial x^1} - \frac{1}{G} v_i \frac{\partial t^i_2}{\partial x^2}. \tag{4.27}$$

By the first identity in (4.25), the first term of the right-hand side of (4.27) is

$$-\frac{1}{E}v_i\frac{\partial t_1^i}{\partial x^1} = -\frac{1}{E}\Gamma_{11}^1(t_1^i v_i) - \frac{1}{E}\Gamma_{11}^2(t_2^i v_i) + a_{11}n^i v_i. \tag{4.28}$$

The first two terms of the right side of the last identity vanish due to the second boundary condition in (4.24), and so

$$-\frac{1}{E}v_i\frac{\partial t_1^i}{\partial x^1} = a_{11}n^i v_i. \tag{4.29}$$

Similarly,

$$-\frac{1}{G}v_i\frac{\partial t_2^i}{\partial x^2} = a_{22}n^i v_i. \tag{4.30}$$

Thus,

$$\tau^{ij}\partial_j v_i = (a_{11} + a_{22})n^i v_i. \tag{4.31}$$

Therefore, $\partial^i v_i = n^i n^j \dot{u}_{ij} + (a_{11}+a_{22})n^i v_i$. By the first equation in (4.1), $\dot{w}_i = v_i$, combined with the definition of the mean curvature, $H = (a_{11}+a_{22})/2$, it follows that $\partial^i v_i = (n^i n^j u_{ij} + 2Hn^i w_i)^{\cdot}$. Finally, from the first boundary condition in (4.24), $C = \partial^i v_i = 0$ *a.e.* on $\partial\Omega$ for all time. This ends the proof of this theorem. □

Theorem 24. *Let Ω be a bounded domain in $\mathbb{R}^3$ with (a.e.) regular boundary and the mean curvature $H \geq 0$ almost everywhere. Given the initial conditions w_i^0, v_i^0, u_{ij}^0 and the forcing terms f_i satisfying the compatibility conditions* (4.4) *and* (4.5) *respectively, the system of differential equations* (4.1)*, together with the boundary conditions* (4.24) *and the initial conditions* (4.2)*, is well-posed. Moreover, its solution satisfies the constraint* (4.3) *for all time.*

Proof. For any solution w_i, v_i, u_{ij} of (4.1) satisfying the boundary conditions (4.24), define the energy

$$E(t) = \frac{1}{2}[\int_\Omega (w_i w^i + v_i v^i + u_{ij}u^{ij})\,dx + 2\int_{\partial\Omega} H(w_i n^i)^2\,ds].$$

Differentiating in time, we obtain

$$\dot{E}(t) = \int_\Omega (\dot{w}_i w^i + \dot{v}_i v^i + \dot{u}_{ij}u^{ij})\,dx + 2\int_{\partial\Omega} H(\dot{w}_i n^i)(w_i n^i)\,ds.$$

By the system (4.1), the derivatives in time can be substituted for spatial derivatives, and so

$$\dot{E}(t) = \int_{\Omega} [v_i w^i + (\partial^j u_{ij})v^i + f_i v^i + (\partial_j v_i) u^{ij} \, dx + 2 \int_{\partial\Omega} H(v_i n^i)(w_i n^i) \, ds.$$

Integrating by parts, it follows that

$$\dot{E}(t) = \int_{\Omega} (v_i w^i + f_i v^i) \, dx + \int_{\partial\Omega} v^i n^j u_{ij} \; ds + 2 \int_{\partial\Omega} H(v_i n^i)(w_i n^i) \, ds.$$

By the orthogonal decomposition of $v^i = n^i n^l v_l + \tau^{il} v_l$, we obtain $v^i n^j u_{ij} = n^i n^j u_{ij} n^l v_l + \tau^{il} v_l n^j u_{ij}$. In view of the second boundary condition in (4.24), the second term of the right-hand side vanishes. Thus, $v^i n^j u_{ij} = n^i n^j u_{ij} n^l v_l$, and so

$$\dot{E}(t) = \int_{\Omega} (v_i w^i + f_i v^i) \, dx + \int_{\partial\Omega} (n^i n^j u_{ij} + 2H w_i n^i)(n^l v_l) \, ds.$$

Due to the second boundary condition in (4.24), the boundary integral vanishes. Therefore,

$$\dot{E}(t) = \int_{\Omega} (v_i w^i + f_i v^i) \, dx.$$

Now, since $H \geq 0$ (*a.e.*), it is easy to see that

$$\dot{E}(t) \leq E(t) + \theta(t),$$

where $\theta(t) = (\int_{\Omega} f_i f^i \, dx)/2$ for all positive time $t > 0$. By the Gronwall inequality, $E(t) \leq e^t [E(0) + \int_0^t e^{-s} \theta(s) \, ds]$. This energy estimate is a key ingredient in proving well-posedness by Galerkin approximations.

The fact that the solution satisfies the constraint (4.3) for all time follows from Theorem 23.

□

4.3 Einstein-Christoffel Formulation

4.3.1 Maximal Nonnegative Constraint-Preserving Boundary Conditions

In this section, we provide maximal nonnegative boundary conditions for the linearized EC system which are constraint-preserving in the sense that the analogue of Theorem 16 is true for the initial–boundary value problem. This will then imply the analogue of Theorem 17. We assume that Ω is a polyhedral domain (see Figure 4.1) and use the notations introduced in Subsection 4.2.1.

First we consider the following boundary conditions on the face:

$$n^i m^j K_{ij} = n^i l^j K_{ij} = n^k n^i n^j f_{kij} = n^k m^i m^j f_{kij} = n^k l^i l^j f_{kij} = n^k m^i l^j f_{kij} = 0. \tag{4.32}$$

These can be written as well:

$$n^i \tau^{jk} K_{ij} = 0, \quad n^k n^i n^j f_{kij} = 0, \quad n^k \tau^{il} \tau^{jm} f_{kij} = 0, \tag{4.33}$$

and so do not depend on the choice of basis for the tangent space. We begin by showing that these boundary conditions are maximal nonnegative for the hyperbolic system (3.36), (3.37), and so, according to the classical theory of [18] and [37], the initial–boundary value problem is well-posed.

Let V denote the vector space of pairs of constant tensors (K_{ij}, f_{kij}) both symmetric with respect to the indices i and j. Thus $\dim V = 24$. The boundary operator A_n associated to the evolution equations (3.36), (3.37) is the symmetric linear operator $V \to V$ given by

$$\tilde{K}_{ij} = n^k f_{kij}, \quad \tilde{f}_{kij} = n_k K_{ij}. \tag{4.34}$$

A subspace N of V is nonnegative for A_n if

$$K_{ij}\tilde{K}^{ij} + f_{kij}\tilde{f}^{kij} \geq 0 \tag{4.35}$$

whenever $(K_{ij}, f_{kij}) \in N$ and $(\tilde{K}_{ij}, \tilde{f}_{kij})$ is defined by (4.34). The subspace is maximal nonnegative if also no larger subspace has this property. The eingenvalues λ of A_n, together with the corresponding eigenvectors $(K_{ij},\, f_{kij})$, are the following:

$\lambda = 0$ with multiplicity 12 and eingenvectors: $(0,\, m_k n_i n_j)$, $(0,\, m_k n_{(i} m_{j)})$, $(0,\, m_k n_{(i} l_{j)})$, $(0,\, m_k m_i m_j)$, $(0,\, m_k m_{(i} l_{j)})$, $(0,\, m_k l_i l_j)$, $(0,\, l_k n_i n_j)$, $(0,\, l_k n_{(i} m_{j)})$, $(0,\, l_k n_{(i} l_{j)})$, $(0,\, l_k m_i m_j)$, $(0,\, l_k m_{(i} l_{j)})$, $(0,\, l_k l_i l_j)$,

$\lambda = -1$ with multiplicity six and eingenvectors: $(n_i n_j,\, -n_k n_i n_j)$, $(m_i m_j,\, -n_k m_i m_j)$, $(l_i l_j,\, -n_k l_i l_j)$, $(n_{(i} m_{j)},\, -n_k n_{(i} m_{j)})$, $(n_{(i} l_{j)},\, -n_k n_{(i} l_{j)})$, $(l_{(i} m_{j)},\, -n_k l_{(i} m_{j)})$,

$\lambda = 1$ with multiplicity six and eingenvectors: $(n_i n_j,\, n_k n_i n_j)$, $(m_i m_j,\, n_k m_i m_j)$, $(l_i l_j,\, n_k l_i l_j)$, $(n_{(i} m_{j)},\, n_k n_{(i} m_{j)})$, $(n_{(i} l_{j)},\, n_k n_{(i} l_{j)})$, $(l_{(i} m_{j)},\, n_k l_{(i} m_{j)})$.

Since A_n has six positive, 12 zero, and six negative eigenvalues, a nonnegative subspace is maximal nonnegative if and only if it has dimension 18. Our claim is that the subspace N defined by (4.32) is maximal nonnegative. The dimension is clearly 18. In view of (4.34), the verification of (4.35) reduces to showing that $n^k f_{kij} K^{ij} \geq 0$ whenever (4.32) holds. In fact, $n^k f_{kij} K^{ij} = 0$, that is, $n^k f_{kij}$ and K_{ij} are orthogonal (when (4.32) holds). To see this, we use orthogonal expansions of each based on the normal and tangential components:

$$K_{ij} = n^l n_i n^m n_j K_{lm} + n^l n_i \tau_j^m K_{lm} + \tau_i^l n^m n_j K_{lm} + \tau_i^l \tau_j^m K_{lm}, \tag{4.36}$$

$$n^k f_{kij} = n^l n_i n^m n_j n^k f_{klm} + n^l n_i \tau_j^m n^k f_{klm} + \tau_i^l n^m n_j n^k f_{klm} + \tau_i^l \tau_j^m n^k f_{klm}. \tag{4.37}$$

In view of the boundary conditions (in the form (4.33)), the two inner terms on the right-hand side of (4.36) and the two outer terms on the right-hand side of (4.37) vanish, and so the orthogonality is evident.

Next we show that the boundary conditions are constraint-preserving. This is based on the following lemma.

Lemma 25. *Suppose that α and β^i vanish. Let K_{ij}, f_{kij} be a solution to the homogeneous hyperbolic system* (3.36), (3.37) *and suppose that the boundary conditions* (4.32) *are satisfied* *on some face of $\partial\Omega$. Let C_j be defined by* (3.34)*. Then*

$$\dot{C}_j n^l \partial_l C^j = 0 \tag{4.38}$$

on the face.

