On the A Priori and A Posteriori Error Analysis in Finite Element Exterior Calculus

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This thesis is dedicated to my parents.

Preface

The present thesis addresses the theoretical and the numerical analysis of the Hodge Laplace equation within the framework of finite element exterior calculus. The content of this thesis is partially based on the following research articles, which have been accepted for publication or submitted for peer review:

[A] M. W. Licht, *Complexes of discrete distributional differential forms and their homology theory.* Accepted for publication in Foundations of Computational Mathematics, (2016), DOI: 10.1007/s10208-016-9315-y.

[B] S. H. Christiansen and M. W. Licht, *Poincaré-Friedrichs inequalities of complexes of discrete distributional differential forms.* Submitted

[C] M. Licht, *Smoothed projections over weakly Lipschitz domains.* Submitted

[D] M. Licht, *Smoothed projections and mixed boundary conditions.* Submitted

[E] M. Licht, *Higher order finite element de Rham complexes, partially localized flux reconstructions, and applications.* Submitted

For the purpose of a streamlined, thorough, and comprehensive exhibition, the content of the submitted versions has been rearranged and some proofs have been expanded. Additionally, the remarks of anonymous referees have been taken into account for the completion of this thesis. Specifically,

- Chapter IV is based on parts of [E],
- Chapter VII is based on [C] and parts of [D],
- Chapter VIII is based on parts of [D],
- Chapter IX is based on [A] and [B],
- and Chapter X is based on parts of [E].

The remaining chapters provide unpublished background material.

In addition, some of the ideas in Chapter IX were written down first for my unpublished Diplom thesis submitted at the University of Bonn in 2012.

[F] M. W. Licht, *Discrete distributional differential forms and their applications.* University of Bonn, August 2012.

Acknowledgement

"Nothing good is ever written. It is re-written."
— scriptwriters' proverb.

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Martin Werner Licht
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Contents

I. Introduction

Partial differential equations relate to almost every area of mathematics, and so a plurality of methods and perspectives has enjoyed a productive history of research. For example, the analysis of partial differential equations in the mathematics of electromagnetism and elasticity has led to the discovery of differential complexes that are composed of the classical differential operators of vector analysis. The study of those differential complexes has revealed structural insights which, in return, have fostered our understanding of the original partial differential equations.

The formalism of differential complexes originated in the field of algebraic topology and inspired the entire branch of mathematics that is now known as homological algebra. Differential geometry and differential topology have driven much of the research on differential complexes throughout the last century and provide a considerable share of the mathematical background of this thesis. This includes in particular the calculus of differential forms, or *exterior calculus*, which enables a unifying perspective on many aspects of classical vector calculus. The *de Rham complex* over a smooth manifold is a prominent example of a differential complex and can be regarded as one of the most extensively studied objects in mathematical analysis. A central result in the theory of the de Rham complex leads back to the origins of differential complexes in algebraic topology: the de Rham cohomology is isomorphic to the singular cohomology of the geometric ambient. Many variations of the de Rham complex have emerged in applications and in interactions with different branches of mathematics.

Mathematical electromagnetism is an especially important and rich field of application for the techniques of exterior calculus. Many insights on the Poisson problem, the $\operatorname{curl}\operatorname{curl}$ equation, and the vector Laplace equation can be obtained from the differential complex of classical vector calculus, which is composed of the gradient operator ∇, the curl operator $\nabla\times$, and the divergence operator $\nabla\cdot$. We can further deepen our mathematical, geometrical, and physical understanding of the topic when moving from classical vector calculus to the calculus of differential forms. Here the aforementioned differential operators are regarded as instances of the exterior derivative, which are constitutive of the de Rham complex. One may argue that the de Rham complex is an indispensable concept in the analysis of these partial differential equations. For example, the dimensions of the de Rham cohomology spaces describe the dimensions of the solution spaces of homogeneous vector Laplace equations. In the larger picture, the solution theory of partial differential equations reflects topological properties of the domain.

Therefore it appears natural to incorporate differential complexes and exterior calculus in the numerical analysis of partial differential equations. Applications in numerical electromagnetism have been a significant motivation for this development, which has gained momentum during the past decades [29, 109]. A publication of Arnold, Falk, and Winther from 2006 [9] has received attention from a broader mathematical audience and has popularized *finite element exterior calculus* as a theoretical framework to unify and complete several research efforts in numerical analysis. Finite element exterior calculus [10, 11] provides a framework to utilize a plethora of results in pure analysis for the theory of finite element methods and has given a comprehensive perspective on the construction of finite element spaces for vector-valued problems.

The topic of this thesis is the numerical analysis of partial differential equations in the framework of finite element exterior calculus. In addition to concepts of the mathematical theory of finite element methods, the framework of finite element exterior calculus involves many branches of mathematics, which include algebraic topology, differential geometry, and functional analysis. We give a brief outline of the theory in order to communicate the underlying ideas and to indicate the starting points of the research in this thesis.

We start with the background in global analysis. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and let $L^2\Lambda^k(\Omega)$ be the Hilbert space of square-integrable differential k-forms over Ω. The exterior derivatives of such differential forms exists in the sense of distributions, and hence we define

$$H\Lambda^k(\Omega) := \left\{ v \in L^2\Lambda^k(\Omega) \mid \mathsf{d}^k v \in L^2\Lambda^{k+1}(\Omega) \right\}. \tag{I.1}$$

This is the Hilbert space of square-integrable differential k-forms whose exterior derivative is square-integrable too. This is precisely the Sobolev space $H^1(\Omega)$ in the special case $k = 0$, but $H\Lambda^k(\Omega)$ generally contains much more than the differential k-forms with coefficients in $H^1(\Omega)$. Since the exterior derivative of an exterior derivative is zero, we obviously have $\mathsf{d}^k H\Lambda^k(\Omega) \subseteq H\Lambda^{k+1}(\Omega)$. As a consequence, we may formulate the L^2 de Rham complex

$$0 \to H\Lambda^0(\Omega) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-2}} H\Lambda^{n-1}(\Omega) \xrightarrow{\mathsf{d}^{n-1}} L^2\Lambda^n(\Omega) \to 0 \tag{I.2}$$

as a first but important example of a differential complex in this thesis. The theory of Hilbert complexes shows that this differential complex satisfies a certain duality relation with another differential complex,

$$0 \leftarrow L^2\Lambda^0(\Omega) \xleftarrow{\delta^1} H_0^*\Lambda^1(\Omega) \xleftarrow{\delta^2} \dots \xleftarrow{\delta^n} H_0^*\Lambda^n(\Omega) \leftarrow 0. \tag{I.3}$$

Here, δ denotes the codifferential and $H_0^*\Lambda^k(\Omega)$ denotes the space of square-integrable k-forms whose codifferential is square-integrable and which in addition satisfy a specific type of boundary conditions along $\partial\Omega$, the details of which we omit at this point. The nature of the aforementioned duality is precisely that δ^{k+1} is the adjoint of d^k in the sense of unbounded operators between L^2 spaces of differential forms.

The Hodge Laplace problem has been studied extensively in analysis. It generalizes many partial differential equations in vector analysis such as the Poisson problem and the vector Laplace problem. Its precise form is as follows: given a square-integrable k-form $f \in L^2\Lambda^k(\Omega)$, the *Hodge Laplace problem* is finding $u \in L^2\Lambda^k(\Omega)$ such that

$$u \in H\Lambda^k(\Omega) \cap H_0^*\Lambda^k(\Omega), \quad \mathsf{d}^k u \in H_0^*\Lambda^{k+1}(\Omega), \quad \delta^k u \in H\Lambda^{k-1}(\Omega), \tag{I.4a}$$

$$\mathsf{d}^{k-1}\delta^k u + \delta^{k+1}\mathsf{d}^k u = f. \tag{I.4b}$$

The first line of these conditions implies regularity of u and its higher derivatives and several boundary conditions, whereas the second line states that u solves the Hodge Laplace equation. The numerical analysis of the Hodge Laplace problem has constituted a major motivation for the development of finite element exterior calculus.

To improve our understanding of the Hodge Laplace problem, we introduce the space $\mathfrak{H}^k(\Omega)$ of *harmonic k-forms*,

$$\mathfrak{H}^k(\Omega) := \left\{ \, p \in H\Lambda^k(\Omega) \cap H_0^*\Lambda^k(\Omega) \;\middle|\; \delta^k p = 0, \ \mathsf{d}^k p = 0 \, \right\}, \tag{I.5}$$

This space is of particular relevance for the analysis of the Hodge Laplace equation because $\mathfrak{H}^k(\Omega)$ is both the kernel of the Hodge Laplace operator and the orthogonal complement of its range. On the other hand, it is a fundamental fact that

$$\mathfrak{H}^k(\Omega) \simeq \frac{\left\{ \, v \in H\Lambda^k(\Omega) \;\middle|\; \mathsf{d}^k v = 0 \, \right\}}{\left\{ \, \mathsf{d}^{k-1} w \mid w \in H\Lambda^{k-1}(\Omega) \, \right\}},$$

where the factor space on the right-hand side is precisely the k-th cohomology space of the L^2 de Rham complex. It can be shown that the dimension of the k-th cohomology space, and thus the dimension of the space of harmonic k-forms, equals the k-th absolute Betti numbers of the domain. This exemplifies a feature of a partial differential equation that reflects properties of the geometric ambient and can be expressed in terms of differential complexes.

Corresponding to the usage of differential complexes in the analysis of partial differential equations, it appears promising to study differential complexes of finite element spaces. Indeed, *finite element de Rham complexes* constitute the foundation of finite element exterior calculus. Given a triangulation $\mathcal{T}$ of the domain Ω, we study finite element de Rham complexes

$$0 \to \mathcal{P}\Lambda^0(\mathcal{T}) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-2}} \mathcal{P}\Lambda^{n-1}(\mathcal{T}) \xrightarrow{\mathsf{d}^{n-1}} \mathcal{P}\Lambda^n(\mathcal{T}) \to 0 \tag{I.6}$$

consisting of piecewise polynomial differential forms that have single-valued traces along inter-element boundaries. These are subcomplexes of the original L^2 de Rham complex. Arnold, Falk and Winther have determined classes of finite element de Rham complexes that realize the k-th absolute Betti numbers on cohomology. The study of finite element de Rham complexes has guided the design of stable and convergent mixed finite element methods for the Hodge Laplace problem.

A first tentative approach to the finite element analysis of the Hodge Laplace equation could begin with switching from the strong formulation (I.4) to a weak formulation, where we seek $u \in H\Lambda^k(\Omega) \cap H_0^*\Lambda^k(\Omega)$ such that

$$\langle \mathsf{d}^k u, \mathsf{d}^k v \rangle + \langle \delta^k u, \delta^k v \rangle = \langle f, v \rangle, \quad v \in H\Lambda^k(\Omega) \cap H_0^*\Lambda^k(\Omega). \tag{I.7}$$

Here the products denote the usual L^2 products. Several issues, however, oppose this as the foundation of a Galerkin method. On the one hand, the kernel of the symmetric bilinear form associated to (I.7) is precisely $\mathfrak{H}^k(\Omega)$, which is generally a non-trivial space unless the domain is contractible. We need a Lagrange multiplier to accommodate the kernel, and in many applications $\mathfrak{H}^k(\Omega)$ can only be approximated in the finite element space. On the other hand, and much more severely, one can show that the piecewise polynomial k-forms in the intersection space $H\Lambda^k(\Omega) \cap H_0^*\Lambda^k(\Omega)$ are generally not a dense subset unless the domain is convex [62]. Consequently, Galerkin methods based on (I.7) will generally not converge to the solution of the original problem.

These problems can be circumvented at the cost of an auxiliary variable for the solution's codifferential. Much research effort has been invested into the analysis of mixed finite element methods where we only need to provide finite element spaces conforming to $H\Lambda^{k-1}(\Omega)$ and $H\Lambda^k(\Omega)$. Specifically, we consider the following mixed formulation of the Hodge Laplace problem: we seek $\sigma \in H\Lambda^{k-1}(\Omega)$, $u \in H\Lambda^k(\Omega)$, and $p \in \mathfrak{H}^k(\Omega)$ such that

$$\langle \sigma, \tau \rangle - \langle u, \mathsf{d}^{k-1}\tau \rangle = 0, \quad \tau \in H\Lambda^{k-1}(\Omega), \tag{I.8}$$

$$\langle \mathsf{d}^{k-1}\sigma, v \rangle + \langle \mathsf{d}^k u, \mathsf{d}^k v \rangle + \langle p, v \rangle = \langle f, v \rangle, \quad v \in H\Lambda^k(\Omega), \tag{I.9}$$

$$\langle u, q \rangle = 0, \quad q \in \mathfrak{H}^k(\Omega). \tag{I.10}$$

One can show that this formulation is uniquely solvable with compact solution operator, and it is easily seen to be equivalent to the original problem except for the introduction of a Lagrange multiplier to handle the space of harmonic forms.

Finite element exterior calculus replicates this variational formulation over the L^2 de Rham complex as a mixed finite element method over the finite element de Rham complex. We seek $\sigma_h \in \mathcal{P}\Lambda^{k-1}(\mathcal{T})$, $u_h \in \mathcal{P}\Lambda^k(\mathcal{T})$, and $p_h \in \mathfrak{H}^k(\mathcal{T})$ such that

$$\langle \sigma_h, \tau_h \rangle - \langle u_h, \mathsf{d}^{k-1}\tau_h \rangle = 0, \quad \tau_h \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}), \tag{I.11}$$

$$\langle \mathsf{d}^{k-1}\sigma_h, v_h \rangle + \langle \mathsf{d}^k u_h, \mathsf{d}^k v_h \rangle + \langle p_h, v_h \rangle = \langle f, v_h \rangle, \quad v_h \in \mathcal{P}\Lambda^k(\mathcal{T}), \tag{I.12}$$

$$\langle u_h, q_h \rangle = 0, \quad q_h \in \mathfrak{H}^k(\mathcal{T}). \tag{I.13}$$

Here $\mathfrak{H}^k(\mathcal{T})$ denotes the space of *discrete harmonic k-forms*,

$$\mathfrak{H}^k(\mathcal{T}) := \left\{ p_h \in \mathcal{P}\Lambda^k(\mathcal{T}) \;\middle|\; \mathsf{d}^{k-1}\mathcal{P}\Lambda^{k-1}(\mathcal{T}) \perp p_h, \ \mathsf{d}^k p_h = 0 \right\}, \tag{I.14}$$

which is a (generally non-conforming) approximation of $\mathfrak{H}^k(\Omega)$. An important structural property of the harmonic k-forms is nevertheless preserved by their discrete counterparts: their dimension equals the k-th absolute Betti numbers and thus corresponds to topological properties of the domain.

We have replicated important structures of the original problem in the discrete setting. But in order to relate both worlds and especially in order to obtain approximation estimates for the Galerkin method, we need a concept of uttermost importance to finite element exterior calculus: commuting uniformly bounded projections. Suppose that we have a commuting diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H\Lambda^0(\Omega) & \xrightarrow{\mathsf{d}} & H\Lambda^1(\Omega) & \xrightarrow{\mathsf{d}} & \dots & \xrightarrow{\mathsf{d}} & L^2\Lambda^n(\Omega) & \longrightarrow & 0 \\ & & {\scriptstyle \pi_h^0}\big\downarrow & & {\scriptstyle \pi_h^1}\big\downarrow & & & & {\scriptstyle \pi_h^n}\big\downarrow & & \\ 0 & \longrightarrow & \mathcal{P}\Lambda^0(\mathcal{T}) & \xrightarrow{\mathsf{d}} & \mathcal{P}\Lambda^1(\mathcal{T}) & \xrightarrow{\mathsf{d}} & \dots & \xrightarrow{\mathsf{d}} & \mathcal{P}\Lambda^n(\mathcal{T}) & \longrightarrow & 0, \end{array}$$

where the vertical operators are idempotent and uniformly L^2 bounded in the relevant discretization parameters. Then a multitude of abstract results is immediately instantiated, including the stability and convergence of mixed finite element methods for the Hodge Laplace problem, corresponding results for eigenvalue problems, upper bounds for Poincaré-Friedrichs constants, and approximation estimates for the harmonic forms.

Significant work has been accomplished on finite element exterior calculus, but several unresolved questions and untapped possibilities have remained at its very foundations. In the course of this thesis we explore such topics. The error analysis of finite element exterior calculus is a thematic priority. The following observations have inspired this research in particular.

- The class of Lipschitz domains, even though a common choice for the geometric ambient in numerical analysis, is unnecessarily restrictive in theory and applications. There is no difficulty in finding polyhedral domains in $\mathbb{R}^3$ that are not Lipschitz domains. We propose the class of *weakly Lipschitz domains* as more natural for the purposes of numerical analysis. Both the L^2 de Rham complex and the finite element de Rham complexes can be formulated on weakly Lipschitz domains in the usual manner, but we need a *smoothed projection* over weakly Lipschitz domains in order to instantiate the abstract Galerkin theory of Hilbert complexes and enable a priori error estimates. Previous works have constructed smoothed projections only over Lipschitz domains.

- The Poisson equation with *mixed boundary conditions* is standard. Much less literature is available on mixed boundary conditions in mathematical and numerical electromagnetism. More generally, the Hodge Laplace equation with mixed boundary conditions has been discussed only in a few selected contributions to global analysis (e.g., [99]). The lack of literature on mixed boundary conditions in pure analysis might be the reason that mixed boundary conditions have not yet been incorporated into finite element exterior calculus: even the Poisson equation with mixed boundary conditions has remained inaccessible. We have incentive to improve this situation since mixed boundary conditions are relevant in theory and practice. For the numerical analysis of the Hodge Laplace equation with mixed boundary conditions, we need to identify a variant of the L^2 de Rham complex with corresponding *partial boundary conditions* and construct the smoothed projection. *Harmonic forms with mixed boundary conditions* are especially interesting here.

- Apart from the a priori error analysis, we are also interested in the *a posteriori error analysis* in finite element exterior calculus. The classical residual error estimator has been studied in finite element exterior calculus by Demlow and Hirani [72]. But several other error estimators in the literature show potential for a productive interaction with finite element exterior calculus. This is especially true for Braess and Schöberl's equilibrated a posteriori error estimator for the curl curl equation [34]. We will study their results in this thesis with particular attention to distributional finite element sequences. Our agenda moreover includes generalizing their intricate error estimator to higher order edge elements. This provides new tools in numerical analysis and solves an open problem of practical interest.

- Last but not least, we are inclined to address the discrete foundations of finite element exterior calculus, namely the construction of finite element spaces and their bases. Despite many contributions to this topic, the theoretical framework cannot be regarded as finished yet, and a comprehensive way of representation is yet to be developed. We illuminate some aspects that might be of interest to a broader audience.

This thesis comprises the research which has evolved from these observations. It is organized into several chapters that are based on accepted publications, submitted articles, and unpublished material. Chapter II addresses several topics related to simplices, simplicial complexes, and chain complexes, which serve as a technical background for subsequent chapters. Chapters III and IV describe the construction of finite element spaces. The remaining chapters treat three different thematic areas that can be read independently of each other. In Chapters VI–VIII we extend the a priori error estimates in finite element exterior calculus. In Chapter IX we develop the notion of discrete distributional differential form. Eventually, in Chapter X we address equilibrated a posteriori error estimates. The thesis is concluded with an outlook to possible future directions of research.

In addition, several basic notions are given in the appendix of this thesis and assumed throughout this work. The reader may briefly consult the appendix before approaching the actual content of this thesis.

Finite Element Differential Forms

The analysis of finite element methods builds upon the notion of finite element spaces. In the case of finite element exterior calculus we build upon the notion of finite element differential forms. A pivotal concept here are geometrically decomposed bases.

We set up the theory of polynomial differential forms on simplices in Chapter III. The finite element spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ over a simplex T have been studied in many publications [9, 10, 29, 52, 53, 57, 107, 108, 109, 153, 154] which build upon previous research on Nédélec and Raviart-Thomas spaces [2, 103, 161, 174]. In Chapter III we give an outline of these spaces over simplices. We do not, however, aim at a complete construction *ab initio*. Hence the reader is strongly assumed to be familiar with prior publications, in particular [9] and [10]. Even though many of these results are known in principle, our way of exposition is new and possibly interesting for experts, and even though Chapter III is not intended as a text book chapter, its content may contribute to popularizing the bases in finite element exterior calculus to practitioners in computational science.

We give a complete derivation of geometrically decomposed bases for the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ and the corresponding spaces with boundary conditions $\mathring{\mathcal{P}}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$. Moreover, we construct extension operators

$$\operatorname{ext}^{k,r}_{F,T} : \mathring{\mathcal{P}}_r\Lambda^k(F) \to \mathcal{P}_r\Lambda^k(T), \quad \operatorname{ext}^{k,r,-}_{F,T} : \mathring{\mathcal{P}}_r^-\Lambda^k(F) \to \mathcal{P}_r^-\Lambda^k(T),$$

from subsimplices F onto a simplex T. This is sufficient to facilitate the geometric decomposition of finite element spaces.

The construction of the basis forms and the geometric decomposition in finite element exterior calculus has so far been distributed over two publications by Arnold, Falk, and Winther [9, 10]. Our manner of presentation is inspired by these works but features also some modest novelties. We first construct bases for $\mathcal{P}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r\Lambda^k(T)$ and derive the geometric decomposition. Analogously we address the spaces $\mathcal{P}_r^-\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ and their geometric decomposition.

A significant innovation in Chapter III is our exposition of the isomorphisms

$$\mathcal{P}_r\Lambda^k(T) \simeq \mathring{\mathcal{P}}^-_{r+n-k+1}\Lambda^{n-k}(T), \quad \mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T) \simeq \mathcal{P}_r^-\Lambda^{n-k}(T),$$

and corresponding duality pairings, which have been used only implicitly in many previous works. A recent publication by Christiansen and Rapetti [57] is a major inspiration here and we generalize their results.

Having studied finite element differential forms over simplices, we turn our attention to finite element differential forms over triangulations in Chapter IV. We begin with a brief review of the classical Whitney forms over a triangulation, which constitute the finite element de Rham complex of lowest order, and their duality to the simplicial chain complex of the triangulation. A minor novelty is that we study the complex of Whitney forms with a general class of boundary conditions.

Proceeding to the construction of higher order finite element spaces, we draw inspiration from the dissertation of Zaglmayr [183] and a publication by Demkowicz, Monk, Vardapetyn, and Rachowicz [69] in the area *hp* finite element methods. We obtain a new description of finite element spaces of higher and possibly non-uniform polynomial order. The basic idea is constructing finite element spaces of higher order through the local augmentation of the space of Whitney forms. Extending that line of thought, we understand finite element de Rham complexes of higher and possibly non-uniform polynomial order as the augmentation of the complex of Whitney forms by local higher order finite element de Rham complexes. Eventually, we devise the finite element interpolant, building upon prior work in the area of *hp* finite element methods. The interpolant is notably different from the canonical interpolant in previous publications on finite element exterior calculus, but it agrees with the *harmonic interpolation* in the theory of finite element systems [56].

Smoothed Projections

A fundamental topic in the theory of finite element methods are *a priori error estimates*, which estimate the approximation error of a Galerkin solution in terms of the data (e.g., its Sobolev norm) and some parameters of the finite element spaces (e.g., the mesh size). Another fundamental topic is the *stability* of finite element methods, which affects their practical solvability by numerical algorithms. Within the Galerkin theory of Hilbert complexes admitted by finite element exterior calculus, stability and convergence can be studied in terms of one single concept: uniformly bounded commuting projections. These are known specifically as *smoothed projections* in this context.

Smoothed projections are critical to finite element exterior calculus, but actually constructing and analyzing smoothed projections is a technically sophisticated endeavor. Building upon earlier works of Christiansen [52] and Schöberl [160], the publication by Arnold, Falk, and Winther [9] approached L^2 de Rham complexes over Lipschitz domains and finite element spaces over quasi-uniform families of triangulations. This was successively extended by Christiansen and Winther [58] who addressed L^2 de Rham complexes over Lipschitz domains with homogeneous boundary conditions and merely shape-uniform families of triangulations.

A major part of this thesis is dedicated to the extension of those contributions. On the one hand, we have incentive to transcend the class of Lipschitz domains and develop finite element exterior calculus over weakly Lipschitz domains, as has been brought to our attention above. Constructing a smoothed projection over weakly Lipschitz domains will accomplish this.

On the other hand, we address the Hodge Laplace equation with *mixed boundary conditions* in finite element exterior calculus. Mixed boundary conditions are standard for the Poisson problem, but the state of research is entirely different for the Hodge Laplace equation and its translations to classical vector analysis. For our understanding of mixed boundary conditions, it is instructive to study the foundational de Rham complexes in the first place: in this case, L^2 de Rham complexes with *partial boundary conditions*. For the analytical background we point to a major publication by Gol'dshtein, Mitrea, and Mitrea [99], whose results we apply to the Hodge Laplace equation with mixed boundary conditions over weakly Lipschitz domains. The challenge is to construct a smoothed projection that preserves partial boundary conditions. This is accomplished in Chapters V–VIII.

We begin with a review of Sobolev spaces of differential forms over domains in Chapter V. For the sake of generality (and for subsequent use) we consider L^p spaces of differential forms. In particular, we introduce the $W^{p,q}$ classes of differential forms, which are L^p integrable differential forms with L^q integrable exterior derivative [100], to the literature of numerical analysis. We study the behavior of those differential forms under pullback along bi-Lipschitz mappings.

Next we discuss the class of weakly Lipschitz domains in Chapter VI. A weakly Lipschitz domain is a domain whose boundary can be flattened locally by a bi-Lipschitz coordinate transformation. Arguably, this class is very large: it contains

Lipschitz domains and also all polyhedral domains in $\mathbb{R}^3$ (see Theorem VI.3.2). With a recourse to Lipschitz topology [134], we introduce the concept of Lipschitz collar (see Theorem VI.1.8) to the numerical literature. Moreover, we prepare our discussion of mixed boundary conditions with the foundational notions of *admissible boundary patch* and *admissible boundary partition*.

The construction and analysis of the smoothed projection is completed in Chapter VII over the course of several stages, reminiscent of the constructions in published literature [9, 52, 58] even though notable changes are made.

First, a commuting extension operator extends the differential form to a neighborhood of the domain. The basic idea is the extension-by-reflection, for which we utilize a Lipschitz collar along the domain boundary. A modification of this idea accommodates for partial boundary conditions: the differential form is extended by zero to a *bulge* attached along the boundary part along which essential boundary conditions are imposed. Subsequently, the pullback along a bi-Lipschitz deformation extends this bulge. The resulting differential form vanishes in a neighborhood of the boundary and the degree of deformation can be controlled locally. This step is crucial for the handling of boundary conditions. In the next stage of the construction, a commuting mollification operator smooths the differential form. The mollification radius is locally controllable.

When we combine the canonical commuting interpolation operator with the aforementioned commuting smoothing operator, and let the mollification radius be controlled by a mesh size function, then we obtain a uniformly bounded commuting mapping from the L^2 de Rham complex with partial boundary conditions to a finite element subcomplex. But this is not yet a projection because it does not leave the finite element space invariant. As a compensation, we employ what is sometimes called the *Schöberl trick*: carefully adjusting the parameters in our construction, we can control the interpolation error over finite element differential forms. This ensures the existence of a uniformly bounded commuting operator that corrects the interpolation error. Putting all this together we recover the projection property.

Estimating the aforementioned interpolation error is far from trivial though. We model our proof after material in earlier publications [9, 58]. To the author's best knowledge and understanding, however, the proofs in those papers are not complete (see Remark VII.8.9). In order to finalize the proof, we utilize an assortment of concepts in geometric measure theory.

The technical effort to derive the smoothed projection is significant, but eventually we instantiate the Galerkin theory of Hilbert complexes, as described in Chapter VIII. We give an outline of Hilbert complexes and then discuss the L^2 de Rham complex with partial boundary conditions. Harmonic forms satisfying mixed boundary conditions are of particular interest because they feature a quality not present in the case of non-mixed boundary conditions: their dimension not only depends on the topology of the domain but also on the topology of the boundary partition. As an example application of the Galerkin theory of Hilbert complexes, we recapitulate the a priori convergence estimates in finite element exterior calculus and apply them to the Hodge Laplace equation with mixed boundary conditions.

De Rham complexes with partial boundary conditions over weakly Lipschitz domains have not been subject of much research yet. As a secondary outcome of our research we obtain a new result on the density of smooth differential forms in Sobolev spaces of differential forms with partial boundary conditions (see Theorem VII.4.3). This has not been available in the literature previously.

Discrete Distributional Differential Forms

After our study of a priori error estimates and smoothed projections we turn the focus towards a very different topic. In Chapter IX we investigate *discrete distributional differential forms.*

This track of research originates from the seminal publication of Braess and Schöberl on equilibrated a posteriori error estimates for edge elements [34]. One of the many novel concepts in their publication have been distributional finite element sequences, which generalize the classical conforming finite element sequences. For a motivating example, let $\mathcal{T}$ be a three-dimensional triangulation and consider a finite element sequence of lowest polynomial order,

$$\mathcal{P}_0^1(\mathcal{T}) \xrightarrow{\text{grad}} \mathbf{Nd}_0^0(\mathcal{T}) \xrightarrow{\text{curl}} \mathbf{RT}_0^0(\mathcal{T}) \xrightarrow{\text{div}} \mathcal{P}_{-1}^0(\mathcal{T}),$$

consisting of the first-order Lagrange space $\mathcal{P}_0^1(\mathcal{T})$ with Dirichlet boundary conditions, the lowest-order Nédélec space $\mathbf{Nd}_0^0(\mathcal{T})$ with homogeneous tangential boundary conditions, the lowest-order Raviart-Thomas space $\mathbf{RT}_0^0(\mathcal{T})$ with homogeneous normal boundary conditions, and the space of piecewise constant functions $\mathcal{P}_{-1}^0(\mathcal{T})$. The subindex -1 in the latter symbol indicates that no continuity conditions are imposed on members of $\mathcal{P}_{-1}^0(\mathcal{T})$ along inter-element faces.

Similarly, let $\mathbf{RT}_{-1}^0(\mathcal{T})$ denote the space of vector fields that are piecewise in the Raviart-Thomas space but which do not necessarily satisfy any normal continuity along inter-element faces or any normal boundary conditions. The divergence of such vector fields in the sense of distributions is contained in the space $\mathcal{P}_{-2}^0(\mathcal{T})$, the space of distributions spanned by integrals over tetrahedra and integrals over faces of the triangulation. We have a well-defined differential complex

$$\mathcal{P}^1(\mathcal{T}) \xrightarrow{\text{grad}} \mathbf{Nd}_0^0(\mathcal{T}) \xrightarrow{\text{curl}} \mathbf{RT}_{-1}^0(\mathcal{T}) \xrightarrow{\text{div}} \mathcal{P}_{-2}^0(\mathcal{T}).$$

We generalize this construction. Let $\mathbf{Nd}_{-1}^0(\mathcal{T})$ denote the space of vector fields that are piecewise in the lowest-order Nédélec space but that do not necessarily satisfy any tangential continuity along inter-element faces or tangential boundary condition along boundary faces. The curl of such a vector field is contained in the space $\mathbf{RT}_{-2}^0(\mathcal{T})$, which is defined as the space of vector-valued distributions spanned by integrals against piecewise Raviart-Thomas vector fields and integrals of the tangential component of a vector field against Nédélec vector fields over faces. The divergence of a distributional vector field in $\mathbf{RT}_{-2}^0(\mathcal{T})$ is contained in $\mathcal{P}_{-3}^0(\mathcal{T})$, which is the direct sum of $\mathcal{P}_{-2}^0(\mathcal{T})$ and the span of integrals over edges. We can assemble the differential complex

$$\mathcal{P}^1(\mathcal{T}) \xrightarrow{\text{grad}} \mathbf{Nd}^0_{-1}(\mathcal{T}) \xrightarrow{\text{curl}} \mathbf{RT}^0_{-2}(\mathcal{T}) \xrightarrow{\text{div}} \mathcal{P}^0_{-3}(\mathcal{T}).$$

Once again, we may repeat this principle and consider the space $\mathcal{P}^1_{-1}(\mathcal{T})$ of piecewise affine functions over $\mathcal{T}$, which is a discontinuous version of the first-order Lagrange elements. Proceeding in a completely analogous manner as above, we generate a differential complex

$$\mathcal{P}^1_{-1}(\mathcal{T}) \xrightarrow{\text{grad}} \mathbf{Nd}^0_{-2}(\mathcal{T}) \xrightarrow{\text{curl}} \mathbf{RT}^0_{-3}(\mathcal{T}) \xrightarrow{\text{div}} \mathcal{P}^0_{-4}(\mathcal{T})$$

of distributional finite element spaces.

In this thesis we translate their notion of distributional finite element sequence into the setting of finite element exterior calculus, which gives rise to the notion of *discrete distributional differential form*. The original contribution of Braess and Schöberl treated only the case of lowest polynomial order over local patches. We develop *discrete distributional de Rham complexes* over arbitrary triangulations and allow for finite element spaces of higher order.

The two major points of investigation are the homology theory of discrete distributional de Rham complexes and Poincaré-Friedrichs inequalities with respect to mesh dependent norms. This work was originally conceived prior to the research on smoothed projections, and one achievement has been the characterization of the cohomology spaces of conforming finite element sequences when partial boundary conditions are imposed.

Flux Reconstruction and A Posteriori Error Estimation

We conclude this thesis with another approach to the work of Braess and Schöberl. In Chapter X we give an affirmative answer to an open question in the area of a posteriori error estimation: *can the equilibrated error estimator for edge elements of lowest order be generalized to the higher order case?*

In order to put this question into proper context, we recall the problem of error estimation in the finite element method. We not only want to compute an approximate solution to a partial differential equation, but we also want to quantitatively estimate the approximation error. The terminology is suggestive of the fact that *a priori* error estimates bound the approximation error *prior* to the computation of the Galerkin solution. By their very nature these error bounds only involve the initial data of the problem.

While a priori error estimates prove the asymptotic convergence of a Galerkin method, they are not as suitable for adaptive finite element methods and reliable error estimation. On the one hand, they typically involve many unknown constants which are difficult to estimate in practice. On the other hand, we may reasonably assume that we can bound the Galerkin error more precisely given the approximate solution as additional information.

This motivates *a posteriori* error estimation, conducted *posterior* to the computation of an approximate solution. Past decades have seen considerable research activity on a posteriori error estimators and many rigorous and heuristic methods have been proposed (see [4, 156, 172] for a small overview). The persistent research interest is especially due to the significant role of a posteriori error estimation in adaptive finite element methods (see, e.g., [46, 111]).

The classical residual error estimator is the prototypical example of an a posteriori error estimator and can be found in many introductory textbooks on finite element methods. For a basic outline let $f \in L^2(\Omega)$ and let $u_h \in H^1_0(\Omega)$ be an approximate solution to the Poisson problem $\Delta u = f$ with Dirichlet boundary conditions. The residual $r_h \in H^{-1}(\Omega)$ is defined as $r_h := f - \Delta u_h$, which means $r_h = \Delta(u - u_h)$ in the sense of distributions. Since $\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ is an isomorphism we conclude that the H^1 norm of the error $u - u_h$ is comparable to the H^{-1} norm of r_h. The latter can be estimated explicitly in terms of a mesh-dependent norm if f and u_h are piecewise polynomial and u_h is the Galerkin solution.

The classical residual error estimator, however, suffers from practical shortcomings that have motivated further research: the estimate involves anonymous constants that are difficult to estimate in practice and it is outcompeted in numerical experiments. This thesis will not explore the classical residual error estimator in finite element exterior calculus in further detail; we refer to the publication of Demlow and Hirani for a detailed exposition [72].

Instead we explore alternative a posteriori error estimators that promise sharper error bounds. Among these, the class of *equilibrated* or *implicit* error estimators has attracted considerable attention [3, 33, 118]. They utilize the *hypercircle method* [170]. For a brief outline of the idea, consider again the Poisson problem $\Delta u = f$ with Dirichlet boundary conditions and let $u_h \in H^1_0(\Omega)$ be any approximate solution. Suppose that $\boldsymbol{\sigma} \in \mathbf{H}(\operatorname{div}, \Omega)$ with $-\operatorname{div} \boldsymbol{\sigma} = f$. The *hypercircle theorem* (or *Prager-Synge theorem*) states that

$$\|\boldsymbol{\sigma} - \operatorname{grad} u_h\|^2_{\mathbf{L}^2} = \|\boldsymbol{\sigma} - \operatorname{grad} u\|^2_{\mathbf{L}^2} + \|\operatorname{grad} u - \operatorname{grad} u_h\|^2_{\mathbf{L}^2}.$$

We obtain a simple $\mathbf{L}^2$ estimate for the error $\operatorname{grad} u - \operatorname{grad} u_h$ in terms of the $\mathbf{L}^2$ norm of $\boldsymbol{\sigma} - \operatorname{grad} u_h$. Practically using the equilibrated error estimator, however, comes at a price: whereas the approximation u_h is assumed to be known from the outset, the flux $\boldsymbol{\sigma}$ needs to be reconstructed with additional computational effort. In principle, $\boldsymbol{\sigma}$ can be obtained as the flux variable in a mixed finite element method for the same Poisson problem, which requires the solution of a global problem. But if u_h is the Galerkin solution over, say, the Lagrange elements of some polynomial order, then this additional structure enables a more efficient flux reconstruction: we can compute a flux $\boldsymbol{\sigma}$ in, say, the Raviart-Thomas space using only *local* computations over patches. Numerical experiments indicate that this error estimator bounds the error more tightly than the classical residual error estimator [47].