Proof. In fact we shall show that $n^j C_j = 0$ (so also $n^j \dot{C}_j = 0$) and $\tau_j^p n^l \partial_l C^j = 0$, which, by (4.13) implies (4.38). First note that

$$C_j = (\delta_j^m \delta^{ik} - \delta_j^k \delta^{im}) \partial_k K_{im} = (\delta_j^m n^i n^k + \delta_j^m \tau^{ik} - \delta_j^k \delta^{im}) \partial_k K_{im},$$

where we have used the first identity in (4.12). Contracting with n^j gives

$$\begin{aligned} n^j C_j &= (n^m n^i n^k + n^m \tau^{ik} - n^k \delta^{im}) \partial_k K_{im} \\ &= -n^m n^i n^k \dot{f}_{kim} + \tau^{il} \tau_l^k n^m \partial_k K_{im} + n^k \delta^{im} \dot{f}_{kim}, \end{aligned}$$

where now we have used the equation (3.37) (with $\beta_i = 0$) for the first and last term and the second identity in (4.12) for the middle term. From the boundary conditions we know that $n^m n^i n^k f_{kim} = 0$, and so the first term on the right-hand side vanishes. Similarly, we know that $\tau^{il} n^m K_{im} = 0$ on the boundary face, and so the second term vanishes as well (since the differential operator $\tau_l^k \partial_k$ is purely tangential). Finally, $n^k \delta^{im} f_{kim} = n^k (n^i n^m + l^i l^m + m^i m^m) f_{kim} = 0$, and so the third term vanishes. We have established that $n^j C_j = 0$ holds on the face.

To show that $\tau_j^p n^l \partial_l C^j = 0$ on the face, we start with the identity

$$\tau_j^p n^l \delta^{mj} \delta^{ik} = \tau^{pm} (n^i n^k + \tau^{ik}) n^l = \tau^{pm} n^i (\delta^{kl} - \tau^{kl}) + \tau^{pm} \tau^{ik} n^l.$$

Similarly

$$\tau_j^p n^l \delta^{kj} \delta^{im} = \tau^{pk} n^l n^i n^m + \tau^{pk} \tau^{im} n^l.$$

Therefore,

$$\begin{aligned}\tau_j^p n^l \partial_l C^j &= \tau_j^p n^l \partial_l (\delta^{mj}\delta^{ik} - \delta^{kj}\delta^{im})\partial_k K_{im} \\ &= (\tau^{pm} n^i \delta^{kl} - \tau^{pm} n^i \tau^{kl} + \tau^{pm}\tau^{ik} n^l - \tau^{pk} n^l n^i n^m - \tau^{pk}\tau^{im} n^l)\partial_k \partial_l K_{im}.\end{aligned}$$

For the last three terms, we again use (3.37) to replace $\partial_l K_{im}$ with $-\dot{f}_{lim}$ and argue as before to see that these terms vanish. For the first term we notice that $\delta^{kl}\partial_k\partial_l K_{im} = \partial^k \partial_k K_{im} = \ddot{K}_{im}$ (from (3.36) and (3.37) with vanishing α and β^i). Since $\tau^{pm} n^i K_{im}$ vanishes on the boundary, this term vanishes. Finally we recognize that the second term is the tangential Laplacian, $\tau^{kl}\partial_k\partial_l$ applied to the quantity $n^i \tau^{pm} K_{im}$, which vanishes. This concludes the proof of (4.38). □

The next theorem asserts that the boundary conditions are constraint-preserving.

Theorem 26. *Let Ω be a polyhedral domain. Given initial data $K_{ij}(0)$, $f_{kij}(0)$ on Ω satisfying the constraints* (3.34) *and* (3.40)*, define K_{ij} and f_{kij} for positive time by the evolution equations* (3.36), (3.37) *and the boundary conditions* (4.32). *Then the constraints* (3.34) *and* (3.40) *are satisfied for all time.*

Proof. Exactly as for Theorem 16 we find that C_j satisfies the wave equation (3.41) and both C_j and $\dot{C}_j$ vanish at the initial time. Define the usual energy

$$E(t) = \frac{1}{2}\int_\Omega (\dot{C}_j \dot{C}^j + \partial^l C_j \partial_l C^j)\, dx.$$

Clearly $E(0) = 0$. From (3.41) and integration by parts

$$\dot{E} = \int_{\partial\Omega} \dot{C}_j n^l \partial_l C^j \, d\sigma. \tag{4.39}$$

Therefore, if $\alpha = 0$ and $\beta^i = 0$, we can invoke Lemma 25, and conclude that E is constant in time. Hence E vanishes identically. Thus C_j is constant, and, since it vanishes at time 0, it vanishes for all time. This establishes the theorem under the additional assumption that α and β^i vanish.

To extend to the case of general α and β^i we use Duhamel's principle. Let $S(t)$ denote the solution operator associated to the homogeneous boundary value problem. That is, given functions $\kappa_{ij}(0)$, $\phi_{kij}(0)$ on Ω, define $S(t)(\kappa_{ij}(0), \phi_{kij}(0)) = (\kappa_{ij}(t), \phi_{kij}(t))$ where κ_{ij}, ϕ_{kij} is the solution to the homogeneous evolution equations

$$\dot{\kappa}_{ij} = -\partial^k \phi_{kij}, \quad \dot{\phi}_{kij} = -\partial_k \kappa_{ij},$$

satisfying the boundary conditions and assuming the given initial values. Then Duhamel's principle represents the solution K_{ij}, f_{kij} of the inhomogeneous initial-boundary value problem (3.36), (3.37), (4.32) as

$$(K_{ij}(t), f_{kij}(t)) = S(t)(K_{ij}(0), f_{kij}(0)) + \int_0^t S(t-s)(-\partial_i \partial_j \alpha(s), L_{kij}(s))\, ds. \tag{4.40}$$

Now it is easy to check that the momentum constraint (3.34) is satisfied when K_{ij} is replaced by $\partial_i \partial_j \alpha(s)$ (for any smooth function α), and the constraint (3.40) is satisfied when f_{kij} is replaced by $L_{kij}(s)$ defined by (3.38) (for any smooth function β^i). Hence the integrand in (4.40) satisfies the constraints by the result for the homogeneous case, as does the first term on the right-hand side, and thus the constraints are indeed satisfied by K_{ij}, f_{kij}. □

The analogue of Theorem 17 for the initial–boundary value problem follows from the preceding theorem exactly as before.

Theorem 27. *Let Ω be a polyhedral domain. Suppose that initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian constraint* (3.33) *and momentum constraint* (3.34)*, respectively, and that initial data $f_{kij}(0)$ is defined by* (3.39)*. Then the unique solution of the linearized EC initial–boundary value problem* (3.31)*,* (3.36)*,* (3.37)*, together with the boundary conditions* (4.32) *satisfies the linearized ADM system* (3.31)–(3.34) *in Ω.*

We close by noting a second set of boundary conditions which are maximal nonnegative and constraint-preserving. These are

$$n^i n^j K_{ij} = m^i m^j K_{ij} = l^i l^j K_{ij} = m^i l^j K_{ij} = n^k n^i m^j f_{kij} = n^k n^i l^j f_{kij} = 0, \tag{4.41}$$

or, equivalently,

$$n^i n^j K_{ij} = 0, \quad \tau^{il}\tau^{jm} K_{ij} = 0, \quad n^k n^i \tau^{jl} f_{kij} = 0.$$

Now when we make an orthogonal expansion as in (4.36), (4.37), the outer terms on the right-hand side of the first equation and the inner terms on the right-hand side of the second equation vanish (it was the reverse before), so we again have the necessary orthogonality to demonstrate that the boundary conditions are maximal nonnegative. Similarly, to prove the analogue of Lemma 25, for these boundary conditions we show that the tangential component of $\dot{C}_j$ vanishes and the normal component of $n^l \partial_l C^j$ vanishes (it was the reverse before). Otherwise the analysis is essentially the same as for the boundary conditions (4.32).

4.3.2 Extended EC System

In this section we indicate an extended initial boundary value problem whose solution solves the linearized ADM system (3.31)–(3.34) in Ω. This approach could present advantages from the numerical point of view since the momentum constraint is "built-in", and so controlled for all time. The new system consists of (3.31), (3.37), and two new sets of equations corresponding to (3.36)

$$\dot{K}_{ij} = -\partial^k f_{kij} + \frac{1}{2}(\partial_i p_j + \partial_j p_i) - \partial^k p_k \delta_{ij} - \partial_i \partial_j \alpha \tag{4.42}$$

and to a new three dimensional vector field p_i defined by

$$\dot{p}_i = \partial^l K_{li} - \partial_i K^l_l, \tag{4.43}$$

respectively. Observe that the additional terms that appear on the right-hand side of (4.42) compared with (3.36) are nothing but the negative components of the formal adjoint of the momentum constraint operator applied to p_i.

Let $\tilde{V}$ be the vector space of quadruples of constant tensors $(g_{ij}, K_{ij}, f_{kij}, p_k)$ symmetric with respect to the indices i and j. Thus $\dim \tilde{V} = 33$. The boundary operator $\tilde{A}_n : \tilde{V} \to \tilde{V}$

in this case is given by

$$\tilde{g}_{ij} = 0, \quad \tilde{K}_{ij} = n^k f_{kij} - \frac{1}{2}(n_i p_j + n_j p_i) + n^k p_k \delta_{ij}, \quad \tilde{f}_{kij} = n_k K_{ij}, \quad \tilde{p}_i = -n^l K_{il} + n_i K^l_l. \tag{4.44}$$

The boundary operator $\tilde{A}_n$ associated to the evolution equations (3.31), (4.42), (3.37), and (4.43) has six positive, 21 zero, and six negative eigenvalues. Therefore, a nonnegative subspace is maximal nonnegative if and only if it has dimension 27. We claim that the following boundary conditions are maximal nonnegative for (3.31), (4.42), (3.37), and (4.43)

$$n^i m^j K_{ij} = n^i l^j K_{ij} = n^k n^i n^j f_{kij} = n^k(m^i m^j f_{kij} + p_k) = n^k(l^i l^j f_{kij} + p_k) = n^k m^i l^j f_{kij} = 0. \tag{4.45}$$

These can be written as well:

$$n^i \tau^{jk} K_{ij} = 0, \quad n^k n^i n^j f_{kij} = 0, \quad n^k(\tau^{il}\tau^{jm} f_{kij} + \tau^{lm} p_k) = 0, \tag{4.46}$$

and so do not depend on the choice of basis for the tangent space.

Let us prove the claim that the subspace $\tilde{N}$ defined by (4.45) is maximal nonnegative. Obviously, $\dim \tilde{N} = 27$. Hence, it remains to be proven that $\tilde{N}$ is also nonnegative. In view of (4.44), the verification of nonnegativity of $\tilde{N}$ reduces to showing that

$$n^k f_{kij} K^{ij} - n^i p^j K_{ij} + n^k p_k K^l_l \geq 0 \tag{4.47}$$

whenever (4.45) holds. In fact, we can prove that the left-hand side of (4.47) vanishes pending (4.45) holds. From the boundary conditions (in the form (4.46)) and the orthogonal expansions (4.36) and (4.37) of K_{ij} and f_{kij}, respectively, the first term on the right-hand side of (4.47) reduces to $n^k \tau^{il}\tau^{jm} f_{kij} K_{lm} = -n^k p_k \tau^{lm} K_{lm}$. Then, combining the first and third terms of the left-hand side of (4.47) gives $-n^k p_k \tau^{ij} K_{ij} + n^k p_k \delta^{ij} K_{ij} = n^k p_k n^i n^j K_{ij}$. Finally, by using the orthogonal decomposition $p^j = n^k p_k n^j + \tau^{kj} p_k$ and the first part of the boundary conditions (4.46) the second term of the left-hand side of (4.47) is $-n^k p_k n^i n^j K_{ij} - p_k n^i \tau^{kj} K_{ij} = -n^k p_k n^i n^j K_{ij}$, which is precisely the negative sum of the first and third terms of the left-hand side of (4.47). This concludes the proof of (4.47).