In the light of those research activities, we are interested in a posteriori error estimators for the curl curl problem. The largest share of previous research has focused on the classical residual error estimator (see, e.g, [16, 144, 160]). A notable exception is Braess and Schöberl's a posteriori error estimator, which has already been mentioned above. For a brief outline, let $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and let $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega)$ be a solution to the partial differential equation $\operatorname{curl}\operatorname{curl} \mathbf{u} = \mathbf{f}$. When $\mathbf{u}_h \in \mathbf{H}(\operatorname{curl}, \Omega)$, then it is reasonable to ask for an $\mathbf{L}^2$ estimate of the error $\operatorname{curl}(\mathbf{u} - \mathbf{u}_h)$.

If the domain is contractible, and $\mathbf{f}$ has vanishing divergence and vanishing normal component along $\partial\Omega$, then there exists $\boldsymbol{\sigma} \in \mathbf{H}(\operatorname{curl}, \Omega)$ with vanishing tangential component along $\partial\Omega$ such that $\operatorname{curl} \boldsymbol{\sigma} = \mathbf{f}$. Under these conditions one can show a vector field analogue to the Prager-Synge identity above,

$$\|\boldsymbol{\sigma} - \operatorname{curl} \mathbf{u}_h\|^2_{\mathbf{L}^2} = \|\boldsymbol{\sigma} - \operatorname{curl} \mathbf{u}\|^2_{\mathbf{L}^2} + \|\operatorname{curl} \mathbf{u} - \operatorname{curl} \mathbf{u}_h\|^2_{\mathbf{L}^2}.$$

Therefore the $\mathbf{H}(\operatorname{curl}, \Omega)$ seminorm of the error $\mathbf{u} - \mathbf{u}_h$ is bounded by the $\mathbf{L}^2$ norm of the vector field $\boldsymbol{\sigma} - \operatorname{curl} \mathbf{u}_h$.

But as before, there is no free lunch: the computation of the flux variable $\boldsymbol{\sigma}$ must precede the equilibrated error estimation. Computing a flux by a mixed method is a global problem, but the computation can be localized under additional conditions: if $\mathbf{u}_h$ is contained in the Nédélec space and satisfies the Galerkin property for the curl curl equation, then computations over local patches recover a flux $\boldsymbol{\sigma}$. In practice, we assume $\mathbf{f}$ in a Raviart-Thomas space with normal boundary conditions and we compute $\boldsymbol{\sigma}$ in a Nédélec space with tangential boundary conditions.

This instance of localized flux reconstruction, however, is considerably more intricate, both mathematically and algorithmically, than for the Poisson problem. Braess and Schöberl have addressed the problem of flux reconstruction for vector-valued finite elements only in the lowest-order case. How to generalize to the higher order case is far from obvious and has remained an unresolved problem for years.

Solving this open problem is the main agenda of Chapter X. We address the topic of *flux reconstruction*, the algorithmic solution of the flux equation $\operatorname{curl} \boldsymbol{\sigma} = \mathbf{f}$ between finite element spaces, and we introduce the *partially localized flux reconstruction* as a novel concept. Here we build upon Chapter IV, where we have constructed the finite element de Rham complexes via local augmentation of the complex of Whitney forms. The partially localized flux reconstruction reduces the flux reconstruction for $\operatorname{curl} \boldsymbol{\sigma} = \mathbf{f}$ between finite element spaces of higher (and possibly non-uniform) polynomial order to the case of lowest polynomial order. This reduction uses only parallelizable local computations.

As a closure of Chapter X we address the problem of equilibrated error estimators for the curl curl problem once again: the fully localized computation of the equilibrated error estimator in the case of higher order finite elements is achieved by combining the partially localized flux reconstruction with the fully localized computations of Braess and Schöberl for the lowest order case. This solves the open problem mentioned above.

This outcome demonstrates how abstract mathematical methods can lead to surprising new insights and practical applications.

II. Simplices and Triangulations

Simplices are studied in many branches of mathematics, such as combinatorics, geometry, or algebraic topology. Simplices also provide the mathematical background for meshes in many finite element methods. In this thesis we draw on these different accesses to simplices in mathematics and have them come together in the study of mixed finite element methods. For that reason, an entire chapter is dedicated to a thorough exposition of simplicial concepts.

We begin with gathering definitions concerning simplices, simplicial complexes, and local patches in Section II.1, in order to establish the combinatorial background. We then turn our attention to quantifying the quality of simplices and simplicial complexes in several sections. In Section II.2 we define the geometric shape measure and relate it to properties of reference transformations. Subsequently, we discuss solid angles of simplices in Section II.3, and show how the geometric shape measure determines a lower bound for the minimum solid angle of a simplex. Having studied the regularity of single simplices, we address the regularity of simplicial triangulations in Section II.4. Finally, we discuss the regularity of reference transformations in Section II.5. Those sections elaborate the technical details of mesh regularity in the theory of finite element methods, which seem to be mathematical folklore. We study those technical details (i) to make the presentation self-contained and fully rigorous, (ii) to make explicit results formally available in the higher-dimensional case, and (iii) because explicit and quantitative estimates are of inherent interest in a computational setting. Lastly, we put simplicial complexes into the perspective of algebraic topology, and study simplicial chain complexes in Section II.6. Here we pay special attention to simplicial chain complexes associated to local patches.

Simplicial complexes have been referred to before in research on finite element differential complexes (e.g., [56, 57, 109, 154]). Throughout this chapter, we discuss combinatorial and algebraic properties of a simplicial complex $\mathcal{T}$ always relative to a subcomplex $\mathcal{U}$ wherever this is applicable. The consideration of subcomplexes is a natural prerequisite for the discussion of de Rham complexes with boundary conditions. Even though a basic concept of algebraic topology, simplicial subcomplexes have not received much attention in numerical literature.

The decomposition of a domain into smaller elements has been the seminal idea of finite element methods, but those elements are not necessarily simplices. Historically, quadrilateral elements have been used since the beginnings of finite element methods [59, 65]. Moreover, prismatic and pyramidal elements appear naturally when connecting tetrahedral and quadrilateral meshes (e.g., [183]). Finite element methods based on discretizations by general polytopes have seen a surge of interest in recent years. There are far too many developments in that area to list them here; two particular examples that we mention are virtual finite element methods [19, 39] and polytopal finite element methods based on generalized barycentric coordinates [90, 91, 96, 137, 173]. Finite element systems are an abstract framework for general polyhedral methods [56]. The combinatorial and algebraic aspects of simplicial complexes carry over with only technical changes to general polyhedral complexes; indeed, general cellular complexes are a standard concept in combinatorial and algebraic topology (e.g. polyhedral complexes [5]). By contrast, the understanding of shape measures for general polytopes is only in its beginnings [95].

II.1. Basic Definitions

Let $m, n \in \mathbb{N}_0$. Points $v_0, \dots, v_m \in \mathbb{R}^n$ are called *affinely independent* if no single of these points is an affine combination of the others. Moreover, if $v_0, \dots, v_m$ are affinely independent, then $m \leq n$.

A set $S \subset \mathbb{R}^n$ is a *closed m-simplex* if it is the convex closure of affinely independent points $v_0, \dots, v_m \in \mathbb{R}^n$, which we call the *vertices* of S. We write $\operatorname{Ver}(S)$ for the vertices of S, and note that $\operatorname{Ver}(S)$ is uniquely determined by S. We say that S is *m-dimensional* or has dimension m. We call S *full-dimensional* if S is an n-simplex. An important simplex is the *m-dimensional reference simplex* $\Delta_m \subset \mathbb{R}^m$,

$$\Delta_m := \operatorname{convex}\{0, e_1, \dots, e_m\}, \tag{II.1}$$

defined as the convex closure of the origin and the m different standard coordinate vectors $e_1, \dots, e_n$ of $\mathbb{R}^m$.

If F is another simplex with $\operatorname{Ver}(F) \subseteq \operatorname{Ver}(S)$, then we call F a *subsimplex* of S, and in turn we call S a *supersimplex* of F. We write $\Delta(S)$ for the set of subsimplices of S. For any $F \in \Delta(S)$ we let $\imath_{F,S} : F \to S$ denote the inclusion. In the sequel, we call 0-simplices also *vertices* and 1-simplices also *edges*. If T is an m-simplex and $F \in \Delta(T)$ is a simplex of dimension $m-1$, then we call F a *face* of T.

Remark II.1.1.
(i) Even though considering the empty set as a simplex is not unheard of in the literature, we do not consider the empty set as a simplex in this thesis. (ii) Our definition of simplex does not allow for "degenerate" simplices whose vertices are not affinely independent. We will not consider those in this thesis. (iii) A 0-simplex is a set containing one single point of $\mathbb{R}^n$, not the point itself, but we will often ignore this difference to simplify the discussion.

A set $\mathcal{T}$ of simplices in $\mathbb{R}^n$ is called a *simplicial complex* if

$$\forall T \in \mathcal{T} : \forall S \in \Delta(T) : S \in \mathcal{T}, \tag{II.2a}$$
$$\forall T, T' \in \mathcal{T} : \big(T \cap T' \neq \emptyset \implies T \cap T' \in \Delta(T) \cap \Delta(T') \big). \tag{II.2b}$$

The first condition means that $\mathcal{T}$ is closed under taking subsimplices, and the second condition means that the intersection of two simplices is either empty or a common subsimplex. A simplicial complex $\mathcal{U}$ with $\mathcal{U} \subseteq \mathcal{T}$ is called a *simplicial subcomplex* of $\mathcal{T}$. For $m \in \mathbb{Z}$ we define

$$\mathcal{T}^m := \{\, T \in \mathcal{T} \mid \dim T = m \,\}, \quad \mathcal{T}^{[m]} := \{\, T \in \mathcal{T} \mid \dim T \leq m \,\}.$$

In other words, $\mathcal{T}^m$, $m \in \mathbb{Z}$, is the set of m-dimensional simplices in $\mathcal{T}$, and $\mathcal{T}^{[m]}$, $m \in \mathbb{Z}$, is the smallest subcomplex of $\mathcal{T}$ that contains all of the m-dimensional simplices of $\mathcal{T}$. The set $\mathcal{T}^{[m]}$ is also known as the m-dimensional *skeleton* of $\mathcal{T}$. We say that $\mathcal{T}$ is *m-dimensional* or *has dimension m* if

$$\forall S \in \mathcal{T} : \exists T \in \mathcal{T}^m : S \subseteq T.$$

We note that $\mathcal{T}^m = \emptyset$ if $m \notin \{0, \dots, \dim \mathcal{T}\}$.

Simplicial complexes appear as discretizations of topological spaces. We write

$$[\mathcal{T}] := \bigcup_{T \in \mathcal{T}} T. \tag{II.3}$$

We then say that $\mathcal{T}$ *triangulates* the closed set $[\mathcal{T}]$. We may sometimes ignore the difference between $\mathcal{T}$ and $[\mathcal{T}]$ in the notation for the sake of simplicity.

Example II.1.2.
(i) If T is an m-simplex, then $\Delta(T)$ is a simplicial complex of dimension m. Then $\Delta(T)^l$ is the set of l-dimensional subsimplices of T for $l \in \mathbb{Z}$. (ii) Any affine triangulation of an m-dimensional topological submanifold of Euclidean space with boundary is an m-dimensional simplicial complex. (iii) Suppose that a simplicial complex $\mathcal{T}$ triangulates a topological manifold with boundary. Then a simplicial subcomplex of $\mathcal{T}$ triangulates the boundary of that manifold. Throughout this thesis, we will consider subcomplexes $\mathcal{U}$ of $\mathcal{T}$ that triangulate a part of the boundary.

We are interested in the local structure of simplicial complexes. For this purpose we introduce notions of patches. To begin with, we define

$$\mathcal{T}(T) := \{\, S \in \Delta(T') \mid T' \in \mathcal{T},\ T \cap T' \neq \emptyset \,\}. \tag{II.4}$$

We call $\mathcal{T}(T)$ the *local patch* or *macropatch* of T in $\mathcal{T}$. It is the smallest simplicial complex which contains all simplices of $\mathcal{T}$ with non-empty intersection with T.

A different notion of simplicial patch is

$$\mathcal{M}(\mathcal{T}, F) := \{\, S \in \mathcal{T} \mid \exists T \in \mathcal{T} : \{S, F\} \subseteq \Delta(T) \,\}. \tag{II.5}$$

Note that $\mathcal{M}(\mathcal{T}, F)$ is the smallest simplicial complex containing all simplices of $\mathcal{T}$ that contain F. We call $\mathcal{M}(\mathcal{T}, F)$ the *micropatch* around F in $\mathcal{T}$.

It will be of interest to construct a subcomplex of $\mathcal{M}(\mathcal{T}, F)$ that models the boundary of the micropatch $\mathcal{M}(\mathcal{T}, F)$. Furthermore, $\mathcal{T}$ triangulates a topological manifold with boundary in our applications, and then it will be of interest to take a subcomplex $\mathcal{U}$ of $\mathcal{T}$ into account in our study of micropatches, where $\mathcal{U}$ triangulates a boundary part of the manifold. Formally, if $\mathcal{U}$ is a simplicial subcomplex of $\mathcal{T}$, then we define

$$\mathcal{N}(\mathcal{T}, \mathcal{U}, F) := \{\, S \in \mathcal{M}(\mathcal{T}, F) \mid F \notin \Delta(S) \text{ or } S \in \mathcal{U} \,\}. \tag{II.6}$$

We call $\mathcal{N}(\mathcal{T}, \mathcal{U}, F)$ the *micropatch boundary* of F in $\mathcal{T}$ relative to $\mathcal{U}$.

Remark II.1.3.
If $\mathcal{T}$ is a simplicial complex of dimension m, then $\mathcal{T}(T)$ and $\mathcal{M}(\mathcal{T}, T)$ are simplicial complexes of dimension m. If additionally $\mathcal{U}$ is a subcomplex of $\mathcal{T}$ of dimension $m-1$, then $\mathcal{N}(\mathcal{T}, \mathcal{U}, F)$ is a simplicial complex of dimension $m-1$.

Remark II.1.4.
We use micropatches to describe the local combinatorial structure of $\mathcal{T}$ relative to $\mathcal{U}$. For example, suppose that $\mathcal{T}$ triangulates a topological manifold M with boundary and that $\mathcal{U}$ triangulates a part of the boundary of that manifold. If $V \in \mathcal{T}^0$ is vertex in the interior of M, then $\mathcal{M}(\mathcal{T}, V)$ is the simplicial ball around V, and $\mathcal{N}(\mathcal{T}, \mathcal{U}, V)$ triangulates the boundary of that simplicial ball; here $\mathcal{N}(\mathcal{T}, \mathcal{U}, V)$ does not depend on $\mathcal{U}$. If V is instead a vertex at the boundary, then $\mathcal{M}(\mathcal{T}, V)$ triangulates a simplicial ball that contains V in its boundary. The simplicial complex $\mathcal{N}(\mathcal{T}, \mathcal{U}, V)$ triangulates a part of the boundary of the simplicial ball; it contains those simplices that do not contain V, and in addition those that are contained in $\mathcal{U}$. In this example, the subcomplex $\mathcal{U}$ enters the definition of $\mathcal{N}(\mathcal{T}, \mathcal{U}, V)$ only for boundary vertices. The micropatch $\mathcal{M}(\mathcal{T}, F)$ and its subcomplex $\mathcal{N}(\mathcal{T}, \mathcal{U}, F)$ appear in our discussion of discrete distributional differential forms.

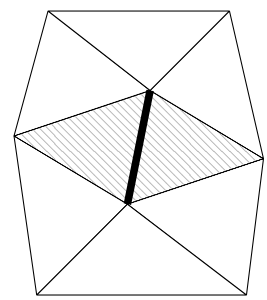

Figure II.1: Illustration of macropatch and micropatch in a triangulation. The macropatch around the thick edge segment is displayed. The hatched area indicates the micropatch of the edge.

II.2. Regularity of Simplices

In this section we introduce regularity criteria for simplices and relations with quantities of interest. We express the shape regularity of a simplex both in geometric terms and in terms of linear algebra. The central notions of this section are the geometric shape measure $\mu(T)$ (see (II.9)) of a simplex T and its relation to the generalized condition number of associated matrices (see (II.21)).

Let $m, n \in \mathbb{N}_0$ with $m \leq n$. Let $T \subset \mathbb{R}^n$ be an m-simplex with vertices $v_0, v_1, \dots, v_m \in \mathbb{R}^n$. We let $\operatorname{diam}(T)$ denote its diameter, and we let $\operatorname{vol}^m(T)$ denote its m-dimensional volume. We observe that $\operatorname{diam}(T)$ is the largest distance between two vertices of T, i.e., the length of the longest edge of T. If T is a vertex, i.e., if

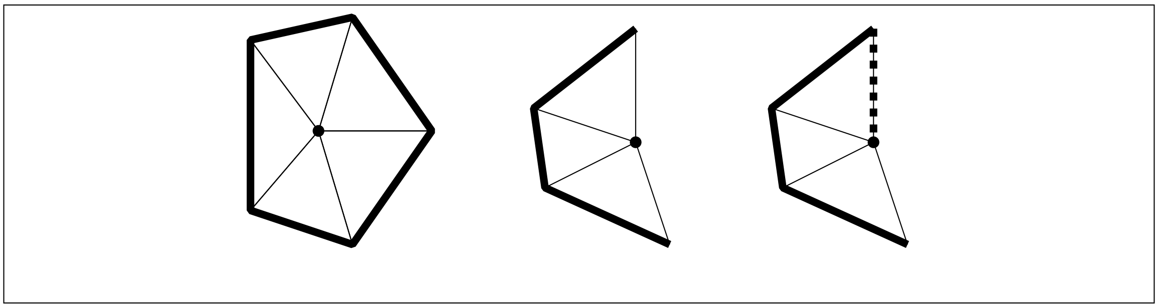

Figure II.2: Illustration of micropatches and micropatch boundaries of a triangulation $\mathcal{T}$ which triangulates a two-dimensional domain. $\mathcal{U}$ is assumed to triangulate a part of the boundary. *Left*: a micropatch around an interior vertex of a triangulation. The thick line indicates the micropatch boundary. *Middle*: a micropatch around a boundary vertex that is not contained in $\mathcal{U}$. Only the boundary edges not adjacent to the vertex are the micropatch boundary. *Right*: a micropatch around the same boundary vertex when the micropatch contains parts of $\mathcal{U}$, which is indicated in dashed lines. The micropatch boundary is the dashed and non-dashed thick lines.

$m = 0$, then $\operatorname{diam}(T) = 0$ and $\operatorname{vol}^0(T) = 1$.

Throughout this section, we assume that T is a simplex of positive dimension, so $1 \leq m$. But we may explicitly mention whenever a result extends formally to the case of zero-dimensional T. The convention $0^0 = 1$ will be useful in this regard.

First we introduce the shape measure of a simplex in purely geometric terms. Assume that T has positive dimension m. For $0 \leq i \leq m$ we let $F_i^T \in \Delta(T)^{m-1}$ denote the face of T opposite to the vertex v_i, and we let h_i^T denote the *height* of F_i^T over v_i, which is the distance of v_i from the affine closure of F_i^T. Applying Cavalieri's principle, we easily find

$$\operatorname{vol}^m(T) = \frac{h_i^T}{m} \operatorname{vol}^{m-1}(F_i). \tag{II.7}$$

Recursive application of this identity gives the estimate

$$\operatorname{vol}^m(T) \leq \frac{\operatorname{diam}(T)^m}{m!}, \tag{II.8}$$

which bounds the volume of T in terms of its diameter. The *geometric shape measure* $\mu(T)$ of T,

$$\mu(T) := \frac{\operatorname{diam}(T)^m}{\operatorname{vol}^m(T)}, \tag{II.9}$$

measures in how far the reverse inequality holds. The idea is that simplices with low geometric shape measure have "good quality" whereas a high geometric shape measure indicates "bad quality".

We note that the geometric shape measure (II.9) of a vertex is well-defined due to $0^0 = 1$. We observe that $\mu(T) = 1$ for simplices of dimension 0 or 1. Generally, (II.8) implies $\mu(T) \geq m!$ as a lower bound.

A simple application of this notion is relating geometric properties of subsimplices of T to the corresponding properties of T itself. By (II.7) we have

$$h_i^T = m \cdot \frac{\operatorname{vol}^m(T)}{\operatorname{vol}^{m-1}(F_i^T)}. \tag{II.10}$$

Using (II.10), (II.9), and (II.8), we obtain a lower bound for h_i^T by

$$\begin{aligned} h_i^T &\geq \frac{m}{\mu(T)} \cdot \frac{\operatorname{diam}(T)^m}{\operatorname{vol}^{m-1}(F_i^T)} && \text{(II.11)} \\ &\geq \frac{m!}{\mu(T)} \cdot \frac{\operatorname{diam}(T)^m}{\operatorname{diam}(F_i^T)^{m-1}} \geq \frac{m!}{\mu(T)} \cdot \operatorname{diam}(T). && \text{(II.12)} \end{aligned}$$

This formalizes that non-degenerate simplices have heights comparable to the simplex diameter.

The diameter of T is an upper bound for the diameter of any subsimplex of T. A converse estimate involves $\mu(T)$ and is a direct consequence of (II.11). For $0 \leq j \leq m$ with $i \neq j$ we have $\|v_i - v_j\| \geq h_i$. Hence for $S \in \Delta(T)$ with positive dimension $\dim S > 0$ we find

$$\operatorname{diam}(S) \geq \frac{m!}{\mu(T)} \operatorname{diam}(T). \tag{II.13}$$

This formalizes that for non-degenerate simplices, the diameter of each subsimplex of positive dimension is comparable to the diameter of the whole simplex.

The geometric shape measure of a simplex T bounds the geometric shape measure of its subsimplices. More precisely, by (II.7) we have

$$\mu(F_i^T) = \frac{\operatorname{diam}(F_i^T)^{m-1}}{\operatorname{vol}^{m-1}(F_i^T)} \leq \frac{h_i^T}{m} \cdot \frac{\operatorname{diam}(T)^{m-1}}{\operatorname{vol}^m(T)} \leq \frac{1}{m} \cdot \mu(T). \tag{II.14}$$

An iteration of (II.14) shows for $0 \leq p \leq m$ and $S \in \Delta(T)^p$ that

$$\mu(S) \leq \frac{p!}{m!} \mu(T). \tag{II.15}$$

A converse inequality does not hold in general, as we easily see from considering a generic triangle.

We have quantified the shape quality of T in geometric terms with $\mu(T)$. Another access to the shape quality of T opens through linear algebra. We assume that $0 \leq i \leq m$ is an index of an arbitrary vertex of T and that $M \in \mathbb{R}^{n \times m}$ is the matrix

$$M = \Big(v_1 - v_i \; \Big| \; \cdots \; \Big| \; v_{i-1} - v_i \; \Big| \; v_{i+1} - v_i \; \Big| \; \cdots \; \Big| \; v_m - v_i \Big).$$

We let $\|M\|_{mc}$ denote the maximum of the ℓ^2 norm of the columns of M. We let $\|M\|_{2,2}$ denote the classical ℓ^2 operator norm of M. An elementary computation shows that

$$\|M\|_{mc} \leq \|M\|_{2,2} \leq m \|M\|_{mc}. \tag{II.16}$$

The triangle inequality implies

$$\|M\|_{mc} \leq \operatorname{diam}(T) \leq 2\|M\|_{mc}. \tag{II.17}$$

We note from this in particular that

$$\frac{1}{m}\|M\|_{2,2} \leq \operatorname{diam}(T) \leq 2\|M\|_{2,2}. \tag{II.18}$$

Hence the diameter of T is comparable to matrix norms of M.

We let $\sigma_1(M), \dots, \sigma_m(M)$ denote the singular values of M in ascending order. We also write $\sigma_{\min}(M) = \sigma_1(M)$ and $\sigma_{\max}(M) = \sigma_m(M)$. Note that $\sigma_{\min}(M) > 0$. It is well-known that

$$\|M\|_{2,2} = \sigma_{\max}(M), \quad \|M^\dagger\|_{2,2} = \sigma_{\min}(M)^{-1}, \tag{II.19}$$

where $M^\dagger$ denotes the Moore-Penrose pseudoinverse of M.

One can show that

$$\operatorname{vol}(T) = \frac{1}{m!}\prod_{i=1}^{m}\sigma_i(M). \tag{II.20}$$

In combination, we observe that

$$\frac{\operatorname{diam}(T)^m}{\operatorname{vol}^m(T)} \leq m!2^m \frac{\sigma_{\max}(M)^m}{\prod_{i=1}^m \sigma_i(M)} \leq m!2^m \left(\frac{\sigma_{\max}(M)}{\sigma_{\min}(M)}\right)^{m-1},$$

and conversely

$$\frac{\operatorname{diam}(T)^m}{\operatorname{vol}^m(T)} \geq \frac{m!}{m^m}\frac{\sigma_{\max}(M)^m}{\prod_{i=1}^m \sigma_i(M)} \geq \frac{m!}{m^m}\frac{\sigma_{\max}(M)}{\sigma_{\min}(M)}.$$

The *generalized condition number* $\kappa(M)$ of M is the quantity

$$\kappa(M) = \sigma_{\max}(M)/\sigma_{\min}(M).$$

The central observation is that

$$\frac{m!}{m^m}\kappa(M) \leq \mu(T) \leq m!2^m\kappa(M)^{m-1}, \tag{II.21}$$

which relates the geometric shape measure of T to the generalized condition number of M.

Remark II.2.1.
Different *shape measures* are used throughout the literature of numerical analysis and computational geometry to quantify the quality of simplices (see [78, 133] for overviews). Another shape measure that is commonly used in finite element literature is the ratio of the diameter and the largest inscribed circle of a simplex (see [37, p.97, Definition (4.2.16)], [32, p.61, Definition 5.1]). Our notion of geometric shape measure equals what is known as *fatness* in differential geometry [50] and is precisely the reciprocal of the *fullness* discussed by Whitney [180]. The *thickness* of a simplex is the ratio of its smallest height to its diameter [146].

Remark II.2.2.
Practical applications have been driving the research on shape regularity for a long time with almost exclusive attention to triangles and tetrahedra (see, e.g., [35, 36, 78, 133]). Considering shape regularity in arbitrary dimension is not only a purely mathematical musing but is also motivated by the emergence of (four-dimensional) space-time methods in recent years (see, e.g., [147, 169]).

II.3. Solid Angles

We dedicate a section to the discussion of solid angles of simplices. We relate the solid angles to the geometric shape measure. Solid angles can be regarded as higher-dimensional generalizations of the classical angle between two vectors. The main result of this section can be paraphrased as follows: *non-degenerate simplices do not have small solid angles.*

Let $n \in \mathbb{N}$. Assume that $T \subset \mathbb{R}^n$ is an n-dimensional simplex. The *solid angle* of T at vertex v_i is the limit

$$\sphericalangle_i(T) := \lim_{\epsilon \to 0} \frac{\operatorname{vol}^n\left(B_\epsilon(v_i) \cap T\right)}{\operatorname{vol}^n\left(B_\epsilon(v_i)\right)}. \tag{II.22}$$

It is easy to see that this limit assumes a constant for $\epsilon > 0$ small enough. For example, the n-dimensional reference simplex Δ_n has a solid angle of 2^{-n} at the origin.

Remark II.3.1.
In the special case $n = 2$, the solid angle coincides with the classical two-dimensional angle of the triangle at vertex v_i when measured as the ratio of radians over 2π. One of the earliest studies of higher-dimensional solid angles was conducted by Euler [85]. Beyond dimension two, their theory is considerably more complex, and very few results are known beyond dimension three (see [81, 157]).

For the purposes of this thesis, the following result gives helpful upper and lower bounds for the solid angle.

Lemma II.3.2.
Let $T = \operatorname{convex}\{v_0, v_1, \dots, v_n\}$ be an n-simplex in $\mathbb{R}^n$. Let $M \in \mathbb{R}^{n \times n}$ denote the matrix with columns $v_1 - v_0, \dots, v_n - v_0$. Then

$$\frac{|\det(M)|}{2^n \sigma_{\max}^n} \leq \sphericalangle_i(T) \leq \frac{|\det(M)|}{2^n \sigma_{\min}^n}. \tag{II.23}$$

Proof. Let $Q = \left(\mathbb{R}_0^+\right)^n$ denote the non-negative quadrant. It is well-known that the radially symmetric function $f(x) = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\|x\|^2\right)$ has unit integral over $\mathbb{R}^n$. By the law of substitution and radial symmetry, we conclude that for every $\sigma > 0$ we have

$$\int_Q \frac{\sigma^n}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{\sigma^2}{2}\|x\|^2\right) \,\mathrm{d}x = 2^{-n}.$$

Next, since f is radially symmetric and has unit integral, we see

$$\sphericalangle_i(T) = \int_{M(Q)} (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\|x\|^2\right) \,\mathrm{d}x.$$

Combining these observations, we obtain

$$\begin{aligned}\sphericalangle_i(T) &= \int_Q (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\langle Mx, Mx\rangle\right) \cdot |\det(M)| \,\mathrm{d}x \\ &\geq \int_Q (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{\sigma_{\max}^2}{2}\|x\|^2\right) \cdot |\det(M)| \,\mathrm{d}x = \frac{|\det(M)|}{2^n\sigma_{\max}^n}.\end{aligned}$$

This proves the lower bound of (II.23). Analogously,

$$\begin{aligned}\sphericalangle_i(T) &= \int_Q (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\langle Mx, Mx\rangle\right) \cdot |\det(M)| \,\mathrm{d}x \\ &\leq \int_Q (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{\sigma_{\min}^2}{2}\|x\|^2\right) \cdot |\det(M)| \,\mathrm{d}x = \frac{|\det(M)|}{2^n\sigma_{\min}^n}\end{aligned}$$

provides the upper bound. The proof is complete. □

We combine Lemma II.3.2 with (II.21) to derive bounds for the solid angle in terms of the geometric shape measure. Let T be the simplex with vertices $v_0, \dots, v_n$ and let $M \in \mathbb{R}^{n\times n}$ be the matrix with columns $v_1 - v_0, \dots, v_n - v_0$. By the definition of the generalized condition number we find

$$2^{-n}\kappa(M)^{-n+1} \leq \sphericalangle_i(T) \leq 2^{-n}\kappa(M)^{n-1}. \tag{II.24}$$

Combining this with (II.21) gives

$$2^{-n}\left(\frac{n^n}{n!}\right)^{1-n}\mu(T)^{1-n} \leq \sphericalangle_i(T) \leq 2^{-n}\left(\frac{n^n}{n!}\right)^{n-1}\mu(T)^{n-1} \tag{II.25}$$

Consequently, we have lower and upper bounds for the solid angles of a simplex in terms of its geometric shape measure.

Remark II.3.3.
Solid angles are generally bounded above by 1 in contrast to the geometric shape measure of a simplex, so it is intuitive that (II.25) does not give a sharp bound for the solid angle. The lower bound, however, shows that "good" simplices do not have small solid angles.

It is intuitive that the reciprocal of the minimum solid angle of a simplex also bounds the geometric shape measure. This has been proven for low dimensions (see [133]). But for the higher-dimensional case, no proof seems to be available in the literature.

Example II.3.4.
An equilateral triangle T with unit side lengths has a volume of $\sqrt{3}/4$, and so $\mu(T) = 4/\sqrt{3}$. Its solid angles are all $1/6$. Inequality (II.25) gives the (trivial) upper bound $2/\sqrt{3} > 1$ and the (non-trivial, non-sharp) lower bound $\sqrt{3}/32$ for the solid angles.

II.4. Regularity of Triangulations

We have discussed regularity criteria for single simplices. In this section we extend that discussion to regularity criteria of entire triangulations and eventually families of meshes. Here, the central notion is the geometric shape measure $\mu(\mathcal{T})$ of a triangulation. Its applications are discussed in Lemmas II.4.1, II.4.3, and II.4.6. These results require stronger conditions on the triangulations, such as being the triangulation of a full-dimensional subset of $\mathbb{R}^n$, as can be seen by simple counterexamples.

Throughout this section, we let $\mathcal{T}$ be an n-dimensional simplicial complex. The *geometric shape measure* $\mu(\mathcal{T})$ of the triangulation $\mathcal{T}$ is defined as

$$\mu(\mathcal{T}) := \sup_{T \in \mathcal{T}} \mu(T), \tag{II.26}$$

and bounds the degeneracy of simplices in $\mathcal{T}$. In the remainder of this section, we show how $\mu(\mathcal{T})$ quantifies further properties of the triangulation.

First, we bound the local combinatorial complexity of the mesh in terms of its geometric shape measure. Formally, we define the quantity $\mu_{\mathrm{N}}(\mathcal{T})$ as

$$\mu_{\mathrm{N}}(\mathcal{T}) := \max_{T \in \mathcal{T}} \# \{ \, S \in \mathcal{T} \mid S \cap T \neq \emptyset \, \} \,. \tag{II.27}$$

This quantity bounds the number of simplices adjacent to a given simplex. We prove that $\mu_{\mathrm{N}}(\mathcal{T})$ can be bounded in terms of the geometric shape measure.

Lemma II.4.1.
Assume that $\mathcal{T}$ triangulates an n-dimensional topological manifold in $\mathbb{R}^n$. Then

$$\mu_{\mathrm{N}}(\mathcal{T}) \leq 2^{n+1}(n+1) \cdot \left\lceil 2^n \left(\frac{n^{\frac{n}{2}}}{n!} \right)^{n-1} \right\rceil \mu(\mathcal{T})^{n-1}. \tag{II.28}$$

Proof. Consider the special case that $V \in \mathcal{T}^0$ is a vertex of $\mathcal{T}$. Then the solid angles of the n-simplices adjacent to V satisfy the lower bound in (II.25). By the additivity of the Hausdorff measure, we obtain the upper bound

$$\# \{ \, T \in \mathcal{T}^n \mid V \in \Delta(T) \, \} \leq \left\lceil 2^n \left(\frac{n^{\frac{n}{2}}}{n!} \right)^{n-1} \right\rceil \mu(T)^{n-1}.$$

More generally, let $T \in \mathcal{T}$. We recall that T has at most $n+1$ vertices, and that every simplex $S \in \mathcal{T}$ adjacent to T has at least one vertex in common with T. Furthermore, every simplex in $\mathcal{T}$ has at most 2^{n+1} subsimplices. The claim follows. □

Remark II.4.2.
The upper bound in Lemma II.4.1 is generally not sharp. It is easy to see that, if $\mathcal{T}$ triangulates a manifold of positive codimension in $\mathbb{R}^n$, then the quantity $\mu_{\mathrm{N}}(\mathcal{T})$ can generally not be bounded in terms of $\mu(\mathcal{T})$ alone.

The second class of inequalities that we consider relates to the question in how far adjacent simplices in $\mathcal{T}$ have comparable diameters. As a preparation, if $T \in \mathcal{T}$ is a simplex of positive dimension, then we set $h_T := \operatorname{diam}(T)$, and if instead $V \in \mathcal{T}^0$ is a vertex of $\mathcal{T}$, then we define

$$h_V := \inf \left\{ h_E \;\middle|\; E \in \mathcal{T}^1,\ V \in \Delta(E)^0 \right\}$$

as the infimum length of the edges of $\mathcal{T}$ adjacent to V. Now, the *local quasi-uniformity constant* $\mu_{\mathrm{lqu}}(\mathcal{T})$ of $\mathcal{T}$ is defined as

$$\mu_{\mathrm{lqu}}(\mathcal{T}) := \sup \left\{ \frac{h_T}{h_S} \;\middle|\; S, T \in \mathcal{T} : S \cap T \neq \emptyset \right\}. \tag{II.29}$$

This can be bounded in terms of the mesh quality.

Lemma II.4.3.
Assume that $\mathcal{T}$ triangulates a topological manifold of dimension at least 2. Then

$$\mu_{\mathrm{lqu}}(\mathcal{T}) \leq \mu(\mathcal{T})^{\mu_{\mathrm{N}}(\mathcal{T})}. \tag{II.30}$$

Proof. Let $T \in \mathcal{T}$ have positive dimension. For $S \in \Delta(T)$ with positive dimension we have $h_S/h_T \leq 1$ and $h_T/h_S \leq \mu(\mathcal{T})$, as follows from a simplification of (II.13). Next, let $T' \in \mathcal{T}$ be adjacent to T and write $S := T \cap T'$. If S has positive dimension, then we observe $h_T/h_{T'} = h_T/h_S \cdot h_S/h_{T'} \leq \mu(\mathcal{T})$. More generally, if $T' \in \mathcal{T}(T)$ has positive dimension and $V \in \mathcal{T}^0$ is a common vertex of T and T', then there exists $L \in \mathbb{N}$ and a finite sequence $T_0, \dots, T_L \in \mathcal{T}$ such that $T_0 = T$ and $T_L = T'$ and such that for all $l \in [1 : L]$ we have $V \in \Delta(T_l)$ and $T_{l-1} \cap T_l$ is a simplex of positive dimension. The existence of such a sequence follows from the assumption that $\mathcal{T}$ triangulates a topological manifold. We have $L \leq \mu_{\mathrm{N}}(\mathcal{T})$, and thus $h_T/h_{T'} \leq \mu(\mathcal{T})^{\mu_{\mathrm{N}}(\mathcal{T})}$ is easily verified. Lastly, if $V \in \Delta(T)^0$, then there exists an edge $E \in \mathcal{T}(T)^1$ such that $h_V = h_E$, and we can simply apply the previous observations. The proof is complete. □

Remark II.4.4.
The previous proof has explicitly used that $\mathcal{T}$ triangulates a manifold. We show that this is generally necessary. Consider a triangulation consisting of two triangles that only meet at one vertex. This triangulation does not triangulate a topological manifold. If we iteratively refine one of the triangles in a uniform manner while not changing the other triangle, then the local quasi-uniformity constants of that sequence of triangulations diverge to infinity although the quality of simplices in the triangulation remains the same. See also Figure II.3 for an illustration. Moreover, the statement does not apply if $\dim M = 1$ as is easily verified.