Theorem 28. *Let Ω be a polyhedral domain. Suppose that the initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian* (3.33) *and momentum constraints* (3.34)*, respectively, $f_{kij}(0)$ is defined by* (3.35)*, and $p_i(0) = 0$. Then the unique solution $(g_{ij}, K_{ij}, f_{kij}, p_i)$ of the initial boundary value problem* (3.31)*,* (4.42)*,* (3.37)*, and* (4.43)*, together with the boundary conditions* (4.45)*, satisfies the properties $p_i = 0$ for all time, and (g_{ij}, K_{ij}) solves the linearized ADM system* (3.31)–(3.34) *in Ω.*

Proof. Observe that the solution of the initial boundary value problem (3.31), (3.36), (3.37), and (4.32) (boundary conditions), together with $p_i = 0$ for all time, is the unique solution of the initial boundary value problem (3.31), (4.42), (3.37), and (4.43), together with the boundary conditions (4.45). The conclusion follows from Theorem 27. □

We close by indicating a second set of maximal nonnegative boundary conditions (corresponding to (4.41)) for (3.31), (4.42), (3.37), and (4.43) for which Theorem 28 holds as well. These are

$$n^i n^j K_{ij} = m^i m^j K_{ij} = m^i l^j K_{ij} = l^i l^j K_{ij} = n^k n^i m^j f_{kij} - m^k p_k = n^k n^i l^j f_{kij} - l^k p_k = 0, \tag{4.48}$$

or, equivalently,

$$n^i n^j K_{ij} = 0, \quad \tau^{il} \tau^{jm} K_{ij} = 0, \quad n^k n^i \tau^{jl} f_{kij} - \tau^{kl} p_k = 0. \tag{4.49}$$

4.3.3 Inhomogeneous Boundary Conditions

In this subsection we provide well-posed constraint-preserving *inhomogeneous* boundary conditions for (3.31), (3.36), and (3.37) corresponding to the two sets of boundary conditions (4.32), and (4.41), respectively. The first set of inhomogeneous boundary conditions corresponds to (4.32) and can be written in the following form

$$n^i m^j \tilde{K}_{ij} = n^i l^j \tilde{K}_{ij} = n^k n^i n^j \tilde{f}_{kij} = n^k m^i m^j \tilde{f}_{kij} = n^k l^i l^j \tilde{f}_{kij} = n^k m^i l^j \tilde{f}_{kij} = 0, \tag{4.50}$$

where $\tilde{K}_{ij} = K_{ij} - \kappa_{ij}$, $\tilde{f}_{kij} = f_{kij} - F_{kij}$, with κ_{ij} and F_{kij} given in $\overline{\Omega}$ for all time and satisfying the constraints (3.34) and (3.40), respectively. The matter of choosing κ_{ij} and F_{kij} is deferred to the Appendix A.

The analogue of Theorem 26 for the inhomogeneous boundary conditions (4.50) is true.

Theorem 29. *Let Ω be a polyhedral domain. Given initial data $K_{ij}(0)$, $f_{kij}(0)$ on Ω satisfying the constraints* (3.34) *and* (3.40)*, define K_{ij} and f_{kij} for positive time by the evolution equations* (3.36)*,* (3.37) *and the boundary conditions* (4.50)*. Then the constraints* (3.34) *and* (3.40) *are satisfied for all time.*

Proof. Observe that $\tilde{K}_{ij}$ and $\tilde{f}_{kij}$ satisfy (3.36) and (3.37) with the forcing terms replaced by $-\partial_i\partial_j\alpha - \partial^k F_{kij} - \dot{\kappa}_{ij}$ and $L_{kij} - \partial_k \kappa_{ij} - \dot{F}_{kij}$, respectively. Exactly as in Theorem 26, it follows that $\tilde{K}_{ij}$ and $\tilde{f}_{kij}$ satisfy (3.34) and (3.40), respectively, for all time. Thus, K_{ij} and f_{kij} satisfy (3.34) and (3.40), respectively, for all time. □

The analogue of Theorem 27 for the case of the inhomogeneous boundary conditions (4.50) follows from the preceding theorem by using the same arguments as in the proof of Theorem 17.

Theorem 30. *Let Ω be a polyhedral domain. Suppose that initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian constraint* (3.33) *and momentum constraint* (3.34)*, respectively, and that initial data $f_{kij}(0)$ is defined by* (3.39)*. Then the unique solution of the linearized EC initial–boundary value problem* (3.31)*,* (3.36)*,* (3.37)*, together with the inhomogeneous boundary conditions* (4.50) *satisfies the linearized ADM system* (3.31)*–*(3.34) *in Ω.*

Note that there is a second set of inhomogeneous boundary conditions corresponding to (4.41) for which Theorem 29 and Theorem 30 remain valid. These are

$$n^i n^j \tilde{K}_{ij} = m^i m^j \tilde{K}_{ij} = l^i l^j \tilde{K}_{ij} = m^i l^j \tilde{K}_{ij} = n^k n^i m^j \tilde{f}_{kij} = n^k n^i l^j \tilde{f}_{kij} = 0, \tag{4.51}$$

where again $\tilde{K}_{ij} = K_{ij} - \kappa_{ij}$, $\tilde{f}_{kij} = f_{kij} - F_{kij}$, with κ_{ij} and F_{kij} given and satisfying the constraints (3.34) and (3.40), respectively.

Similar considerations can be made for the extended system introduced in the previous section. There are two sets of *inhomogeneous* boundary conditions for which the extended system produces solutions of the linearized ADM system (3.31)–(3.34) on a polyhedral domain Ω. These are

$$n^i m^j \tilde{K}_{ij} = n^i l^j \tilde{K}_{ij} = n^k n^i n^j \tilde{f}_{kij} = n^k(m^i m^j \tilde{f}_{kij} + p_k) = n^k(l^i l^j \tilde{f}_{kij} + p_k) = n^k m^i l^j \tilde{f}_{kij} = 0 \tag{4.52}$$

and, respectively,

$$n^i n^j \tilde{K}_{ij} = m^i m^j \tilde{K}_{ij} = m^i l^j \tilde{K}_{ij} = l^i l^j \tilde{K}_{ij} = n^k n^i m^j \tilde{f}_{kij} - m^k p_k = n^k n^i l^j \tilde{f}_{kij} - l^k p_k = 0, \tag{4.53}$$

where $\tilde{K}_{ij}$ and $\tilde{f}_{kij}$ are defined as before.

The next theorem is an extension of Theorem 28 to the case of inhomogeneous boundary conditions.

Theorem 31. *Let Ω be a polyhedral domain. Suppose that the initial data $g_{ij}(0)$ and $K_{ij}(0)$ are given satisfying the Hamiltonian* (3.33) *and momentum constraints* (3.34)*, respectively, $f_{kij}(0)$ is defined by* (3.35)*, and $p_i(0) = 0$. Then the unique solution $(g_{ij}, K_{ij}, f_{kij}, p_i)$ of the initial boundary value problem* (3.31)*,* (4.42)*,* (3.37)*, and* (4.43)*, together with the inhomogeneous boundary conditions* (4.52) *(or* (4.53)*), satisfies the properties $p_i = 0$ for all time, and (g_{ij}, K_{ij}) solves the linearized ADM system* (3.31)*–*(3.34) *in Ω.*

Proof. Note that the solution of the initial boundary value problem (3.31), (3.36), (3.37), and (4.50) (or (4.51), respectively), together with $p_i = 0$ for all time, is the unique solution of the initial boundary value problem (3.31), (4.42), (3.37), and (4.43), together with the boundary conditions (4.52) (or (4.53), respectively). The conclusion follows from Theorem 30. □

4.4 Alekseenko-Arnold Formulation

4.4.1 Maximal Nonnegative Constraint-Preserving Boundary Conditions

The equivalence of (3.50), (3.51) and the linearized ADM system has been studied in the second section of [3] for the case of pure initial value problem with the result that for given initial data satisfying the constraints, the unique solution of the linearized AA evolution equations satisfies the linearized ADM system, and so the linearized ADM system and the linearized AA system are equivalent (see [3], Theorem 1.). Our interest is in the case when the spatial domain is bounded and appropriate boundary conditions are imposed. In this subsection, we provide maximal nonnegative boundary conditions for the linearized AA system which are constraint-preserving. This will then imply the analogue of the equivalence result proven in [3] (Theorem 1.) for the case of bounded domains. The ideas and techniques used in this section have many points in common to those used in Section 4.3 for the case of EC formulation.

Assume that Ω is a polyhedral domain. Consider an arbitrary face of $\partial\Omega$ and let n^i denote its exterior unit normal. Denote by m^i and l^i two additional vectors which together n^i form an orthonormal basis (see Figure 4.1).

First we consider the following boundary conditions on the face:

$$n^i m^j \kappa_{ij} = n^i l^j \kappa_{ij} = n^k m^i m^j \lambda_{kij} = n^k l^i l^j \lambda_{kij} = n^k m^i l^j \lambda_{kij} = 0. \tag{4.54}$$

These are equivalent to:

$$n^i \tau^{jk} \kappa_{ij} = 0, \quad n^k \tau^{il} \tau^{jm} \lambda_{kij} = 0, \tag{4.55}$$

where $\tau_i^j := m_i m^j + l_i l^j$ is the projection operator to n^i, and so do not depend on the choice of basis for the tangent space. We begin by showing that these boundary conditions are maximal nonnegative for the hyperbolic system (3.50), (3.51), and so, according to the classical theory of [18] and [37], the initial boundary value problem is well-posed.

Let V denote the vector space of pairs of constant tensors $(\kappa_{ij}, \lambda_{kij})$. Here κ_{ij} is symmetric with respect to the indices i and j, and λ_{kij} is a third-order constant tensor which is antisymmetric with respect to the first two indices and satisfies the cyclic identity $\lambda_{kij} + \lambda_{jki} + \lambda_{ijk} = 0$. Thus $\dim V = 14$. The boundary operator A_n associated to the evolution equations (3.36), (3.37) is the symmetric linear operator $V \to V$ given by

$$\tilde{\kappa}_{ij} = -n^k \lambda_{k(ij)}, \quad \tilde{\lambda}_{kij} = -n_{[k}\kappa_{i]j}. \tag{4.56}$$

A subspace N of V is called nonnegative for A_n if

$$\kappa_{ij}\tilde{\kappa}^{ij} + \lambda_{kij}\tilde{\lambda}^{kij} \geq 0 \tag{4.57}$$

whenever $(\kappa_{ij}, \lambda_{kij}) \in N$ and $(\tilde{\kappa}_{ij}, \tilde{\lambda}_{kij})$ is defined by (4.56). The subspace is maximal nonnegative if also no larger subspace has this property. Since A_n has five positive, four zero, and five negative eigenvalues, a nonnegative subspace is maximal nonnegative if and only if it has dimension nine. Our claim is that the subspace N defined by (4.54) is maximal nonnegative. The dimension is clearly nine. In view of (4.56), the verification of (4.57) reduces to showing that $n^k\lambda_{kij}\kappa^{ij} \leq 0$ whenever (4.32) holds. In fact, $n^k\lambda_{kij}\kappa^{ij} = 0$, that is, $n^k\lambda_{kij}$ and κ_{ij} are orthogonal (when (4.54) holds). To see this, we use orthogonal expansions of each based on the normal and tangential components:

$$\kappa_{ij} = n^l n_i n^m n_j \kappa_{lm} + n^l n_i \tau_j^m \kappa_{lm} + \tau_i^l n^m n_j \kappa_{lm} + \tau_i^l \tau_j^m \kappa_{lm}, \tag{4.58}$$

$$n^k\lambda_{kij} = n^l n_i n^m n_j n^k \lambda_{klm} + n^l n_i \tau_j^m n^k \lambda_{klm} + \tau_i^l n^m n_j n^k \lambda_{klm} + \tau_i^l \tau_j^m n^k \lambda_{klm}. \tag{4.59}$$

In view of the boundary conditions (in the form (4.55)), the two inner terms on the right-hand side of (4.58) and the last term of the right-hand side of (4.59) vanish. Also, the first two terms of the right-hand side of (4.59) vanish due to the antisymmetry of λ_{kij} with respect to the first two indices. Thus, the orthogonality is evident.