Remark II.4.5.
The definition of h_V enables us to assign a "length" to each vertex $V \in \mathcal{T}^0$. The important property is that h_V is comparable to the local mesh size. We could have chosen slightly different definitions, such as the average or maximum diameter of edges adjacent to V, and that would serve our purpose equally. Associating a local mesh size to vertices has been a useful formalism in several publications (e.g., [34])

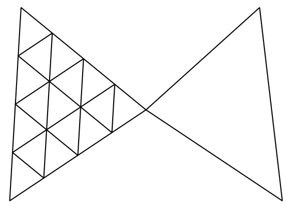

Figure II.3: Two triangles that do not triangulate a manifold. If the left triangle is subject to iterated uniform refinement, then the produced triangles do not degenerate, but the length between adjacent triangles become incomparable.

Our third quantity considers the diameter of neighborhoods of a simplex. We will quantify in terms of the geometric shape measure in how far local patches around a simplex $T \in \mathcal{T}$ contain an Euclidean neighborhood of T whose size is proportional to the diameter of T. Specifically, we define the *neighborhood constant* as

$$\mu_{\mathrm{r}}(\mathcal{T}) := \sup_{T \in \mathcal{T}} \sup \left\{ r > 0 \;\middle|\; \begin{array}{l} B_{r \cdot \mathrm{diam}(T)}(T) \cap M \text{ is contained in the} \\ \text{interior of } [\mathcal{T}(T)] \text{ relative to } M. \end{array} \right\}. \tag{II.31}$$

We use the following bound on the neighborhood constant.

Lemma II.4.6.
Let $\mathcal{T}$ be a triangulation. Then

$$\mu_{\mathrm{r}}(\mathcal{T}) \leq \frac{n!}{2n^{\frac{n+1}{2}} \mu_{\mathrm{lqu}}(\mathcal{T}) \mu(\mathcal{T})}.$$

Proof. Let $m \in \mathbb{N}$. For $0 \leq l \leq m-1$ we let S^m_l be the smallest subsimplex of Δ_m that contains the origin and the first l unit vectors, and let F^m_l be the smallest subsimplex of Δ_m containing the remaining $m-l$ unit vectors. It is evident that F^m_0, the face of Δ_m opposite to the origin, has distance $1/\sqrt{m}$ from the origin. More generally, the distance of S^m_l and F^m_l is $1/\sqrt{m-l}$.

Let $T \in \mathcal{T}$ and let $T' \in \mathcal{T}(T)^n$ with $T \neq T'$. We denote the vertices of T' by $v_0, \dots, v_n$ and assume that $v_0, \dots, v_l$ are precisely those vertices that T' has in common with T. We let $\varphi : \Delta_n \to T'$ be the unique affine mapping that maps the origin to v_0 and the unit vectors $e_i \in \mathbb{R}^n$ to v_i for $1 \leq i \leq n$. We may write $\varphi(x) = Mx + v_0$, where the i-th column of $M \in \mathbb{R}^{n \times n}$ is $v_i - v_0$. Consequently,

$$\varphi(S^n_l) = \mathrm{convex}\{v_0, \dots, v_l\}, \quad \varphi(F^n_l) = \mathrm{convex}\{v_{l+1}, \dots, v_m\}.$$

By (II.18) we see that the distance between these two sets is at least $\sigma_{\min}(M)/\sqrt{n}$, where $\sigma_{\min}(M) \geq \kappa(M)^{-1}\mathrm{diam}(T')/2$. We recall $\kappa(M) \leq \frac{n^{\frac{n}{2}}}{n!}\mu(\mathcal{T})$ from (II.21). Furthermore, we have $\mathrm{diam}(T') \geq \mu_{\mathrm{lqu}}(\mathcal{T})^{-1}\mathrm{diam}(T)$.

This implies that a closed r-neighborhood of $\varphi(S^n_l)$ has positive distance from $\varphi(F^n_l)$, where

$$r < \frac{n!}{2n^{\frac{n+1}{2}} \mu_{\mathrm{lqu}}(\mathcal{T}) \mu(\mathcal{T})} \mathrm{diam}(T)$$

This implies in particular that an r-neighborhood of T in $[\mathcal{T}(T)]$ is compactly contained in $[\mathcal{T}(T)]$. $\square$

Remark II.4.7.
The constant $\mu_{\mathrm{r}}(\mathcal{T})$ has the following alternative interpretation: for every $T \in \mathcal{T}$, the set $[\mathcal{T}(T)]$ contains the closed $\mu_{\mathrm{r}}(\mathcal{T})h_T$-environment around T with respect to the *inner path metric* of the space $[\mathcal{T}]$. The distance of two points $x, y \in [\mathcal{T}]$ with respect to the inner path metric is the infimum of the lengths of rectifiable paths in $[\mathcal{T}]$ between x and y. If this path metric and the Euclidean metric are comparable over $[\mathcal{T}]$, then it is possible to modify the definition of $\mu_{\mathrm{r}}(\mathcal{T})$ such that for every $T \in \mathcal{T}$, the set $[\mathcal{T}(T)]$ contains the Euclidean closed $\mu_{\mathrm{r}}(\mathcal{T})h_T$-environment around T. It is easy to see that the equivalence of the Euclidean metric and the local path metric depends on global properties of the metric space $[\mathcal{T}]$ which cannot be "felt" by the local geometry of $\mathcal{T}(T)$. The set $[-1,1]^2 \setminus ([0,1] \times \{0\})$ is a compact space where these two metrics are not equivalent.

Remark II.4.8.
In applications we typically consider families $(\mathcal{T}_h)_h$ of simplicial complexes that triangulate a fixed topological manifold, where h ranges over some set of indices, typically the mesh size. In this remark we relate the results above, which consider regularity criteria for single fixed simplicial complexes, to regularity criteria for such families of simplicial complexes, which can be found in finite element literature (e.g. [37, Definition (4.4.13)], [32, Definition 5.1]).

Let $(\mathcal{T}_h)_h$ be a family of simplicial complexes. We call $(\mathcal{T}_h)_h$ *shape-uniform* if the geometric shape measures $(\mu(\mathcal{T}_h))_h$ satisfy a uniform upper bound. For example, if a sequence of simplicial complexes is constructed from an initial simplicial complex $\mathcal{T}_0$ by successively applying local refinement of simplices via newest-vertex-bisection, then the resulting family is shape-uniform (see Maubauch [135]). In two and three dimensions one can implement local mesh refinement alternatively with red-green refinement (see [13, 142]).

We call a family of simplicial complexes $(\mathcal{T}_h)_h$ *quasi-uniform* if their geometric shape measures $(\mu(\mathcal{T}))_h$ satisfy a uniform upper bound and if additionally

$$\sup_h \sup_{S,T \in \mathcal{T}_h} \frac{h_T}{h_S} < \infty. \tag{II.32}$$

This means that the simplices have comparable diameters in each single simplicial complex $\mathcal{T}_h$. For example, if a sequence of simplicial complexes is constructed by successive global uniform refinement of an initial simplicial complex $\mathcal{T}_h$, then the resulting family is quasi-uniform (see Bey [26]).

II.5. Regularity of Reference Transformations

Scaling arguments are the most important application of measures of shape regularity in finite element theory. The basic idea is to transform between the local geometry and a reference geometry. The reference geometry should not depend on the concrete mesh, and the quality of the reference transformations should depend only on the mesh quality. We implement this idea for simplices, local patches, and micropatches. The main point of this section is to define reference transformations and establish their regularity. In this section, suppose that the simplicial complex

$\mathcal{T}$ triangulates a compact topological manifold.

For every simplex $T \in \mathcal{T}$ of dimension m there exists $M_T \in \mathbb{R}^{n\times m}$ and $b_T \in \mathbb{R}^n$ such that the affine mapping $\varphi_T(x) = M_T x + b_T$ maps the reference simplex Δ_m onto the simplex T. This implies that $b_T \in \operatorname{Ver}(T)$ is a vertex of T and that the edge vectors from b_T to the remaining vertices of T constitute the columns of M_T. Consequently, the results of Section II.2 apply and we can relate the generalized condition number $\kappa(M_T)$ to the geometric shape measure of T by (II.21). We call such an affine mapping φ_T a *reference transformation* of T.

In general, there are up to $(n+1)!$ different reference transformations of a simplex. Henceforth we fix a reference transformation φ_T for each simplex T. With a slight abuse of notation, we identify this affine mapping with its restriction to the reference m-simplex

$$\varphi_T : \Delta_m \to T. \tag{II.33}$$

This is a diffeomorphism of manifolds with corners (see [127]) whose Jacobian $\mathrm{D}\,\varphi_T$ is constant. We define

$$\kappa(\mathcal{T}) := \sup_{T\in\mathcal{T}} \kappa\left(\mathrm{D}\,\varphi_T\right). \tag{II.34}$$

Via (II.18) and (II.19) we easily find

$$\sigma_{\max}\left(\mathrm{D}\,\varphi_T\right) \leq n\operatorname{diam}(T), \quad \sigma_{\min}\left(\mathrm{D}\,\varphi_T\right)^{-1} \leq 2\kappa\left(\mathcal{T}\right)\operatorname{diam}(T)^{-1}, \tag{II.35}$$

$$\kappa\left(\mathcal{T}\right) \leq \frac{n^n}{n!}\mu(\mathcal{T}). \tag{II.36}$$

In particular, if the triangulation $\mathcal{T}$ has a low geometric shape measure, then the singular values of all reference transformations are comparable to the diameters of the associated simplices.

We now attend to the construction of reference patches. We first construct a reference patch and a reference transformation for the macropatch $\mathcal{T}(T)$ around a simplex $T \in \mathcal{T}$.

We observe that for every finite triangulation the macropatch $\mathcal{T}(T)$ has one of finitely many combinatorial structures, the number of which can be bounded in terms of $\mu_{\mathrm{N}}(\mathcal{T})$. Thus there exist a finite number of simplicial complexes $\mathcal{S}_1, \dots, \mathcal{S}_N$ in $\mathbb{R}^n$ such that for every $T \in \mathcal{T}$ there exists $1 \leq i_T \leq N$ such that $\mathcal{T}(T)$ has the same combinatorial structure as $\mathcal{S}_{i_T}$. We write $\widehat{\mathcal{T}}(T) := \mathcal{S}_{i_T}$ in the sequel and call this the *reference macropatch*.

Consequently, there exists a homeomorphism

$$\Phi_T : \left[\widehat{\mathcal{T}}(T)\right] \to \left[\mathcal{T}(T)\right] \tag{II.37}$$

such that the restriction $\Phi_{T|S}$ to $S \in \widehat{\mathcal{T}}(T)$ is affine. We conclude that the Jacobians $\mathrm{D}\,\Phi_T$ and $\mathrm{D}\,\Phi_T^{-1}$ exist almost everywhere, and that

$$\kappa\left(\mathrm{D}\,\Phi_{T|S}\right) \leq \mu_\Phi \cdot \mu(\mathcal{T}), \quad S \in \widehat{\mathcal{T}}(T), \tag{II.38}$$

where μ_Φ can be bounded in terms of n, $\mu_{\mathrm{N}}(\mathcal{T})$, and $\kappa(\mathcal{T})$.

In a similar manner, we construct the *reference micropatch* $\widehat{\mathcal{M}}(\mathcal{T}, F)$ of $F \in \mathcal{T}$ and define the reference mapping

$$\Psi_F : \left[\widehat{\mathcal{M}}(\mathcal{T}, F)\right] \to [\mathcal{M}(\mathcal{T}, F)]. \tag{II.39}$$

This homeomorphism restricts to a diffeomorphism on each simplex. We have

$$\kappa\left(\mathrm{D}\,\Psi_{F|S}\right) \leq \mu_\Psi \cdot \mu(\mathcal{T}), \quad S \in \widehat{\mathcal{M}}(\mathcal{T}, F) \tag{II.40}$$

where μ_Ψ can be bounded in terms of n, $\mu_{\mathrm{N}}(\mathcal{T})$, and $\kappa(\mathcal{T})$. Additionally, we let $\widehat{\mathcal{N}}(\mathcal{T},\mathcal{U}, F) \subseteq \widehat{\mathcal{M}}(\mathcal{T}, F)$ denote the simplicial subcomplex which Ψ_F maps onto $\mathcal{N}(\mathcal{T},\mathcal{U}, F)$. We call $\widehat{\mathcal{N}}(\mathcal{T},\mathcal{U}, F)$ the *reference micropatch boundary* of F.

II.6. Chain Complexes

We finish this chapter with a discussion of simplicial complexes from the point of view of algebraic topology. We refer to specialized literature (e.g., [31, 93, 122, 126, 127, 168]) for further background. While most of this section covers standard material, it also contains some concepts particular to this thesis.

To begin with, we introduce the notion of orientation of simplices. This can be done in different ways; we use a purely combinatorial definition here, which is equivalent to the notion of orientation of manifolds (with corners) known in differential geometry (see Lee [127]).

Let $S \subseteq \mathbb{R}^n$ be a simplex of dimension $m = \dim S$. If $m = 0$, then we define an *orientation* over S as a choice of sign in $\{-1, 1\}$. If $m > 0$, then an *orientation* of S is defined as an equivalence class of enumerations of the vertices of S, where two enumerations are considered equivalent if they can be transferred into each other by a permutation of positive sign. An *oriented simplex* is a simplex equipped with a choice of orientation.

If $S = \{v_0\}$ is a vertex, then we let $[v_0]$ denote the oriented vertex with positive orientation. If S has positive dimension m, and $\rho : [0 : m] \to \mathrm{Ver}(S)$ is an enumeration of the vertices of S, then we write $[\rho(0), \dots, \rho(m)]$ for the oriented m-simplex with vertices $\mathrm{Ver}(S)$ and the orientation induced by ρ.

If $[v_0, \dots, v_m]$ is an oriented m-simplex, then we write $-[v_0, \dots, v_m]$ for the oriented m-simplex with the same vertices but the opposite orientation. For every permutation $\pi \in \mathrm{Perm}(0 : m)$ we have

$$\left[v_{\pi(0)}, \dots, v_{\pi(m)}\right] = \mathrm{sgn}(\pi)[v_0, \dots, v_m]. \tag{II.41}$$

Lastly, if $S \in \mathcal{T}$ with $\mathrm{Ver}(S) = \{v_0, \dots, v_m\}$, then we say that the orientation of the oriented simplex $[v_0, \dots, v_m]$ induces the orientation of the oriented subsimplex $(-1)^i[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_m]$.

Often we understand each simplex with a fixed orientation. In that case we may identify a simplex S with the corresponding oriented simplex in order to simplify the notation. If $S \in \mathcal{T}^m$ and $F \in \Delta(S)^{m-1}$, then we let $o(F,S) = 1$ in the case that the fixed orientation of S induces the fixed orientation of F, and let $o(F,S) = -1$ in the contrary case.

Let $\mathcal{T}$ be a simplicial complex in $\mathbb{R}^n$. For $m \in \mathbb{Z}$, the space of *simplicial m-chains* of $\mathcal{T}$ is the real vector space $\mathcal{C}_m(\mathcal{T})$ generated by the oriented m-simplices $[v_0,\dots,v_m]$ with $\{v_0,\dots,v_m\} \in \mathcal{T}^m$, where we make the identification

$$[v_{\pi(0)},\dots,v_{\pi(m)}] = \operatorname{sgn}(\pi)[v_0,\dots,v_m], \quad \pi \in \operatorname{Perm}(0:m).$$

Note that the set $\mathcal{T}^m$ is empty for negative m or $m > n$, in which case $\mathcal{C}_m(\mathcal{T})$ is the zero vector space. The *simplicial boundary operator*

$$\partial_m : \mathcal{C}_m(\mathcal{T}) \to \mathcal{C}_{m-1}(\mathcal{T}) \tag{II.42}$$

is the linear operator that is defined by setting

$$\partial_m[v_0,\dots,v_m] := \sum_{i=0}^{m}(-1)^i[v_0,\dots,v_{i-1},v_{i+1},\dots,v_m], \quad [v_0,\dots,v_m] \in \mathcal{C}_m(\mathcal{T})$$

and taking the linear extension. The following observation is fundamental.

Lemma II.6.1.
Let $m \in \mathbb{Z}$. Then $\partial_{m-1}\partial_m S = 0$ for $S \in \mathcal{T}^m$.

Proof. It suffices to consider the case $m \geq 2$. Fix $[v_0,\dots,v_m] \in \mathcal{C}_m(\mathcal{T})$. Then

$$\begin{aligned}
&\partial_{m-1}\partial_m[v_0,\dots,v_m] \\
&= \sum_{i=0}^{m}(-1)^i\partial_{m-1}[v_0,\dots,v_{i-1},v_{i+1},\dots,v_m] \\
&= \sum_{i=0}^{m}\sum_{j=0}^{i-1}(-1)^{i+j}\partial_{m-1}[v_0,\dots,v_{j-1},v_{j+1},\dots,v_{i-1},v_{i+1},\dots,v_m] \\
&\quad - \sum_{i=0}^{m}\sum_{j=i+1}^{m}(-1)^{i+j}\partial_{m-1}[v_0,\dots,v_{i-1},v_{i+1},\dots,v_{j-1},v_{j+1},\dots,v_m].
\end{aligned}$$

Rearranging the sum, we obtain

$$\begin{aligned}
&\partial_{m-1}\partial_m[v_0,\dots,v_m] \\
&= \sum_{0\leq j<i\leq m}(-1)^{i+j}\partial_{m-1}[v_0,\dots,v_{j-1},v_{j+1},\dots,v_{i-1},v_{i+1},\dots,v_m] \\
&\quad - \sum_{0\leq i<j\leq m}(-1)^{i+j}\partial_{m-1}[v_0,\dots,v_{i-1},v_{i+1},\dots,v_{j-1},v_{j+1},\dots,v_m] = 0.
\end{aligned}$$

By linear extension, the desired result follows. □

Remark II.6.2.
If a fixed orientation is understood for each simplex $S \in \mathcal{T}$, then these oriented

simplices constitute a canonical basis of $\mathcal{C}_m(\mathcal{T})$, and the boundary operator can be written as

$$\partial_m S = \sum_{F \in \Delta(S)^{m-1}} o(F,S)F, \quad S \in \mathcal{T}^m.$$

The boundary operator and the spaces of simplicial chains can be assembled into a differential complex, the *simplicial chain complex* of the triangulation $\mathcal{T}$,

$$\dots \xrightarrow{\partial_{m+1}} \mathcal{C}_m(\mathcal{T}) \xrightarrow{\partial_m} \mathcal{C}_{m-1}(\mathcal{T}) \xrightarrow{\partial_{m-1}} \dots \tag{II.43}$$

Let $\mathcal{U} \subseteq \mathcal{T}$ be a simplicial subcomplex of $\mathcal{T}$. This induces the corresponding spaces $\mathcal{C}_m(\mathcal{U})$ of simplicial chains and simplicial chain complex over $\mathcal{U}$,

$$\dots \xrightarrow{\partial_{m+1}} \mathcal{C}_m(\mathcal{U}) \xrightarrow{\partial_m} \mathcal{C}_{m-1}(\mathcal{U}) \xrightarrow{\partial_{m-1}} \dots \tag{II.44}$$

Since $\mathcal{C}_m(\mathcal{U})$ is a subspace of $\mathcal{C}_m(\mathcal{T})$, we may consider the quotient space

$$\mathcal{C}_m(\mathcal{T},\mathcal{U}) := \mathcal{C}_m(\mathcal{T})/\mathcal{C}_m(\mathcal{U}).$$

We call $\mathcal{C}_m(\mathcal{T},\mathcal{U})$ the space of *simplicial m-chains of $\mathcal{T}$ relative to $\mathcal{U}$*. Since

$$\partial_m \mathcal{C}_m(\mathcal{T}) \subseteq \mathcal{C}_{m-1}(\mathcal{T}), \quad \partial_m \mathcal{C}_m(\mathcal{U}) \subseteq \mathcal{C}_{m-1}(\mathcal{U})$$

we conclude that we have a well-defined operator

$$\partial_m : \mathcal{C}_m(\mathcal{T},\mathcal{U}) \to \mathcal{C}_{m-1}(\mathcal{T},\mathcal{U})$$

that satisfies the *differential property*

$$\partial_{m-1}\partial_m C = 0, \quad C \in \mathcal{C}_m(\mathcal{T},\mathcal{U}).$$

Remark II.6.3.
If orientations on the simplices in $\mathcal{T}$ are understood, then the canonical basis of $\mathcal{C}_m(\mathcal{T},\mathcal{U})$ is given by (the equivalence classes of) the m-simplices in $\mathcal{T}^m \setminus \mathcal{U}^m$.

Next we approach the homology theory of these differential complexes. We introduce the quotient spaces

$$\mathcal{H}_m(\mathcal{T},\mathcal{U}) := \frac{\ker\Big(\partial_m : \mathcal{C}_m(\mathcal{T},\mathcal{U}) \to \mathcal{C}_{m-1}(\mathcal{T},\mathcal{U})\Big)}{\operatorname{ran}\Big(\partial_{m+1} : \mathcal{C}_{m+1}(\mathcal{T},\mathcal{U}) \to \mathcal{C}_m(\mathcal{T},\mathcal{U})\Big)}. \tag{II.45}$$

We call $\mathcal{H}_m(\mathcal{T},\mathcal{U})$ the *m-th simplicial homology space of $\mathcal{T}$ relative to $\mathcal{U}$*. If $\mathcal{U} = \emptyset$, then we call

$$\mathcal{H}_m(\mathcal{T}) := \mathcal{H}_m(\mathcal{T},\emptyset)$$

the *m-th (absolute) simplicial homology space of $\mathcal{T}$*. The dimensions of the simplicial homology spaces,

$$b_m(\mathcal{T},\mathcal{U}) := \dim \mathcal{H}_m(\mathcal{T},\mathcal{U}), \quad b_m(\mathcal{T}) := \dim \mathcal{H}_m(\mathcal{T}),$$

are of particular interest. We call $b_m(\mathcal{T},\mathcal{U})$ the *m-th simplicial Betti number* of $\mathcal{T}$ relative to $\mathcal{U}$, and we call $b_m(\mathcal{T})$ the *m-th absolute simplicial Betti number* of $\mathcal{T}$. Note that $b_m(\mathcal{T}) = b_m(\mathcal{T},\emptyset)$.

The homology spaces of simplicial chain complexes reflect the topological features of the triangulated topological spaces. Assume that M is a topological space and that Γ is a topological subspace. The *m-th topological Betti number* $b_m(M,\Gamma)$ *of* M *relative to* Γ is defined as the dimension of the m-th singular homology space of M relative to Γ. We refer to [168, Chapter 4, Section 4] for the details of this concept. In the case $\Gamma = \emptyset$ we call $b_m(M) := b_m(M,\emptyset)$ the *m-th absolute topological Betti number* of M.

In the presence of a triangulation of M by a simplicial complex $\mathcal{T}$, the topological Betti numbers can be expressed in combinatorial terms. Assume that $\mathcal{T}$ is a simplicial complex that triangulates M and that $\mathcal{U} \subseteq \mathcal{T}$ is a simplicial subcomplex that triangulates Γ. In that case we have the identity

$$b_m(\mathcal{T},\mathcal{U}) = b_m(M,\Gamma), \quad m \in \mathbb{Z}. \tag{II.46}$$

This is Theorem 8 in [168, Chapter 4, Section 6]. If the simplicial complex $\mathcal{T}$ is finite, then this implies in particular that the topological Betti numbers are effectively computable.

Example II.6.4.
The following topological Betti numbers are of frequent interest. Let $p \in \mathbb{N}_0$ and $m \in \mathbb{Z}$.

(i) All Betti numbers $b_m(B^p)$ of the p-ball B^p vanish except for $b_0(B^p) = 1$.

(ii) All Betti numbers $b_m(S^p)$ of the p-sphere S^p vanish except for $b_p(S^p) = 1$ and $b_0(S^p) = 1$.

(iii) All Betti numbers $b_m(B^p,\partial B^p)$ of the p-ball relative to its boundary vanish except for $b_p(B^p,\partial B^p) = 1$.

(iv) If $D^{p-1} \subsetneq \partial B^p$ is homeomorphic to B^{p-1}, then all Betti numbers $b_m(B^p, D^{p-1})$ of the p-ball relative to a disk on the boundary vanish.

Remark II.6.5.
The Betti numbers have a geometric interpretation. For example, the 0-th absolute Betti number equals the number of path-connected components of a topological space (Corollary 8 in [168, Chapter 4, Section 4]).

For a two-dimensional bounded domain, the 0-th absolute Betti number counts the number of path-connected components, and the first Betti number counts the number of holes inside the domain.

More complicated examples are possible in higher dimensions. Consider a three-dimensional cube inside of which a “doughnut” has been cut out. The 0-th Betti number is still the number of path-connected components. The first Betti number is 1, and the first homology space can be represented by a circle wrapped around the doughnut and piercing through the hole. The second Betti number is 1, and the second homology space can be represented by the internal spherical surface that encloses the doughnut.

The presence of a boundary patch leads to more complex situations too. Consider the unit cube in dimension two and pick a non-trivial boundary patch. The first Betti number then equals the number of path-connected components of the boundary patch minus one. The homology space is represented by line segments that lead from one boundary patch to the other one.

II.7. Homology of Micropatches

We have introduced the notion of micropatch in Section II.1. We close this chapter with a study of simplicial chain complexes associated to micropatches. The micropatches of a simplicial complex encode its local combinatorial structure, which can be analyzed in terms of chain complexes. We will revisit this notion later in our study of discrete distributional differential forms.

Let $\mathcal{T}$ be a simplicial complex and let $F \in \mathcal{T}$. Then $\mathcal{M}(\mathcal{T}, F)$ induces the simplicial chain complex

$$\dots \xrightarrow{\partial_{m+1}} \mathcal{C}_m(\mathcal{M}(\mathcal{T}, F)) \xrightarrow{\partial_m} \mathcal{C}_{m-1}(\mathcal{M}(\mathcal{T}, F)) \xrightarrow{\partial_{m-1}} \dots \tag{II.47}$$

Assume that $\mathcal{U}$ is a simplicial subcomplex. Then $\mathcal{N}(\mathcal{T},\mathcal{U}, F)$ induces the simplicial chain complex

$$\dots \xrightarrow{\partial_{m+1}} \mathcal{C}_m(\mathcal{N}(\mathcal{T},\mathcal{U}, F)) \xrightarrow{\partial_m} \mathcal{C}_{m-1}(\mathcal{N}(\mathcal{T},\mathcal{U}, F)) \xrightarrow{\partial_{m-1}} \dots \tag{II.48}$$

We will study the simplicial chain complex of $\mathcal{M}(\mathcal{T}, F)$ relative to $\mathcal{N}(\mathcal{T},\mathcal{U}, F)$. In order to simplify the notation, we write

$$\mathcal{C}^F_m(\mathcal{T},\mathcal{U}) := \mathcal{C}_m\big(\mathcal{M}(\mathcal{T}, F), \mathcal{N}(\mathcal{T},\mathcal{U}, F)\big), \tag{II.49}$$

$$\mathcal{H}^F_m(\mathcal{T},\mathcal{U}) := \mathcal{H}_m\big(\mathcal{M}(\mathcal{T}, F), \mathcal{N}(\mathcal{T},\mathcal{U}, F)\big), \tag{II.50}$$

$$b^F_m(\mathcal{T},\mathcal{U}) := b_m\big(\mathcal{M}(\mathcal{T}, F), \mathcal{N}(\mathcal{T},\mathcal{U}, F)\big), \tag{II.51}$$

for the spaces of simplicial chains, the simplicial homology spaces, and the simplicial Betti numbers, respectively, of $\mathcal{M}(\mathcal{T}, F)$ relative to $\mathcal{N}(\mathcal{T},\mathcal{U}, F)$. We consider the simplicial chain complex

$$\dots \xrightarrow{\partial_{m+1}} \mathcal{C}^F_m(\mathcal{T},\mathcal{U}) \xrightarrow{\partial_m} \mathcal{C}^F_{m-1}(\mathcal{T},\mathcal{U}) \xrightarrow{\partial_{m-1}} \dots \tag{II.52}$$

Our goal is to determine the simplicial Betti numbers of the simplicial complex $\mathcal{M}(\mathcal{T}, F)$ relative to its subcomplex $\mathcal{N}(\mathcal{T},\mathcal{U}, F)$. This can be complicated in general. For the purpose of this thesis, it is sufficient to carry out the analysis in the following special case.

Lemma II.7.1.
Assume $\mathcal{T}$ triangulates an n-dimensional topological manifold M with boundary, and that $\mathcal{U}$ triangulates a topological submanifold Γ of ∂M of dimension $n-1$ with boundary. Let $\mathcal{V} \subset \mathcal{T}$ be the simplicial subcomplex that triangulates the closure of the complement of Γ in $\partial\Omega$. Then

$$b^F_m(\mathcal{T},\mathcal{U}) = \begin{cases} \delta^{nm} & \text{if } F \notin \mathcal{V}, \\ 0 & \text{if } F \in \mathcal{V}, \end{cases} \qquad F \in \mathcal{T}. \tag{II.53}$$

Proof. The proof uses the identity of simplicial and topological Betti numbers (II.46). It thus remains to determine the topological Betti numbers of $[\mathcal{M}(\mathcal{T},F)]$ relative to $[\mathcal{N}(\mathcal{T},\mathcal{U},F)]$. We accomplish this by reducing the question to the instances Example II.6.4, for which the Betti numbers are known.

It is easy to see that the closed set $[\mathcal{M}(\mathcal{T},F)]$ is homeomorphic to a topological n-ball. We let $\mathcal{B}(\mathcal{T},F)$ denote the subcomplex of $\mathcal{M}(\mathcal{T},F)$ that triangulates the topological boundary of $[\mathcal{M}(\mathcal{T},F)]$. Then $\mathcal{N}(\mathcal{T},\mathcal{U},F)$ is a subcomplex of $\mathcal{B}(\mathcal{T},F)$. We make a case distinction.

1. Suppose that $F \nsubseteq \Gamma$. Then $\mathcal{B}(\mathcal{T},F) = \mathcal{N}(\mathcal{T},\mathcal{U},F)$. The relevant case are the Betti numbers of a ball relative to its boundary, and so $b_m^F(\mathcal{T},\mathcal{U}) = \delta^{nm}$.
2. Similarly, suppose that $F \subseteq \Gamma$ but $F \notin \mathcal{V}$. Again, $\mathcal{B}(\mathcal{T},F) = \mathcal{N}(\mathcal{T},\mathcal{U},F)$, and so $b_m^F(\mathcal{T},\mathcal{U}) = \delta^{nm}$.
3. Finally, suppose that $F \subseteq \Gamma$ with $F \in \mathcal{V}$. Then $\mathcal{N}(\mathcal{T},\mathcal{U},F)$ triangulates a topological ball of dimension $n-1$ embedded in $[\mathcal{B}(\mathcal{T},F)]$. We conclude that $b_m^F(\mathcal{T},\mathcal{U}) = 0$.

This shows (II.53). The proof is complete. □

Another helpful observation is that the homology spaces of the simplicial chain complexes of micropatches are well-behaved with respect to restriction to subcomplexes. Unfolding definitions, we observe that $\mathcal{C}_m^F(\mathcal{T},\mathcal{U})$ is spanned by the (oriented) simplices of $\mathcal{T} \setminus \mathcal{U}$ that contain F as a subcomplex:

$$\mathcal{C}_m^F(\mathcal{T},\mathcal{U}) \simeq \operatorname{span}\{\, C \in \mathcal{T}^m \setminus \mathcal{U}^m \mid \exists T \in \mathcal{T} : F \in \Delta(F),\ C \in \Delta(T) \,\}, \quad F \in \mathcal{T}^{[m]}.$$

Consequently, we only need to consider m-simplices in the definition of $\mathcal{C}_m^F(\mathcal{T},\mathcal{U})$. In particular,

$$\mathcal{C}_{m-1}^F(\mathcal{T},\mathcal{U}) = \mathcal{C}_{m-1}^F\left(\mathcal{T}^{[m-1]},\mathcal{U}^{[m-1]}\right), \quad F \in \mathcal{T}^{[m-1]}. \tag{II.54}$$

As a consequence, the lower homology spaces of $\mathcal{M}(\mathcal{T},F)$ relative to $\mathcal{N}(\mathcal{T},\mathcal{U},F)$ can be calculated from considering the lower-dimensional skeletons only.

It will be of interest to us that a result similar to Lemma II.7.1 holds for lower dimensional skeletons provided that the original simplicial complex satisfies a generalization of (II.53). This is a simple consequence of (II.54).

Lemma II.7.2.
Let $\mathcal{T}$ be an m-dimensional simplicial complex and let $\mathcal{U}$ be a subcomplex of $\mathcal{T}$. Let $F \in \mathcal{T}^{[m-1]}$. If

$$b_p^F(\mathcal{T},\mathcal{U}) = 0, \quad p < m,$$

then

$$b_p^F(\mathcal{T}^{[m-1]},\mathcal{U}^{[m-1]}) = 0, \quad p < m-1.$$

Proof. This is verified by linear algebra. If the simplicial chain complex of $\mathcal{M}(\mathcal{T},F)$ relative to $\mathcal{N}(\mathcal{T},\mathcal{U},F)$ is exact at indices $k < m$, then the corresponding simplicial chain complex of the $(m-1)$-skeleton is exact at indices $k < m-1$. □

III. Finite Element Spaces over Simplices

A particular achievement of finite element exterior calculus has been the identification of spaces of polynomial differential forms invariant under affine transformations, and subsequently the construction of *finite element de Rham complexes*. This chapter is dedicated to the study of finite element spaces of polynomial differential forms over simplices. Our main goal in this chapter is constructing geometrically decomposed bases for the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ over a simplex T.

We regard the exposition [9, Chapter 4] of Arnold, Falk, and Winther as our starting point. First they review geometrically decomposed bases for the spaces $\mathcal{P}_r\Lambda^0(T)$ and $\mathcal{P}_1^-\Lambda^k(T)$ and their degrees of freedom. Then they develop a preliminary basis for $\mathcal{P}_r^-\Lambda^k(T)$. Towards geometric decompositions, they subsequently determine a geometrically decomposed basis of the degrees of freedom of $\mathcal{P}_r\Lambda^k(T)$, and a geometrically decomposed basis of the degrees of freedom of $\mathcal{P}_r^-\Lambda^k(T)$. Finally, they give geometrically decomposed bases for $\mathcal{P}_r^-\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$, the latter implicitly, and next for the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r\Lambda^k(T)$, again implicitly in the latter case. Their derivation of the geometrically decomposed bases for the finite element spaces utilizes isomorphisms

$$\mathcal{P}_r\Lambda^k(T) \simeq \mathring{\mathcal{P}}_{r+n-k+1}^-\Lambda^{n-k}(T), \quad \mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T) \simeq \mathcal{P}_r^-\Lambda^{n-k}(T). \tag{III.1}$$

A subsequent publication of Arnold, Falk, and Winther [10] has extended these studies and is another major point of reference to us. There they give explicit bases for the spaces with vanishing trace $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r\Lambda^k(T)$. The geometrically decomposed bases in [10] provide additional algebraic conditions that are of independent interest: not only the basis forms are associated to a subsimplex (vertices, edges, ...) of T each, but also the spaces themselves are decomposed into subspaces associated to a subsimplex each. The construction is more complex for the spaces $\mathcal{P}_r\Lambda^k(T)$ than for the spaces $\mathcal{P}_r^-\Lambda^k(T)$.

There are still incentives for reapproaching geometrically decomposed bases in finite element exterior calculus. The foundational parts of the theory are distributed over two research articles. A new systematical approach may help a larger audience access the theory. Even though we do not aim at a completely self-contained exposition in this thesis and still assume some basic familiarity with the theory of finite element differential forms, it seems reasonable to illuminate some new approaches to the topic. Another motivation is that the 2006 publication of Arnold, Falk, and Winther [9] already provides very simple geometrically decomposed bases in finite element exterior calculus; we give a new presentation of those results.