Next we show that the boundary conditions are constraint-preserving. This is based on the following lemma.

Lemma 32. *Suppose that α and β^i vanish. Let κ_{ij}, λ_{kij} be a solution to the homogeneous* *hyperbolic system* (3.50), (3.51) *and suppose that the boundary conditions* (4.54) *are satisfied on some face of* $\partial\Omega$*. Let* C_j *be defined by* (3.10)*. Then*

$$\dot{C}_j n^l \partial_l C^j + n^j \dot{C}_j \partial_l C^l = 0 \tag{4.60}$$

on the face.

Proof. In fact we shall show that $n^j C_j = 0$ (so also $n^j \dot{C}_j = 0$) and $\tau_j^p n^l \partial_l C^j = 0$, which, by (4.13) implies (4.60). First note that

$$C_j = (\delta_j^m \delta^{ik} - \delta_j^k \delta^{im}) \partial_k \kappa_{im}$$

Contracting with n^j and using the first identity of (4.12) give

$$\begin{aligned} n^j C_j &= (n^m \delta^{ik} - n^k \delta^{im}) \partial_k \kappa_{im} \\ &= [n^m (n^i n^k + \tau^{ik}) - n^k (n^i n^m + \tau^{im})] \partial_k \kappa_{im} \\ &= (n^m \tau^{ik} - n^k \tau^{im}) \partial_k \kappa_{im} \\ &= n^m \tau^{ik} \partial_k \kappa_{im} - n^k \tau^{im} (\sqrt{2} \dot{\lambda}_{kim} + \partial_i \kappa_{km}), \end{aligned}$$

where now we have used the equation (3.51) (with $\beta_i = 0$) for the last term. From the boundary conditions we know that $n^m \tau^{ik} \partial_k \kappa_{im} = 0$, and so the first term on the right-hand side vanishes (since the differential operator $\tau^{ik} \partial_k$ is purely tangential). Similarly, $n^k \tau^{im} \lambda_{kim} = n^k \tau^{im} \partial_i \kappa_{km} = 0$ on the boundary face, and so the second term vanishes as well. We have established that $n^j C_j = 0$ holds on the face.

To show that $\tau_j^p n^l \partial_l C^j = 0$ on the face, we start with the identity

$$\tau_j^p n^l \delta^{mj} \delta^{ik} = \tau^{pm} (n^i n^k + \tau^{ik}) n^l = \tau^{pm} n^i (\delta^{kl} - \tau^{kl}) + \tau^{pm} \tau^{ik} n^l.$$

Similarly

$$\tau_j^p n^l \delta^{kj} \delta^{im} = \tau^{pk} n^l n^i n^m + \tau^{pk} \tau^{im} n^l.$$

Therefore,

$$\begin{aligned}\tau_j^p n^l \partial_l C^j &= \tau_j^p n^l \partial_l (\delta^{mj}\delta^{ik} - \delta^{kj}\delta^{im})\partial_k \kappa_{im} \\ &= (\tau^{pm} n^i \delta^{kl} - \tau^{pm} n^i \tau^{kl} + \tau^{pm}\tau^{ik} n^l - \tau^{pk} n^l n^i n^m - \tau^{pk}\tau^{im} n^l)\partial_k \partial_l \kappa_{im}.\end{aligned}$$

The second term of the right-hand side vanishes since it involves the tangential Laplacian $\tau^{kl}\partial_k\partial_l$ applied to the quantity $\tau^{pm}n^i\kappa_{im}$, which vanishes. For the third and last terms, we again use (3.51) to replace $\partial_l\kappa_{im}$ with $\sqrt{2}\dot{\lambda}_{lim} + \partial_i\kappa_{lm}$ and argue as before to see that the resulting terms vanish. For the first term we notice that $\delta^{kl}\partial_k\partial_l\kappa_{im} = \partial^k\partial_k\kappa_{im} = \ddot{\kappa}_{im} + \partial_{(i}C_{m)} - \partial_i\partial_m\kappa_l^l$ (from (3.50) and (3.51) with vanishing α and β^i). So, $\tau^{pm}n^i\delta^{kl}\partial_k\partial_l\kappa_{im} = \tau^{pm}n^i\ddot{\kappa}_{im} + \frac{1}{2}\tau^{pm}n^i\partial_i C_m + \frac{1}{2}\tau^{pm}n^i\partial_m C_i - \tau^{pm}n^i\partial_i\partial_m\kappa_l^l$. From $\tau^{pm}n^i\kappa_{im} = 0$ and $n^iC_i = 0$ on the face, this identity reduces to $\tau^{pm}n^i\delta^{kl}\partial_k\partial_l\kappa_{im} = \frac{1}{2}\tau^{pm}n^i\partial_i C_m - \tau^{pm}n^i\partial_i\partial_m\kappa_l^l$.

Finally, $\tau^{pk}n^l n^i n^m\partial_k\partial_l\kappa_{im} = \tau^{pk}(\delta^{li} - \tau^{li})n^m\partial_k\partial_l\kappa_{im} = \tau^{pk}n^m\partial_k\partial^l\kappa_{lm} - \tau^{pk}\tau^{li}n^m\partial_k\partial_l\kappa_{im} = \tau^{pk}n^m\partial_k C_m - \tau^{pk}n^m\partial_k\partial_m\kappa_l^l - \tau^{pk}\tau^{li}n^m\partial_k\partial_l\kappa_{im}$. Again, from $n^mC_m = 0$ on the face and the tangentiality of $\tau^{pk}\partial_k$, the first term vanishes. Also, from the boundary conditions and the tangentiality of $\tau^{pk}\tau^{li}\partial_k\partial_l$, the last term vanishes. So, $\tau^{pk}n^l n^i n^m\partial_k\partial_l\kappa_{im} = -\tau^{pk}n^m\partial_k\partial_m\kappa_l^l$.

From above, $\tau_j^p n^l\partial_l C^j = \frac{1}{2}\tau^{pm}n^i\partial_i C_m$, which implies $\tau_j^p n^l\partial_l C^j = 0$ on the face. This concludes the proof of (4.60). □

The next theorem asserts that the boundary conditions are constraint-preserving.

Theorem 33. *Let Ω be a polyhedral domain. Given $\gamma_{ij}(0)$, $\kappa_{ij}(0)$ on Ω satisfying the constraints* (3.33) *and* (3.34)*, define $\lambda_{kij}(0)$ by* (3.53)*. Having $\kappa_{ij}(0)$ and $\lambda_{kij}(0)$, define κ_{ij} and λ_{kij} for positive time by the evolution equations* (3.50), (3.51)*, and the boundary conditions* (4.54)*. Then the constraints* (3.10) *are satisfied for all time.*

Proof. Exactly as in the proof of Theorem 1. of [3], we find that C_j satisfies satisfies the elastic wave equation

$$\ddot{C}_j = \partial^l\partial_{(l}C_{j)} \tag{4.61}$$

and both C_j and $\dot{C}_j$ vanish at the initial time. Define the usual energy

$$E(t) = \frac{1}{2}\int_\Omega \{\dot{C}_j\dot{C}^j + \frac{1}{2}[\partial^l C_j \partial_l C^j + (\partial_l C^l)^2]\}\, dx.$$

Clearly $E(0) = 0$. From (4.61) and integration by parts

$$\dot{E} = \int_{\partial\Omega} (\dot{C}_j n^l \partial_l C^j + n^j \dot{C}_j \partial_l C^l) d\sigma. \tag{4.62}$$

Therefore, if $\alpha = 0$ and $\beta^i = 0$, we can invoke Lemma 32, and conclude that E is constant in time. Hence E vanishes identically. Thus C_j is constant, and, since it vanishes at time 0, it vanishes for all time. This establishes the theorem under the additional assumption that α and β^i vanish.

To extend to the case of general α and β^i we use Duhamel's principle. Let $S(t)$ denote the solution operator associated to the homogeneous boundary value problem. That is, given functions $\theta_{ij}(0)$, $\phi_{kij}(0)$ on Ω, define $S(t)(\theta_{ij}(0), \phi_{kij}(0)) = (\theta_{ij}(t), \phi_{kij}(t))$ where θ_{ij}, ϕ_{kij} is the solution to the homogeneous evolution equations

$$\frac{1}{\sqrt{2}}\dot{\theta}_{ij} = \partial^k \phi_{k(ij)}, \quad \frac{1}{\sqrt{2}}\dot{\phi}_{kij} = \partial_{[k}\theta_{i]j},$$

satisfying the boundary conditions and assuming the given initial values. Then Duhamel's principle represents the solution κ_{ij}, λ_{kij} of the inhomogeneous initial boundary value problem (3.50), (3.51), (4.54) as

$$(\kappa_{ij}(t), \lambda_{kij}(t)) = S(t)(\kappa_{ij}(0), \lambda_{kij}(0)) + \int_0^t S(t-s)(-\frac{1}{\sqrt{2}}\partial_i\partial_j\alpha(s), \eta_{kij}(s))\, ds. \tag{4.63}$$

Now it is easy to check that the momentum constraint (3.10) is satisfied when κ_{ij} is replaced by $-\frac{1}{\sqrt{2}}\partial_i\partial_j\alpha(s)$ (for any smooth function α), and, from (3.50), $\dot{C}_j(0) = \sqrt{2}\partial^k(\partial^l \eta_{k(lj)} - \partial_j {\eta_{kl}}^l) = 0$ for any smooth shift vector β^i. Hence the integrand in (4.63) satisfies the constraints by the result for the homogeneous case, as does the first term on the right-hand side, and thus the constraints are indeed satisfied by κ_{ij}, λ_{kij}. □

The analogue of Theorem 1. of [3] for the initial boundary value problem follows from the preceding theorem exactly as there.