We give a brief outline of the calculus of differential forms over simplices in Section III.1 and introduce polynomial differential forms over simplices in Section III.2. The subsequent Section III.3 is dedicated to several auxiliary lemmas which have appeared in publications on finite element exterior calculus and which we frequently use throughout this chapter. In Section III.4 we introduce the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ and some of their basic properties. This also includes the subspaces $\mathring{\mathcal{P}}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ with vanishing traces along the simplex boundary.

We then focus our attention to the major topic of this chapter: the construction of geometrically decomposed bases in Sections III.5 and III.6. In contrast to prior expositions, we first derive geometrically decomposed bases for the spaces $\mathcal{P}_r\Lambda^k(T)$ and then independently for the spaces $\mathcal{P}_r^-\Lambda^k(T)$. In particular, this naturally produces *consistent extension operators* in the sense of [10, Section 4] and bases for the spaces $\mathring{\mathcal{P}}_r\Lambda^k(T)$, and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$. Moreover, our construction does not involve the degrees of freedom or the isomorphisms (III.1) mentioned above.

For the space $\mathcal{P}_r^-\Lambda^k(T)$ our basis is the same as given in [9, Subsections 4.4, 4.7] and in [10, Theorem 6.1, Section 7]). By contrast, our basis for the space $\mathcal{P}_r\Lambda^k(T)$ coincides with the one given in [9] and is thus different from the one in [10, Section 8]. An advantage of the basis of the space $\mathcal{P}_r\Lambda^k(T)$ in [9] is its simplicity, but a disadvantage is that the subspaces associated to different subsimplices generally depend on the numbering of the vertices. No such trade off is made for the geometrically decomposed bases of $\mathcal{P}_r^-\Lambda^k(T)$.

Even though we establish geometrically decomposable bases, the spanning sets are still of interest. We show that the isomorphisms (III.1) have a natural expression in terms of the canonical spanning sets. Thus we can transfer linear dependencies between the canonical spanning sets of the spaces in each isomorphic pair. For the first isomorphic pair, this follows from Proposition 3.1 of [57], which has been a major inspiration for this research. With different techniques, we reproduce the result and its analogon for the second isomorphic pair.

Duality pairings that correspond to these isomorphisms are another concept in the seminal publication of Arnold, Falk, and Winther, which only recently has been identified as a subject worth independent study by Christiansen and Rapetti [57]. They have discovered more details about the first isomorphism in (III.1) and the corresponding duality pairing. Their results for the second isomorphic pair are less extensive. Using different methods, we reproduce and refine their result on the first isomorphic pair and give an analogous result for the second isomorphic pair.

Our primary sources for this chapter are the expositions by Arnold, Falk, and Winther [9, 10]. Their work is preceded by several contributions in numerical analysis that address the construction of bases for finite element spaces of vector fields. (e.g., [2, 103, 107, 108, 109, 174]). We refer to [153] and [154] for additional algebraic approaches.

III.1. Smooth Differential Forms over Simplices

We commence this chapter with a discussion of smooth differential forms over simplices. This is only a review of basic notions, with particular attention to affine diffeomorphisms between simplices and traces onto subsimplices. For a rigorous discussion of smooth differential forms on simplices, we refer to the treatment of *manifolds with corners* in Lee's monograph [127].

Let $T \subset \mathbb{R}^N$ be a simplex of dimension n. We let $C^\infty(T)$ denote the space of restrictions of smooth functions over $\mathbb{R}^N$ onto T. More generally, for $k \in \mathbb{Z}$ we let $C^\infty\Lambda^k(T)$ denote the space of traces of smooth differential k-forms on $\mathbb{R}^N$ onto T. We have $C^\infty\Lambda^0(T) = C^\infty(T)$ and $C^\infty\Lambda^k(T) = \{0\}$ for $k \notin \{0, \dots, n\}$.

When $\omega \in C^\infty\Lambda^k(T)$ and $\eta \in C^\infty\Lambda^l(T)$, then $\omega \wedge \eta \in C^\infty\Lambda^{k+l}(T)$ denotes their exterior product . We recall that $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$. Furthermore, we recall the *exterior derivative*

$$\mathsf{d}^k_T : C^\infty\Lambda^k(T) \to C^\infty\Lambda^{k+1}(T). \tag{III.2}$$

It is well-known that for $\omega \in C^\infty\Lambda^k(T)$ and $\omega \in C^\infty\Lambda^l(T)$ we have

$$\mathsf{d}^{k+l}_T(\omega \wedge \eta) = \mathsf{d}^k_T\omega \wedge \eta + (-1)^k\omega \wedge \mathsf{d}^l_T\eta.$$

We also recall that the integral of an n-form over T is well-defined provided that an orientation of T is fixed.

Suppose that S is another simplex of dimension n and that $\varphi_{S,T} : S \to T$ is an affine diffeomorphism from S onto T. Then the pullback induces linear mappings

$$\varphi^*_{S,T} : C^\infty\Lambda^k(T) \to C^\infty\Lambda^k(S).$$

These commute with the exterior derivative, and distribute over the exterior product,

$$\begin{aligned} \varphi^*_{S,T}\mathsf{d}^k_T\omega &= \mathsf{d}^k_S\varphi^*_{S,T}\omega, \quad \omega \in C^\infty\Lambda^k(T), \\ \varphi^*_{S,T}(\omega \wedge \eta) &= \varphi^*_{S,T}\omega \wedge \varphi^*_{S,T}\eta, \quad \omega \in C^\infty\Lambda^k(T), \quad \omega \in C^\infty\Lambda^l(T). \end{aligned}$$

For the pullback along the inverse $\varphi^{-1}_{S,T} : T \to S$ we use the special notation $\varphi^{-*}_{S,T}$. Moreover, the integral transformation

$$\int_S \varphi^*_{S,T}\omega = o(\varphi_{S,T})\int_T \omega, \quad \omega \in C^\infty\Lambda^n(T)$$

holds for n-forms, where $o(\varphi_{S,T}) = 1$ if $\varphi_{S,T}$ is orientation preserving and $o(\varphi) = -1$ if $\varphi_{S,T}$ is orientation reversing.

We also consider the trace operator onto subsimplices. For every m-dimensional subsimplex $F \in \Delta(T)$ of T, we have the inclusion $\imath_{F,T} : F \to T$, and the pullback along that inclusion defines the *trace operators*

$$\operatorname{tr}^k_{F,T} : C^\infty\Lambda^k(T) \to C^\infty\Lambda^k(F), \quad k \in \mathbb{Z}.$$

Since $\imath_{F,T}\imath_{f,F} = \imath_{f,T}$ for $F \in \Delta(T)$ and $f \in \Delta(F)$, we also have $\operatorname{tr}^k_{F,f}\operatorname{tr}^k_{T,F} = \operatorname{tr}^k_{T,f}$. As with the pullback along affine diffeomorphisms, we observe

$$\begin{aligned} \operatorname{tr}^{k+1}_{T,F}\mathsf{d}^k_T\omega &= \mathsf{d}^k_F\operatorname{tr}^k_{T,F}\omega, \quad \omega \in C^\infty\Lambda^k(T), \\ \operatorname{tr}^{k+l}_{T,F}(\omega \wedge \eta) &= \operatorname{tr}^k_{T,F}\omega \wedge \operatorname{tr}^l_{T,F}\eta, \quad \omega \in C^\infty\Lambda^k(T), \quad \omega \in C^\infty\Lambda^l(T). \end{aligned}$$

We note at this point that $\operatorname{tr}^k_{T,F} \omega = 0$ for every $\omega \in C^\infty\Lambda^k(T)$ if $k > \dim F$. In particular, if $F \neq T$ then $\operatorname{tr}^n_{T,F} \omega = 0$ for every n-form $\omega \in C^\infty\Lambda^n(T)$.

Having discussed the exterior derivative and traces, we recall a version of Stokes' theorem that states

$$\int_T \mathsf{d}^{n-1}\omega = \sum_{F \in \Delta(T)^{m-1}} o(F,T) \int_F \operatorname{tr}^{n-1}_{T,F} \omega, \quad \omega \in C^\infty\Lambda^{n-1}(T). \tag{III.3}$$

Sometimes we will use the existence of Riemannian metric over simplices. We refer to the literature on differential geometry for further information on this topic (cf. Agricola and Friedrich [92]). At this point, we merely remark that there exists a bilinear pairing

$$B_T : C^\infty\Lambda^k(T) \times C^\infty\Lambda^k(T) \to \mathbb{R}$$

for simplex T that is scalar product. In particular, $B_T(\omega,\omega) > 0$ for each $\omega \in C^\infty\Lambda^k(T)$ holds true. The choice of B_T is generally not canonical: it depends on a Riemannian metric and the orientation.

III.2. Polynomial Differential Forms over Simplices

A specific class of differential forms over simplices are polynomial differential forms. Whereas polynomial differential forms on $\mathbb{R}^n$ can be discussed easily within the canonical Euclidean coordinate system, polynomial differential forms over a simplex can be discussed with the help of barycentric coordinates. In this section we develop polynomial differential forms primarily with barycentric coordinates.

We recall that T is the convex closure of its $n+1$ different vertices, which we enumerate by $v_0^T, \dots, v_n^T$. The *barycentric coordinates* $\lambda_0^T, \lambda_1^T, \dots, \lambda_n^T \in C^\infty(T)$ are the unique affine functions over T that satisfy the *Lagrange property*

$$\lambda_i^T(v_j) = \delta_{ij}, \quad i,j \in [0:n]. \tag{III.4}$$

The exterior derivatives $\mathsf{d}\lambda_0^T, \mathsf{d}\lambda_1^T, \dots, \mathsf{d}\lambda_n^T \in C^\infty\Lambda^1(T)$ of the barycentric coordinates are constant 1-forms, corresponding to the gradients of the barycentric coordinates. The Lagrange property of the barycentric coordinates implies the linear independence of the barycentric coordinate functions and that they constitute a partition of unity, i.e. $1 = \lambda_0^T + \dots + \lambda_n^T$ over T. As a consequence, we have the partition of zero $0 = \mathsf{d}\lambda_0^T + \dots + \mathsf{d}\lambda_n^T$ of their exterior derivatives. It can be shown that this is the only linear independence up to scaling between the exterior derivatives of the barycentric coordinate functions.

Lemma III.2.1.
Let $c_i \in \mathbb{R}$ for $i \in [0:n]$ and assume that $0 = c_0\mathsf{d}\lambda_0 + \dots + c_n\mathsf{d}\lambda_n$. Then $c_0 = \dots = c_n$.

Proof. From the assumption we have $c = c_0\lambda_0 + \dots + c_n\lambda_n$ for some $c \in \mathbb{R}$. Via the Lagrange property we find that $c = c_0 = \dots = c_n$. □

The barycentric coordinates and their exterior derivatives can be combined to new objects. For a multiindex $\alpha \in A(0:n)$ we define the *barycentric polynomial*

$$\lambda_T^\alpha := \prod_{i=0}^{n} (\lambda_i^T)^{\alpha(i)}. \tag{III.5}$$

For $k \in \mathbb{N}_0$ and $\sigma \in \Sigma(1:k, 0:n)$ we define the *basic k-alternator* as

$$\mathsf{d}\lambda_\sigma^T := \mathsf{d}\lambda_{\sigma(1)}^T \wedge \cdots \wedge \mathsf{d}\lambda_{\sigma(k)}^T. \tag{III.6}$$

Note that $\mathsf{d}\lambda_\emptyset^T = 1$. Moreover, for $k \in \mathbb{N}_0$ and $\rho \in \Sigma(0:k, 0:n)$ we introduce the *Whitney k-form*

$$\phi_\rho^T := \sum_{p \in [\rho]} \epsilon(p, \rho - p) \lambda_p^T \mathsf{d}\lambda_{\rho-p}^T. \tag{III.7}$$

We henceforth agree to the convention that the sums and exterior products of differential forms of the form (III.5), (III.6), and (III.7) are called *polynomial differential forms*. We will use products and sums of these objects to construct spaces of polynomial differential forms.

Remark III.2.2.
In the sequel, we may simplify the notation by

$$\lambda_i \equiv \lambda_i^T, \quad \lambda^\alpha \equiv \lambda_T^\alpha, \quad \mathsf{d}\lambda_\sigma \equiv \mathsf{d}\lambda_\sigma^T, \quad \phi_\rho \equiv \phi_\rho^T,$$

whenever the simplex T is fixed and understood,

In the previous section we have studied the transformation behavior of differential forms over simplices, and we give special scrutiny to the case of polynomial differential forms. As above, we suppose that S is another n-dimensional simplex and that $\varphi_{S,T} : S \to T$ is an affine diffeomorphism from S onto T. We first observe that $\varphi_{S,T}$ necessarily maps the vertices $v_0^S, \dots, v_n^S$ of S bijectively to the vertices $v_0^T, \dots, v_n^T$ of T. Hence, with a mild abuse of notation, we introduce a permutation $\varphi_{S,T} : [0:n] \to [0:n]$ by setting $v_{\varphi_{S,T}(i)}^T = v_i^S$ for $i \in [0:n]$. Now it is easy to observe that

$$\begin{aligned}
\varphi_{S,T}^* \lambda_T^\alpha &= \lambda_S^{\alpha\varphi_{S,T}}, \quad \alpha \in A(0:n), \\
\varphi_{S,T}^* \mathsf{d}\lambda_\sigma^T &= \mathsf{d}\lambda_{\varphi_{S,T}\sigma}^S, \quad \sigma \in \Sigma(1:k, 0:n), \\
\varphi_{S,T}^* \phi_\rho^T &= \phi_{\varphi_{S,T}\rho}^S, \quad \rho \in \Sigma(0:k, 0:n),
\end{aligned}$$

This shows how to transform polynomial differential forms along affine transformations of simplices.

Similarly, we observe that polynomial differential forms are preserved by taking traces onto subsimplices. Let $F \in \Delta(T)$ be an m-dimensional subsimplex of T. We assume to have fixed enumerations of the vertices $v_0^T, \dots, v_n^T$ of T and of the vertices $v_0^F, \dots, v_m^F$ of F such that there exists $\imath_{F,T} \in \Sigma(0:m, 0:n)$ with $v_i^F = v_{\imath_{F,T}(i)}^T$ for

$i \in [0:m]$. Note that such a mapping exists if and only if our ordering of the vertices of T restricts to our ordering of the vertices of F. Via the Lagrange property we easily obtain for $i \in [0:n]$ that

$$\operatorname{tr}^0_{T,F} \lambda^T_i = \begin{cases} \lambda^F_j & \text{if } i \in [\imath_{F,T}],\ i = \imath_{F,T}(j), \\ 0 & \text{if } i \notin [\imath_{F,T}]. \end{cases}$$

It is easily observed that for $\alpha \in A(0:n)$ we have $[\alpha] \subseteq [\imath_{F,T}]$ if and only if there exists $\alpha' \in A(0:m)$ with $\alpha' = \alpha\imath_{F,T}$. In that case, $|\alpha'| = |\alpha\imath_{F,T}|$. Thus, for every multiindex $\alpha \in A(0:n)$ we observe that

$$\operatorname{tr}^0_{T,F} \lambda^\alpha_T = \begin{cases} \lambda^{\alpha'}_T & \text{if } [\alpha] \subseteq [\imath_{F,T}],\ \alpha' \in A(0:m),\ \alpha' = \alpha\imath_{F,T}, \\ 0 & \text{if } [\alpha] \nsubseteq [\imath_{F,T}]. \end{cases}$$

For $\sigma \in \Sigma(1:k, 0:m)$ we have $\imath_{F,T}\sigma \in \Sigma(1:k,0:n)$. Conversely, for $\sigma \in \Sigma(1:k,0:n)$ we have $[\sigma] \subseteq [\imath_{F,T}]$ if and only if there exists $\sigma' \in \Sigma(1:k,0:m)$ with $\sigma = \imath_{F,T}\sigma'$. In that case, σ' is unique. Analogous statements hold for any $\rho \in \Sigma(0:k,0:n)$. Thus we observe

$$\operatorname{tr}^k_{T,F} \mathsf{d}\lambda^T_\sigma = \begin{cases} \mathsf{d}\lambda^{\sigma'}_F & \text{if } [\sigma] \subseteq [\imath_{F,T}],\ \sigma' \in \Sigma(1:k,0:m),\ [\imath_{F,T}\sigma'] = [\sigma], \\ 0 & \text{if } [\sigma] \nsubseteq [\imath_{F,T}], \end{cases}$$

$$\operatorname{tr}^k_{T,F} \phi^T_\rho = \begin{cases} \phi^T_{\rho'} & \text{if } [\rho] \subseteq [\imath_{F,T}],\ \rho' \in \Sigma(0:k,0:m),\ [\imath_{F,T}\rho'] = [\rho], \\ 0 & \text{if } [\rho] \nsubseteq [\imath_{F,T}]. \end{cases}$$

These basic relations describe the behavior of polynomial differential forms under taking traces to subsimplices.

Having established these basic definitions and results, we introduce some differential forms of particular interest in the sequel. We let $1_T \in C^\infty(T)$ denote the function over T with constant value 1, and we let $\mathrm{vol}_T \in C^\infty\Lambda^n(T)$ denote the constant n-form over T whose integral $\int_T \mathrm{vol}_T = \mathrm{vol}^n(T)$ over T equals the volume $\mathrm{vol}^n(T)$. The n-form vol_T is also known as the *volume form* of T. There are different ways to represent the volume form in terms of polynomial differential forms.

Lemma III.2.3.
Let $\sigma \in \Sigma(1:n,0:n)$ and let $p \in [0:n] \setminus [\sigma]$. Then

$$\mathsf{d}\lambda_\sigma = \frac{\epsilon(p,\sigma)}{n! \cdot \mathrm{vol}^n(T)} \mathrm{vol}_T .$$

Proof. We let $\varphi_T : \Delta_n \to T$ be the unique affine diffeomorphism which maps 0 to v_0 and e_i to v_i for $1 \leq i \leq n$. We then find that

$$\int_T \mathsf{d}\lambda^T_\sigma = \int_T \varphi_T^{-*} \mathsf{d}\lambda^{\Delta_n}_\sigma = s_0 \cdot \int_{\Delta_n} \mathsf{d}\lambda^{\Delta_n}_\sigma = \frac{s_0 s_1}{n!},$$

where $s_0, s_1 \in \{-1,1\}$ are specified as follows. Let $\tau \in \Sigma(1:n,0:n)$ satisfy $[\tau] = [1:n]$. If $\sigma = \tau$, then we let $s_1 = \epsilon(0,\sigma) = 1$. Otherwise, $0 \in [\sigma]$ and there exists a unique $p \in [1:n] \setminus [\sigma]$. We then let $s_1 = \epsilon(p,\sigma)$. Lastly, we let $s_0 = 1$ if φ_T preserves the orientation and $s_0 = -1$ otherwise. The proof is complete. $\square$

Lemma III.2.4.
Let $\rho \in \Sigma(0:n, 0:n)$. Then

$$\phi_\rho^T = \frac{\mathrm{vol}_T}{n! \cdot \mathrm{vol}^n(T)}.$$

Proof. Using Lemma III.2.3, the identity

$$\phi_\rho = \sum_{p \in [\rho]} \epsilon(p, \rho - p) \lambda_p^T \mathsf{d}\lambda_{\rho-p}^T = \sum_{p \in [\rho]} \lambda_p^T \mathsf{d}\lambda_{\rho-0}^T = \mathsf{d}\lambda_{\rho-0}^T = \frac{1}{n! \cdot \mathrm{vol}^n(T)} \mathrm{vol}_T$$

is easily verified. □

We finish this section with some special notation and results considering Whitney k-forms. Any enumeration of the vertices of T induces an enumeration of the vertices of each subsimplex of T. Suppose that we have fixed enumerations of the subsimplices of T that are all compatible with each other. For every $\rho \in \Sigma(0:k, 0:n)$ we let $F_\rho^T \in \Delta(T)$ be the unique k-dimensional subsimplex of T with $[\imath_{F,T}] = [\rho]$. In other words, F_ρ^T is the subsimplex of T whose vertices have the indices indicated by ρ. The enumeration of the vertices of T thus yields a bijective mapping between $\Delta(T)^k$ and $\Sigma(0:k, 0:n)$. We then let $\phi_F^T := \phi_\rho^T$, where $F = F_\rho^T$.

We first verify that $\mathrm{tr}_{T,F}^k \phi_f^T = \phi_f^F$ for all $F \in \Delta(T)$ and $f \in \Delta(F)$. With our observations about the traces of Whitney forms and Lemma III.2.4, we get

$$\int_F \phi_G^T = \begin{cases} 1/n! & \text{if } F = G, \\ 0 & \text{if } F \neq G, \end{cases} \qquad F, G \in \Delta(T)^k. \tag{III.8}$$

III.3. Auxiliary Lemmas

We give elementary proofs for some auxiliary lemmas concerning polynomial differential forms. All of these results have been proven in the literature several times. In this section, let T be an n-simplex assumed be understood.

Consider $\sigma \in \Sigma(1:k, 0:n)$ for some $k \in [1:n]$ and $p \in [\sigma]$. We then have

$$\mathsf{d}\lambda_\sigma = \epsilon(p, \sigma - p) \mathsf{d}\lambda_p \wedge \mathsf{d}\lambda_{\sigma-p}. \tag{III.9}$$

This follows from the definition of $\mathsf{d}\lambda_\sigma$ and properties of the alternating product. We interpret this as a recursive formula for the basic alternators. The following lemma gives a recursive formula for the Whitney forms.

Lemma III.3.1.
Let $k \in [0:n]$. If $\rho \in \Sigma(0:k, 0:n)$ and $q \in [0:n] \setminus [\rho]$, then

$$\epsilon(q, \rho) \phi_{\rho+q} = \lambda_q \mathsf{d}\lambda_\rho - \mathsf{d}\lambda_q \wedge \phi_\rho. \tag{III.10}$$

Proof. Let k, ρ, and q be as in the statement of the lemma. Unfolding definitions gives

$$\begin{aligned} \epsilon(q, \rho)\phi_{\rho+q} - \lambda_q \mathsf{d}\lambda_\rho &= \epsilon(q, \rho) \sum_{l \in [\rho+q]} \epsilon(l, \rho + q - l) \lambda_l \mathsf{d}\lambda_{\rho+q-l} - \lambda_q \mathsf{d}\lambda_\rho \\ &= \epsilon(q, \rho) \sum_{l \in [\rho]} \epsilon(l, \rho + q - l) \lambda_l \mathsf{d}\lambda_{\rho+q-l}. \end{aligned}$$

A simple calculation yields

$$\begin{aligned}
&\epsilon(q,\rho)\sum_{l\in[\rho]}\epsilon(l,\rho+q-l)\lambda_l \mathsf{d}\lambda_{\rho+q-l}\\
&=\epsilon(q,\rho)\sum_{l\in[\rho]}\epsilon(l,\rho+q-l)\epsilon(q,\rho-l)\lambda_l \mathsf{d}\lambda_q\wedge\mathsf{d}\lambda_{\rho-l}\\
&=\epsilon(q,\rho)\sum_{l\in[\rho]}\epsilon(l,q)\epsilon(q,l)\epsilon(l,\rho-l)\epsilon(q,\rho)\lambda_l \mathsf{d}\lambda_q\wedge\mathsf{d}\lambda_{\rho-l}\\
&=-\mathsf{d}\lambda_q\wedge\sum_{l\in[\rho]}\epsilon(l,\rho-l)\lambda_l\mathsf{d}\lambda_{\rho-l}\\
&=-\mathsf{d}\lambda_q\wedge\phi_\rho,
\end{aligned}$$

where we have used

$$\epsilon(q,\sigma-l)=\epsilon(q,\sigma)\epsilon(q,l),\quad \epsilon(l,\sigma+q-l)=\epsilon(l,q)\epsilon(l,\sigma-l),$$

valid for $l\in[\sigma]$. This completes the proof. □

Lemma III.3.2 (Proposition 3.4 in [57], Equation (6.6) in [10])**.**
Let $k\in[0:n]$ and $\rho\in\Sigma(0:k,0:n)$. Then

$$\mathsf{d}^k\phi_\rho=(k+1)\mathsf{d}\lambda_\rho. \tag{III.11}$$

Proof. For $k\in[0:n]$ and $\rho\in\Sigma(0:k,0:n)$ we find

$$\mathsf{d}^k\phi_\rho=\sum_{p\in[\rho]}\epsilon(p,\rho-p)\mathsf{d}\lambda_p^T\wedge\mathsf{d}\lambda_{\rho-p}^T=(k+1)\mathsf{d}\lambda_\rho,$$

which is the desired result. □

Remark III.3.3.
The preceding observation motivates the notation $\lambda_\rho:=\phi_\rho$ for the Whitney forms, which can be found in several publications (e.g. Christiansen and Rapetti [57]).

Lemma III.3.4 (Proposition 3.4 in [57])**.**
Let $k\in[0:n]$ and $\sigma\in\Sigma(0:k,0:n)$. Then

$$\mathsf{d}\lambda_\sigma=\sum_{q\in[\sigma^c]}\epsilon(q,\sigma)\phi_{\sigma+q}. \tag{III.12}$$

Proof. Via Lemma III.3.1 we find

$$\begin{aligned}
\sum_{q\in[\sigma^c]}\epsilon(q,\sigma)\phi_{\sigma+q}&=\sum_{q\in[\sigma^c]}\left(\lambda_q\mathsf{d}\lambda_\sigma-\mathsf{d}\lambda_q\wedge\phi_\sigma\right)\\
&=\left(\sum_{q\in[\sigma^c]}\lambda_q\right)\mathsf{d}\lambda_\sigma-\left(\sum_{q\in[\sigma^c]}\mathsf{d}\lambda_q\right)\wedge\phi_\sigma.
\end{aligned}$$

Using the partition of zero, the definition of the Whitney forms, Equation (III.9), and eventually the partition of unity, we derive

$$\begin{aligned}
\sum_{q\in[\sigma^c]} \epsilon(q,\sigma)\phi_{\sigma+q} &= \sum_{q\in[\sigma^c]} \lambda_p \mathsf{d}\lambda_\sigma + \sum_{p\in[\sigma]} \mathsf{d}\lambda_p \wedge \phi_\sigma \\
&= \sum_{q\in[\sigma^c]} \lambda_p \mathsf{d}\lambda_\sigma + \sum_{p\in[\sigma]} \mathsf{d}\lambda_p \wedge \epsilon(p,\sigma-p)\lambda_p \mathsf{d}\lambda_{\sigma-p} \\
&= \sum_{q\in[\sigma^c]} \lambda_p \mathsf{d}\lambda_\sigma + \sum_{p\in[\sigma]} \lambda_p \mathsf{d}\lambda_\sigma = \sum_{i=0}^{n} \lambda_i \mathsf{d}\lambda_\sigma = \mathsf{d}\lambda_\sigma.
\end{aligned}$$

This had to be shown. □

The following identity describes an elementary linear dependence between Whitney forms of higher order.

Lemma III.3.5 (Proposition 3.3 in [57], Equation (6.5) in [10])**.**
Let $k \in [0:n]$ and $\rho \in \Sigma(0:k, 0:n)$. Then

$$\sum_{p\in[\rho]} \epsilon(p,\rho-p)\lambda_p \phi_{\rho-p} = 0. \tag{III.13}$$

Proof. Using (III.7), we expand the left-hand side of (III.13) to see

$$\begin{aligned}
\sum_{p\in[\rho]} \epsilon(p,\rho-p)\lambda_p \phi_{\rho-p} &= \sum_{p\in[\rho]} \epsilon(p,\rho-p)\lambda_p \sum_{s\in[\rho-p]} \lambda_s \epsilon(s,\rho-p-s)\mathsf{d}\lambda_{\rho-p-s} \\
&= \sum_{\substack{p,s\in[\rho]\\ p\neq s}} \epsilon(p,\rho-p)\epsilon(s,\rho-p-s)\lambda_p\lambda_s \mathsf{d}\lambda_{\rho-p-s} \\
&= \sum_{\substack{p,s\in[\rho]\\ p\neq s}} \epsilon(p,\rho-p)\epsilon(s,\rho-s)\epsilon(s,p)\lambda_p\lambda_s \mathsf{d}\lambda_{\rho-p-s}.
\end{aligned}$$

It is evident that the last expression vanishes, since the sum contains for each summand also its negative. □

III.4. Finite Element Spaces

We are now in the position to introduce spaces of polynomial differential forms and discuss some basic properties. Let T be an n-dimensional simplex. For $r, k \in \mathbb{Z}$ we define the space $\mathcal{P}_r\Lambda^k(T)$ by

$$\mathcal{P}_r\Lambda^k(T) := \operatorname{span}\left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \alpha \in A(r,0:n),\ \sigma \in \Sigma(1:k,0:n) \right\}, \tag{III.14}$$

and we define the space $\mathcal{P}_r^-\Lambda^k(T)$ by

$$\mathcal{P}_r^-\Lambda^k(T) := \operatorname{span}\left\{ \lambda_T^\alpha \phi_\sigma^T \;\middle|\; \alpha \in A(r-1,0:n),\ \sigma \in \Sigma(0:k,0:n) \right\}. \tag{III.15}$$

We first study the transformation properties of these spaces. Suppose that T' is another n-dimensional simplex and let $\varphi_{T',T} : T' \to T$ be an affine diffeomorphism. In that case we have isomorphisms

$$\varphi^*_{T',T} : \mathcal{P}_r\Lambda^k(T) \to \mathcal{P}_r\Lambda^k(T'), \quad \varphi^*_{T',T} : \mathcal{P}^-_r\Lambda^k(T) \to \mathcal{P}^-_r\Lambda^k(T'),$$

for $r, k \in \mathbb{Z}$, as follows from the discussion in Section III.2. We also consider traces to subsimplices. Let $F \in \Delta(T)$ be a subsimplex of T. It is easy to verify that the traces preserve the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}^-_r\Lambda^k(T)$ and that they are even surjective. We have

$$\mathcal{P}_r\Lambda^k(F) = \operatorname{tr}^k_{T,F} \mathcal{P}_r\Lambda^k(T), \quad \mathcal{P}^-_r\Lambda^k(F) = \operatorname{tr}^k_{T,F} \mathcal{P}^-_r\Lambda^k(T),$$

for $r, k \in \mathbb{Z}$. It is now of particular interest to consider the subspaces of $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}^-_r\Lambda^k(T)$ with vanishing traces along the boundary. We introduce

$$\mathring{\mathcal{P}}_r\Lambda^k(T) := \operatorname{span}\left\{ \omega \in \mathcal{P}_r\Lambda^k(T) \;\middle|\; \forall F \in \Delta(T) \setminus \{T\} : \operatorname{tr}^k_{T,F}\, \omega = 0 \right\}, \qquad \text{(III.16)}$$

$$\mathring{\mathcal{P}}^-_r\Lambda^k(T) := \operatorname{span}\left\{ \omega \in \mathcal{P}^-_r\Lambda^k(T) \;\middle|\; \forall F \in \Delta(T) \setminus \{T\} : \operatorname{tr}^k_{T,F}\, \omega = 0 \right\}. \qquad \text{(III.17)}$$

Similar as above, if T' is another n-dimensional simplex and $\varphi_{T',T} : T' \to T$ is an affine diffeomorphism, then we have isomorphisms

$$\varphi^*_{T',T} : \mathring{\mathcal{P}}_r\Lambda^k(T) \to \mathring{\mathcal{P}}_r\Lambda^k(T'), \quad \varphi^*_{T',T} : \mathring{\mathcal{P}}^-_r\Lambda^k(T) \to \mathring{\mathcal{P}}^-_r\Lambda^k(T'),$$

for $r, k \in \mathbb{Z}$.

We note that for some combinations of parameters $r, k, n \in \mathbb{Z}$, the above spaces are linear hulls of the empty set, in which case the respective spaces are the zero-dimensional vector space. In particular, we have

$$\mathcal{P}_r\Lambda^k(T) = \mathcal{P}^-_r\Lambda^k(T) = \mathring{\mathcal{P}}_r\Lambda^k(T) = \mathring{\mathcal{P}}^-_r\Lambda^k(T) = \{0\}$$

if $k \notin [0 : n]$ or if $r < 0$. We also note at this point that $\mathcal{P}_0\Lambda^0(T) = \operatorname{span}\{1_T\}$ but $\mathcal{P}^-_0\Lambda^0(T) = \{0\}$ in our definition (see also Remark III.4.2 below).

We consider some inclusion properties. We have

$$\mathcal{P}_{r-1}\Lambda^k(T) \subseteq \mathcal{P}^-_r\Lambda^k(T) \subseteq \mathcal{P}_r\Lambda^k(T), \quad r, k \in \mathbb{Z},$$

by Lemma III.3.4 and an iteration of Lemma III.3.1. We immediately obtain

$$\mathring{\mathcal{P}}_{r-1}\Lambda^k(T) \subseteq \mathring{\mathcal{P}}^-_r\Lambda^k(T) \subseteq \mathring{\mathcal{P}}_r\Lambda^k(T), \quad r, k \in \mathbb{Z}. \qquad \text{(III.18)}$$

From definitions we have

$$\mathcal{P}_r\Lambda^0(T) = \mathcal{P}^-_r\Lambda^0(T), \quad \mathring{\mathcal{P}}_r\Lambda^0(T) = \mathring{\mathcal{P}}^-_r\Lambda^0(T)$$

in the case $r \geq 1$. On the other hand,

$$\mathcal{P}_{r-1}\Lambda^n(T) = \mathcal{P}^-_r\Lambda^n(T) = \mathring{\mathcal{P}}_{r-1}\Lambda^n(T) = \mathring{\mathcal{P}}^-_r\Lambda^n(T)$$

is implied by Lemma III.2.4. But for $r \geq 0$ and $k \in [1 : n-1]$, the inclusions (III.18) are strict.

Remark III.4.1.
In this section we have described the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ only in terms of barycentric coordinates, but we can define the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ alternatively as the traces of $\mathcal{P}_r\Lambda^k(\mathbb{R}^N)$ and $\mathcal{P}_r^-\Lambda^k(\mathbb{R}^N)$ onto T, where $\mathcal{P}_r\Lambda^k(\mathbb{R}^N)$ and $\mathcal{P}_r^-\Lambda^k(\mathbb{R}^N)$ are spaces of polynomial differential forms over $\mathbb{R}^N$. Our definition of $\mathcal{P}_r^-\Lambda^k(T)$ is equivalent to the definition of $\mathcal{P}_r^-\Lambda^k(T)$ in the literature, where the Koszul operator is used (see [9]).

Remark III.4.2.
The case $r = 0$ is the only instance where $\mathcal{P}_r\Lambda^0(T)$ and $\mathcal{P}_r^-\Lambda^0(T)$ differ. This is noted explicitly in the seminal paper of Arnold, Falk, and Winther [9, p.34].

The exterior derivative gives rise to linear mappings between these spaces of polynomial differential forms. We have

$$
\begin{aligned}
\mathsf{d}_T^k : \mathcal{P}_r\Lambda^k(T) \to \mathcal{P}_{r-1}\Lambda^{k+1}(T), \quad \mathsf{d}_T^k : \mathcal{P}_r\Lambda^k(T) \to \mathcal{P}_r^-\Lambda^{k+1}(T),\\
\mathsf{d}_T^k : \mathcal{P}_r^-\Lambda^k(T) \to \mathcal{P}_{r-1}\Lambda^{k+1}(T), \quad \mathsf{d}_T^k : \mathcal{P}_r^-\Lambda^k(T) \to \mathcal{P}_r^-\Lambda^{k+1}(T),
\end{aligned}
$$

and corresponding mappings between spaces with boundary conditions,

$$
\begin{aligned}
\mathsf{d}_T^k : \mathring{\mathcal{P}}_r\Lambda^k(T) \to \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(T), \quad \mathsf{d}_T^k : \mathring{\mathcal{P}}_r\Lambda^k(T) \to \mathring{\mathcal{P}}_r^-\Lambda^{k+1}(T),\\
\mathsf{d}_T^k : \mathring{\mathcal{P}}_r^-\Lambda^k(T) \to \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(T), \quad \mathsf{d}_T^k : \mathring{\mathcal{P}}_r^-\Lambda^k(T) \to \mathring{\mathcal{P}}_r^-\Lambda^{k+1}(T),
\end{aligned}
$$

Moreover, one can show that

$$\mathsf{d}_T^k\mathcal{P}_r^-\Lambda^k(T) = \mathsf{d}_T^k\mathcal{P}_r\Lambda^k(T), \tag{III.19a}$$
$$\mathsf{d}_T^k\mathring{\mathcal{P}}_r^-\Lambda^k(T) = \mathsf{d}_T^k\mathring{\mathcal{P}}_r\Lambda^k(T), \tag{III.19b}$$

and that

$$\ker \mathsf{d}_T^k \cap \mathcal{P}_r^-\Lambda^k(T) = \ker \mathsf{d}_T^k \cap \mathcal{P}_{r-1}\Lambda^k(T), \tag{III.20a}$$
$$\ker \mathsf{d}_T^k \cap \mathring{\mathcal{P}}_r^-\Lambda^k(T) = \ker \mathsf{d}_T^k \cap \mathring{\mathcal{P}}_{r-1}\Lambda^k(T). \tag{III.20b}$$

These identities have been proven in [9], and we will prove them in Section III.9. They will not be used in the remaining sections of this chapter.

The constant function 1_T spans the kernel of $\mathsf{d}_T^k : C^\infty\Lambda^0(T) \to C^\infty\Lambda^1(T)$. On the other hand, $C^\infty\Lambda^n(T)$ is the direct sum of $\mathsf{d}^{n-1}C^\infty\Lambda^{n-1}(T)$ and the span of vol_T. Analogous statements hold for polynomial differential forms. It will be convenient to introduce notation for spaces with those special differential forms removed. Let $\int_T : C^\infty\Lambda^0(T) \to \mathbb{R}$ and $\int_T : C^\infty\Lambda^n(T) \to \mathbb{R}$ denote the respective integral mappings of 0- and n-forms over T. We set

$$\underline{\mathcal{P}}_r\Lambda^k(T) := \begin{cases} \mathcal{P}_r\Lambda^0(T) \cap \ker \int_T & \text{if } k = 0, \\ \mathcal{P}_r\Lambda^k(T) & \text{otherwise,} \end{cases} \tag{III.21}$$

$$\underline{\mathcal{P}}_r^-\Lambda^k(T) := \begin{cases} \mathcal{P}_r\Lambda^0(T) \cap \ker \int_T & \text{if } k = 0, \\ \mathcal{P}_r^-\Lambda^k(T) & \text{otherwise,} \end{cases} \tag{III.22}$$

$$\mathring{\underline{\mathcal{P}}}_r\Lambda^k(T) := \begin{cases} \mathring{\mathcal{P}}_r\Lambda^n(T) \cap \ker \int_T & \text{if } k = n, \\ \mathring{\mathcal{P}}_r\Lambda^k(T) & \text{otherwise,} \end{cases} \tag{III.23}$$

$$\mathring{\underline{\mathcal{P}}}_r^-\Lambda^k(T) := \begin{cases} \mathring{\mathcal{P}}_r^-\Lambda^n(T) \cap \ker \int_T & \text{if } k = n, \\ \mathring{\mathcal{P}}_r^-\Lambda^k(T) & \text{otherwise.} \end{cases} \tag{III.24}$$

We obviously have for $r \geq 0$ the direct sum decompositions

$$\mathcal{P}_r\Lambda^0(T) = \underline{\mathcal{P}}_r\Lambda^0(T) \oplus \operatorname{span}\{1_T\}, \tag{III.25}$$

$$\mathcal{P}_{r+1}^-\Lambda^0(T) = \underline{\mathcal{P}}_{r+1}^-\Lambda^0(T) \oplus \operatorname{span}\{1_T\}, \tag{III.26}$$

$$\mathring{\mathcal{P}}_r\Lambda^n(T) = \mathring{\underline{\mathcal{P}}}_r\Lambda^n(T) \oplus \operatorname{span}\{\mathrm{vol}_T\}, \tag{III.27}$$

$$\mathring{\mathcal{P}}_{r+1}^-\Lambda^n(T) = \mathring{\underline{\mathcal{P}}}_{r+1}^-\Lambda^n(T) \oplus \operatorname{span}\{\mathrm{vol}_T\}, \tag{III.28}$$

and no changes in the other cases. With these definitions at our disposal, we may concisely state that

$$\forall \omega \in \underline{\mathcal{P}}_r\Lambda^k(T) : \left(\mathsf{d}^k\omega = 0 \implies \exists \eta \in \underline{\mathcal{P}}_{r+1}^-\Lambda^{k-1} : \mathsf{d}^{k-1}\eta = \omega\right), \tag{III.29a}$$

$$\forall \omega \in \mathring{\underline{\mathcal{P}}}_r\Lambda^k(T) : \left(\mathsf{d}^k\omega = 0 \implies \exists \eta \in \mathring{\underline{\mathcal{P}}}_{r+1}^-\Lambda^{k-1} : \mathsf{d}^{k-1}\eta = \omega\right). \tag{III.29b}$$

The implications (III.29) will be proven in Section III.9 and will not be used in the remaining sections of this chapter.

III.5. Basis construction for $\mathcal{P}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r\Lambda^k(T)$

Let T be a simplex of dimension n and let $r, k \in \mathbb{Z}$. In this section we review bases for the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r\Lambda^k(T)$. This includes geometric decompositions for the space $\mathcal{P}_r\Lambda^k(T)$ and extension operators.

The canonical spanning set for $\mathcal{P}_r\Lambda^k(T)$ is given by

$$\mathcal{SP}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \alpha \in A(r, 0:n),\ \sigma \in \Sigma(1:k, 0:n) \right\}. \tag{III.30}$$

The members of $\mathcal{SP}_r\Lambda^k(T)$ are not linearly independent in general, but we can specify linearly independent subsets. A first possible choice is

$$\mathcal{B}_0\mathcal{P}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{c} \alpha \in A(r, 0:n), \\ \sigma \in \Sigma(1:k, 0:n),\ \min[\sigma] > 0 \end{array} \right\}. \tag{III.31}$$

In the case that $k = 0$ we have $[\sigma] = \emptyset$ and by convention $\min[\sigma] = \infty$. So the basis for the barycentric polynomials over T is included as a special case. Formally we prove the following two lemmas.

Lemma III.5.1.
Let $r \in \mathbb{Z}$. The set $\mathcal{B}_0\mathcal{P}_r\Lambda^0(T)$ is a basis for $\mathcal{P}_r\Lambda^0(T)$.

Proof. By definition, $\mathcal{B}_0\mathcal{P}_r\Lambda^0(T)$ is a spanning set for $\mathcal{P}_r\Lambda^0(T)$, so it remains to prove the linear independence of the members of $\mathcal{B}_0\mathcal{P}_r\Lambda^0(T)$. Suppose that $(c_\alpha)_\alpha$ are real numbers indexed over $A(r,n)$ and suppose that

$$0 = \sum_{\alpha \in A(r,n)} c_\alpha \lambda_T^\alpha. \tag{III.32}$$

We prove that $c_\alpha = 0$ for all $\alpha \in A(r,n)$ by induction along the dimension of the simplex. If T is a vertex, $\dim T = 0$, then $A(r,n)$ has only a single member and the statement follows. Next, suppose that the statement holds true over simplices with dimension strictly smaller than T. We know that $\operatorname{tr}^0_{T,F} \mathcal{B}_0\mathcal{P}_r\Lambda^0(T) = \mathcal{B}_0\mathcal{P}_r\Lambda^0(F)$. By the induction assumption we conclude that $c_\alpha = 0$ for all $\alpha \in A(r,n)$ with $[\alpha] \neq [0:n]$. It remains to show the linear independence of the interior basis functions.

To complete the proof, we use another induction argument along the polynomial order. If $r < n+1$, then $[\alpha] \neq [0:n]$ for all $\alpha \in A(r,n)$ and the claim is already proven. Next, suppose the claim is already proven for polynomial orders strictly smaller than r and that $r \geq n+1$. Then for each $\alpha \in A(r,n)$ there exists $\alpha' \in A(r-n-1,n)$ such that $\lambda_T^\alpha = \lambda_0^T \lambda_1^T \cdots \lambda_n^T \cdot \lambda_T^{\alpha'}$. Note that $\lambda_0^T \lambda_1^T \cdots \lambda_n^T$ is positive over the relative interior of T. The linear independence of $\mathcal{B}_0\mathcal{P}_r\Lambda^0(T)$ is a consequence of the linear independence of $\mathcal{B}_0\mathcal{P}_{r-n-1}\Lambda^0(T)$. This completes the second induction argument. $\square$

Lemma III.5.2.
Let $k, r \in \mathbb{Z}$. The set $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$ is a basis for $\mathcal{P}_r\Lambda^k(T)$.

Proof. Let $\alpha \in A(r, 0:n)$ and $\sigma \in \Sigma(1:k, 0:n)$. If $0 \notin [\sigma]$, then $\lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \in \mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$. If instead $0 \in [\sigma]$, then $\lambda_T^\alpha \mathsf{d}\lambda_\sigma^T$ is in the span of $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$, as can be seen by the partition of zero property. We conclude that $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$ is a spanning set for $\mathcal{P}_r\Lambda^k(T)$.

The case $k = 0$ has been treated above. For the case $k > 1$ we recall that the constant 1-forms $\mathsf{d}\lambda_1^T, \dots, \mathsf{d}\lambda_n^T$ span the cotangent space at each point. Consequently, the constant k-forms $\mathsf{d}\lambda_\sigma^T$ with $\sigma \in \Sigma(1:k, 1:n)$ span the k-th exterior power of the cotangent space at each point. Since the monomials λ_T^α for $\alpha \in A(r, 0:n)$ are linearly independent, so are the members of $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$. The proof is complete. $\square$

The basis $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$ is straight-forward to derive, but there are disadvantages when working with $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$. For example, traces of members of $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$ onto a face $F \in \Delta(T)$ generally do not contain $\mathcal{P}_r\Lambda^k(F)$ unless $k = 0$. We introduce a basis that has better properties and is almost as easy to describe. However, we need to impose the restriction that $r \geq 1$. We define

$$\mathcal{B}\mathcal{P}_r\Lambda^k(T) := \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{c} \alpha \in A(r, 0:n),\ \sigma \in \Sigma(1:k, 0:n), \\ \min[\alpha] \notin [\sigma] \end{array} \right\}. \tag{III.33}$$

Theorem III.5.3.
If $r \geq 1$, then the set $\mathcal{B}\mathcal{P}_r\Lambda^k(T)$ is a basis of $\mathcal{P}_r\Lambda^k(T)$.

Proof. First we show that $\mathcal{BP}_r\Lambda^k(T)$ spans $\mathcal{P}_r\Lambda^k(T)$. Let $\alpha \in A(r, 0 : n)$ and $\sigma \in \Sigma(1 : k, 0 : n)$ with $\lfloor\alpha\rfloor \in [\sigma]$. We find

$$\begin{aligned}
\lambda_T^\alpha \mathsf{d}\lambda_\sigma^T &= \epsilon(\lfloor\alpha\rfloor, \sigma - \lfloor\alpha\rfloor)\lambda_T^\alpha \mathsf{d}\lambda_{\lfloor\alpha\rfloor}^T \wedge \mathsf{d}\lambda_{\sigma-\lfloor\alpha\rfloor}^T \\
&= -\epsilon(\lfloor\alpha\rfloor, \sigma - \lfloor\alpha\rfloor)\lambda_T^\alpha \sum_{q\in[\sigma^c]} \mathsf{d}\lambda_q^T \wedge \mathsf{d}\lambda_{\sigma-\lfloor\alpha\rfloor}^T \\
&= -\epsilon(\lfloor\alpha\rfloor, \sigma - \lfloor\alpha\rfloor)\lambda_T^\alpha \sum_{q\in[\sigma^c]} \epsilon(q, \sigma - \lfloor\alpha\rfloor)\mathsf{d}\lambda_{\sigma-\lfloor\alpha\rfloor+q}^T \\
&= \sum_{q\in[\sigma^c]} \epsilon(\lfloor\alpha\rfloor, \sigma - \lfloor\alpha\rfloor)\epsilon(q, \sigma)\lambda_T^\alpha \mathsf{d}\lambda_{\sigma-\lfloor\alpha\rfloor+q}^T.
\end{aligned}$$

Hence the spanning set property is shown. Suppose that the members of $\mathcal{BP}_r\Lambda^k(T)$ are linearly dependent. Then there exist coefficients $c_{\alpha,\sigma} \in \mathbb{R}$ such that

$$0 = \sum_{\substack{\alpha\in A(r,0:n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor\notin[\sigma]}} c_{\alpha,\sigma}\lambda_T^\alpha \mathsf{d}\lambda_\sigma^T.$$

We define the constant k-forms

$$V_\alpha := \sum_{\substack{\sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor\notin[\sigma]}} c_{\alpha,\sigma}\mathsf{d}\lambda_\sigma^T, \quad \alpha \in A(r, 0 : n).$$

We note for each $\alpha \in A(r, 0 : n)$ that $V_\alpha = 0$ if and only if for all $\sigma \in \Sigma(1 : k, 0 : n)$ with $\lfloor\alpha\rfloor \notin [\sigma]$ we have $c_{\alpha,\sigma} = 0$. Suppose there exists $\beta \in A(r, 0 : n)$ with $V_\alpha \neq 0$. Then we let V^β denote the constant k-vector field over T such that $V_\beta(V^\beta) = 1$ everywhere over T. By assumption, we have

$$0 = \sum_{\alpha\in A(r,0:n)} \lambda_T^\alpha V_\alpha(V^\beta) = \lambda_T^\beta + \sum_{\substack{\alpha\in A(r,0:n)\\ \alpha\neq\beta}} \lambda_T^\alpha V_\alpha(V^\beta).$$

But this contradicts the linear independence of the barycentric monomials. Hence no such β exists, and we conclude that all coefficients $c_{\alpha,\sigma}$ vanish. This shows linear independence of $\mathcal{BP}_r\Lambda^k(T)$ and completes the proof. □

Our next objective is to relate this basis to the combinatorial structure of the simplex T. Specifically, we show that $\mathcal{BP}_r\Lambda^k(T)$ is well-behaved under taking the trace to subsimplices. This is formalized in the following lemma.

Lemma III.5.4.
Let $F \in \Delta(T)$ and $m = \dim F$. For each $\lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \in \mathcal{BP}_r\Lambda^k(T)$ we either have $[\alpha] \cup [\sigma] \nsubseteq [\imath_{F,T}]$ and

$$\operatorname{tr}_{T,F}^k \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T = 0,$$

or we have $[\alpha] \cup [\sigma] \subseteq [\imath_{F,T}]$, in which case there exist $\alpha' \in A(r, 0 : m)$ and $\sigma' \in \Sigma(1 : k, 0 : m)$ with

$$\operatorname{tr}_{T,F}^k \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T = \lambda_F^{\alpha'} \mathsf{d}\lambda_{\sigma'}^F, \quad \lambda_F^{\alpha'} \mathsf{d}\lambda_{\sigma'}^F \in \mathcal{BP}_r\Lambda^k(F),$$
$$\alpha' = \alpha\imath_{F,T}, \quad \imath_{F,T}\sigma' = \sigma.$$

Conversely, if $\lambda_F^\alpha \mathsf{d}\lambda_\sigma^F \in \mathcal{B}\mathcal{P}_r\Lambda^k(F)$, then there exist $\alpha' \in A(r, 0:n)$ and $\sigma' \in \Sigma(1:k, 0:n)$ such that

$$\lambda_T^{\alpha'} \mathsf{d}\lambda_{\sigma'}^T \in \mathcal{B}\mathcal{P}_r\Lambda^k(T), \quad \operatorname{tr}_{T,F}^k \lambda_T^{\alpha'} \mathsf{d}\lambda_{\sigma'}^T = \lambda_F^\alpha \mathsf{d}\lambda_\sigma^F,$$
$$\alpha = \alpha' \imath_{F,T}, \quad \imath_{F,T}\sigma = \sigma'.$$

Proof. This follows from the results of Section III.2. □

Having discussed spanning sets and bases for $\mathcal{P}_r\Lambda^k(T)$, we now address a spanning set and a basis for $\mathring{\mathcal{P}}_r\Lambda^k(T)$. Ideally, the basis should be a subset of the basis $\mathcal{B}\mathcal{P}_r\Lambda^k(T)$ for $\mathcal{P}_r\Lambda^k(T)$. We introduce

$$\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T) = \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{c} \alpha \in A(r, 0:n),\ \sigma \in \Sigma(1:k, 0:n), \\ [\alpha] \cup [\sigma] = [0:n] \end{array} \right\}. \tag{III.34}$$

and

$$\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) = \left\{ \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T \;\middle|\; \begin{array}{c} \alpha \in A(r, 0:n),\ \sigma \in \Sigma(1:k, 0:n), \\ \min[\alpha] \notin [\sigma],\ [\alpha] \cup [\sigma] = [0:n] \end{array} \right\}. \tag{III.35}$$

It is evident that

$$\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{S}\mathcal{P}_r\Lambda^k(T), \quad \mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{B}\mathcal{P}_r\Lambda^k(T), \quad \mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T) \subseteq \mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T).$$

Moreover, the following is verified easily.

Theorem III.5.5.
The set $\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T)$ is a basis for $\mathring{\mathcal{P}}_r\Lambda^k(T)$, and $\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T)$ is a spanning set for that space.

Proof. Let $\omega \in \mathring{\mathcal{P}}_r\Lambda^k(T)$. Then $\omega \in \mathcal{P}_r\Lambda^k(T)$, and thus there exist unique coefficients $c_{\alpha,\sigma}$ such that

$$\omega = \sum_{\substack{\sigma \in \Sigma(1:k,0:n) \\ \lfloor\alpha\rfloor \notin [\sigma]}} c_{\alpha,\sigma} \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T.$$

For $m \in [0:n-1]$ and $F \in \Delta(T)^m$ we then find

$$0 = \operatorname{tr}_{T,F}^k \omega = \sum_{\substack{\sigma \in \Sigma(1:k,0:n) \\ \lfloor\alpha\rfloor \notin [\sigma]}} c_{\alpha,\sigma} \operatorname{tr}_{T,F}^k \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T = \sum_{\substack{\sigma \in \Sigma(1:k,0:n) \\ \lfloor\alpha\rfloor \notin [\sigma] \\ [\alpha]\cup[\sigma] \subseteq [\imath_{F,T}]}} c_{\alpha,\sigma} \operatorname{tr}_{T,F}^k \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T.$$

By Lemma III.5.4 we thus have $c_{\alpha,\sigma} = 0$ whenever $[\alpha] \cup [\sigma] \neq [0:n]$. We thus find that $\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T)$ is a spanning set of $\mathring{\mathcal{P}}_r\Lambda^k(T)$, and a fortiori $\mathcal{S}\mathring{\mathcal{P}}_r\Lambda^k(T)$ is a spanning set too. Furthermore, $\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T)$ is linearly independent, being a subset of $\mathcal{B}\mathcal{P}_r\Lambda^k(T)$. Thus $\mathcal{B}\mathring{\mathcal{P}}_r\Lambda^k(T)$ is a basis of $\mathring{\mathcal{P}}_r\Lambda^k(T)$. The proof is complete. □

We are now in a position to address the geometric decomposition of finite element spaces. This means that we decompose the space $\mathcal{P}_r\Lambda^k(T)$ into the direct sum of subspaces associated to the subsimplices of T. The key ingredient for this are extension operators. For every subsimplex $F \in \Delta(T)$ of T we introduce the operator

$$\operatorname{ext}^{k,r}_{F,T} : \mathring{\mathcal{P}}_r\Lambda^k(F) \to \mathcal{P}_r\Lambda^k(T), \tag{III.36}$$

by taking the linear extension of setting

$$\operatorname{ext}^{k,r}_{F,T} \lambda_F^\alpha \mathsf{d}\lambda_\sigma^F := \lambda_T^{\alpha'} \mathsf{d}\lambda_{\sigma'}^T, \quad \lambda_F^\alpha \mathsf{d}\lambda_\sigma^F \in \mathcal{B}\mathcal{P}_r\Lambda^k(F),$$

where $\sigma' = \imath_{F,T}\sigma$ and $\alpha' \in A(r, 0:n)$ is $\alpha\imath_{F,T}^{-1}$ over $[\imath_{F,T}]$ and zero elsewhere. Since $\mathring{\mathcal{B}}\mathcal{P}_r\Lambda^k(F)$ is a basis of $\mathring{\mathcal{P}}_r\Lambda^k(F)$, this is well-defined.

Lemma III.5.6.
The following observations hold.

(i) For all $T \in \mathcal{T}$ we have

$$\operatorname{ext}^{k,r}_{T,T} \omega = \omega, \quad \omega \in \mathring{\mathcal{P}}_r\Lambda^k(T).$$

(ii) For all $T \in \mathcal{T}$, $F \in \Delta(T)$, and $f \in \Delta(F)$ we have

$$\operatorname{ext}^{k,r}_{f,F} \omega = \operatorname{tr}^k_{T,F} \operatorname{ext}^{k,r}_{f,T} \omega, \quad \omega \in \mathring{\mathcal{P}}_r\Lambda^k(f).$$

(iii) For all $T \in \mathcal{T}$ and $f, F \in \Delta(T)$ with $f \notin \Delta(F)$ we have

$$\operatorname{tr}^k_{T,F} \operatorname{ext}^{k,r}_{f,T} \omega = 0, \quad \omega \in \mathring{\mathcal{P}}_r\Lambda^k(f).$$

Proof. This follows again from Lemma III.5.4 and Theorem III.5.5. □

Theorem III.5.7.
For every $\omega \in \mathcal{P}_r\Lambda^k(T)$ there exist unique $\mathring{\omega}_F \in \mathring{\mathcal{P}}_r\Lambda^k(F)$ for $F \in \Delta(T)$ such that

$$\omega = \sum_{F \in \Delta(T)} \operatorname{ext}^{k,r}_{F,T} \mathring{\omega}_F.$$

Proof. According to Theorem III.5.3 there exist unique coefficients $c_{\alpha,\sigma}$ such that

$$\omega = \sum_{\substack{\sigma \in \Sigma(1:k,0:n) \\ \lfloor\alpha\rfloor \notin [\sigma]}} c_{\alpha,\sigma} \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T.$$

We define $\mathring{\omega}_F \in \mathring{\mathcal{P}}\Lambda^k(F)$ for $F \in \Delta(T)$ by

$$\mathring{\omega}_F := \sum_{\substack{\sigma \in \Sigma(1:k,0:n) \\ \lfloor\alpha\rfloor \notin [\sigma] \\ [\alpha]\cup[\sigma] = [\imath_{F,T}]}} c_{\alpha,\sigma} \operatorname{tr}^k_{T,F} \lambda_T^\alpha \mathsf{d}\lambda_\sigma^T.$$

These terms satisfy the required relation. The proof is complete. □

Remark III.5.8.
The definition of $\operatorname{ext}^{k,r}_{F,T}$ depends on the enumeration of the vertices of F and T. Though this dependence is perhaps not desirable, it is sufficient for the purpose of this thesis. We refer to Section 8 of [10] for extension operators that do not depend on the enumeration of vertices.

III.6. Basis construction for $\mathcal{P}_r^-\Lambda^k(T)$ and $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$

The agenda of the previous section for the $\mathcal{P}_r$-family of spaces is repeated in this section for the $\mathcal{P}_r^-$-family of spaces. Let T be a simplex and let $k, r \in \mathbb{Z}$ with $r \geq 1$. We consider the sets

$$\mathcal{S}\mathcal{P}_r^-\Lambda^k(T) = \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \alpha \in A(r-1,n),\ \rho \in \Sigma(0:k,0:n) \right\}, \tag{III.37}$$

and

$$\mathcal{B}\mathcal{P}_r^-\Lambda^k(T) = \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \begin{array}{c} \alpha \in A(r-1,n),\ \rho \in \Sigma(0:k,0:n), \\ \lfloor\alpha\rfloor \geq \lfloor\rho\rfloor \end{array} \right\}. \tag{III.38}$$

We easily observe that

$$\mathcal{B}\mathcal{P}_r^-\Lambda^k(T) \subseteq \mathcal{S}\mathcal{P}_r^-\Lambda^k(T).$$

By construction, $\mathcal{S}\mathcal{P}_r^-\Lambda^k(T)$ is a spanning set for $\mathcal{P}_r^-\Lambda^k(T)$. Our goal is to show that the subset $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$ is even a basis of $\mathcal{P}_r^-\Lambda^k(T)$. In a first step, we prove that it is a smaller spanning set.

Lemma III.6.1.
The set $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$ is a spanning set of $\mathcal{P}_r^-\Lambda^k(T)$.

Proof. We need to show that any $\lambda_T^\alpha \phi_\rho^T$ can be written as a linear combination of elements of $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$. If $r = 1$, then there is nothing to show, so let us assume that $r > 1$. To see this, suppose that $\lambda_T^\alpha \phi_\rho^T \in \mathcal{S}\mathcal{P}_r^-\Lambda^k(T)$ such that $\lfloor\alpha\rfloor < \lfloor\rho\rfloor$. There exists $\beta \in A(r-2,n)$ such that $\lambda_T^\alpha = \lambda_T^\beta \lambda_{\lfloor\alpha\rfloor}^T$. Using Lemma III.3.5, we find that

$$\lambda_T^\alpha \phi_\rho^T = \lambda_T^\beta \lambda_{\lfloor\alpha\rfloor}^T \phi_\rho^T = \lambda_T^\beta \sum_{j=0}^{k} (-1)^j \lambda_{\rho(j)}^T \phi_{\rho+\lfloor\alpha\rfloor-\rho(j)}^T.$$

This shows the desired result. □

Similarly as in the preceding section, we study the behavior of $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$ under taking traces.

Lemma III.6.2.
Let $F \in \Delta(T)^m$. For each $\lambda_T^\alpha \phi_\rho^T \in \mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$ we either have $[\alpha] \cup [\rho] \nsubseteq [\imath_{F,T}]$ and

$$\operatorname{tr}_{T,F}^k \lambda_T^\alpha \phi_\rho^T = 0,$$

or we have $[\alpha] \cup [\rho] \subseteq [\imath_{F,T}]$, in which case there exist $\alpha' \in A(r-1, 0:m)$ and $\rho' \in \Sigma(0:k, 0:m)$ with

$$\operatorname{tr}_{T,F}^k \lambda_T^\alpha \phi_\rho^T = \lambda_F^{\alpha'} \phi_{\rho'}^F, \quad \lambda_F^{\alpha'} \phi_{\rho'}^F \in \mathcal{B}\mathcal{P}_r^-\Lambda^k(F),$$
$$\alpha' = \alpha \imath_{F,T}, \quad \imath_{F,T}\rho' = \rho.$$

Conversely, if $\lambda_F^\alpha \phi_\rho^F \in \mathcal{B}\mathcal{P}_r^-\Lambda^k(F)$, then there exist $\alpha' \in A(r-1, 0:n)$ and $\rho' \in \Sigma(0:k, 0:n)$ such that

$$\lambda_T^{\alpha'} \phi_{\rho'}^T \in \mathcal{B}\mathcal{P}_r^-\Lambda^k(T), \quad \operatorname{tr}_{T,F}^k \lambda_T^{\alpha'} \phi_{\rho'}^T = \lambda_F^\alpha \phi_\rho^F,$$
$$\alpha = \alpha' \imath_{F,T}, \quad \imath_{F,T}\rho = \rho'.$$

Proof. This is again a consequence of the results of Section III.2. □

It remains to prove that the members of $\mathcal{BP}_r^-\Lambda^k(T)$ are linearly independent. To this end, we make a small detour and consider the set

$$\mathcal{B}_0\mathcal{P}_r^-\Lambda^k(T) = \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \begin{array}{c} \alpha \in A(r-1,n),\ \rho \in \Sigma(0:k,0:n), \\ \lfloor\rho\rfloor = 0 \end{array} \right\}. \tag{III.39}$$

Clearly, we have the inclusion

$$\mathcal{B}_0\mathcal{P}_r^-\Lambda^k(T) \subseteq \mathcal{BP}_r^-\Lambda^k(T).$$

We prove the following auxiliary result first.

Lemma III.6.3.
The set $\mathcal{B}_0\mathcal{P}_r^-\Lambda^k(T)$ is linearly independent.

Proof. Let $\omega \in \mathcal{P}_r^-\Lambda^k(T)$ be in the span of $\mathcal{B}_0\mathcal{P}_r^-\Lambda^k(T)$. Then there exist coefficients $c_{\alpha,\rho}$ with

$$\omega = \sum_{\substack{\alpha\in A(r,0:n)\\ \rho\in\Sigma(0:k,0:n)\\ \lfloor\rho\rfloor=0}} c_{\alpha,\rho}\lambda_T^\alpha \phi_\rho^T.$$

We observe that $\omega = \omega_0 + \omega_+$, where

$$\omega_0 := \sum_{\substack{\alpha\in A(r,0:n)\\ \rho\in\Sigma(0:k,0:n)\\ \lfloor\rho\rfloor=0}} c_{\alpha,\rho}\lambda_T^\alpha \lambda_0^T \mathsf{d}\lambda_{\rho-0}^T,$$

$$\omega_+ := \sum_{i=1}^{n} \mathsf{d}\lambda_i^T \wedge \sum_{\substack{\alpha\in A(r,0:n)\\ \rho\in\Sigma(0:k,0:n)\\ \lfloor\rho\rfloor=0}} \sum_{p\in[\rho-0]} \epsilon(p,\rho-0-p) c_{\alpha,\rho}\lambda_p^T \lambda_T^\alpha \mathsf{d}\lambda_{\rho-0-p}^T.$$

From the definitions of ω_0 and ω_+ we obtain a descriptions of ω_0 and ω_+ in terms of the basis $\mathcal{BP}_r\Lambda^k(T)$ of $\mathcal{P}_r\Lambda^k(T)$.

Let us assume $\omega = 0$. We prove that all coefficients $c_{\alpha,\rho}$ vanish by induction. First, it is evident that $c_{\alpha,\rho} = 0$ for $\alpha(0) = r-1$. Now let us assume that $s \in [1:r-1]$ such that $c_{\alpha,\rho} = 0$ for all $\alpha(0) \in [s:r-1]$. Since the terms $\lambda_T^\alpha\lambda_0$ with $\alpha(s) = s-1$ in the definition of ω_0 always have a higher exponent in index 0 than the terms $\lambda_p^T\lambda_T^\alpha$ in the definition of ω_+ we conclude that $c_{\alpha,\rho} = 0$ for $\alpha(0) = s-1$. Eventually we derive $c_{\alpha,\rho} = 0$ for all coefficients. The proof is complete. □

Theorem III.6.4.
The set $\mathcal{BP}_r^-\Lambda^k(T)$ is a basis of $\mathcal{P}_r^-\Lambda^k(T)$.

Proof. Since $\mathcal{BP}_r^-\Lambda^k(T)$ is a spanning set of $\mathcal{P}_r^-\Lambda^k(T)$, it only remains to prove its linear independence. We prove the claim by induction on the dimension of T. We start with the case $\dim T = k$. Here we notice that $\Sigma(0:k,0:k)$ has only one single member, and so the claim holds due to the linear independence of the barycentric polynomials.

Next, let $\dim T = m+1$ and let us assume that the claim holds the face $F \in \Delta(T)^m$. We have

$$\mathcal{P}_r^-\Lambda^k(T) = \operatorname{span} \mathcal{B}_0\mathcal{P}_r^-\Lambda^k(T) + \operatorname{span} \mathcal{B}\mathcal{P}_r^-\Lambda^k(T) \setminus \mathcal{B}_0\mathcal{P}_r^-\Lambda^k(T)$$

It is now evident that $\mathcal{B}_0\mathcal{P}_r^-\Lambda^k(T) \subseteq \ker \operatorname{tr}^k_{T,F}$. On the other hand, $\operatorname{tr}^k_{T,F}$ yields an isomorphism between the span of $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T) \setminus \mathcal{B}_0\mathcal{P}_r^-\Lambda^k$ and the span of $\mathcal{B}\mathcal{P}_r^-\Lambda^k(F)$. This implies the desired result. □

Next we study bases and spanning sets for the spaces with homogeneous boundary conditions. We first introduce the sets

$$\mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T) = \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \begin{array}{c} \alpha \in A(r-1,n),\ \rho \in \Sigma(0:k,0:n), \\ [\alpha] \cup [\rho] = [0:n] \end{array} \right\} \tag{III.40}$$

and

$$\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) = \left\{ \lambda_T^\alpha \phi_\rho^T \;\middle|\; \begin{array}{c} \alpha \in A(r-1,n),\ \rho \in \Sigma(0:k,0:n), \\ \lfloor \rho \rfloor = 0,\ [\alpha] \cup [\rho] = [0:n] \end{array} \right\}. \tag{III.41}$$

We observe the inclusions

$$\mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T) \subseteq \mathcal{S}\mathcal{P}_r^-\Lambda^k(T), \quad \mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T) \subseteq \mathcal{B}\mathcal{P}_r^-\Lambda^k(T).$$

Theorem III.6.5.
The set $\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ is a basis of $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$, and $\mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ is a spanning set for $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$.

Proof. Let $\omega \in \mathring{\mathcal{P}}_r^-\Lambda^k(T)$. Then $\omega \in \mathcal{P}_r^-\Lambda^k(T)$, and thus there exist unique coefficients $c_{\alpha,\rho}$ such that

$$\omega = \sum_{\substack{\alpha \in A(r-1,n) \\ \rho \in \Sigma(0:k,0:n) \\ \lfloor \alpha \rfloor \geq \lfloor \rho \rfloor}} c_{\alpha,\rho} \lambda_T^\alpha \phi_\rho^T.$$

When F is a face of T, then we find

$$0 = \operatorname{tr}^k_{T,F} \omega = \sum_{\substack{\alpha \in A(r-1,n) \\ \rho \in \Sigma(0:k,0:n) \\ \lfloor \alpha \rfloor \geq \lfloor \rho \rfloor}} c_{\alpha,\rho} \operatorname{tr}^k_{T,F} \lambda_T^\alpha \phi_\rho^T = \sum_{\substack{\alpha \in A(r-1,n) \\ \rho \in \Sigma(0:k,0:n) \\ \lfloor \alpha \rfloor \geq \lfloor \rho \rfloor \\ [\alpha] \cup [\rho] \subseteq [\imath_{F,T}]}} c_{\alpha,\rho} \operatorname{tr}^k_{T,F} \lambda_T^\alpha \phi_\rho^T.$$

We thus find that

$$\omega = \sum_{\substack{\alpha \in A(r-1,n) \\ \rho \in \Sigma(0:k,0:n) \\ \lfloor \alpha \rfloor \geq \lfloor \rho \rfloor \\ [\alpha] \cup [\rho] = [0:n]}} c_{\alpha,\rho} \lambda_T^\alpha \phi_\rho^T.$$

Hence $\mathcal{B}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ is a spanning set of $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$, and moreover it is linearly independent, being a subset of $\mathcal{B}\mathcal{P}_r^-\Lambda^k(T)$. It follows that $\mathcal{S}\mathring{\mathcal{P}}_r^-\Lambda^k(T)$ is a spanning set of $\mathring{\mathcal{P}}_r^-\Lambda^k(T)$. The proof is complete. □

Eventually, we can define an extension operator that facilitates a geometric decomposition. Whenever F is a subsimplex of T, we consider the operator

$$\operatorname{ext}^{r,k,-}_{F,T} : \mathring{\mathcal{P}}^{-}_{r}\Lambda^{k}(F) \to \mathcal{P}^{-}_{r}\Lambda^{k}(T),$$

which is defined by setting

$$\operatorname{ext}^{r,k,-}_{F,T} \lambda^{\alpha}_{F}\phi^{F}_{\rho} = \lambda^{\iota(F,T)\alpha}_{T}\phi^{T}_{\iota(F,T)\rho}, \quad \lambda^{\alpha}_{F}\phi^{F}_{\rho} \in \mathcal{B}\mathcal{P}^{-}_{r}\Lambda^{k}(F).$$

where $\rho' = \imath_{F,T}\rho$ and $\alpha' \in A(r-1,0:n)$ is $\alpha\imath^{-1}_{F,T}$ over $[\imath_{F,T}]$ and zero elsewhere. Since $\mathring{\mathcal{B}}\mathcal{P}^{-}_{r}\Lambda^{k}(F)$ is a basis of $\mathring{\mathcal{P}}^{-}_{r}\Lambda^{k}(F)$, this is well-defined.

Lemma III.6.6.
The following observations hold.

(i) For all $T \in \mathcal{T}$ we have

$$\operatorname{ext}^{k,r,-}_{T,T}\, \omega = \omega, \quad \omega \in \mathring{\mathcal{P}}^{-}_{r}\Lambda^{k}(T).$$

(ii) For all $T \in \mathcal{T}$, $F \in \Delta(T)$, and $f \in \Delta(F)$ we have

$$\operatorname{ext}^{k,r,-}_{f,F}\, \omega = \operatorname{tr}^{k}_{T,F} \operatorname{ext}^{k,r,-}_{f,T}\, \omega, \quad \omega \in \mathring{\mathcal{P}}^{-}_{r}\Lambda^{k}(f).$$

(iii) For all $T \in \mathcal{T}$ and $f, F \in \Delta(T)$ with $f \notin \Delta(F)$ we have

$$\operatorname{tr}^{k}_{T,F} \operatorname{ext}^{k,r,-}_{f,T}\, \omega = 0, \quad \omega \in \mathring{\mathcal{P}}^{-}_{r}\Lambda^{k}(f).$$

Proof. This follows again from Lemma III.5.4 and Theorem III.5.5. □

Theorem III.6.7.
For every $\omega \in \mathcal{P}^{-}_{r}\Lambda^{k}(T)$ there exist unique $\mathring{\omega}_{F} \in \mathring{\mathcal{P}}^{-}_{r}\Lambda^{k}(F)$ for $F \in \Delta(T)$ such that

$$\omega = \sum_{F \in \Delta(T)} \operatorname{ext}^{k,r,-}_{F,T}\, \mathring{\omega}_{F}.$$

Proof. According to Theorem III.6.4 there exist unique coefficients $c_{\alpha,\rho}$ such that

$$\omega = \sum_{\substack{\rho \in \Sigma(0:k,0:n) \\ \lfloor \alpha \rfloor \notin [\rho]}} c_{\alpha,\rho} \lambda^{\alpha}_{T} \mathsf{d}\lambda^{T}_{\rho}.$$

We define $\mathring{\omega}_{F} \in \mathring{\mathcal{P}}\Lambda^{k}(F)$ for $F \in \Delta(T)$ by

$$\mathring{\omega}_{F} := \sum_{\substack{\rho \in \Sigma(0:k,0:n) \\ \lfloor \alpha \rfloor \notin [\rho] \\ [\alpha] \cup [\rho] = [\imath_{F,T}]}} c_{\alpha,\rho} \operatorname{tr}^{k}_{T,F} \lambda^{\alpha}_{T} \mathsf{d}\lambda^{T}_{\rho}.$$

These terms satisfy the required relation. The proof is complete. □

Remark III.6.8.
The bases for $\mathcal{P}^{-}_{r}\Lambda^{k}(T)$ and $\mathring{\mathcal{P}}^{-}_{r}\Lambda^{k}(T)$ are identical to the bases presented or implied in Section 4 of [9] or in [10]. Moreover, our extension operator can easily be generalized to the extension operator in Section 8 of [10]. In particular, it does not depend on the enumeration of vertices.