Theorem 34. *Let* Ω *be a polyhedral domain. Suppose that initial data* $\gamma_{ij}(0)$ *and* $\kappa_{ij}(0)$ *are given satisfying the Hamiltonian constraint* (3.9) *and momentum constraint* (3.10)*, respectively, and that initial data* $\lambda_{kij}(0)$ *is defined by* (3.53)*. Then the unique solution of the linearized AA initial boundary value problem* (3.50)*,* (3.51)*, and* γ *defined by* (3.54) *together with the boundary conditions* (4.54) *satisfies the linearized ADM system* (3.7)*–*(3.10) *in* Ω*.*

We close by noting a second set of boundary conditions which are maximal nonnegative and constraint-preserving. These are

$$n^i n^j \kappa_{ij} = m^i m^j \kappa_{ij} = l^i l^j \kappa_{ij} = m^k n^i n^j \lambda_{kij} = l^k n^i n^j \lambda_{kij} = 0. \tag{4.64}$$

Now when we make an orthogonal expansion as in (4.58), (4.59), the first term on the right-hand side of the first equation and the first two terms on the right-hand side of the second equation vanish. The necessary orthogonality to demonstrate that the boundary conditions are maximal nonnegative follows from the boundary conditions and the antisymmetry of λ_{kij} with respect to its first two indices. Similarly, to prove the analogue of Lemma 32 for these boundary conditions, we show that the tangential component of $\dot{C}_j$ vanishes and the normal component of $n^l \partial_l C^j$ vanishes (it was the reverse before). The analysis is essentially the same as for the boundary conditions (4.54). However, we need to do some more work here since the vanishing of both tangential component of $\dot{C}_j$ and normal component of $n^l \partial_l C^j$ is not enough for proving (4.60). We prove that $\partial_j C^j$ vanishes on the face. From (3.51), observe that $\dot{\lambda}_l{}^{jl} = C^j/\sqrt{2}$. So, from (3.50) and the antisymmetry of λ_{kij} in k and i, $\partial_j C^j = \sqrt{2}\partial_j \dot{\lambda}_l{}^{jl} = -\sqrt{2}\partial_j \dot{\lambda}^{jl}{}_l = -\ddot{\kappa}_l^l = -\delta^{ij}\ddot{\kappa}_{ij} = -(n^i n^j + m^i m^j + l^i l^j)\ddot{\kappa}_{ij}$. From (4.64), it follows that the divergence of C^j vanishes on the face. Otherwise the approach follows exactly the ideas and techniques used for the boundary conditions (4.54).

4.4.2 Extended AA System

By using similar ideas and techniques as in Subsection 4.3.2, we indicate an extended initial boundary value problem corresponding to the AA formulation whose solution solves the linearized ADM system (3.7)–(3.10) in Ω. The new system consists of (3.7) for γ_{ij}, (3.51), and two new sets of equations, one replacing (3.50)

$$\frac{1}{\sqrt{2}}\dot{\kappa}_{ij} = \partial^k \lambda_{k(ij)} + \frac{1}{2}(\partial_i p_j + \partial_j p_i) - \partial^k p_k \delta_{ij} - \frac{1}{\sqrt{2}}\partial_i \partial_j \alpha, \tag{4.65}$$

and the other one corresponding to a new three dimensional vector field p_i defined by

$$\dot{p}_i = \partial^l \kappa_{li} - \partial_i \kappa_l^{\ l}. \tag{4.66}$$

Observe that the additional terms that appear on the right-hand side of (4.65) compared with (3.50) are precisely the negative components of the formal adjoint of the momentum constraint operator (3.10) applied to p_i.

Let $\tilde{V}$ be the vector space of quadruples of constant tensors $(\gamma_{ij}, \kappa_{ij}, \lambda_{kij}, p_k)$, where γ_{ij}, κ_{ij} are symmetric with respect to the indices i and j and λ_{kij} belong to the 8-dimensional space $\mathbb{T}$ defined in Subsection 3.4.3. Thus $\dim \tilde{V} = 23$. The boundary operator $\tilde{A}_n : \tilde{V} \to \tilde{V}$ in this case is given by

$$\tilde{\gamma}_{ij} = 0, \quad \tilde{\kappa}_{ij} = -n^k \lambda_{k(ij)} - \frac{1}{2}(n_i p_j + n_j p_i) + n^k p_k \delta_{ij}, \quad \tilde{\lambda}_{kij} = -n_{[k}\kappa_{i]j}, \quad \tilde{p}_i = -n^l \kappa_{il} + n_i \kappa_l^{\ l}. \tag{4.67}$$

The boundary operator $\tilde{A}_n$ associated to the evolution equations (3.7), (4.65), (3.51), and (4.66) has five positive, 13 zero, and five negative eigenvalues. Therefore, a nonnegative subspace is maximal nonnegative if and only if it has dimension 18. We claim that the following boundary conditions are maximal nonnegative for (3.7), (4.65), (3.51), and (4.66)

$$n^i m^j \kappa_{ij} = n^i l^j \kappa_{ij} = n^k (m^i m^j \lambda_{kij} - p_k) = n^k (l^i l^j \lambda_{kij} - p_k) = n^k m^i l^j \lambda_{kij} = 0. \tag{4.68}$$

These can be written as well:

$$n^i\tau^{jk}\kappa_{ij} = 0, \quad n^k(\tau^{il}\tau^{jm}\lambda_{kij} - \tau^{lm}p_k) = 0, \tag{4.69}$$

and so do not depend on the choice of basis for the tangent space.

Let us prove the claim that the subspace $\tilde{N}$ defined by (4.68) is maximal nonnegative. Obviously, $\dim \tilde{N} = 18$. Hence, it remains to be proven that $\tilde{N}$ is also nonnegative. In view of (4.67), the verification of nonnegativity of $\tilde{N}$ reduces to showing that

$$\kappa_{ij}\tilde{\kappa}^{ij} + \lambda_{kij}\tilde{\lambda}^{kij} + p_i\tilde{p}^i \geq 0 \tag{4.70}$$

whenever (4.68) holds. In fact, we can prove that the left-hand side of (4.70) vanishes pending that (4.68) holds:

$$\begin{aligned}
&\kappa_{ij}\tilde{\kappa}^{ij} + \lambda_{kij}\tilde{\lambda}^{kij} + p_i\tilde{p}^i = \\
&= -2n^k\kappa^{ij}\lambda_{kij} - 2n^ip^j\kappa_{ij} + 2n^ip_i\kappa^l_l \\
&= -2(n^ln^in^mn^j\kappa_{lm} + \tau^{li}\tau^{mj}\kappa_{lm})(\tau^l_in^mn_jn^k\lambda_{klm} + \tau^l_i\tau^m_jn^k\lambda_{klm}) - 2n^ip^j\kappa_{ij} + 2n^ip_i\kappa^l_l \\
&= -2n^k\tau^{rl}\tau pm\kappa_{rp}\lambda_{klm} - 2n^ip^j\kappa_{ij} + 2n^ip_i\kappa^l_l \\
&= -2n^kp_k\tau^{rp}\kappa_{rp} - 2n^ip^j\kappa_{ij} + 2n^ip_i\kappa^l_l \\
&= -2n^kp_k(\delta^{rp} - n^rn^p)\kappa_{rp} - 2n^ip^j\kappa_{ij} + 2n^ip_i\kappa^l_l \\
&= 2n^kp_kn^rn^p\kappa_{rp} - 2n^ip^j\kappa_{ij} \\
&= 2n^kp_kn^rn^p\kappa_{rp} - 2n^i(n^kp_kn^j + \tau^{kj}p_k)\kappa_{ij} \\
&= -2n^i\tau^{kj}\kappa_{ij}p_k = 0.
\end{aligned} \tag{4.71}$$

This concludes the proof of (4.70).

Theorem 35. *Let Ω be a polyhedral domain. Suppose that the initial data $\gamma_{ij}(0)$ and $\kappa_{ij}(0)$ are given satisfying the Hamiltonian* (3.9) *and momentum constraints* (3.10)*, respectively, $\lambda_{kij}(0)$ is defined by* (3.53)*, and $p_i(0) = 0$. Then the unique solution $(\gamma_{ij}, \kappa_{ij}, \lambda_{kij}, p_i)$ of the initial boundary value problem* (3.7)*,* (4.65)*,* (3.51)*, and* (4.66)*, together with the boundary conditions* (4.68)*, satisfies the properties $p_i = 0$ for all time, and $(\gamma_{ij}, \kappa_{ij})$ solves* *the linearized ADM system* (3.7)–(3.10) *in* Ω.

Proof. Observe that the solution of the initial boundary value problem (3.7), (3.50), (3.51), and (4.54) (boundary conditions), together with $p_i = 0$ for all time, is the unique solution of the initial boundary value problem (3.7), (4.65), (3.51), and (4.66), together with the boundary conditions (4.68). The conclusion follows from Theorem 34. □

We close by indicating a second set of maximal nonnegative boundary conditions (corresponding to (4.64)) for (3.7), (4.65), (3.51), and (4.66) for which Theorem 35 holds as well. These are

$$n^i n^j \kappa_{ij} = m^i m^j \kappa_{ij} = l^i l^j \kappa_{ij} = m^k n^i n^j \lambda_{kij} + m^k p_k = l^k n^i n^j \lambda_{kij} + l^k p_k = 0. \qquad (4.72)$$

4.5 Arnold Formulation

Throughout this section, we will use the notations and results introduced in Subsection 3.4.4. In this part we unveil the technique that led us to the finding of maximal nonnegative constraint preserving boundary conditions for the three formulations, EC, AA, and A, respectively, analyzed in this thesis from this point of view.

4.5.1 Maximal Nonnegative Constraint Preserving Boundary Conditions

In this subsection we exhibit in detail our technique for finding maximal nonnegative constraint preserving boundary conditions for Arnold's formulation. Everything that follows can be done for general polyhedral domains, but for the sake of simplicity let Ω be a parallelepiped (a box) with faces parallel to the coordinate planes. Again, by m^i and l^i we denote two additional vectors which together n^i form an orthonormal basis (see Figure 4.3).

Denote by $\underset{=}{v} = \underset{=}{M}\underset{\equiv}{\nu}$, and $\underset{=}{w} = \underset{=}{M}\underset{\equiv}{\mu}$. Then, from (3.61), we have that $\underset{=}{v}$ and $\underset{=}{w}$ are solutions

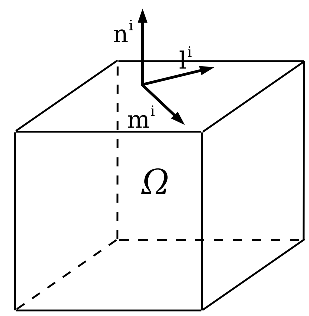

Figure 4.3: A parallelipipedic domain.

of the following problem

$$\dot{\underline{v}} = -\frac{1}{2}\,\underline{\mathrm{curl}}\,\underline{w}, \tag{4.73}$$

$$\dot{\underline{w}} = \frac{1}{2}\,\underline{\mathrm{curl}}\,\underline{v}, \tag{4.74}$$

$$\underline{v}(0) = \underline{w}(0) = 0. \tag{4.75}$$

Observe that (4.73), (4.74) is a first order symmetric hyperbolic system. Its boundary matrix has the following eigenvalues (given with the corresponding eigenvectors):

$\lambda_1 = 0: \ (\underline{n},\ 0)^T,\ (0,\ \underline{n})^T,$

$\lambda_2 = -\frac{1}{2}: \ (\underline{m},\ \underline{l})^T,\ (\underline{l},\ -\underline{m})^T,$

$\lambda_3 = \frac{1}{2}: \ (\underline{l},\ \underline{m})^T\ (-\underline{m},\ \underline{l})^T,$

where $\underline{n}$ is the exterior unit normal and $\underline{m}$, $\underline{l}$ are chosen so that $\underline{m} \perp \underline{n}$ and $\underline{l} = \underline{n} \times \underline{m}$ (see Figure 4.3).