III.7. Linear Dependencies

The two major families of finite element differential forms over simplices can be related in a curious manner. Arnold, Falk and Winther's study of bases and spanning sets in [9] utilizes isomorphisms

$$\mathcal{P}_r\Lambda^k(T) \simeq \mathring{\mathcal{P}}^-_{r+k}\Lambda^{n-k}(T), \quad \mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T) \simeq \mathcal{P}^-_{r+1}\Lambda^{n-k}(T).$$

Even though these isomorphisms play a central role in [9], not much research has been invested so far. We will elaborate several aspects of these mappings and show how they expose linear independencies in the canonical spanning sets of the finite element spaces. Throughout this section we let T be an n-dimensional simplex. Moreover, in this and the next section we write

$$\begin{aligned} \lambda^\sigma &:= \lambda_{\sigma(1)} \cdots \lambda_{\sigma(k)}, \quad \sigma \in \Sigma(1:k, 0:n), \quad k \in \mathbb{Z}, \\ \lambda^\rho &:= \lambda_{\rho(0)} \cdots \lambda_{\rho(k)}, \quad \rho \in \Sigma(0:k, 0:n), \quad k \in \mathbb{Z}. \end{aligned}$$

First, we consider a relation between $\mathcal{P}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}^-_{r+k}\Lambda^k(T)$. Note that a part of the following statement is implied already by Proposition 3.7 of [57].

Lemma III.7.1.
Let $k, r \in \mathbb{Z}$. Let $\omega_{\alpha,\sigma} \in \mathbb{R}$ for $\sigma \in \Sigma(1:k, 0:n)$ and $\alpha \in A(r, n)$. Then

$$\sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n)}} \omega_{\alpha,\sigma}\lambda^\alpha \mathsf{d}\lambda_\sigma = 0 \quad \Longleftrightarrow \quad \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n)}} \epsilon(\sigma, \sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\lambda^\sigma\phi_{\sigma^c} = 0, \tag{III.42}$$

which is the case if and only if the condition

$$\omega_{\alpha,\sigma} - \sum_{p \in [\sigma]} \epsilon(p, \sigma - p)\omega_{\alpha,\sigma-p+0} = 0 \tag{III.43}$$

holds for $\alpha \in A(r, n)$ and $\sigma \in \Sigma(1:k, 0:n)$ with $0 \notin [\sigma]$.

Proof. In the special case $k = 0$, the statement is trivial. We prove the statement for the case $1 \leq k \leq n$. Let $\sigma \in \Sigma(1:k, 0:n)$ with $0 \in [\sigma]$.

For $q \in [\sigma^c]$, it is an elementary fact that $\epsilon(q, \sigma - 0) = -\epsilon(q, \sigma)$. By combinatorial arguments and the partition of zero of the barycentric differentials, we first find

$$\begin{aligned} \mathsf{d}\lambda_\sigma = \mathsf{d}\lambda_0 \wedge \mathsf{d}\lambda_{\sigma-0} &= -\sum_{q \in [\sigma^c]} \mathsf{d}\lambda_q \wedge \mathsf{d}\lambda_{\sigma-0} \\ &= -\sum_{q \in [\sigma^c]} \epsilon(q, \sigma - 0)\mathsf{d}\lambda_{\sigma-0+q} = \sum_{q \in [\sigma^c]} \epsilon(q, \sigma)\mathsf{d}\lambda_{\sigma-0+q}. \end{aligned}$$

We calculate that

$$
\begin{aligned}
S_L &:= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)}} \omega_{\alpha,\sigma}\lambda^\alpha \mathsf{d}\lambda_\sigma \\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\notin[\sigma]}} \omega_{\alpha,\sigma}\lambda^\alpha \mathsf{d}\lambda_\sigma + \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\in[\sigma]}} \omega_{\alpha,\sigma}\lambda^\alpha \mathsf{d}\lambda_\sigma \\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\notin[\sigma]}} \omega_{\alpha,\sigma}\lambda^\alpha \mathsf{d}\lambda_\sigma + \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\in[\sigma]}} \omega_{\alpha,\sigma}\lambda^\alpha \sum_{q\in[\sigma^c]} \epsilon(q,\sigma)\mathsf{d}\lambda_{\sigma-0+q} \\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\notin[\sigma]}} \left(\omega_{\alpha,\sigma} + \sum_{p\in[\sigma]} \epsilon(p,\sigma-p+0)\omega_{\alpha,\sigma-p+0} \right) \lambda^\alpha \mathsf{d}\lambda_\sigma \\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\notin[\sigma]}} \left(\omega_{\alpha,\sigma} - \sum_{p\in[\sigma]} \epsilon(p,\sigma-p)\omega_{\alpha,\sigma-p+0} \right) \lambda^\alpha \mathsf{d}\lambda_\sigma .
\end{aligned}
$$

This is an expression in terms of the basis $\mathcal{B}_0\mathcal{P}_r\Lambda^k(T)$ of $\mathcal{P}_r\Lambda^k(T)$.

Next, using Lemma III.3.5 we find

$$
\lambda^\sigma \phi_{\sigma^c} = \lambda^{\sigma-0}\lambda^0\phi_{\sigma^c} = \lambda^{\sigma-0} \sum_{q\in[\sigma^c]} \epsilon(q,\sigma)\lambda^q \phi_{\sigma^c-q+0}.
$$

We then calculate

$$
\begin{aligned}
&\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\in[\sigma]}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\lambda^\sigma\phi_{\sigma^c}, \\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\in[\sigma]}} \sum_{q\in[\sigma^c]} \epsilon(\sigma,\sigma^c)\epsilon(q,\sigma^c-q)\omega_{\alpha,\sigma}\lambda^\alpha\lambda^{\sigma-0+q}\phi_{\sigma^c-q+0},
\end{aligned}
$$

so that

$$
\begin{aligned}
S_R &:= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\lambda^\sigma\phi_{\sigma^c} \\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\notin[\sigma]}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\lambda^\sigma\phi_{\sigma^c} + \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\in[\sigma]}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\lambda^\sigma\phi_{\sigma^c} \\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ 0\notin[\sigma]}} \left(\epsilon(\sigma,\sigma^c)u_\sigma + \sum_{p\in[\sigma]} \epsilon(\sigma-p+0,\sigma^c+p-0)\epsilon(p,\sigma^c-0)u_{\sigma-p+0} \right) \lambda^\alpha\lambda^\sigma\phi_{\sigma^c}
\end{aligned}
$$

holds. This is an expression in terms of the basis $\mathcal{B}\mathring{\mathcal{P}}^-_{r+k}\Lambda^k(T)$ of $\mathring{\mathcal{P}}^-_{r+k}\Lambda^k(T)$.

The proof is finished if we prove $S_L = S_R$. The combinatorial observation

$$\begin{aligned}
-\epsilon(\sigma,\sigma^c)\epsilon(p,\sigma-p) &= (-1)^k\epsilon(\sigma,\sigma^c)\epsilon(\sigma-p,p)\\
&= -\epsilon(\sigma,\sigma^c)\epsilon(\sigma-p,p)\epsilon(\sigma-p,0)\\
&= -\epsilon(\sigma,\sigma^c)\epsilon(\sigma-p,p)\epsilon(0,\sigma^c-0)\epsilon(\sigma-p,0)\\
&= \epsilon(\sigma-p+0,\sigma^c+p-0)\epsilon(p,\sigma^c-0)
\end{aligned}$$

for $\sigma \in \Sigma(1:k,0:n)$ with $0 \notin [\sigma]$ and $p \in [\sigma]$ accomplishes that. □

With very similar methods we prove a relation between the finite element spaces $\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T)$ and $\mathcal{P}^-_{r+1}\Lambda^{n-k}(T)$. This statement is an expectable but new analogue of Proposition 3.7 in [57].

Lemma III.7.2.
Let $k, r \in \mathbb{Z}$. Let $\omega_{\alpha,\sigma} \in \mathbb{R}$ for $\sigma \in \Sigma(1:k,0:n)$ and $\alpha \in A(r,n)$. Then

$$\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)}} \omega_{\alpha,\sigma}\lambda^\alpha\lambda^\sigma \mathsf{d}\lambda_\sigma = 0 \quad\Longleftrightarrow\quad \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\phi_{\sigma^c} = 0, \tag{III.44}$$

which is the case if and only if the condition

$$\omega_{\alpha,\sigma} - \sum_{q\in[\sigma]\cap[\alpha]} \epsilon(\lfloor\sigma^c\rfloor,\sigma-q)\epsilon(q,\sigma-q)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q} = 0 \tag{III.45}$$

holds for $\alpha \in A(r,n)$ and $\sigma \in \Sigma(1:k,0:n)$ with $\lfloor\alpha\rfloor \geq \lfloor\sigma^c\rfloor$.

Proof. In the special case $k = 0$, the statement is trivial. So let us assume $k > 0$. Using Lemma III.3.5 again, the computation

$$\begin{aligned}
\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor<\lfloor\sigma^c\rfloor}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\phi_{\sigma^c} &= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor<\lfloor\sigma^c\rfloor}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^{\alpha-\lfloor\alpha\rfloor}\lambda^{\lfloor\alpha\rfloor}\phi_{\sigma^c}\\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor<\lfloor\sigma^c\rfloor}} \sum_{\substack{q\in[\sigma^c]\\ q\neq\lfloor\alpha\rfloor}} \epsilon(\sigma,\sigma^c)\epsilon(q,\sigma^c-q)\omega_{\alpha,\sigma}\lambda^{\alpha-\lfloor\alpha\rfloor+q}\phi_{\sigma^c+\lfloor\alpha\rfloor-q}
\end{aligned}$$

is easily verified. From this we conclude on the one hand that

$$\begin{aligned}
S_L &:= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\phi_{\sigma^c}\\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor\geq\lfloor\sigma^c\rfloor}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\phi_{\sigma^c} + \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor<\lfloor\sigma^c\rfloor}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\phi_{\sigma^c}\\
&= \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor\geq\lfloor\sigma^c\rfloor}} \epsilon(\sigma,\sigma^c)\omega_{\alpha,\sigma}\lambda^\alpha\phi_{\sigma^c}\\
&\quad + \sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor<\lfloor\sigma^c\rfloor}} \sum_{\substack{q\in[\sigma^c]\\ q\neq\lfloor\alpha\rfloor}} \epsilon(\sigma,\sigma^c)\epsilon(q,\sigma^c-q)\omega_{\alpha,\sigma}\lambda^{\alpha-\lfloor\alpha\rfloor+q}\phi_{\sigma^c+\lfloor\alpha\rfloor-q}.
\end{aligned}$$

For $\sigma\in\Sigma(1:k,0:n)$, $\alpha\in A(r,n)$ and $q\in[\sigma]\cap[\alpha]$ with $\lfloor\alpha\rfloor\geq\lfloor\sigma^c\rfloor$ we find

$$\begin{aligned}
&\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor<\lfloor\sigma^c\rfloor}} \sum_{\substack{q\in[\sigma^c]\\ q\neq\lfloor\alpha\rfloor}} \epsilon(\sigma,\sigma^c)\epsilon(q,\sigma^c-q)\omega_{\alpha,\sigma}\lambda^{\alpha-\lfloor\alpha\rfloor+q}\phi_{\sigma^c+\lfloor\alpha\rfloor-q},\\
= &\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor\geq\lfloor\sigma^c\rfloor\\ q\in[\sigma]\cap[\alpha]}} \epsilon(\sigma+\lfloor\sigma^c\rfloor-q,\sigma^c-\lfloor\sigma^c\rfloor+q)\epsilon(q,\sigma^c-\lfloor\sigma^c\rfloor)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q}\lambda^\alpha\phi_{\sigma^c}.
\end{aligned}$$

The combinatorial observation

$$\begin{aligned}
&\epsilon(\sigma+\lfloor\sigma^c\rfloor-q,\sigma^c-\lfloor\sigma^c\rfloor+q)\epsilon(q,\sigma^c-\lfloor\sigma^c\rfloor)\\
&= -\epsilon(\sigma,\sigma^c)\epsilon(\sigma-q,q)\epsilon(\lfloor\sigma^c\rfloor,\sigma^c-\lfloor\sigma^c\rfloor)\epsilon(\sigma-q,\lfloor\sigma^c\rfloor)\epsilon(q,\sigma^c-\lfloor\sigma^c\rfloor)\epsilon(q,\sigma^c-\lfloor\sigma^c\rfloor)\\
&= -\epsilon(\sigma,\sigma^c)\epsilon(\sigma-q,q)\epsilon(\sigma-q,\lfloor\sigma^c\rfloor)
\end{aligned}$$

proves that S_L equals

$$\sum_{\substack{\alpha\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ \lfloor\alpha\rfloor\geq\lfloor\sigma^c\rfloor}} \epsilon(\sigma,\sigma^c)\left(\omega_{\alpha,\sigma} - \sum_{q\in[\sigma]\cup[\alpha]} \epsilon(\sigma-q,q)\epsilon(\sigma-q,\lfloor\sigma^c\rfloor)\omega_{\alpha+\lfloor\sigma^c\rfloor-q,\sigma+\lfloor\sigma^c\rfloor-q}\right)\lambda^\alpha\phi_{\sigma^c}.$$

This an expression in terms of the basis $\mathcal{BP}_r^-\Lambda^{n-k}(T)$ of $\mathcal{P}_r^-\Lambda^{n-k}(T)$.

On the other hand, we find

$$
\begin{aligned}
&- \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n) \\ \lfloor \alpha \rfloor < \lfloor \sigma^c \rfloor}} \omega_{\alpha,\sigma} \lambda^\alpha \lambda^{\sigma^c} \mathsf{d}\lambda_\sigma \\
&= - \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n) \\ \lfloor \alpha \rfloor < \lfloor \sigma^c \rfloor}} \omega_{\alpha,\sigma} \lambda^\alpha \lambda^{\sigma^c} \epsilon(\lfloor \alpha \rfloor, \sigma - \lfloor \alpha \rfloor) \mathsf{d}\lambda_{\lfloor \alpha \rfloor} \wedge \mathsf{d}\lambda_{\sigma - \lfloor \alpha \rfloor} \\
&= \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n) \\ \lfloor \alpha \rfloor < \lfloor \sigma^c \rfloor}} \sum_{\substack{q \in [\sigma^c] \\ q \neq \lfloor \alpha \rfloor}} \omega_{\alpha,\sigma} \lambda^\alpha \lambda^{\sigma^c} \epsilon(\lfloor \alpha \rfloor, \sigma - \lfloor \alpha \rfloor) \epsilon(q, \sigma - \lfloor \alpha \rfloor) \mathsf{d}\lambda_{\sigma - \lfloor \alpha \rfloor + q} \\
&= \sum_{\substack{\beta \in A(r,n) \\ \rho \in \Sigma(1:k,0:n) \\ \lfloor \beta \rfloor \geq \lfloor \rho^c \rfloor}} \sum_{q \in [\rho] \cap [\beta]} \omega_{\beta + \lfloor \rho^c \rfloor - q, \rho + \lfloor \rho^c \rfloor - q} \lambda^{\beta + \lfloor \rho^c \rfloor - q} \lambda^{\rho^c - \lfloor \rho^c \rfloor + q} \epsilon(\lfloor \rho^c \rfloor, \rho - q) \epsilon(q, \rho - q) \mathsf{d}\lambda_\rho \\
&= \sum_{\substack{\beta \in A(r,n) \\ \rho \in \Sigma(1:k,0:n) \\ \lfloor \beta \rfloor \geq \lfloor \rho^c \rfloor}} \sum_{q \in [\rho] \cap [\beta]} \omega_{\beta + \lfloor \rho^c \rfloor - q, \rho + \lfloor \rho^c \rfloor - q} \lambda^{\beta} \lambda^{\rho^c} \epsilon(\lfloor \rho^c \rfloor, \rho - q) \epsilon(q, \rho - q) \mathsf{d}\lambda_\rho.
\end{aligned}
$$

We thus infer

$$
\begin{aligned}
S_R &:= \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n)}} \omega_{\alpha,\sigma} \lambda^\alpha \lambda^{\sigma^c} \mathsf{d}\lambda_\sigma \\
&= \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n) \\ \lfloor \alpha \rfloor \geq \lfloor \sigma^c \rfloor}} \left(\omega_{\alpha,\sigma} - \sum_{q \in [\sigma] \cap [\alpha]} \epsilon(\lfloor \sigma^c \rfloor, \sigma - q) \epsilon(q, \sigma - q) \omega_{\alpha + \lfloor \sigma^c \rfloor - q, \sigma + \lfloor \sigma^c \rfloor - q} \right) \lambda^\alpha \lambda^{\sigma^c} \mathsf{d}\lambda_\sigma.
\end{aligned}
$$

This is an expression in terms of a basis of $\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T)$. Thus the desired statement $S_L = S_R$ follows. □

These results give correspondences between the linear dependencies of the canonical spanning sets of $\mathcal{P}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T)$, and between the linear dependencies of the canonical spanning sets of $\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T)$ and $\mathcal{P}^-_{r+1}\Lambda^{n-k}(T)$. An immediate application is the well-definedness of the following isomorphisms.

We have a linear isomorphism $\mathcal{P}_r\Lambda^k(T)$ to $\mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T)$ which is defined via

$$
\sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n)}} \omega_{\alpha,\sigma} \lambda^\alpha \mathsf{d}\lambda_\sigma \mapsto \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n)}} \omega_{\alpha,\sigma} \lambda^\alpha \lambda^\sigma \phi_{\sigma^c} \tag{III.46}
$$

and a linear isomorphism from $\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T)$ to $\mathcal{P}_{r+1}\Lambda^{n-k}(T)$ which is defined via

$$
\sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n)}} \omega_{\alpha,\sigma} \lambda^\alpha \phi_{\sigma^c} \mapsto \sum_{\substack{\alpha \in A(r,n) \\ \sigma \in \Sigma(1:k,0:n)}} \omega_{\alpha,\sigma} \lambda^\alpha \lambda^{\sigma^c} \mathsf{d}\lambda_\sigma. \tag{III.47}
$$

Lemma III.7.1 and Lemma III.7.2 imply the well-definedness of those mappings.

These results produce conditions under which a finite element differential form vanishes, expressed in the canonical spanning set. We finish this section with two auxiliary results that provide coefficient conditions equivalent to the ones encountered in the previous two lemmas but which are easier to handle in some situations, including the next chapter.

Lemma III.7.3.
Let $k, r \in \mathbb{Z}$ and let $\omega_{\alpha,\sigma} \in \mathbb{R}$ for $\alpha \in A(r,n)$ and $\sigma \in \Sigma(1:k, 0:n)$. Then the condition

$$\omega_{\alpha,\sigma} - \sum_{p \in [\sigma]} \epsilon(p, \sigma - p)\omega_{\alpha,\sigma-p+0} = 0 \tag{III.48}$$

holds for $\alpha \in A(r,n)$ and $\sigma \in \Sigma(1:k, 0:n)$ with $0 \notin [\sigma]$ if and only the condition

$$\sum_{p \in [\theta]} \epsilon(p, \theta - p)\omega_{\alpha,\theta-p} = 0 \tag{III.49}$$

holds for $\alpha \in A(r,n)$ and $\theta \in \Sigma(1:k+1, 0:n)$.

Proof. The lemma is trivial if $k = 0$, so let us assume that $1 \leq k \leq n$. Clearly, the second claim implies the first, so we assume the first claim holds. Then the second claim holds for all θ with $0 \in [\theta]$. If instead $0 \notin [\theta]$, then we find

$$\begin{aligned}\sum_{p \in [\theta]} \epsilon(p, \theta - p)\omega_{\alpha,\theta-p} &= \sum_{p \in [\theta]} \sum_{s \in [\theta - p]} \epsilon(p, \theta - p)\epsilon(s, \theta - p - s)\omega_{\alpha,\theta-p-s+0} \\ &= \sum_{p \in [\theta]} \sum_{s \in [\theta - p]} \epsilon(p, s)\epsilon(p, \theta - p)\epsilon(s, \theta - p - s)\omega_{\alpha,\theta-p-s+0}.\end{aligned}$$

Antisymmetry implies that this sum vanishes. The lemma is proven. □

We devise an analogous result that extends Lemma III.7.2

Lemma III.7.4.
Let $k, r \in \mathbb{Z}$ and let $\omega_{\alpha,\sigma} \in \mathbb{R}$ for $\alpha \in A(r,n)$ and $\sigma \in \Sigma(1:k, 0:n)$. The condition

$$\omega_{\alpha,\sigma} - \sum_{q \in [\sigma] \cap [\alpha]} \epsilon(\lfloor \sigma^c \rfloor, \sigma - q)\epsilon(q, \sigma - q)\omega_{\alpha+\lfloor \sigma^c \rfloor - q, \sigma + \lfloor \sigma^c \rfloor - q} = 0$$

holds for $\alpha \in A(r,n)$ and $\sigma \in \Sigma(1:k, 0:n)$ with $\lfloor \alpha \rfloor \geq \lfloor \sigma \rfloor$ if and only the condition

$$\sum_{\beta \in A(r+1,n)} \sum_{p \in [\theta]} \epsilon(\theta - p, p)\omega_{\beta,\theta-p} = 0$$

holds for $\beta \in A(r+1,n)$ and $\theta \in \Sigma(1:k+1, 0:n)$.

Proof. The lemma is trivial if $k = 0$, so let us assume that $1 \leq k \leq n$. The first condition has several equivalent formulations:

$$\begin{aligned}
& \omega_{\alpha,\sigma} - \sum_{q \in [\sigma] \cap [\alpha]} \epsilon(\lfloor \sigma^c \rfloor, \sigma - q)\epsilon(q, \sigma - q)\omega_{\alpha + \lfloor \sigma^c \rfloor - q, \sigma + \lfloor \sigma^c \rfloor - q} = 0 \\
\Longleftrightarrow \quad & \omega_{\alpha,\sigma} - \sum_{q \in [\sigma] \cap [\alpha]} \epsilon(\lfloor \sigma^c \rfloor, \sigma)\epsilon(\lfloor \sigma^c \rfloor, q)\epsilon(q, \sigma - q)\omega_{\alpha + \lfloor \sigma^c \rfloor - q, \sigma + \lfloor \sigma^c \rfloor - q} = 0 \\
\Longleftrightarrow \quad & \epsilon(\lfloor \sigma^c \rfloor, \sigma)\omega_{\alpha,\sigma} + \sum_{q \in [\sigma] \cap [\alpha]} \epsilon(q, \lfloor \sigma^c \rfloor)\epsilon(q, \sigma - q)\omega_{\alpha + \lfloor \sigma^c \rfloor - q, \sigma + \lfloor \sigma^c \rfloor - q} = 0 \\
\Longleftrightarrow \quad & \epsilon(\lfloor \sigma^c \rfloor, \sigma)\omega_{\alpha,\sigma} + \sum_{q \in [\sigma] \cap [\alpha]} \epsilon(q, \sigma + \lfloor \sigma^c \rfloor - q)\omega_{\alpha + \lfloor \sigma^c \rfloor - q, \sigma + \lfloor \sigma^c \rfloor - q} = 0 \\
\Longleftrightarrow \quad & \sum_{q \in [\sigma + \lfloor \sigma^c \rfloor] \cap [\alpha + \lfloor \sigma^c \rfloor]} \epsilon(q, \sigma + \lfloor \sigma^c \rfloor - q)\omega_{\alpha + \lfloor \sigma^c \rfloor - q, \sigma + \lfloor \sigma^c \rfloor - q} = 0.
\end{aligned}$$

It is now obvious that the second condition implies the first condition.

Let us assume in turn that the first condition holds, and derive the second condition. From the first condition we conclude that the second condition already holds for $\beta \in A(r+1, n)$ and $\theta \in \Sigma(1 : k+1, 0 : n)$ for which there exists $\sigma \in \Sigma(1 : k, 0 : n)$ and $\alpha \in A(r, n)$ such that $\theta = \sigma + \lfloor \sigma^c \rfloor$ and $\beta = \alpha + \lfloor \sigma^c \rfloor$.

But we that $\theta = \sigma + \lfloor \sigma^c \rfloor$ if and only if $0 \in [\theta]$ and $\lfloor \sigma^c \rfloor = 0$. So it remains to show the second condition for the case $0 \notin [\theta] \cap [\beta]$. For such θ and β, we find

$$\begin{aligned}
& \sum_{p \in [\theta] \cap [\beta]} \epsilon(\theta - p, p)\omega_{\beta - p, \theta - p} \\
&= - \sum_{p \in [\theta] \cap [\beta]} \sum_{s \in [\theta] \cap [\beta] \setminus \{p\}} \epsilon(\theta - p, p)\epsilon(s, \theta - p + 0 - s)\omega_{\beta - p + 0 - s, \theta - p + 0 - s} \\
&= (-1)^k \sum_{p \in [\theta] \cap [\beta]} \sum_{s \in [\theta] \cap [\beta] \setminus \{p\}} \epsilon(\theta - p, p)\epsilon(s, \theta - p + 0 - s)\omega_{\beta - p + 0 - s, \theta - p + 0 - s}
\end{aligned}$$

using the first condition. But with the combinatorial observation

$$\begin{aligned}
\epsilon(\theta - p, p)\epsilon(s, \theta - p + 0 - s) &= \epsilon(\theta + 0 - p, p)\epsilon(s, \theta - p + 0 - s) \\
&= -\epsilon(\theta + 0 - p, p)\epsilon(s, p)\epsilon(s, \theta + 0 - s)
\end{aligned}$$

we eventually tell that the sum vanishes if and only if

$$0 = \sum_{\substack{s, p \in [\theta] \cap [\beta] \\ p \neq s}} \epsilon(\theta + 0 - p, p)\epsilon(s, p)\epsilon(s, \theta + 0 - s)\omega_{\beta - p + 0 - s, \theta - p + 0 - s}.$$

This holds because the terms in the sum cancel. The statement is proven. □

III.8. Duality Pairings

Our next goal is to refine the results of the preceding section. Corresponding to the isomorphisms

$$\mathcal{P}_r\Lambda^k(T) \simeq \mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T), \quad \mathcal{P}^-_{r+1}\Lambda^{n-k}(T) \simeq \mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T),$$

there exist two non-degenerate bilinear pairings: the first between $\mathcal{P}_r\Lambda^k(T)$ and $\mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T)$, and the second between $\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T)$ and $\mathcal{P}^-_{r+1}\Lambda^{n-k}(T)$. Similar to the isomorphisms, those bilinear pairings have already been used in the seminal publication of Arnold, Falk, and Winther [9], but not much further study has been applied. The first pairing is provided by our Theorem III.8.2, which is a refinement of Proposition 3.1 in [57].

We continue to assume that T is an n-dimensional simplex. We begin with a technical auxiliary result.

Lemma III.8.1.
Let $k, r \in \mathbb{Z}$ and $\sigma, \rho \in \Sigma(1:k, 0:n)$. Then the following holds true.

- If $[\sigma] \cap [\rho^c]$ has cardinality greater than one, then
$$\mathsf{d}\lambda_\sigma \wedge \phi_{\rho^c} = 0. \tag{III.50a}$$
- If $[\sigma] \cap [\rho^c] = \emptyset$, then
$$\mathsf{d}\lambda_\sigma \wedge \phi_{\rho^c} = (-1)^k \epsilon(\sigma, \sigma^c) \sum_{q \in [\sigma^c]} \lambda_q \phi_T. \tag{III.50b}$$
- If $[\sigma] \cap [\rho^c]$ contains exactly one element, then
$$\mathsf{d}\lambda_\sigma \wedge \phi_{\rho^c} = (-1)^{k+1} \epsilon(\rho, \rho^c) \epsilon(p, \sigma - p) \epsilon(q, \sigma - p) \lambda_p \phi_T, \tag{III.50c}$$
where $q \in [\sigma^c]$ and $p \in [\sigma]$ are the unique solutions of $\rho = \sigma - p + q$.

Proof. Let $\sigma, \rho \in \Sigma(1:k, 0:n)$, so $\rho^c \in \Sigma(0:n-k, 0:n)$. The three cases of (III.50) are disjoint and their disjunction is true. Also, if $[\sigma] \cap [\rho^c] = \{p\}$ for some $p \in [\sigma]$, then $\rho = \sigma - p + q$ for some $q \in [\sigma^c]$, and $\rho^c = \sigma^c - q + p$. In particular, $[\sigma^c] \cap [\rho] = \{q\}$. We see that the right-hand side of (III.50c) is well-defined.

Firstly, if $[\sigma] \cap [\rho^c]$ has more than one element, then it is easy to verify that
$$\mathsf{d}\lambda_\sigma \wedge \phi_{\rho^c} = 0. \tag{III.51}$$
This can be seen by expanding the Whitney form ϕ_{ρ^c} according to (III.7) and then using the definition of the alternating product.

Secondly, if $\sigma = \rho$, or equivalently, $[\sigma] \cap [\rho^c] = \emptyset$, then we see, using (III.7), (III.9) and Lemma III.3.4, that
$$\begin{aligned}
\mathsf{d}\lambda_\sigma \wedge \phi_{\sigma^c} &= \mathsf{d}\lambda_\sigma \wedge \sum_{q \in [\sigma^c]} \lambda_q \epsilon(q, \sigma^c - q) \mathsf{d}\lambda_{\sigma^c - q} \\
&= \sum_{q \in [\sigma^c]} \lambda_q \epsilon(q, \sigma^c - q) \epsilon(\sigma, \sigma^c - q) \mathsf{d}\lambda_{\sigma + \sigma^c - q} \\
&= \sum_{q \in [\sigma^c]} \lambda_q \epsilon(q, \sigma^c - q) \epsilon(\sigma, \sigma^c - q) \epsilon(q, \sigma + \sigma^c - q) \phi_T.
\end{aligned}$$
From the combinatorial observation that
$$\epsilon(q, \sigma^c - q) \epsilon(\sigma, \sigma^c - q) \epsilon(q, \sigma + \sigma^c - q) = (-1)^k \epsilon(\sigma, \sigma^c),$$

we conclude the desired expression for $\mathsf{d}\lambda_\sigma \wedge \phi_{\sigma^c}$.

Eventually, we consider the case that $[\sigma] \cap [\rho^c]$ has exactly one element. Then there exist unique $p, q \in [0 : n]$ such that $[\sigma] \cap [\rho^c] = \{p\}$ and $[\sigma^c] \cap [\rho] = \{q\}$ and $\rho^c = \sigma^c - q + p$. We find, similar as above, that

$$\begin{aligned}
\mathsf{d}\lambda_\sigma \wedge \phi_{\rho^c} &= \mathsf{d}\lambda_\sigma \wedge \phi_{\sigma^c - q + p} \\
&= \epsilon(p, \sigma^c - q)\lambda_p \mathsf{d}\lambda_\sigma \wedge \mathsf{d}\lambda_{\sigma^c - q} \\
&= \epsilon(p, \sigma^c - q)\epsilon(\sigma, \sigma^c - q)\lambda_p \mathsf{d}\lambda_{\sigma + \sigma^c - q} \\
&= \epsilon(p, \sigma^c - q)\epsilon(\sigma, \sigma^c - q)\epsilon(q, \sigma + \sigma^c - q)\lambda_p \phi_T \\
&= (-1)^k \epsilon(p, \sigma^c - q)\epsilon(\sigma, \sigma^c)\epsilon(q, \sigma^c - q)\lambda_p \phi_T.
\end{aligned}$$

With the combinatorial observation

$$\epsilon(\sigma - p + q, \sigma^c - q + p) = \epsilon(\sigma, \sigma^c)\epsilon(\sigma - p, p)\epsilon(q, \sigma^c - q)(-1)\epsilon(\sigma - p, q)\epsilon(p, \sigma^c - q),$$

we derive

$$\begin{aligned}
&(-1)^k \epsilon(p, \sigma^c - q)\epsilon(\sigma, \sigma^c)\epsilon(q, \sigma^c - q) \\
&\qquad = (-1)^{k+1} \epsilon(\sigma - p + q, \sigma^c - q + p)\epsilon(p, \sigma - p)\epsilon(q, \sigma - p).
\end{aligned}$$

From this, the identity

$$\begin{aligned}
\mathsf{d}\lambda_\sigma \wedge \epsilon(\rho, \rho^c)\phi_{\rho^c} &= (-1)^{k+1} \epsilon(\sigma - p + q, \sigma^c - q + p)\epsilon(p, \sigma - p)\epsilon(q, \sigma - p)\lambda_p \phi_T \\
&= (-1)^{k+1} \epsilon(\rho, \rho^c)\epsilon(p, \sigma - p)\epsilon(q, \sigma - p)\lambda_p \phi_T
\end{aligned}$$

follows. The proof is complete. □

This auxiliary lemma has the following implications, which we utilize in the proofs of this section's main results. For $k \in \mathbb{Z}$ and $\sigma \in \Sigma(1 : k, 0 : n)$ we find

$$\mathsf{d}\lambda_\sigma \wedge \epsilon(\sigma, \sigma^c)\lambda^\sigma \phi_{\sigma^c} = (-1)^k \lambda^\sigma \sum_{q \in [\sigma^c]} \lambda_q \phi_T. \tag{III.52}$$

If furthermore $\rho = \sigma - p + q$ for $p \in [\sigma]$ and $q \in [\sigma^c]$, then

$$\mathsf{d}\lambda_\sigma \wedge \epsilon(\rho, \rho^c)\lambda^\rho \phi_{\rho^c} = (-1)^{k+1} \epsilon(p, \sigma - p)\epsilon(q, \sigma - p)\lambda^\rho \lambda_p \phi_T \tag{III.53}$$

on the one hand, while

$$\begin{aligned}
\mathsf{d}\lambda_\rho \wedge \epsilon(\sigma, \sigma^c)\lambda^\sigma \phi_{\sigma^c} &= (-1)^{k+1} \epsilon(q, \rho - q)\epsilon(p, \rho - q)\lambda^\sigma \lambda_q \phi_T \\
&= (-1)^{k+1} \epsilon(p, \sigma - p)\epsilon(q, \sigma - p)\lambda^\rho \lambda_p \phi_T
\end{aligned} \tag{III.54}$$

on the other hand; the symmetry result

$$\mathsf{d}\lambda_\sigma \wedge \epsilon(\rho, \rho^c)\lambda^\rho \phi_{\rho^c} = \mathsf{d}\lambda_\rho \wedge \epsilon(\sigma, \sigma^c)\lambda^\sigma \phi_{\sigma^c} \tag{III.55}$$

follows thence.

Analogously, for $\sigma \in \Sigma(1:k,0:n)$ we have

$$\lambda^{\sigma^c} \mathsf{d}\lambda_\sigma \wedge \epsilon(\sigma,\sigma^c)\phi_{\sigma^c} = (-1)^k \lambda^{\sigma^c} \sum_{q\in[\sigma^c]} \lambda_q \phi_T. \tag{III.56}$$

If $\sigma \in \Sigma(1:k,0:n)$ and $\rho = \sigma - p + q$ with $p \in [\sigma]$ and $q \in [\sigma^c]$, then

$$\lambda^{\sigma^c} \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c)\phi_{\rho^c} = (-1)^{k+1} \epsilon(p,\sigma-p)\epsilon(q,\sigma-p)\lambda^{\sigma^c}\lambda_p\phi_T. \tag{III.57}$$

If $\sigma,\rho \in \Sigma(1:k,0:n)$ with $[\sigma] \cup [\rho^c]$ having cardinality greater than one, then

$$\lambda^{\sigma^c} \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c)\phi_{\rho^c} = 0. \tag{III.58}$$

We have an analogous symmetry result

$$\lambda^{\sigma^c} \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c)\phi_{\rho^c} = \lambda^{\rho^c} \mathsf{d}\lambda_\rho \wedge \epsilon(\sigma,\sigma^c)\phi_{\sigma^c} \tag{III.59}$$

for $\sigma,\rho \in \Sigma(1:k,0:n)$.

We have the technical preparation ready to prove the first main result of this section.

Theorem III.8.2.
Let $k, r \in \mathbb{Z}$ and let $\omega_{\alpha,\sigma} \in \mathbb{R}$ for $\alpha \in A(r,n)$ and $\sigma \in \Sigma(1:k,0:n)$. Then

$$\begin{aligned}
&\sum_{\alpha,\beta\in A(r,n)} \sum_{\sigma,\rho\in\Sigma(1:k,0:n)} \int_T \omega_{\alpha,\sigma}\lambda^\alpha \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c)\omega_{\beta,\rho}\lambda^\beta\lambda^\rho\phi_{\rho^c} \\
&= (-1)^k \sum_{\theta\in\Sigma(1:k+1,0:n)} \int_T \lambda^\theta \left(\sum_{\alpha\in A(r,n)} \sum_{p\in[\theta]} \epsilon(p,\theta-p)\lambda^\alpha \omega_{\alpha,\theta-p} \right)^2.
\end{aligned}$$

In particular, this term is zero if and only one of the equivalent conditions of Lemma III.7.1 and Lemma III.7.3 is satisfied.

Proof. Let us write

$$S(\theta,\alpha,\omega) := \sum_{p\in[\theta]} \epsilon(p,\theta-p)\omega_{\alpha,\theta-p}, \quad \theta \in \Sigma(1:k+1,0:n), \quad \alpha \in A(r,n).$$

We moreover write

$$\begin{aligned}
S(\omega) &:= \sum_{\alpha,\beta\in A(r,n)} \sum_{\sigma,\rho\in\Sigma(1:k,0:n)} \int_T \omega_{\alpha,\sigma}\lambda^\alpha \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c)\omega_{\beta,\rho}\lambda^\beta\lambda^\rho\phi_{\rho^c}, \\
S_d(\omega) &:= \sum_{\alpha,\beta\in A(r,n)} \int_T \lambda^{\alpha+\beta} \sum_{\sigma\in\Sigma(1:k,0:n)} \omega_{\alpha,\sigma}\omega_{\beta,\sigma}\lambda^\sigma \mathsf{d}\lambda_\sigma \wedge \epsilon(\sigma,\sigma^c)\phi_{\sigma^c}, \\
S_o(\omega) &:= \sum_{\alpha,\beta\in A(r,n)} \int_T \lambda^{\alpha+\beta} \sum_{\substack{\sigma,\rho\in\Sigma(1:k,0:n)\\ \sigma\neq\rho}} \omega_{\alpha,\sigma}\omega_{\beta,\rho} \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c)\lambda^\rho\phi_{\rho^c}.
\end{aligned}$$

So $S(\omega) = S_d(\omega) + S_d(\omega)$ splits into a *diagonal part* $S_d(\omega)$ and an *off-diagonal part* $S_o(\omega)$. We apply our previous observations and find that $S(\omega)$ equals

$$\sum_{\substack{\alpha,\beta\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ q\in[\sigma^c]}} \int_T (-1)^k \lambda^{\alpha+\beta}\lambda^{\sigma+q}\omega_{\alpha,\sigma}\left(\omega_{\beta,\sigma} - \sum_{p\in[\sigma]} \epsilon(p,\sigma-p)\epsilon(q,\sigma-p)\omega_{\beta,\sigma-p+q}\right)\phi_T.$$

With the combinatorial observation

$$\begin{aligned}\epsilon(p,\sigma-p)\epsilon(q,\sigma-p) &= \epsilon(p,\sigma+q-p)\epsilon(p,q)\epsilon(q,\sigma)\epsilon(q,p)\\ &= -\epsilon(p,\sigma+q-p)\epsilon(\sigma,q),\end{aligned}$$

we simplify the sum further to

$$\begin{aligned}&\sum_{\substack{\alpha,\beta\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ q\in[\sigma^c]}} (-1)^k \int_T \lambda^{\alpha+\beta}\lambda^{\sigma+q}\omega_{\alpha,\sigma}\epsilon(q,\sigma)\left(\sum_{p\in[\sigma+q]} \epsilon(p,\sigma-p+q)\omega_{\beta,\sigma+q-p}\right)\phi_T\\ &= \sum_{\alpha,\beta\in A(r,n)} \int_T \lambda^{\alpha+\beta} \sum_{\substack{\sigma\in\Sigma(1:k,0:n)\\ q\in[\sigma^c]}} (-1)^k\lambda^{\sigma+q}\omega_{\alpha,\sigma}\epsilon(q,\sigma)S(\sigma+q,\beta,\omega)\phi_T\\ &= (-1)^k \sum_{\alpha,\beta\in A(r,n)} \int_T \lambda^{\alpha+\beta} \sum_{\theta\in\Sigma(1:k+1,0:n)} \lambda^\theta \sum_{p\in[\theta]} \omega_{\alpha,\theta-p}\epsilon(p,\theta-p)S(\theta,\beta,\omega)\phi_T\\ &= (-1)^k \sum_{\theta\in\Sigma(1:k+1,0:n)} \int_T \lambda^\theta \sum_{\alpha,\beta\in A(r,n)} \lambda^{\alpha+\beta}S(\theta,\alpha,\omega)S(\theta,\beta,\omega)\phi_T\\ &= (-1)^k \sum_{\theta\in\Sigma(1:k+1,0:n)} \int_T \lambda^\theta \left(\sum_{\alpha\in A(r,n)} \lambda^\alpha S(\theta,\alpha,\omega)\right)^2 \phi_T.\end{aligned}$$

The integrand is non-negative. Hence the integral vanishes if and only if

$$0 = \sum_{\alpha\in A(r,n)} \lambda^\alpha S(\theta,\alpha,\omega), \quad \theta\in\Sigma(1:k+1,0:n).$$

Since the λ^α are linearly independent for $\alpha\in A(r,n)$, this holds if and only if one of the equivalent conditions of Lemma III.7.1 and Lemma III.7.3 is satisfied. $\square$

We apply Theorem III.8.2 in the following manner. Consider the bilinear form

$$(\omega,\eta) \mapsto \sum_{\alpha,\beta\in A(r,n)} \sum_{\sigma,\rho\in\Sigma(1:k,0:n)} \int_T \omega_{\alpha,\sigma}\lambda^\alpha \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c)\eta_{\beta,\rho}\lambda^\beta\lambda^\rho\phi_{\rho^c}$$

defined for $\omega,\eta\in\mathbb{R}^{A(r,n)\times\Sigma(1:k,0:n)}$. We have shown in this section that this bilinear form is symmetric and semidefinite. Its degeneracy space is exactly the linear space of coefficients that satisfy the conditions of Lemma III.7.1 and Lemma III.7.3. This implies in particular that the bilinear form

$$(\omega, \eta) \mapsto \int_T \omega \wedge \eta, \quad \omega \in \mathcal{P}_r\Lambda^k(T), \quad \eta \in \mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T),$$

is non-degenerate.

Theorem III.8.3.
Let $k, r \in \mathbb{Z}$ and $\omega_{\alpha,\sigma} \in \mathbb{R}$ for $\alpha \in A(r,n)$ and $\sigma \in \Sigma(1:k, 0:n)$. Then

$$\begin{aligned}
&\sum_{\alpha,\beta \in A(r,n)} \sum_{\sigma,\rho \in \Sigma(1:k,0:n)} \int_T \omega_{\alpha,\sigma} \lambda^\alpha \lambda^{\sigma^c} \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c) \omega_{\beta,\rho} \lambda^\beta \phi_{\rho^c} \\
&= (-1)^k \sum_{\theta \in \Sigma(1:k+1,0:n)} \int_T \lambda^{\theta^c} \left(\sum_{\alpha \in A(r,n)} \sum_{p \in [\theta]} \epsilon(p, \theta - p) \lambda^\alpha \lambda_p \omega_{\alpha,\theta-p} \right)^2 .
\end{aligned}$$

In particular, this term is zero if and only one of the equivalent conditions of Lemma III.7.2 and Lemma III.7.4 is satisfied.

Proof. Let us consider

$$S(\omega) := \sum_{\alpha,\beta \in A(r,n)} \sum_{\sigma,\rho \in \Sigma(1:k,0:n)} \int_T \omega_{\alpha,\sigma} \lambda^\alpha \lambda^{\sigma^c} \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c) \omega_{\beta,\rho} \lambda^\beta \phi_{\rho^c}.$$

We can split the sum into two parts. On the one hand, for the *diagonal part*,

$$\begin{aligned}
S_d(\omega) &:= \sum_{\alpha,\beta \in A(r,n)} \sum_{\sigma \in \Sigma(1:k,0:n)} \int_T \omega_{\alpha,\sigma} \lambda^\alpha \lambda^{\sigma^c} \mathsf{d}\lambda_\sigma \wedge \epsilon(\sigma,\sigma^c) \omega_{\beta,\sigma} \lambda^\beta \phi_{\sigma^c} \\
&= \sum_{\alpha,\beta \in A(r,n)} \sum_{\sigma \in \Sigma(1:k,0:n)} \int_T \omega_{\alpha,\sigma} \omega_{\beta,\sigma} \lambda^{\alpha+\beta} \lambda^{\sigma^c} (-1)^k \sum_{q \in [\sigma]} \lambda_q \phi_T,
\end{aligned}$$

while on the other hand, for the *off-diagonal part*,

$$\begin{aligned}
S_o(\omega) &:= \sum_{\alpha,\beta \in A(r,n)} \sum_{\substack{\sigma,\rho \in \Sigma(1:k,0:n) \\ \sigma \neq \rho}} \int_T \omega_{\alpha,\sigma} \lambda^\alpha \lambda^{\sigma^c} \mathsf{d}\lambda_\sigma \wedge \epsilon(\rho,\rho^c) \omega_{\beta,\rho} \lambda^\beta \phi_{\rho^c} \\
&= \sum_{\substack{\sigma \in \Sigma(1:k,0:n) \\ \alpha,\beta \in A(r,n)}} \sum_{\substack{p \in [\sigma] \\ q \in [\sigma^c]}} \int_T \omega_{\alpha,\sigma} \omega_{\beta,\sigma-p+q} \lambda^{\alpha+\beta} \lambda^{\sigma^c} (-1)^{k+1} \epsilon(p,\sigma-p) \epsilon(q,\sigma-p) \lambda_p \phi_T.
\end{aligned}$$

Since $S(\omega) = S_d(\omega) + S_o(\omega)$, we combine that $(-1)^k S(\omega)$ equals

$$\sum_{\substack{\alpha,\beta\in A(r,n)\\ \sigma\in\Sigma(1:k,0:n)\\ q\in[\sigma^c]}} \int_T \lambda^{\alpha+\beta}\omega_{\alpha,\sigma}\lambda^{\sigma^c}\left(\omega_{\beta,\rho}\lambda_q - \sum_{p\in[\sigma]}\epsilon(p,\sigma-p)\epsilon(q,\sigma-p)\omega_{\beta,\sigma-p+q}\lambda_p\right)\phi_T$$

$$= \sum_{\alpha,\beta\in A(r,n)} \int_T \lambda^{\alpha+\beta} \sum_{\substack{\sigma\in\Sigma(1:k,0:n)\\ q\in[\sigma^c]}} \omega_{\alpha,\sigma}\lambda^{\sigma^c}\epsilon(q,\sigma)\left(\sum_{p\in[\sigma+q]}\epsilon(p,\sigma-p+q)\omega_{\beta,\sigma-p+q}\lambda_p\right)\phi_T$$

$$= \sum_{\alpha,\beta\in A(r,n)} \int_T \lambda^{\alpha+\beta} \sum_{\substack{\theta\in\Sigma(1:k+1,0:n)\\ p\in[\theta^c]}} \epsilon(p,\theta-p)\omega_{\alpha,\theta-p}\lambda^{\theta^c}\lambda_p\left(\sum_{p\in[\theta]}\epsilon(p,\theta-p)\omega_{\beta,\theta-p}\lambda_p\right)\phi_T$$

$$= \sum_{\theta\in\Sigma(1:k+1,0:n)} \int_T \lambda^{\theta^c}\left(\sum_{\alpha\in A(r,n)}\sum_{p\in[\theta]}\epsilon(p,\theta-p)\omega_{\alpha,\theta-p}\lambda^{\alpha}\lambda_p\right)^2\phi_T$$

$$= \sum_{\theta\in\Sigma(1:k+1,0:n)} \int_T \lambda^{\theta^c}\left(\sum_{\beta\in A(r+1,n)}\sum_{p\in[\theta]}\epsilon(\theta-p,p)\omega_{\beta-p,\theta-p}\lambda^{\beta}\right)^2\phi_T.$$

The integrand is non-negative. Moreover, we see that it vanishes if and only if the conditions of Lemma III.7.2 and Lemma III.7.4. This completes the proof. □

Similar as before, we utilize Theorem III.8.3 for our understanding of bilinear pairings. We define

$$(\omega,\eta)\mapsto \sum_{\alpha,\beta\in A(r,n)}\sum_{\sigma,\rho\in\Sigma(1:k,0:n)}\int_T \omega_{\alpha,\sigma}\lambda^{\alpha}\lambda^{\sigma^c}\mathsf{d}\lambda_\sigma\wedge\epsilon(\rho,\rho^c)\eta_{\beta,\rho}\lambda^{\beta}\phi_{\rho^c}$$

for $\omega,\eta\in\mathbb{R}^{A(r,n)\times\Sigma(1:k,0:n)}$. This bilinear form is symmetric and semidefinite. Its degeneracy space is exactly the linear space of coefficients that satisfy the conditions of Lemma III.7.2 and Lemma III.7.4. The non-degeneracy of the bilinear form

$$(\omega,\eta)\mapsto\int_T\omega\wedge\eta,\quad \omega\in\mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T),\quad \eta\in\mathcal{P}^-_{r+1}\Lambda^{n-k}(T),$$

is an important consequence.

III.9. Flux Reconstruction over a Simplex

At the end of this chapter we make up for a promise given earlier: we prove the statements (III.19), (III.20), and (III.29). We begin with the identities concerning the spaces without boundary conditions. Specifically, (III.19a), (III.20a) and (III.29) are evident by the following theorem.

Theorem III.9.1.
Let T be an n-simplex and let $r,k\in\mathbb{Z}$. Then the following holds:

- If $k > 0$ and $\omega \in \mathcal{P}_r\Lambda^k(T)$ with $\mathsf{d}^k\omega = 0$, then there exists $\xi \in \mathcal{P}_{r+1}^-\Lambda^{k-1}(T)$ such that $\mathsf{d}^{k-1}\xi = \omega$.
- If $\omega \in \mathcal{P}_r^-\Lambda^k(T)$ with $\mathsf{d}^k\omega = 0$, then $\omega \in \mathcal{P}_{r-1}\Lambda^k(T)$.

Proof. Using the partition of unity property of the barycentric coordinates over T and the partition of zero property of their differentials, it is easily proven that

$$\mathcal{AP}_r\Lambda^k(T) := \left\{ \lambda^\alpha \mathsf{d}\lambda_\sigma \;\middle|\; \begin{array}{c} \alpha \in A(1:n),\ |\alpha| \leq r, \\ \sigma \in \Sigma(1:k, 1:n) \end{array} \right\}$$

is a basis of $\mathcal{P}_r\Lambda^k(T)$. We define a linear mapping $P^k : \mathcal{P}_r\Lambda^k(T) \to \mathcal{P}_{r+1}^-\Lambda^{k-1}(T)$ by

$$P^k\left(\lambda^\alpha \mathsf{d}\lambda_\sigma\right) := (r+k)^{-1}\lambda^\alpha \sum_{i=1}^{k} (-1)^i \lambda_{\sigma(i)} \mathsf{d}\lambda_{\sigma-\sigma(i)}, \quad \lambda^\alpha \mathsf{d}\lambda_\sigma \in \mathcal{AP}_r\Lambda^k(T).$$

One can moreover show that

$$\mathsf{d}^{k-1}P^k\omega + P^{k+1}\mathsf{d}^k\omega = (r+k)\omega. \tag{III.60}$$

In particular, $(r+k)\omega = \mathsf{d}^{k-1}P^k\omega$ if $\mathsf{d}^k\omega = 0$, and $(r+k)\omega = P^{k+1}\mathsf{d}^k\omega$ if $P^k\omega = 0$. The first statement of the lemma is an easy consequence.

To prove (III.60), we fix $\alpha \in A(1:n)$ and $\sigma \in \Sigma(1:k, 0:n)$. We observe

$$\mathsf{d}^0\left(\lambda^\alpha\right) = \sum_{i \in [\alpha]} \alpha(i)\lambda^{\alpha-i}\mathsf{d}\lambda_i.$$

On the one hand,

$$\mathsf{d}^{k-1}P^k\left(\lambda^\alpha \mathsf{d}\lambda_\sigma\right) = k \cdot \lambda^\alpha \mathsf{d}\lambda_\sigma + \mathsf{d}^0\left(\lambda^\alpha\right) \wedge \phi_\rho,$$

and

$$\begin{aligned}
\mathsf{d}^0\left(\lambda^\alpha\right) \wedge \phi_\rho &= \sum_{\substack{i \in [\alpha] \\ p \in [\sigma]}} \alpha(i)\lambda^{\alpha-i}\mathsf{d}\lambda_i \wedge \epsilon(p,\sigma)\lambda_p \mathsf{d}\lambda_{\sigma-p} \\
&= \sum_{i \in [\alpha] \cap [\sigma]} \alpha(i)\lambda^\alpha \mathsf{d}\lambda_\sigma + \sum_{\substack{i \in [\alpha] \setminus [\sigma] \\ p \in [\sigma]}} \alpha(i)\lambda^{\alpha+p-i}\epsilon(p,\sigma)\epsilon(i,\sigma-p)\mathsf{d}\lambda_{\sigma-p+i}.
\end{aligned}$$

On the other hand,

$$P^{k+1}\mathsf{d}^k\left(\lambda^\alpha \mathsf{d}\lambda_\sigma\right) = \sum_{i \in [\alpha] \setminus [\sigma]} \alpha(i)\lambda^\alpha \mathsf{d}\lambda_\sigma + \sum_{\substack{i \in [\alpha] \setminus [\sigma] \\ p \in [\sigma]}} \alpha(i)\lambda^{\alpha+p-i}\epsilon(i,\sigma)\epsilon(p,\sigma-i)\mathsf{d}\lambda_{\sigma+i-p}.$$

For $i \in [\alpha] \setminus [\sigma]$ and $p \in [\sigma]$ we make the combinatorial observation

$$\epsilon(p,\sigma)\epsilon(i,\sigma-p) = \epsilon(p,\sigma)\epsilon(i,p)\epsilon(i,\sigma) = -\epsilon(p,\sigma)\epsilon(p,i)\epsilon(i,\sigma) = -\epsilon(i,\sigma)\epsilon(p,\sigma-i).$$

This in combination yields (III.60).

For proving the second statement, we need a fact about the space $\mathcal{P}_r^-\Lambda^k(T)$. For $\alpha \in A(r,n)$ and $\rho \in \Sigma(0:k, 1:n)$ we see

$$\mathsf{d}^k\left(\lambda^\alpha \phi_\rho\right) = \sum_{i=1}^{n} \alpha(i)\lambda^{\alpha - i}\mathsf{d}\lambda_i \wedge \phi_\rho + \lambda^\alpha \wedge \mathsf{d}^k \phi_\rho.$$

We have $\mathsf{d}^k\phi_\rho = (k+1)\mathsf{d}\lambda_\rho$ and one can show that $P^{k+1}\mathsf{d}\lambda_\rho$ differs from ϕ_ρ only by constant scaling and by addition of a constant k-form. It is revealed in combination that

$$\mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_{r-1}\Lambda^k(T) + P^{k+1}\mathcal{P}_{r-1}\Lambda^{k+1}(T).$$

Let us suppose that $\omega \in \mathcal{P}_r^-\Lambda^k(T)$. Then there exist $\omega_0 \in \mathcal{P}_{r-1}\Lambda^k(T)$ and $\omega_1 \in \mathcal{P}_{r-1}\Lambda^{k+1}(T)$ such that $\omega = \omega_0 + P^{k+1}\omega_1$. If $\mathsf{d}^k\omega = 0$, then $\mathsf{d}^k\omega_0 + \mathsf{d}^{k-1}P^k\omega_1 = 0$. But upon representing ω_0 and ω_1 in terms of the basis families $\mathcal{A}\mathcal{P}_r\Lambda^k(T)$ it is obvious that we may assume $\mathsf{d}^k\omega_0 = 0$ and $\mathsf{d}^{k-1}P^k\omega_1 = 0$ without loss of generality. But then (III.60) implies $\omega_1 = P^{k+1}\omega_2$ for some $\omega_2 \in \mathcal{P}_{r-1}\Lambda^{k+2}(T)$. Consequently, $P^k\omega_1 = 0$ and $\omega = \omega_0$, which proves the desired claim. $\square$

Proving (III.19a), (III.20a) and (III.29) relies on the duality pairings discussed in the preceding section.

Theorem III.9.2.
Let T be an n-simplex and let $r, k \in \mathbb{Z}$. Then the following holds:

- If $k > 0$ and $\omega \in \mathring{\mathcal{P}}_r\Lambda^k(T)$ with $\mathsf{d}^k\omega = 0$, then there exists $\xi \in \mathring{\mathcal{P}}_{r+1}^-\Lambda^{k-1}(T)$ such that $\mathsf{d}^{k-1}\xi = \omega$.
- If $\omega \in \mathring{\mathcal{P}}_r^-\Lambda^k(T)$ with $\mathsf{d}^k\omega = 0$, then $\omega \in \mathring{\mathcal{P}}_{r-1}\Lambda^k(T)$.

Proof. It suffices to show that the two differential complexes

$$\mathring{\mathcal{P}}_{r+1}^-\Lambda^{k-1}(T) \xrightarrow{\mathsf{d}^{k-1}} \mathring{\mathcal{P}}_r\Lambda^k(T) \xrightarrow{\mathsf{d}^k} \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(T) \tag{III.61a}$$

$$\mathring{\mathcal{P}}_{r+1}^-\Lambda^{k-1}(T) \xrightarrow{\mathsf{d}^{k-1}} \mathring{\mathcal{P}}_{r+1}^-\Lambda^k(T) \xrightarrow{\mathsf{d}^k} \mathring{\mathcal{P}}_r\Lambda^{k+1}(T) \tag{III.61b}$$

are exact at the middle terms. This is the case if and only if the dual complexes

$$\mathring{\mathcal{P}}_{r+1}^-\Lambda^{k-1}(T)' \xleftarrow{\left(\mathsf{d}^{k-1}\right)'} \mathring{\mathcal{P}}_r\Lambda^k(T)' \xleftarrow{\left(\mathsf{d}^k\right)'} \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(T)' \tag{III.62a}$$

$$\mathring{\mathcal{P}}_{r+1}^-\Lambda^{k-1}(T)' \xleftarrow{\left(\mathsf{d}^{k-1}\right)'} \mathring{\mathcal{P}}_{r+1}^-\Lambda^k(T)' \xleftarrow{\left(\mathsf{d}^k\right)'} \mathring{\mathcal{P}}_r\Lambda^{k+1}(T)' \tag{III.62b}$$

are exact at the middle terms. We show that this is the case if and only if the differential complexes

$$\mathcal{P}_{r-n+k-1}\Lambda^{n-k+1}(T) \xleftarrow{\mathsf{d}^{k-1}} \mathcal{P}_{r-n+k}^-\Lambda^{n-k}(T) \xleftarrow{\mathsf{d}^k} \mathcal{P}_{r-n+k}^-\Lambda^{k-k-1}(T) \tag{III.63a}$$

$$\mathcal{P}_{r-n+k-1}\Lambda^{n-k+1}(T) \xleftarrow{\mathsf{d}^{k-1}} \mathcal{P}_{r-n+k}\Lambda^{n-k}(T) \xleftarrow{\mathsf{d}^k} \mathcal{P}_{r-n+k+1}^-\Lambda^{n-k-1}(T) \tag{III.63b}$$

are exact at the middle terms. Indeed, these two differential complexes are exact at the middle term, as follows by Theorem III.9.1.

To complete the proof, we recall the isomorphisms

$$\mathcal{P}_r\Lambda^k(T) \simeq \mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T), \quad \mathcal{P}^-_{r+1}\Lambda^{n-k}(T) \simeq \mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T),$$

and the corresponding duality pairings. Reindexing gives us isomorphisms

$$\mathcal{P}_{r-n+k-1}\Lambda^{n-k}(T) \simeq \mathring{\mathcal{P}}^-_r\Lambda^k(T), \quad \mathcal{P}^-_{r-n+k}\Lambda^{n-k}(T) \simeq \mathring{\mathcal{P}}_r\Lambda^k(T),$$

and corresponding duality pairings. With an integration by parts formula derived from Stokes' theorem over the simplex (III.3) it is now easily verified that (III.62a) is isomorphic to (III.63a) and that (III.62b) is isomorphic to (III.63b). □

Remark III.9.3.
There are many different routes that prove the exactness of finite element differential complexes over simplices. Theorem (III.9.1) uses a variant of the Poincaré mapping, which has been discussed in different forms in finite element literature [9, 109]. The situation is considerably more complicated when boundary conditions are imposed, and this chapter has provided a new proof. An alternative method of proof can employ smoothed projections over simplex with boundary conditions and relies on the exactness of the L^2 de Rham complex with boundary conditions [11, 58]. On the other hand, the exactness of the finite element differential complex with boundary conditions over simplex follows by an induction argument over the dimension that utilizes long exact sequences on cohomology and finite element de Rham complexes over the boundary triangulation of a simplex [53, 56].

IV. Finite Element de Rham Complexes

In the previous chapters, we have first discussed simplices and then differential forms over simplices. We proceed with spaces of differential forms over simplicial complexes. The topic of this chapter is the construction of finite element de Rham complexes of higher and possibly non-uniform polynomial order. We also provide a commuting interpolant.

We begin with the most basic example and discuss a finite element de Rham complex of lowest polynomial order in Section IV.1. Of course, this is just the differential complex of Whitney forms. Whitney forms have been discussed in Whitney's monograph on geometric measure theory [180], and they have been subject of research in numerical analysis for decades (see [29, 109]). A fundamental result is the duality of the differential complex of Whitney forms to the simplicial chain complex of the underlying triangulation. This determines the cohomology spaces of this differential complex: the Whitney form cohomology realizes the simplicial Betti numbers of the triangulation. We consider a general class of boundary conditions and describe the finite element interpolant onto the Whitney forms.

The construction of finite element differential forms of higher polynomial order is a topic rich in structure and results which this thesis approaches with the following intuition: *global* properties of a finite element de Rham complex are described entirely by its lowest-order contributions, whereas the higher-order contributions are only local. The dissertation of Sabine Zaglmayr [183] systematically applies that idea: to build the higher order finite element differential complex, she starts with a lowest-order finite element differential complex and associates exact finite element differential complexes of higher polynomial order to each simplex with support in the local patch of the respective simplex. This method allows for a simple construction of finite element de Rham complexes of non-uniform polynomial order.

We put that intuition to work a different manner. As a preparation for the construction of higher order finite element de Rham complexes, we introduce the notion of *admissible sequence type* in Section IV.2. We recall that the $\mathcal{P}_r$-type and $\mathcal{P}_r^-$-type spaces of finite element differential forms can be composed in different manners to finite element de Rham complexes (see Section 3 of [9]) over single simplices, triangulations, or $\mathbb{R}^n$, and one may or may not impose boundary conditions. The notion of admissible sequence type abstracts the choice of $\mathcal{P}_r$- and $\mathcal{P}_r^-$-type spaces from the specific geometric ambient.

This notion also allows us to associate particular choices of finite element de Rham complexes to single simplices, which commences the construction of higher order finite element de Rham complexes in Section IV.3. We generalize a common practice in the theory of hp-adaptive finite element methods: an H^1-conforming finite element space of non-uniform polynomial order is defined by associating a polynomial order to each simplex. Similarly, we define a finite element de Rham complex of non-uniform polynomial order by associating a admissible sequence type to each simplex such that a compatibility condition holds.

We finish this chapter with the commuting interpolant from piecewise smooth de Rham complexes onto the finite element de Rham complexes in Section IV.4. In the case of uniform finite element spaces, such an interpolant has been given by Arnold, Falk, and Winther [9]. We use different techniques in this thesis. Specifically, our construction follows the ideas of Demkowicz et. al. [69], whose key idea is a Hodge decomposition of the degrees of freedom. This construction principle was recast in the calculus of differential forms within the framework of element systems (see e.g. [56, Proposition 5.44]), where we the resulting interpolant was called *harmonic interpolant*.

Research efforts in finite element exterior calculus have focused on spaces of uniform polynomial order [9, 10] but have given considerably less attention to spaces with spatially varying polynomial order (but see [56, 108]). Finite element spaces of the latter kind, however, are constitutive for p-adaptive and hp-adaptive finite element methods (p-FEM and hp-FEM, [68, 152, 165]). We recall that h-adaptive methods refine the mesh locally but keep the polynomial order fixed, that p-adaptive methods keep the mesh fixed but locally increase the polynomial order, and that hp-adaptive methods combine local mesh refinement and variation of the polynomial order. The latter form of adaptivity allows for efficient approximation of functions with spatially varying smoothness or isolated singularities, for example by Lagrange elements with non-uniform polynomial order. The theory of hp-adaptive mixed finite element methods in numerical electromagnetism utilizes differential complexes of spaces of non-uniform polynomial order, which include generalizations of Nédélec elements and Raviart-Thomas elements [1, 67, 143, 161]. For the most part, these research efforts have been formalized in terms of classical vector calculus.

Our construction of finite element de Rham complexes of non-uniform polynomial order may serve as a preparation towards the study of hp-adaptive methods in finite element exterior calculus, but this is not a part of this thesis. Instead, we regard the aforementioned principle of constructing higher order finite element de Rham complexes by local augmentation of a global differential complex to be of general interest in the theory of finite element methods. Support for this assessment will be provided in Chapter X of this thesis with an application in a posteriori error estimation. Considering finite element spaces of non-uniform polynomial order is then only small addition once the basic idea has been established.

IV.1. The Complex of Whitney Forms

In this section we introduce the complex of Whitney forms as the principle example of a finite element de Rham complex. In particular, we discuss its relation to the simplicial chain complex and develop a commuting interpolant. Throughout this section, we let $\mathcal{T}$ be a simplicial complex and let $\mathcal{U} \subseteq \mathcal{T}$ be a simplicial subcomplex.

Before we discuss Whitney forms, we first consider piecewise smooth differential forms. For $k \in \mathbb{Z}$ we define

$$C^\infty\Lambda^k(\mathcal{T}) := \left\{ (\omega_T)_{T\in\mathcal{T}} \in \bigoplus_{T\in\mathcal{T}} C^\infty\Lambda^k(T) \;\middle|\; \forall T \in \mathcal{T} : \forall F \in \Delta(T) : \operatorname{tr}^k_{T,F}\omega_T = \omega_F \right\}.$$

We can identify this with the space of differential k-forms that are piecewise smooth with respect to $\mathcal{T}$ and that have single-valued traces along simplex boundaries. Henceforth, we may also write $\operatorname{tr}^k_T \omega := \omega_T$ for $\omega \in C^\infty\Lambda^k(\mathcal{T})$ and $T \in \mathcal{T}$.

Since the exterior derivatives on simplices commute with the trace operators, we have a well-defined exterior derivative

$$\mathsf{d}^k : C^\infty\Lambda^k(\mathcal{T}) \to C^\infty\Lambda^{k+1}(\mathcal{T}), \quad (\omega_T)_{T\in\mathcal{T}} \mapsto \left(\mathsf{d}^k_T\omega_T\right)_{T\in\mathcal{T}}. \tag{IV.1}$$

Since $\mathsf{d}^{k+1}\mathsf{d}^k\omega = 0$ for every $\omega \in C^\infty\Lambda^k(\mathcal{T})$, we may compose a differential complex

$$\dots \xrightarrow{\mathsf{d}^{k-1}} C^\infty\Lambda^k(\mathcal{T}) \xrightarrow{\mathsf{d}^k} C^\infty\Lambda^{k+1}(\mathcal{T}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{IV.2}$$

In order to formalize boundary conditions, we furthermore define

$$C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) := \left\{ \omega \in C^\infty\Lambda^k(\mathcal{T}) \;\middle|\; \forall F \in \mathcal{U} : \omega_F = 0 \right\}. \tag{IV.3}$$

It is easily verified that

$$\mathsf{d}^k\left(C^\infty\Lambda^k(\mathcal{T},\mathcal{U})\right) \subseteq C^\infty\Lambda^{k+1}(\mathcal{T},\mathcal{U}). \tag{IV.4}$$

In particular, we may compose the differential complex

$$\dots \xrightarrow{\mathsf{d}^{k-1}} C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^k} C^\infty\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{IV.5}$$

Remark IV.1.1.
Constructions similar to our definition of $C^\infty\Lambda^k(\mathcal{T})$ have appeared in mathematics before. Our definition is a special case of a *finite element system* in the terminology of [56]. Another variant of the idea is exemplified by *Sullivan forms* in global analysis [79], which are piecewise *flat* differential forms in the sense of geometric measure theory with single-valued traces along simplex boundaries.

Example IV.1.2.
We motivate these definitions by a practical illustration. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded polyhedral domain triangulated by a simplicial complex $\mathcal{T}$. Then the members of $C^\infty\Lambda^k(\mathcal{T})$ correspond to the differential k-forms over Ω that are piecewise smooth with respect to $\mathcal{T}$ and have single-valued traces on subsimplices. Moreover, suppose we have a subset of the boundary $\Gamma \subseteq \partial\Omega$ such that a simplicial subcomplex $\mathcal{U} \subset \mathcal{T}$ triangulates Γ. Then $C^\infty\Lambda^k(\mathcal{T},\mathcal{U})$ is the subspace of $C^\infty\Lambda^k(\mathcal{T})$ whose members have vanishing traces along Γ. In that way, $\mathcal{U}$ may be used to model homogeneous boundary conditions.

We next discuss an important relation between the simplicial chains and the piecewise smooth differential forms. Suppose that $\omega \in C^\infty\Lambda^k(\mathcal{T},\mathcal{U})$ and $T \in \mathcal{T}^k \backslash \mathcal{U}^k$. We then write

$$\int_T \omega := \int_T \operatorname{tr}^k_T \omega_T$$

for the integral of ω over the oriented simplex T. By linear extension we obtain a bilinear pairing

$$\int : C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) \times \mathcal{C}_k(\mathcal{T},\mathcal{U}) \to \mathbb{R}, \quad (\omega, S) \to \int_S \omega. \tag{IV.6}$$

We easily observe that

$$\int_{\partial_{k+1} S} \omega = \int_S \mathsf{d}^k \omega, \quad \omega \in C^\infty\Lambda^k(\mathcal{T},\mathcal{U}), \quad S \in \mathcal{C}_{k+1}(\mathcal{T},\mathcal{U}).$$

The linear pairing (IV.6) is degenerate in general.

We will identify a differential subcomplex of (IV.5) restricting to which in the first variable makes the bilinear pairing (IV.6) non-degenerate. Specifically, we employ a finite element de Rham complex of lowest polynomial order. To that end, we define the spaces of *Whitney forms* over $\mathcal{T}$ by

$$\mathcal{W}\Lambda^k(\mathcal{T}) := \left\{ \omega \in C^\infty\Lambda^k(\mathcal{T}) \;\middle|\; \forall T \in \mathcal{T} : \omega_T \in \mathcal{P}^-_1\Lambda^k(T) \right\}, \tag{IV.7}$$

and the space of Whitney forms over $\mathcal{T}$ relative to $\mathcal{U}$ by

$$\mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) := \mathcal{W}\Lambda^k(\mathcal{T}) \cap C^\infty\Lambda^k(\mathcal{T},\mathcal{U}). \tag{IV.8}$$

It is an immediate consequence of definitions that we have a well-defined operator

$$\mathsf{d}^k : \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U}),$$

and consequently the *differential complex of Whitney forms*

$$\dots \xrightarrow{\mathsf{d}^{k-1}} \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^k} \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{IV.9}$$

The notion of Whitney forms was originally motivated by their duality to the simplicial chains, which we discuss soon.

It is of interest to point out an explicit basis for the spaces $\mathcal{W}\Lambda^k(\mathcal{T})$. We make recourse to the basis forms which originally have been called *Whitney forms*. For every $f \in \mathcal{T}^k$ we define the Whitney k-form associated to $\phi^{\mathcal{T}}_f \in \mathcal{W}\Lambda^k(\mathcal{T})$ by setting

$$\operatorname{tr}^k_T \phi^{\mathcal{T}}_f := \begin{cases} \phi^T_f & \text{if } f \in \Delta(T), \\ 0 & \text{otherwise,} \end{cases}$$

for each $T \in \mathcal{T}$ Indeed, when $T \in \mathcal{T}$, $f \in \Delta(T)^k$, and $F \in \Delta(T)$, then either we have $f \notin \Delta(F)$, in which case $\operatorname{tr}^k_{T,F} \phi^T_f = 0$, or instead we have $f \in \Delta(F)$, in which case $\operatorname{tr}^k_{T,F} \phi^T_f = \phi^F_f$. Hence $\phi^{\mathcal{T}}_f \in \mathcal{W}\Lambda^k(\mathcal{T})$.

Lemma IV.1.3.
Let $f, g \in \mathcal{T}^k$. Then $\operatorname{tr}^k_g \phi^{\mathcal{T}}_f \neq 0$ if and only if $f \neq g$.