If we consider maximal nonnegative boundary conditions for (4.73)–(4.75), then the resulting initial boundary value problem has the unique solution $\underline{v} = 0,\ \underline{w} = 0$.

From Proposition 10, the general form of maximal nonnegative boundary conditions on the face having as exterior unit normal vector $\underline{n} = (1, 0, 0)^T$ is given by

$$(1-\beta)v_2 + \alpha v_3 + \alpha w_2 + (1+\beta)w_3 = 0, \tag{4.76}$$

$$-\delta v_2 + (1+\gamma)v_3 + (\gamma - 1)w_2 + \delta w_3 = 0, \tag{4.77}$$

with

$$|| \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} || \leq 1.$$

From

$$v_2 = \partial_x \nu_{12} + \partial_y(-\nu_{11} - \nu_{33}) + \partial_z \nu_{23},$$

$$v_3 = \partial_x \nu_{13} + \partial_y \nu_{23} + \partial_z(-\nu_{11} - \nu_{22}),$$

$$w_2 = \partial_x \mu_{12} + \partial_y(-\mu_{11} - \mu_{33}) + \partial_z \mu_{23},$$

$$w_3 = \partial_x \mu_{13} + \partial_y \mu_{23} + \partial_z(-\mu_{11} - \mu_{22}),$$

the conditions (4.76)–(4.77) become

$$[(1-\beta)\partial_x \nu_{12} + \alpha \partial_x \nu_{13} + \alpha \partial_x \mu_{12} + (1+\beta)\partial_x \mu_{13}]+ \tag{4.78}$$

$$\partial_y[-(1-\beta)\nu_{11} - (1-\beta)\nu_{33} - -\alpha\mu_{11} - \alpha\mu_{33} + (1+\beta)\mu_{23}]+$$

$$\partial_z[(1-\beta)\nu_{23} - \alpha\nu_{11} - \alpha\nu_{22} + \alpha\mu_{23} - (1+\beta)\mu_{11} - (1+\beta)\mu_{22}] = 0,$$

$$[-\delta\partial_x \nu_{12} + (\gamma+1)\partial_x \nu_{13} + (\gamma-1)\partial_x \mu_{12} + \delta\partial_x \mu_{13}]+ \tag{4.79}$$

$$\partial_y[\delta\nu_{11} + \delta\nu_{33} + (\gamma+1)\nu_{23} + (1-\gamma)\mu_{11} + (1-\gamma)\mu_{33} + \delta\mu_{23}]+$$

$$\partial_z[-\delta\nu_{23} - (\gamma+1)\nu_{11} - (\gamma+1)\nu_{22} + (\gamma-1)\mu_{23} - \delta\mu_{11} - \delta\mu_{22}] = 0$$

With no loss of generality (via Duhamel's principle), we can assume that the forcing terms in (3.77),(3.78) vanish.

From the system (3.77)–(3.78), it follows that

$$\partial_x \nu_{13} = \partial_y \nu_{23} + \partial_z(\nu_{11} - \nu_{22}) - 2\dot{\mu}_{12}, \tag{4.80}$$

$$\partial_x \nu_{12} = \partial_y(\nu_{11} - \nu_{33}) + \partial_z \nu_{23} + 2\dot{\mu}_{13}, \tag{4.81}$$

$$\partial_x \mu_{13} = \partial_y \mu_{23} + \partial_z(\mu_{11} - \mu_{22}) + 2\dot{\nu}_{12}, \tag{4.82}$$

$$\partial_x \mu_{12} = \partial_y(\mu_{11} - \mu_{33}) + \partial_z \mu_{23} - 2\dot{\nu}_{13}. \tag{4.83}$$

By introducing (4.80)–(4.83) into (4.78)–(4.79), we get

$$\begin{gathered}\partial_t[(1-\beta)\mu_{13} - \alpha\mu_{12} - \alpha\nu_{13} + (1+\beta)\nu_{12}]+ \\ \partial_y[(\beta-1)\nu_{33} + \alpha\nu_{23} - \alpha\mu_{33} + (1+\beta)\mu_{23}]+ \\ \partial_z[-\alpha\nu_{22} + (1-\beta)\nu_{23} - (1+\beta)\mu_{22} + \alpha\mu_{23}] = 0,\end{gathered} \tag{4.84}$$

$$\begin{gathered}\partial_t[-\delta\mu_{13} - (\gamma+1)\mu_{12} - (\gamma-1)\nu_{13} + \delta\nu_{12}]+ \\ \partial_y[\delta\nu_{33} + (\gamma+1)\nu_{23} + (1-\gamma)\mu_{33} + \delta\mu_{23}]+ \\ \partial_z[-(\gamma+1)\nu_{22} - \delta\nu_{23} - \delta\mu_{22} - (1-\gamma)\mu_{23}] = 0,\end{gathered} \tag{4.85}$$

on the part of the boundary having $(1,0,0)^T$ as the unit exterior normal vector.

Next, observe that we can get from the system (3.77)–(3.78) four relations that do not involve the transverse derivative ∂_x

$$\partial_t \nu_{11} + \partial_y \mu_{13} - \partial_z \mu_{12} = 0, \tag{4.86}$$

$$\partial_t(\nu_{22} + \nu_{33}) - \partial_y \mu_{13} + \partial_z \mu_{12} = 0, \tag{4.87}$$

$$\partial_t \mu_{11} - \partial_y \nu_{13} + \partial_z \nu_{12} = 0, \tag{4.88}$$

$$\partial_t(\mu_{22} + \mu_{33}) + \partial_y \nu_{13} - \partial_z \nu_{12} = 0. \tag{4.89}$$

We will not use the relations (4.86) and (4.88) since they seem to introduce only complications in what follows.

By adding combinations of (4.87) and (4.89) to both (4.84) and (4.85), we get

$$\partial_t[(1-\beta)\mu_{13} - \alpha\mu_{12} - \alpha\nu_{13} + (1+\beta)\nu_{12} + m\nu_{22} + m\nu_{33} + n\mu_{22} + n\mu_{33}] + \tag{4.90}$$

$$\partial_y[(\beta-1)\nu_{33} + \alpha\nu_{23} - \alpha\mu_{33} + (1+\beta)\mu_{23} - m\mu_{13} + n\nu_{13}] +$$

$$\partial_z[-\alpha\nu_{22} + (1-\beta)\nu_{23} - (1+\beta)\mu_{22} + \alpha\mu_{23} + m\mu_{12} - n\nu_{12}] = 0$$

and

$$\partial_t[-\delta\mu_{13} - (\gamma+1)\mu_{12} - (\gamma-1)\nu_{13} + \delta\nu_{12} + p\nu_{22} + p\nu_{33} + q\mu_{22} + q\mu_{33}] + \tag{4.91}$$

$$\partial_y[\delta\nu_{33} + (\gamma+1)\nu_{23} + (1-\gamma)\mu_{33} + \delta\mu_{23} - p\mu_{13} + q\nu_{13}] +$$

$$\partial_z[-(\gamma+1)\nu_{22} - \delta\nu_{23} - \delta\mu_{22} - (1-\gamma)\mu_{23} + p\mu_{12} - q\nu_{12}] = 0$$

on the part of the boundary having $(1,0,0)^T$ as the unit exterior normal vector.

It is easy to see now that (4.84)–(4.85) are satisfied if the following conditions hold:

$$m\nu_{22} + m\nu_{33} - \alpha\nu_{13} + (1+\beta)\nu_{12} + n\mu_{22} + n\mu_{33} + (1-\beta)\mu_{13} - \alpha\mu_{12} = 0, \tag{4.92}$$

$$(\beta-1)\nu_{33} + \alpha\nu_{23} + n\nu_{13} - \alpha\mu_{33} + (1+\beta)\mu_{23} - m\mu_{13} = 0, \tag{4.93}$$

$$\alpha\nu_{22} + (\beta-1)\nu_{23} + n\nu_{12} + (1+\beta)\mu_{22} - \alpha\mu_{23} - m\mu_{12} = 0, \tag{4.94}$$

$$p\nu_{22} + p\nu_{33} + (1-\gamma)\nu_{13} + \delta\nu_{12} + q\mu_{22} + q\mu_{33} - \delta\mu_{13} - (\gamma+1)\mu_{12} = 0, \tag{4.95}$$

$$\delta\nu_{33} + (\gamma+1)\nu_{23} + q\nu_{13} + (1-\gamma)\mu_{33} + \delta\mu_{23} - p\mu_{13} = 0, \tag{4.96}$$

$$(\gamma+1)\nu_{22} + \delta\nu_{23} + q\nu_{12} + \delta\mu_{22} + (1-\gamma)\mu_{23} - p\mu_{12} = 0. \tag{4.97}$$

The matrix A corresponding to this linear homogeneous system is:

$$A = \begin{pmatrix} m & m & 0 & -\alpha & 1+\beta & n & n & 0 & 1-\beta & -\alpha \\ 0 & \beta-1 & \alpha & n & 0 & 0 & -\alpha & 1+\beta & -m & 0 \\ \alpha & 0 & \beta-1 & 0 & n & 1+\beta & 0 & -\alpha & 0 & -m \\ p & p & 0 & 1-\gamma & \delta & q & q & 0 & -\delta & -1-\gamma \\ 0 & \delta & 1+\gamma & q & 0 & 0 & 1-\gamma & \delta & -p & 0 \\ 1+\gamma & 0 & \delta & 0 & q & \delta & 0 & 1-\gamma & 0 & -p \end{pmatrix}.$$

Before we go any further, let us characterize the maximal nonnegative boundary conditions for (3.77)–(3.78).

The boundary matrix corresponding to this symmetric hyperbolic system has the following eigenvalues, given with the corresponding eigenvectors:

$$\lambda = 0: \begin{pmatrix} \underline{n} \odot \underline{n} \\ \underline{\underline{0}} \end{pmatrix}, \begin{pmatrix} \underline{\underline{0}} \\ \underline{n} \odot \underline{n} \end{pmatrix}, \begin{pmatrix} \underline{m} \odot \underline{m} + \underline{l} \odot \underline{l} \\ \underline{\underline{0}} \end{pmatrix}, \begin{pmatrix} \underline{\underline{0}} \\ \underline{m} \odot \underline{m} + \underline{l} \odot \underline{l} \end{pmatrix},$$

$$\lambda = \frac{1}{2}: \begin{pmatrix} \underline{l} \odot \underline{n} \\ \underline{m} \odot \underline{n} \end{pmatrix}, \begin{pmatrix} \underline{m} \odot \underline{n} \\ -\underline{n} \odot \underline{l} \end{pmatrix},$$

$$\lambda = 1: \begin{pmatrix} \frac{1}{2}(\underline{l} \odot \underline{l} - \underline{m} \odot \underline{m}) \\ \underline{m} \odot \underline{l} \end{pmatrix}, \begin{pmatrix} \underline{m} \odot \underline{l} \\ \frac{1}{2}(\underline{m} \odot \underline{m} - \underline{l} \odot \underline{l}) \end{pmatrix},$$

$$\lambda = -\frac{1}{2}: \begin{pmatrix} \underline{m} \odot \underline{n} \\ \underline{l} \odot \underline{n} \end{pmatrix}, \begin{pmatrix} \underline{n} \odot \underline{l} \\ -\underline{m} \odot \underline{n} \end{pmatrix},$$

$$\lambda = -1: \begin{pmatrix} \underline{m} \odot \underline{l} \\ \frac{1}{2}(\underline{l} \odot \underline{l} - \underline{m} \odot \underline{m}) \end{pmatrix}, \begin{pmatrix} \frac{1}{2}(\underline{m} \odot \underline{m} - \underline{l} \odot \underline{l}) \\ \underline{m} \odot \underline{l} \end{pmatrix}.$$

From Proposition 10, we know that if $M \in \mathbb{R}^{4\times 4}$ and the norm of the matrix

$$\begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is less or equal to 1, then the following boundary conditions are maximal nonnegative:

$$\begin{pmatrix} (\underline{m}\odot\underline{n})\underline{\underline{\nu}} + (\underline{l}\odot\underline{n})\underline{\underline{\mu}} \\ (\underline{n}\odot\underline{l})\underline{\underline{\nu}} - (\underline{m}\odot\underline{n})\underline{\underline{\mu}} \\ (\underline{m}\odot\underline{l})\underline{\underline{\nu}} + \underline{\underline{\mu}}(\underline{l}\odot\underline{l} - \underline{m}\odot\underline{m})/2 \\ \underline{\underline{\nu}}(\underline{m}\odot\underline{m} - \underline{l}\odot\underline{l})/2 + (\underline{m}\odot\underline{l})\underline{\underline{\mu}} \end{pmatrix} + M \begin{pmatrix} (\underline{l}\odot\underline{n})\underline{\underline{\nu}} + (\underline{m}\odot\underline{n})\underline{\underline{\mu}} \\ (\underline{m}\odot\underline{n})\underline{\underline{\nu}} - (\underline{n}\odot\underline{l})\underline{\underline{\mu}} \\ \underline{\underline{\nu}}(\underline{l}\odot\underline{l} - \underline{m}\odot\underline{m})/2 + (\underline{m}\odot\underline{l})\underline{\underline{\mu}} \\ (\underline{m}\odot\underline{l})\underline{\underline{\nu}} + \underline{\underline{\mu}}(\underline{m}\odot\underline{m} - \underline{l}\odot\underline{l})/2 \end{pmatrix} = 0. \tag{4.98}$$

For $\underline{n} = (1,0,0)^T$, $\underline{m} = (0,1,0)^T$, and $\underline{l} = (0,0,1)^T$, (4.98) is equivalent to

$$B \begin{pmatrix} \nu_{22} \\ \vdots \\ \mu_{12} \end{pmatrix} = 0,$$

where

$$B = \begin{pmatrix} -m_{13} & m_{13} & 2m_{14} & 2m_{11} & 2(1+m_{12}) & m_{14} & -m_{14} & 2m_{13} & 2(1-m_{12}) & 2m_{11} \\ -m_{23} & m_{23} & 2m_{24} & 2(1+m_{21}) & 2m_{22} & m_{24} & -m_{24} & 2m_{23} & -2m_{22} & 2(-1+m_{21}) \\ -m_{33} & m_{33} & 2(1+m_{34}) & 2m_{31} & 2m_{32} & -1+m_{34} & 1-m_{34} & 2m_{33} & -2m_{32} & 2m_{31} \\ 1-m_{43} & -1+m_{43} & 2m_{44} & 2m_{41} & 2m_{42} & m_{44} & -m_{44} & 2(1+m_{43}) & -2m_{42} & 2m_{41} \end{pmatrix}. \tag{4.99}$$

Now, we want a match between the systems (4.92)–(4.97) and (4.98). First of all, observe that the first two columns of B coincide except the sign and, since we want each row of A to be a linear combination of rows of B, this induces

$$\alpha = 0;\ \beta = 1;\ \gamma = -1;\ \delta = 0;\ m = 0;\ p = 0.$$

Also, we have to worry about the rank of the matrix A, which is a 6×10–matrix. Obviously, the rank of A cannot exceed four. Fortunately, we know that the rank of A equals the rank of AA^T which is a 6×6–matrix. Using this fact, we can see that A has rank four for the above values of the other parameters if and only if $n^2 + q^2 = 4$ (otherwise A has rank greater or equal to five). From here, we get that the rows of B must be linear combinations of the rows of A too and it turns out that M must be

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

So, the maximal nonnegative boundary conditions on the face are:

$$\nu_{12} = 0, \tag{4.100}$$

$$\nu_{13} = 0, \tag{4.101}$$

$$-\mu_{22} + \mu_{33} = 0, \tag{4.102}$$

$$\mu_{23} = 0. \tag{4.103}$$

Same kind of conditions could be obtained for each face of Ω. In fact, we can write the boundary conditions for all faces in the following form:

$$\begin{pmatrix} \underline{n} \odot \underline{m} \\ \underline{\underline{0}} \end{pmatrix} : \begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \end{pmatrix} = 0, \tag{4.104}$$

$$\begin{pmatrix} \underline{n} \odot \underline{l} \\ \underline{\underline{0}} \end{pmatrix} : \begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \end{pmatrix} = 0, \tag{4.105}$$

$$\begin{pmatrix} \underline{\underline{0}} \\ \underline{l} \odot \underline{l} - \underline{m} \odot \underline{m} \end{pmatrix} : \begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \end{pmatrix} = 0, \tag{4.106}$$

$$\begin{pmatrix} \underline{\underline{0}} \\ \underline{m} \odot \underline{l} \end{pmatrix} : \begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \end{pmatrix} = 0, \tag{4.107}$$

or

$$\begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \end{pmatrix}_{|\partial\Omega} \in X, \tag{4.108}$$

where

$$X = \mathrm{Span}\{ \begin{pmatrix} \underline{n} \odot \underline{n} \\ \underline{\underline{0}} \end{pmatrix}, \begin{pmatrix} \underline{m} \odot \underline{l} \\ \underline{\underline{0}} \end{pmatrix}, \begin{pmatrix} \underline{m} \odot \underline{m} \\ \underline{\underline{0}} \end{pmatrix}, \begin{pmatrix} \underline{m} \odot \underline{m} - \underline{l} \odot \underline{l} \\ \underline{\underline{0}} \end{pmatrix}, \begin{pmatrix} \underline{\underline{0}} \\ \underline{n} \odot \underline{n} \end{pmatrix},$$

$$\begin{pmatrix} \underline{\underline{0}} \\ \underline{n} \odot \underline{m} \end{pmatrix}, \begin{pmatrix} \underline{\underline{0}} \\ \underline{n} \odot \underline{l} \end{pmatrix}, \begin{pmatrix} \underline{\underline{0}} \\ \underline{m} \odot \underline{m} + \underline{l} \odot \underline{l} \end{pmatrix} \}.$$

Observe that X is a maximal nonnegative (in fact null) subspace for the boundary operator

$$A_n = \begin{pmatrix} \underline{\underline{0}} & K_n \\ -K_n & \underline{\underline{0}} \end{pmatrix},$$

where by definition

$$K_n \underline{\underline{u}} = \frac{1}{2}[\underline{\underline{\mathrm{Skw}}}(\underline{n})\underline{\underline{u}} - \underline{\underline{u}}\,\underline{\underline{\mathrm{Skw}}}(\underline{n})],$$

for all $\underline{\underline{u}} \in \mathbb{R}^{3\times 3}_{sym}$.

Also, it is easy to see that the boundary conditions (4.104)–(4.107) can be written as:

$$n^i \tau^{jk} \nu_{ij} = 0, \quad (2\tau^{ik}\tau^{jl} - \tau^{kl}\tau^{ij})\mu_{ij} = 0, \tag{4.109}$$

where $\tau^{ij} := m^i m^j + l^i l^j$ is the projection operator orthogonal to n^i, and so, there is no dependence on the particular choice of m^i and l^i.

Define

$$E(t) = \int_\Omega |\underline{v}|^2 + |\underline{w}|^2 dx.$$

Then,

$$\dot{E}(t) = \int_{\partial\Omega} \underline{w}(\underline{n} \times \underline{v}) d\sigma = -\int_{\partial\Omega} \underline{v}(\underline{n} \times \underline{w}) d\sigma,$$

where $\underline{n}$ is the exterior unit normal vector.

By using (4.104)–(4.107), it is easy to prove that $\dot{E}(t) = 0$ and this implies again that $\underline{M}\underline{\underline{\nu}} = 0$ and $\underline{M}\underline{\underline{\mu}} = 0$ for all time.

By the stucture of the main system of differential equations (3.77)–(3.78), there is a second set of maximal nonnegative constraint preserving boundary conditions

$$n^i \tau^{jk} \mu_{ij} = 0, \quad (2\tau^{ik}\tau^{jl} - \tau^{kl}\tau^{ij})\nu_{ij} = 0. \tag{4.110}$$

Moreover, by our above analysis there are no other sets of maximal nonnegative constraint preserving boundary conditions corresponding to the system (3.77)–(3.78).

4.5.2 Equivalent Unconstrained Initial Boundary Value Problem

Consider the constrained initial boundary valu problem (2.20)–(2.22) together with maximal nonnegative boundary conditions

$$u_{|\partial\Omega} \in Y, \tag{4.111}$$

where Y is a maximal nonnegative boundary space. Also, consider the associated extended problem (2.23)–(2.24) with boundary conditions

$$\begin{pmatrix} u \\ z \end{pmatrix}_{|\partial\Omega} \in \overline{Y} = Y \times (B_n Y)^{\perp}, \tag{4.112}$$

where $B_n(x) = -\Sigma_{j=1}^{N} B^j n_j$ is the boundary matrix corresponding to the first order differential operator B at $x \in \partial\Omega$.

From Corollary 15, $\overline{Y}$ is nonnegative for the boundary matrix $\overline{A}_n$ associated to (2.23) if and only if Y is nonnegative for the boundary matrix A_n associated to (2.20).

Furthermore, assume $\overline{Y}$ is maximal nonnegative for $\overline{A}_n$. Let v be a solution of (2.20)–(2.22) and (4.111), and $(u, z)^T$ be a solution of (2.23)–(2.24) and (4.112).

A formal calculation shows that

$$\frac{1}{2}\frac{d}{dt}(\|u - v\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(\Omega)}) = -\frac{1}{2}\int_{\partial\Omega}(u - v)^T A_n (u - v)\, d\sigma \leq 0.$$

From here, it follows that $z = 0$ and $u = v$. In other words, the initial boundary value problems (2.20)–(2.22) with (4.111) and (2.23)–(2.24) with (4.112) are equivalent.

Returning to our problem, observe that the boundary matrix of the extended system (3.90) reads

$$\overline{A}_n = \begin{pmatrix} A_n & B_n^T \\ B_n & 0 \end{pmatrix},$$

where

$$B_n = \begin{pmatrix} M_n & 0 \\ 0 & M_n \end{pmatrix},$$

with $M_n \underline{\underline{u}} = \underline{\underline{u}}\, \underline{n} - (\mathrm{tr}\, \underline{\underline{u}})\, \underline{n},\ \forall \underline{\underline{u}} \in \mathbb{R}^{3\times 3}_{sym}$.

Straighforward computations show that

$$B_nX = \mathrm{Span}\{\begin{pmatrix} \underline{n} \\ \underline{0} \end{pmatrix};\begin{pmatrix} \underline{m} \\ \underline{0} \end{pmatrix};\begin{pmatrix} \underline{l} \\ \underline{0} \end{pmatrix};\begin{pmatrix} \underline{0} \\ \underline{n} \end{pmatrix}\},$$

and so,

$$(B_nX)^\perp = \mathrm{Span}\{\begin{pmatrix} \underline{0} \\ \underline{m} \end{pmatrix};\begin{pmatrix} \underline{0} \\ \underline{l} \end{pmatrix}\}.$$

Therefore, $\overline{X} = X \times (B_nX)^\perp$ has dimension ten and it is nonnegative for $\overline{A}_n$. Since $\overline{A}_n$ has exactly ten nonnegative and eight strictly negative eigenvalues, it follows that $\overline{X}$ is also maximal nonnegative. Thus, the following boundary conditions are suitable for the extended system (3.90)

$$\begin{pmatrix} \underline{n} \odot \underline{m} \\ \underline{\underline{0}} \end{pmatrix} : \begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \end{pmatrix} = 0, \tag{4.113}$$

$$\begin{pmatrix} \underline{n} \odot \underline{l} \\ \underline{\underline{0}} \end{pmatrix} : \begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \end{pmatrix} = 0, \tag{4.114}$$

$$\begin{pmatrix} \underline{\underline{0}} \\ \underline{l} \odot \underline{l} - \underline{m} \odot \underline{m} \end{pmatrix} : \begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \end{pmatrix} = 0, \tag{4.115}$$

$$\begin{pmatrix} \underline{\underline{0}} \\ \underline{m} \odot \underline{l} \end{pmatrix} : \begin{pmatrix} \underline{\underline{\nu}} \\ \underline{\underline{\mu}} \end{pmatrix} = 0, \tag{4.116}$$

$$\underline{p} = 0, \tag{4.117}$$

$$\underline{q}\ \underline{n} = 0, \tag{4.118}$$

or,

$$n^i \tau^{jk} \nu_{ij} = 0, \quad (2\tau^{ik}\tau^{jl} - \tau^{kl}\tau^{ij})\mu_{ij} = 0, \quad p_i = 0, \quad n^i q_i = 0. \tag{4.119}$$

Note the existence of a second set of boundary conditions for the extended system (3.90) related to (4.110):

$$n^i \tau^{jk} \mu_{ij} = 0, \quad (2\tau^{ik}\tau^{jl} - \tau^{kl}\tau^{ij})\nu_{ij} = 0, \quad p_i = 0, \quad n^i q_i = 0. \tag{4.120}$$

Appendix A

General Solution for the Linearized Momentum Constraints

In order to apply the considerations of Subsection 4.2.3 to the EC problem, it would be useful to solve the linearized momentum constraint equations (3.34) for symmetric matrix fields defined on the entire space.

Define the symbol ϵ_{ijk} by

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (ijk) \text{ is either (123), (231), or (321),} \\ -1 & \text{if } (ijk) \text{ is either (321), (213), or (132),} \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

Note that

$$\epsilon_{ik}{}^{m}\epsilon_{psm} = \delta_{ip}\delta_{ks} - \delta_{is}\delta_{kp}. \tag{A.2}$$

From $C_j(\kappa) := \partial^l \kappa_{jl} - \partial_j \kappa^l_l = \partial^l(\kappa_{jl} - \kappa^s_s \delta_{jl}) = 0$, there exists a matrix field σ_{ij} so that $\kappa_{jk} - \kappa^l_l \delta_{jk} = \epsilon_k{}^{il}\partial_i \sigma_{jl}$. Therefore,

$$\kappa_{jk} = \epsilon_k{}^{il}\partial_i \sigma_{jl} - \frac{1}{2}\epsilon^{pil}\partial_i \sigma_{pl}\delta_{jk}. \tag{A.3}$$

Since κ_{ij} is symmetric,

$$\epsilon_k{}^{il}\partial_i \sigma_{jl} = \epsilon_j \, \partial_i \sigma_{kl}. \tag{A.4}$$

It is not difficult to see that (A.4) is equivalent to

$$\partial^l \sigma_{li} - \partial_i \sigma_s^s = 0. \tag{A.5}$$

The proof of this fact follows easily by multiplying (A.4) with ϵ^{kji}, and using the identity (A.2).

From (A.5) and $\partial^l \sigma_{li} - \partial_i \sigma_s^s = \partial^l(\sigma_{li} - \sigma_s^s \delta_{li})$, there exists a matrix field η_{ij} such that

$$\sigma_{ij} - \sigma_s^s \delta_{ij} = \epsilon_i{}^{kl} \partial_k \eta_{lj}, \tag{A.6}$$

and so,

$$\sigma_{ij} = \epsilon_i{}^{kl} \partial_k \eta_{lj} - \frac{1}{2} \epsilon^{skl} \partial_k \eta_{ls} \delta_{ij}. \tag{A.7}$$

By substituting (A.7) into (A.3) we get the general solution of the linearized momentum constraint (3.34).

Appendix B

The 4–D Covariant Formulation

Assume that the metric $g_{\mu\nu}$ is a slight perturbation of the Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \ |h_{\mu\nu}| \ll 1.$$

To first order in h, the Christoffel symbols are given by

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}(h_{\rho\nu,\mu} + h_{\rho\mu,\nu} - h_{\mu\nu,\rho}), \tag{B.1}$$

and so, to first order in h, the Ricci tensor is

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\lambda\mu,\nu} = \frac{1}{2}(-\Box h_{\mu\nu} + h^{\lambda}_{\nu,\lambda\mu} + h^{\lambda}_{\mu,\lambda\nu} - h^{\lambda}_{\lambda,\mu\nu}), \tag{B.2}$$

where $\Box$ is the d'Alembertian.

Therefore, the Einstein equations in first approximation read

$$\Box h_{\mu\nu} - h^{\lambda}_{\nu,\lambda\mu} - h^{\lambda}_{\mu,\lambda\nu} + h^{\lambda}_{\lambda,\mu\nu} = -16\pi G S_{\mu\nu}, \tag{B.3}$$

where

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\lambda}_{\lambda}$$

is the source term.

Here $T_{\mu\nu}$ is taken to lowest order in $h_{\mu\nu}$ and satisfies the ordinary conservation conditions $T^{\mu}_{\nu,\mu} = 0$.

We can not expect a field equation such as (B.3) to provide unique solutions because given any solution, we can always generate other solutions by performing coordinate transformations. The most general coordinate transformation that leaves the field weak is of the form

$$x^{\mu} \longrightarrow x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x), \tag{B.4}$$

where $\epsilon^{\mu}_{,\nu}$ is at most of the same order of magnitude as $h_{\mu\nu}$.

From

$$g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\rho}} g^{\lambda\rho}, \tag{B.5}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{B.6}$$

$$g'^{\mu\nu} = \eta^{\mu\nu} - h'^{\mu\nu}, \tag{B.7}$$

it follows that

$$h'_{\mu\nu} = h_{\mu\nu} - \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu}, \tag{B.8}$$

where $\epsilon_{\mu} = \eta_{\mu\nu}\epsilon^{\nu}$.

By direct computations, we can prove that (B.8) is also a solution of (B.3). To remove this non-uniqueness of solution, we have to work in some particular gauge. It turns out that one of the most convenient choice is the *harmonic gauge*, for which

$$\Gamma^{\lambda} = g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0. \tag{B.9}$$

Let us prove that there is always a coordinate system in which (B.9) holds.

Recall that the Christoffel symbols transforms as

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}}. \tag{B.10}$$

Contracting this with $g'^{\mu\nu}$, we get

$$\Gamma'^{\lambda} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \Gamma^{\rho} - g^{\rho\sigma} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}}. \tag{B.11}$$

Thus, if Γ^{λ} does not vanish, we can always find a new coordinate system x'^{ν} by solving the second–order partial differential equations

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \Gamma^{\rho} - g^{\rho\sigma} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} = 0, \tag{B.12}$$

which gives $\Gamma'^{\lambda} = 0$ in the x'–coordinate system.

So, without loss of generality, assume that the chosen gauge is harmonic. Then, since $\Gamma^{\lambda} = 0$, it follows that the coordinates are harmonic functions

$$\Box x^{\mu} = (g^{\lambda k} x^{\mu}_{;\lambda})_{;k} = g^{\lambda k} \frac{\partial^2 x^{\mu}}{\partial x^{\lambda} \partial x^{k}} - \Gamma^{\lambda} \frac{\partial x^{\mu}}{\partial x^{\lambda}} = -\Gamma^{\mu} = 0. \tag{B.13}$$

This explains the term *harmonic gauge*.

For a harmonic gauge, the equation (B.3) is

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu}. \tag{B.14}$$

One solution of this equation is the *retarded potential*

$$h_{\mu\nu}(x,t) = 4G \int \frac{S_{\mu\nu}(x', t - |x - x'|)}{|x - x'|} dx'. \tag{B.15}$$

To this solution, we can add any solution of the corresponding homogeneous equation

$$\Box h_{\mu\nu} = 0, \tag{B.16}$$

which also satisfies

$$\frac{\partial}{\partial x^\mu} h^\mu_\nu = \frac{1}{2} \frac{\partial}{\partial x^\nu} h^\mu_\mu, \tag{B.17}$$

which comes from $\Gamma^\lambda = 0$ to first order.

Observe that (B.15) automatically satisfies the harmonic condition (B.17) (at least for a compactly supported source $S_{\mu\nu}$), since

$$\frac{\partial}{\partial x^\mu} S^\mu_\nu = \frac{1}{2} \frac{\partial}{\partial x^\nu} S^\mu_\mu$$

that follows from the conservation law

$$\frac{\partial}{\partial x^\mu} T^\mu_\nu = 0$$

and

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T^\lambda_\lambda.$$

Interpretation: The solution (B.15) is interpreted as the gravitational radiation produced by the source $S_{\mu\nu}$, whereas any additional term (solution of (B.16), and (B.17)) represents the gravitational radiation coming from infinity. The term $t - |x - x'|$ in (B.15) shows that the gravitational effects propagate with the speed of light.

The drawback of this method is the fact that the gauge condition $\Gamma^\lambda = 0$ can be imposed only locally [11].

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