Proof. This follows immediately from Equation (III.8). □

Lemma IV.1.4.
The Whitney forms $\phi^{\mathcal{T}}_f, f \in \mathcal{T}^k$, are a basis of $\mathcal{W}\Lambda^k(\mathcal{T})$.

Proof. The linear independence of the $\phi^{\mathcal{T}}_f, f \in \mathcal{T}^k$, is an immediate consequence of Lemma IV.1.3. To complete the proof, let $\phi \in \mathcal{W}\Lambda^k(\mathcal{T})$ be arbitrary but fixed. There exist unique $c^T_f \in \mathbb{R}$ for each $T \in \mathcal{T}$ and $f \in \Delta(T)$ such that $\operatorname{tr}^k_T \phi = \sum_{f\in\Delta(T)^k} c^T_f \phi^T_f$. For $T \in \mathcal{T}$ and $g \in \Delta(T)^k$ we have $\int_g \operatorname{tr}^k_{T,g} \operatorname{tr}^k_T \phi = c^T_g \int_g \phi^g_g = c^T_g/(k!)$, and so we conclude that $c^T_g = c^{T'}_g$ for $T, T' \in \mathcal{T}$ with $g \in \Delta(T) \cap \Delta(T')$. This means that there exist $c_f \in \mathbb{R}$ for $f \in \Delta(T)$ such that for all $T \in \mathcal{T}$ we have $\operatorname{tr}^k_T \phi = \sum_{f\in\Delta(T)^k} c_f \phi^T_f$. But then $\phi = \sum_{f\in\mathcal{T}^k} c_f \phi_f$. The proof is complete. □

Lemma IV.1.5.
Let $\phi \in \mathcal{W}\Lambda^k(\mathcal{T})$ and $T \in \mathcal{T}$. Then $\operatorname{tr}^k_T \phi = 0$ if and only if for all $f \in \Delta(T)$ we have $\operatorname{tr}^k_f \phi = 0$.

Proof. There exist $c_f \in \mathbb{R}$ for each $f \in \mathcal{T}^k$ such that $\phi = \sum_{f\in\mathcal{T}^k} c_f \phi_f$. Consequently we have $\operatorname{tr}^k_T \phi = \sum_{f\in\Delta(T)^k} c_f \phi^T_f$. Since the k-forms ϕ^T_f for $f \in \Delta(T)^k$ are linearly independent, we verify $c_f = 0$ for $f \in \Delta(T)$. The proof is complete. □

Lemma IV.1.6.
The Whitney forms $\phi^{\mathcal{T}}_f, f \in \mathcal{T}^k \setminus \mathcal{U}^k$, are a basis for $\mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$.

Proof. This is a combination of Lemma IV.1.4 and Lemma IV.1.5. □

Lemma IV.1.7.
The bilinear pairing

$$\int : \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \times \mathcal{C}_k(\mathcal{T},\mathcal{U}) \to \mathbb{R}, \quad (\omega, S) \mapsto \int_S \operatorname{tr}^k_S \omega \tag{IV.10}$$

is non-degenerate.

Proof. This is a combination of Lemma IV.1.6 with Lemma IV.1.3. □

We determine the dimension of the cohomology spaces of the complex of Whitney forms. $\mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$ is linearly isomorphic to the dual space of $\mathcal{C}_k(\mathcal{T},\mathcal{U})$ by Lemma IV.1.7. The exterior derivative $\mathsf{d}^k : \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U})$ transforms into the dual of the simplicial boundary operator $\partial_{k+1} : \mathcal{C}_{k+1}(\mathcal{T},\mathcal{U}) \to \mathcal{C}_k(\mathcal{T},\mathcal{U})$ along that isomorphism. In summary, the complex of Whitney forms over $\mathcal{T}$ relative to $\mathcal{U}$ is isomorphic to the dual complex of the simplicial chain complex of $\mathcal{T}$ relative to $\mathcal{U}$.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\mathsf{d}^{k-1}} & \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^k} & \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots \\ & & \simeq\big\downarrow & & \simeq\big\downarrow & & \\ \dots & \xrightarrow{\partial'_k} & \mathcal{C}_k(\mathcal{T},\mathcal{U})' & \xrightarrow{\partial'_{k+1}} & \mathcal{C}_{k+1}(\mathcal{T},\mathcal{U})' & \xrightarrow{\partial'_{k+2}} & \dots \end{array} \tag{IV.11}$$

These two differential complexes are isomorphic and thus their cohomology spaces are isomorphic. On the other hand, the cohomology spaces of the bottom complex in (IV.11) have the same dimension as homology spaces of the simplicial chain complex of $\mathcal{T}$ relative to $\mathcal{U}$, and in particular their dimensions realize the simplicial Betti numbers. In combination,

$$\begin{aligned}\dim \frac{\ker\left(\mathsf{d}^k : \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U})\right)}{\operatorname{ran}\left(\mathsf{d}^{k-1} : \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^{k}(\mathcal{T},\mathcal{U})\right)} \\ = \dim \frac{\ker\left(\partial'_{k+1} : \mathcal{C}_k(\mathcal{T},\mathcal{U})' \to \mathcal{C}_{k+1}(\mathcal{T},\mathcal{U})'\right)}{\operatorname{ran}\left(\partial'_{k} : \mathcal{C}_{k-1}(\mathcal{T},\mathcal{U})' \to \mathcal{C}_{k}(\mathcal{T},\mathcal{U})'\right)} = b_k(\mathcal{T},\mathcal{U}).\end{aligned}$$

This determines the dimension of the cohomology spaces of the complex of Whitney forms: the complex of Whitney forms realizes the simplicial Betti numbers of $\mathcal{T}$ relative to $\mathcal{U}$ on cohomology.

Example IV.1.8.
We revisit Example IV.1.2 above, where $\mathcal{T}$ triangulates a compact topological manifold M and $\mathcal{U}$ triangulates a subset $\Gamma \subseteq \partial M$ of the boundary. As already mentioned in Chapter II, the topological and simplicial Betti numbers coincide, which means $b_k(\mathcal{T},\mathcal{U}) = b_k(M,\Gamma)$ for all $k \in \mathbb{Z}$. Consequently, the differential complex of Whitney forms (IV.9) realizes the Betti numbers of M relative to Γ on cohomology.

We are now in a position to provide the *canonical finite element interpolant* from the space $C^\infty\Lambda^k(\mathcal{T})$ onto the space $\mathcal{W}\Lambda^k(\mathcal{T})$. We define

$$I^k_{\mathcal{W}} : C^\infty\Lambda^k(\mathcal{T}) \to \mathcal{W}\Lambda^k(\mathcal{T}) \tag{IV.12}$$

by requiring

$$\int_S I^k_{\mathcal{W}}\omega = \int_S \omega, \quad \omega \in C^\infty\Lambda^k(\mathcal{T}), \quad S \in \mathcal{C}_k(\mathcal{T}).$$

With Lemma IV.1.7 we see that this is well-defined. We also observe that

$$I^k_{\mathcal{W}}\omega = \omega, \quad \omega \in \mathcal{W}\Lambda^k(\mathcal{T}).$$

So the operator $I^k_{\mathcal{W}}$ acts as the identity on Whitney forms.

Moreover, $I^k_{\mathcal{W}}$ is local in the sense that for $T \in \mathcal{T}$ and $\omega \in C^\infty\Lambda^k(\mathcal{T})$ we have

$$\omega_T = 0 \implies \operatorname{tr}^k_T I^k_{\mathcal{W}}\omega = 0.$$

This is a consequence of Lemma IV.1.5. Hence, by restricting the interpolant we obtain a well-defined mapping

$$I^k_{\mathcal{W}} : C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}).$$

The interpolation operator commutes with the exterior derivative,

$$\mathsf{d}^k I^k_{\mathcal{W}}\omega = I^{k+1}_{\mathcal{W}}\mathsf{d}^k\omega, \quad \omega \in C^\infty\Lambda^k(\mathcal{T}).$$

This is verified by

$$\int_S I^{k+1}_{\mathcal{W}} \mathsf{d}^k \omega = \int_S \mathsf{d}^k \omega = \int_{\partial_{k+1} S} \omega = \int_{\partial_{k+1} S} I^k_{\mathcal{W}} \omega = \int_S \mathsf{d}^k I^k_{\mathcal{W}} \omega$$

for $S \in \mathcal{C}_{k+1}(\mathcal{T})$ and $\omega \in C^\infty\Lambda^k(\mathcal{T})$. So the diagram

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\mathsf{d}^{k-1}} & C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^k} & C^\infty\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots \\
 & & \Big\downarrow{\scriptstyle I^k_{\mathcal{W}}} & & \Big\downarrow{\scriptstyle I^{k+1}_{\mathcal{W}}} & & \\
\dots & \xrightarrow{\mathsf{d}^{k-1}} & \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^k} & \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots
\end{array}$$

commutes. In particular, $I^k_{\mathcal{W}}$ is a morphism of differential complexes.

IV.2. Polynomial de Rham Complexes over Simplices

The goal of this chapter is to develop finite element de Rham complexes of higher polynomial order over triangulations. The previous section has served our understanding of the lowest-order case. Before we develop the higher order case, we gather some results concerning higher order finite element de Rham complexes on single simplices. First we make the informal observation that differential complexes of similar *type* appear throughout finite element exterior calculus in different variants. For example, a differential complex of trimmed polynomial differential forms of fixed order r appears as differential complex over a single simplex, over a triangulation, or with boundary conditions. It is of interest to turn the idea of sequences having a *type* into a rigorous mathematical notion. A particular motivation are differential complexes in the theory of *hp*-adaptive methods, composed of finite element spaces of non-uniform polynomial order. In that application we wish to assign types of polynomial de Rham complexes to each simplex to describe the local order of approximation.

We first introduce a set of formal symbols

$$\mathscr{S} := \left\{ \dots, \mathcal{P}_{r-1}, \mathcal{P}^-_r, \mathcal{P}_r, \mathcal{P}^-_{r+1}, \dots \right\}. \tag{IV.13}$$

The set $\mathscr{S}$ is endowed with a total order $\leq$ that is defined by $\mathcal{P}^-_r \leq \mathcal{P}_r$ and $\mathcal{P}_r \leq \mathcal{P}^-_{r+1}$ for each $r \in \mathbb{Z}$.

An *admissible sequence type* is a mapping $\mathcal{P} : \mathbb{Z} \to \mathscr{S}$ that satisfies the condition

$$\mathcal{P}(k) \in \left\{ \mathcal{P}_r, \mathcal{P}^-_r \right\} \implies \mathcal{P}(k+1) \in \left\{ \mathcal{P}^-_r, \mathcal{P}_{r-1} \right\}. \tag{IV.14}$$

for all $k \in \mathbb{Z}$. We let $\mathscr{A}$ denote the set of admissible sequence types. The total order on $\mathscr{S}$ induces a partial order $\leq$ on $\mathscr{A}$, where for all $\mathcal{P}, \mathcal{S} \in \mathscr{A}$ we have $\mathcal{P} \leq \mathcal{S}$ if and only if for all $k \in \mathbb{Z}$ we have $\mathcal{P}(k) \leq \mathcal{S}(k)$.

If $\mathcal{P} \in \mathscr{A}$ is an admissible sequence type and T is an n-simplex, then we define for each $k \in \mathbb{Z}$ the spaces

$$\mathcal{P}\Lambda^k(T) := \begin{cases} \mathcal{P}_r\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r, \\ \mathcal{P}_r^-\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r^-, \end{cases} \tag{IV.15}$$

$$\mathring{\mathcal{P}}\Lambda^k(T) := \begin{cases} \mathring{\mathcal{P}}_r\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r, \\ \mathring{\mathcal{P}}_r^-\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r^-, \end{cases} \tag{IV.16}$$

$$\underline{\mathcal{P}}\Lambda^k(T) := \begin{cases} \underline{\mathcal{P}}_r\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r, \\ \underline{\mathcal{P}}_r^-\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r^-, \end{cases} \tag{IV.17}$$

$$\mathring{\underline{\mathcal{P}}}\Lambda^k(T) := \begin{cases} \mathring{\underline{\mathcal{P}}}_r\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r, \\ \mathring{\underline{\mathcal{P}}}_r^-\Lambda^k(T) & \text{if } \mathcal{P}(k) = \mathcal{P}_r^-. \end{cases} \tag{IV.18}$$

The terminology already suggests that the symbols $\mathscr{S}$ describe finite element spaces, whereas the admissible sequence types $\mathscr{A}$ describe finite element differential complexes. To make this idea rigorous, we begin with an easy observation that follows from (IV.14). For each admissible sequence type $\mathcal{P} \in \mathscr{A}$, $k \in \mathbb{Z}$, and m-dimensional simplex $T \subset \mathbb{R}^n$ we have

$$\mathsf{d}^k\left(\mathcal{P}\Lambda^k(T)\right) \subseteq \mathcal{P}\Lambda^{k+1}(T), \quad \mathsf{d}^k\left(\mathring{\mathcal{P}}\Lambda^k(T)\right) \subseteq \mathring{\mathcal{P}}\Lambda^{k+1}(T),$$
$$\mathsf{d}^k\left(\underline{\mathcal{P}}\Lambda^k(T)\right) \subseteq \underline{\mathcal{P}}\Lambda^{k+1}(T), \quad \mathsf{d}^k\left(\mathring{\underline{\mathcal{P}}}\Lambda^k(T)\right) \subseteq \mathring{\underline{\mathcal{P}}}\Lambda^{k+1}(T).$$

In the light of this, each admissible sequence type describes the composition of a differential complex. Suppose that T is a simplex and that $\mathcal{P} \in \mathscr{A}$ is an admissible sequence type. Then we have a polynomial de Rham complex over T,

$$0 \to \mathbb{R} \longrightarrow \mathcal{P}\Lambda^0(T) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \mathcal{P}\Lambda^n(T) \to 0, \tag{IV.19}$$

and a polynomial de Rham complex over T with boundary conditions,

$$0 \to \mathring{\mathcal{P}}\Lambda^0(T) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \mathring{\mathcal{P}}\Lambda^n(T) \xrightarrow{\int_T} \mathbb{R} \to 0. \tag{IV.20}$$

We will also consider the reduced differential complexes

$$0 \to \underline{\mathcal{P}}\Lambda^0(T) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \underline{\mathcal{P}}\Lambda^n(T) \to 0, \tag{IV.21}$$

$$0 \to \mathring{\underline{\mathcal{P}}}\Lambda^0(T) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \mathring{\underline{\mathcal{P}}}\Lambda^n(T) \to 0. \tag{IV.22}$$

We establish the exactness of these differential complexes.

Lemma IV.2.1.
Let T be a simplex and let $\mathcal{P} \in \mathscr{A}$ be an admissible sequence type. If $1_T \in \mathcal{P}\Lambda^0(T)$, then (IV.19) is well-defined and exact. If $\mathrm{vol}_T \in \mathring{\mathcal{P}}\Lambda^n(T)$, then (IV.20) is exact.

Proof. With regards to the first sequence, it is obvious that $\ker \mathsf{d}^0 \cap \mathcal{P}\Lambda^0(T)$ is spanned by 1_T. Let $k \in \{1, \dots, n\}$ and $\omega \in \mathcal{P}\Lambda^k(T)$ with $\mathsf{d}^k\omega = 0$. Then there exists $r \in \mathbb{Z}$ with $\omega \in \mathcal{P}_r\Lambda^k(T)$. By (III.29a), there exists $\xi \in \mathcal{P}_{r+1}^-\Lambda^{k-1}(T)$ with $\mathsf{d}^{k-1}\xi = \omega$. Since $\mathcal{P}_{r+1}^-\Lambda^{k-1}(T) \subseteq \mathcal{P}\Lambda^{k-1}(T)$, the exactness of the first sequence follows.

With regards to the second sequence, it is obvious that $\ker \mathsf{d}^0 \cap \mathring{\mathcal{P}}\Lambda^0(T)$ is the trivial vector space. Now let $k \in \{1, \dots, n\}$ and $\omega \in \mathcal{P}\Lambda^k(T)$ with $\mathsf{d}^k\omega = 0$. If $k = n$, the we assume additionally $\int_T \omega = 0$. There exists $r \in \mathbb{Z}$ such that $\omega \in \mathring{\mathcal{P}}_r\Lambda^k(T)$. By (III.29b), we obtain $\eta \in \mathring{\mathcal{P}}^-_{r+1}\Lambda^{k-1}(T)$ with $\mathsf{d}^{k-1}\eta = \omega$. But we also have $\mathring{\mathcal{P}}^-_{r+1}\Lambda^{k-1}(T) \subseteq \mathring{\mathcal{P}}\Lambda^{k-1}(T)$. This completes the proof. □

Lemma IV.2.2.
Let T be a simplex and let $\mathcal{P}$ be an admissible sequence type. Then (IV.21) and (IV.22) are exact sequences. □

Proof. If $1_T \in \mathcal{P}\Lambda^0(T)$, then $\mathcal{P}\Lambda^0(T) = \operatorname{span}\{1_T\} \oplus \underline{\mathcal{P}}\Lambda^0(T)$, and if $\operatorname{vol}_T \in \mathring{\mathcal{P}}\Lambda^n(T)$, then $\mathcal{P}\Lambda^n(T) = \operatorname{span}\{\operatorname{vol}_T\} \oplus \mathring{\underline{\mathcal{P}}}\Lambda^n(T)$. The claim now follows immediately from the preceding result. □

Example IV.2.3.
The admissible sequence types describe the finite element de Rham complexes of finite element exterior calculus. Over a triangle $T \subseteq \mathbb{R}^2$, these take one of the forms

$$\begin{aligned}
\mathcal{P}_r\Lambda^0(T) &\longrightarrow \mathcal{P}^-_r\Lambda^1(T) \longrightarrow \mathcal{P}_{r-1}\Lambda^2(T),\\
\mathcal{P}_r\Lambda^0(T) &\longrightarrow \mathcal{P}_{r-1}\Lambda^1(T) \longrightarrow \mathcal{P}_{r-2}\Lambda^2(T),
\end{aligned}$$

and over a tetrahedron $T \subseteq \mathbb{R}^3$, these take one of the forms

$$\begin{aligned}
\mathcal{P}_r\Lambda^0(T) &\longrightarrow \mathcal{P}^-_r\Lambda^1(T) \longrightarrow \mathcal{P}^-_r\Lambda^2(T) \longrightarrow \mathcal{P}_{r-1}\Lambda^3(T),\\
\mathcal{P}_r\Lambda^0(T) &\longrightarrow \mathcal{P}^-_r\Lambda^1(T) \longrightarrow \mathcal{P}_{r-1}\Lambda^2(T) \longrightarrow \mathcal{P}_{r-2}\Lambda^3(T),\\
\mathcal{P}_r\Lambda^0(T) &\longrightarrow \mathcal{P}_{r-1}\Lambda^1(T) \longrightarrow \mathcal{P}^-_{r-1}\Lambda^2(T) \longrightarrow \mathcal{P}_{r-2}\Lambda^3(T),\\
\mathcal{P}_r\Lambda^0(T) &\longrightarrow \mathcal{P}_{r-1}\Lambda^1(T) \longrightarrow \mathcal{P}_{r-2}\Lambda^2(T) \longrightarrow \mathcal{P}_{r-3}\Lambda^3(T).
\end{aligned}$$

In general, when the polynomial order of the space of 0-forms is fixed, then there are 2^{n-1} different differential complexes in the framework of finite element exterior calculus over an n-dimensional simplex.

Now we move our attention towards dual spaces and their representations. This prepares the discussion of the degrees of freedom of finite element de Rham complexes in the next section. Let T be a simplex and let g be a smooth Riemannian metric over T. This induces a positive definite bilinear form (cf. Agricola and Friedrich [92] or the discussion in the previous chapter)

$$B_g : C^\infty\Lambda^k(T) \times C^\infty\Lambda^k(T) \to \mathbb{R}, \quad (\omega, \eta) \mapsto \int_T \langle \omega, \eta \rangle_g.$$

The restriction of this bilinear form to any finite-dimensional subspace of $C^\infty\Lambda^k(T)$ gives a Hilbert space structure on that subspace. We apply this idea to the spaces $\mathring{\underline{\mathcal{P}}}\Lambda^k(T)$, since this is the special case needed in the sequel. The following lemma, however, generalizes to the spaces of the form $\mathcal{P}\Lambda^k(T)$, $\mathring{\mathcal{P}}\Lambda^k(T)$ and $\underline{\mathcal{P}}\Lambda^k(T)$ with minimal changes.

Lemma IV.2.4.
Let $\mathcal{P} \in \mathscr{A}$, let $k \in \mathbb{Z}$, and let $T \subset \mathbb{R}^n$ be a simplex. For every linear functional $\Psi : \mathring{\underline{\mathcal{P}}}\Lambda^k(T) \to \mathbb{R}$ there exist $\rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(T)$ and $\beta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(T)$ such that

$$\Psi(\omega) = \int_T \langle \omega, \mathsf{d}^{k-1}\rho \rangle_g + \int_T \langle \mathsf{d}^k \omega, \mathsf{d}^k \beta \rangle_g, \quad \omega \in \mathring{\underline{\mathcal{P}}}\Lambda^k(T).$$

Proof. Let $\Psi : \mathring{\underline{\mathcal{P}}}\Lambda^k(T) \to \mathbb{R}$ be linear and let $\omega \in \mathring{\underline{\mathcal{P}}}\Lambda^k(T)$ be arbitrary. Since B_g induces a Hilbert space structure on a finite-dimensional vector space, the Riesz representation theorem ensures the existence of $\eta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(T)$ such that $\Psi(\omega) = B_g(\omega, \eta)$. We write $A_0 = \mathring{\underline{\mathcal{P}}}\Lambda^k(T) \cap \ker \mathsf{d}^k$ and let A_1 denote the orthogonal complement of A_0 in $\mathring{\underline{\mathcal{P}}}\Lambda^k(T)$ with respect to the scalar product B_g. We have an orthogonal decomposition $\mathring{\underline{\mathcal{P}}}\Lambda^k(T) = A_0 \oplus A_1$, and unique decompositions $\omega = \omega_0 + \omega_1$ and $\eta = \eta_0 + \eta_1$ with $\omega_0, \eta_0 \in A_0$ and $\omega_1, \eta_1 \in A_1$. Thus

$$\Psi(\omega) = \int_T \langle \omega, \eta \rangle_g = \int_T \langle \omega_0, \eta_0 \rangle_g + \int_T \langle \omega_1, \eta_1 \rangle_g.$$

By the exactness of (IV.22) there exists $\rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(T)$ such that $\eta_0 = \mathsf{d}^{k-1}\rho$. Since the bilinear form $B_g\left(\mathsf{d}^k \cdot, \mathsf{d}^k \cdot\right)$ is a scalar product over A_1 equivalent to B_g, we may use the Riesz representation theorem again to obtain $\beta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(T)$ with $B_g\left(\mathsf{d}^k\omega_1, \mathsf{d}^k\beta\right) = B_g\left(\omega_1, \eta_1\right)$. The proof is complete. $\square$

IV.3. Higher Order Finite Element Complexes

We are in a position now to discuss the finite element de Rham complexes of higher and possibly non-uniform polynomial order over a simplicial complex.

Let $\mathcal{T}$ be a simplicial complex and let $\mathcal{U}$ be a (possibly empty) subcomplex of $\mathcal{T}$. We let $\mathcal{P} : \mathcal{T} \to \mathscr{A}$ be a mapping that associates to each simplex $T \in \mathcal{T}$ an admissible sequence type $\mathcal{P}_T : \mathbb{Z} \to \mathscr{S}$. We then define

$$\mathcal{P}\Lambda^k(\mathcal{T}) := \left\{ \omega \in C^\infty\Lambda^k(\mathcal{T}) \;\middle|\; \forall T \in \mathcal{T} : \omega_T \in \mathcal{P}_T\Lambda^k(T) \right\}. \tag{IV.23}$$

By construction, the exterior derivative preserves this class of differential forms,

$$\mathsf{d}^k\mathcal{P}\Lambda^k(\mathcal{T}) \subseteq \mathcal{P}\Lambda^{k+1}(\mathcal{T}), \tag{IV.24}$$

and in particular, we have a differential complex

$$\dots \xrightarrow{\mathsf{d}^{k-1}} \mathcal{P}\Lambda^k(\mathcal{T}) \xrightarrow{\mathsf{d}^k} \mathcal{P}\Lambda^{k+1}(\mathcal{T}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{IV.25}$$

We furthermore define the subspaces

$$\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) := \mathcal{P}\Lambda^k(\mathcal{T}) \cap C^\infty\Lambda^k(\mathcal{T}),$$

which constitute the differential complex

$$\dots \xrightarrow{\mathsf{d}^{k-1}} \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^k} \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{IV.26}$$

Having associated an admissible sequence type $\mathcal{P}_T$ to each $T \in \mathcal{T}$, we say that the *hierarchy condition holds* if

$$\forall T \in \mathcal{T} : \forall F \in \Delta(T) : \mathcal{P}_F \leq \mathcal{P}_T. \tag{IV.27}$$

We call $\mathcal{P}$ *hierarchical* if the hierarchy condition holds. We assume the hierarchy condition throughout this section; if $\mathcal{P} : \mathcal{T} \to \mathscr{A}$ is not hierarchical, then one can find $\tilde{\mathcal{P}} : \mathcal{T} \to \mathscr{A}$ satisfying the hierarchy condition and yielding the same finite element spaces. In order to simplify the notation, we will write $\mathcal{P}\Lambda^k(T) := \mathcal{P}_T\Lambda^k(T)$ from here on.

Example IV.3.1.
The admissible sequence types associated to each simplex describe the order of approximation associated to each simplex. If we choose the same admissible sequence for every simplex, then the resulting spaces $\mathcal{P}\Lambda^k(\mathcal{T})$ are finite element spaces of uniform polynomial order of the kind considered originally in finite element exterior calculus. The most simple example is obtained by choosing for each $T \in \mathcal{T}$ the admissible sequence type $\mathcal{P} \in \mathscr{A}$ with $\mathcal{P}(k) = \mathcal{P}_1^-$ for all $k \in \mathbb{Z}$. In the sequel we will see that this choice leads to differential complexes of lowest order.

Remark IV.3.2.
The general idea of the hierarchy condition is that the polynomial order associated to a simplex is at least the polynomial order associated to any subsimplex. Imposing such a condition is common in the literature on *hp* finite element methods [68]. Indeed, if $(\mathcal{P}_T)_{T\in\mathcal{T}}$ violates the hierarchy condition, then there exists a family of sequence types $(\mathcal{S}_T)_{T\in\mathcal{T}}$ that satisfies the hierarchy condition and yields the same space $\mathcal{P}\Lambda^k(\mathcal{T})$. This is analogous to what is called *minimum rule* in the literature (see [69]). We refer also to [56] for the corresponding concept in the theory of element systems.

The geometric decomposition of finite element spaces is a concept of paramount importance. To establish geometric decompositions for the spaces $\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$, we recur to an idea of Chapter III and discuss extension operators. Specifically, we assume to have linear *local extension operators*

$$\operatorname{ext}^k_{F,T} : \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \to \mathcal{P}\Lambda^k(T) \tag{IV.28}$$

for every $F \in \Delta(T)$ with $T \in \mathcal{T}$ such that the following properties hold:

(i) for all $F \in \mathcal{T}$ we have

$$\operatorname{ext}^k_{F,F}\omega = \omega, \quad \omega \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F), \tag{IV.29a}$$

(ii) for all $T \in \mathcal{T}$ with $F \in \Delta(T)$ and $f \in \Delta(F)$ we have

$$\operatorname{tr}^k_{T,F}\operatorname{ext}^k_{f,T} = \operatorname{ext}^k_{f,F}, \tag{IV.29b}$$

(iii) for all $T \in \mathcal{T}$ and $F, G \in \Delta(T)$ with $F \notin \Delta(G)$ we have

$$\operatorname{tr}^k_{T,G}\operatorname{ext}^k_{F,T} = 0. \tag{IV.29c}$$

For each $F \in \mathcal{T}$ we then define the associated *global extension operator*,

$$\operatorname{Ext}_F^k : \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \to C^\infty\Lambda^k(\mathcal{T}), \quad \mathring{\omega} \mapsto \bigoplus_{\substack{T \in \mathcal{T} \\ F \in \Delta(T)}} \operatorname{ext}_{F,T}^k \mathring{\omega}. \tag{IV.30}$$

It follows from (IV.29b) that this mapping indeed takes values in $C^\infty\Lambda^k(\mathcal{T})$. Moreover, definitions imply

$$\operatorname{Ext}_F^k \left(\mathring{\underline{\mathcal{P}}}\Lambda^k(F) \right) \subseteq \mathcal{P}\Lambda^k(\mathcal{T}). \tag{IV.31}$$

We note that $\operatorname{Ext}_F^k \omega$ for $\omega \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ vanishes on all simplices of $\mathcal{T}$ that do not contain F as a subsimplex.

Example IV.3.3.
We recall the extension operators introduced for geometric decompositions of the spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$, which were introduced in Chapter III. These were

$$\operatorname{ext}_{F,T}^{k,r} : \mathring{\mathcal{P}}_r\Lambda^k(F) \to \mathcal{P}_r\Lambda^k(T), \quad \operatorname{ext}_{F,T}^{k,r,-} : \mathring{\mathcal{P}}_r^-\Lambda^k(F) \to \mathcal{P}_r^-\Lambda^k(T).$$

These extension operators satisfy the required properties. They are a possible choice for the local extension operators in this section, in accordance to whether $\mathcal{P}_F(k) = \mathcal{P}_r$ or $\mathcal{P}_F(k) = \mathcal{P}_r^-$.

We can describe the geometric decomposition of $\mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$ in terms of the extension operators. The hierarchy condition is critical for that. For every $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ we define $\omega^{\mathcal{W}} \in \mathcal{P}\Lambda^k(\mathcal{T})$ by

$$\omega^{\mathcal{W}} := \sum_{F \in \mathcal{T}^k} \operatorname{vol}(F)^{-1} \left(\int_F \operatorname{tr}_F^k \omega \right) \operatorname{Ext}_F^k \operatorname{vol}_F. \tag{IV.32}$$

We then define recursively for every $m \in \{k, \dots, n\}$

$$\mathring{\omega}_F := \operatorname{tr}_F^k \left(\omega - \omega^{\mathcal{W}} - \sum_{l=k}^{m-1} \omega^l \right), \quad F \in \mathcal{T}^m, \tag{IV.33}$$

$$\omega^m := \sum_{F \in \mathcal{T}^m} \operatorname{Ext}_F^k \mathring{\omega}_F. \tag{IV.34}$$

The following theorem shows that these definitions are well-defined and give a decomposition of ω.

Theorem IV.3.4.
Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. Then we have $\mathring{\omega}_F \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ for every $F \in \mathcal{T}$ and

$$\omega = \omega^{\mathcal{W}} + \sum_{m=k}^{n} \omega^m. \tag{IV.35}$$

Proof. By construction of $\omega^{\mathcal{W}}$, we have

$$\int_F \operatorname{tr}_F^k \omega^{\mathcal{W}} = \int_F \operatorname{tr}_F^k \omega, \quad F \in \mathcal{T}^k.$$

By definition, $\operatorname{tr}_F^k\left(\omega - \omega^{\mathcal{W}}\right) \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ for every $F \in \mathcal{T}^k$. With ω^k as defined above, we see

$$\operatorname{tr}_F^k\left(\omega - \omega^{\mathcal{W}} - \omega^k\right) = 0, \quad F \in \mathcal{T}^k.$$

Let us now suppose that for some $m \in \{k, \dots, n-1\}$ we have shown

$$\operatorname{tr}_f^k\left(\omega - \omega^{\mathcal{W}} - \sum_{l=k}^{m} \omega^l\right) = 0, \quad f \in \mathcal{T}^m.$$

By definition we have $\mathring{\underline{\mathcal{P}}}\Lambda^k(F) = \mathring{\mathcal{P}}\Lambda^k(F)$ for $F \in \mathcal{T}^{m+1}$, and $\mathring{\omega}_F \in \mathring{\mathcal{P}}\Lambda^k(F)$ for $F \in \mathcal{T}^{m+1}$. We conclude that ω^{m+1} is well-defined and that

$$\operatorname{tr}_F^k\left(\omega - \omega^{\mathcal{W}} - \sum_{l=k}^{m+1} \omega^l\right) = 0, \quad F \in \mathcal{T}^{m+1}.$$

An induction argument then provides (IV.35). The proof is complete. □

Lemma IV.3.5.
Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ and $F \in \mathcal{T}$. Then we have $\omega_F = 0$ if and only if $\operatorname{tr}_f^k \omega^{\mathcal{W}} = 0$ for all $f \in \Delta(F)$ and $\mathring{\omega}_f = 0$ for all $f \in \Delta(F)^k$.

Proof. For any $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ and $F \in \mathcal{T}^m$ we observe

$$\begin{aligned}
\omega_F &= \operatorname{tr}_F^k \omega^{\mathcal{W}} + \sum_{k \leq m \leq n} \sum_{f \in \mathcal{T}^m} \operatorname{tr}_F^k \operatorname{Ext}_f^k \mathring{\omega}_f \\
&= \sum_{f \in \Delta(F)^k} \operatorname{vol}(F)^{-1} \left(\int_f \operatorname{tr}_f^k \omega\right) \operatorname{ext}_{f,F}^k \operatorname{vol}_F + \sum_{f \in \Delta(F)} \operatorname{ext}_{f,F}^k \mathring{\omega}_f.
\end{aligned}$$

If $k = m$, then $\omega_F = \operatorname{tr}_F^k \omega^{\mathcal{W}} + \mathring{\omega}_F$, and the claims follows by this being a direct sum. If $k < m$, let us assume that the claim holds for all $f \in \mathcal{T}$ with $k \leq \dim f < m$. Then $\omega_F = \mathring{\omega}_F$, which again proves the claim. The lemma now follows from an induction argument. □

Lemma IV.3.6.
For $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ we have $\omega \in \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$ if and only if $\mathring{\omega}_F = 0$ for all $F \in \mathcal{U}$ and $\omega_F^{\mathcal{W}} = 0$ for all $F \in \mathcal{U}^k$.

Proof. This is a simple consequence of Lemma IV.3.5. □

Lemma IV.3.7.
For $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ we have $\omega = 0$ if and only if $\mathring{\omega}_F = 0$ for all $F \in \mathcal{T}$ and $\omega_F^{\mathcal{W}} = 0$ for all $F \in \mathcal{T}^k$.

Proof. This follows from Lemma IV.3.6 applied to the case $\mathcal{U} = \mathcal{T}$. □

Theorem IV.3.8.
We have

$$\mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U}) = \mathcal{W}\Lambda^k(\mathcal{T}, \mathcal{U}) \oplus \bigoplus_{F \in \mathcal{T} \setminus \mathcal{U}} \operatorname{Ext}_F^k \mathring{\underline{\mathcal{P}}}\Lambda^k(F).$$

Proof. The claim is implied by Theorem IV.3.4 and Lemma IV.3.7. □

A modification of the geometric decomposition will be helpful in the sequel.

Lemma IV.3.9.
Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. Then there exist unique $\mathring{\omega}_F \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ for $F \in \mathcal{T}$ such that

$$\omega = I^k_{\mathcal{W}}\omega + \sum_{m=k}^{n} \sum_{F \in \mathcal{T}^m} \operatorname{Ext}^k_F \mathring{\omega}^m_F. \tag{IV.36}$$

Proof. Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. The trace of $I^k_{\mathcal{W}}\omega - \omega$ over any simplex $F \in \mathcal{T}^k$ has vanishing integral. The claim follows from applying Theorem IV.3.4 to $I^k_{\mathcal{W}} - \omega$. □

IV.4. Commuting Interpolants

We finish this chapter with the finite element interpolant and study some of its properties. The basic ideas have already been used in prior literature [56, 69], but we apply some modifications and extensions. Our construction explicitly calculates the geometric decomposition of the interpolating differential form. First we define

$$J^k_{\mathcal{W}} : C^\infty\Lambda^k(\mathcal{T}) \to \mathcal{P}\Lambda^k(\mathcal{T}), \quad \omega \mapsto \sum_{F \in \mathcal{T}^k} \operatorname{vol}(F)^{-1} \left(\int_F \omega \right) \operatorname{Ext}^k_F \operatorname{vol}_F . \tag{IV.37}$$

Subsequently for $m \in \{k, \dots, n\}$ we make the recursive definition

$$J^k_m : C^\infty\Lambda^k(\mathcal{T}) \to \mathcal{P}\Lambda^k(\mathcal{T}), \quad \omega \mapsto \sum_{F \in \mathcal{T}^m} \operatorname{Ext}^k_F J^k_F\omega, \tag{IV.38}$$

where for each $F \in \mathcal{T}^m$ we define

$$J^k_F : C^\infty\Lambda^k(\mathcal{T}) \to \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \tag{IV.39}$$

by requiring $J^k_F\omega$ for $\omega \in C^\infty\Lambda^k(\mathcal{T})$ to be the unique solution of

$$\int_F \left\langle J^k_F\omega, \mathsf{d}^{k-1}\rho \right\rangle_g = \int_F \left\langle \operatorname{tr}^k_F \left(\omega - J^k_{\mathcal{W}}\omega - \sum_{k=l}^{m-1} J^k_l \omega \right), \mathsf{d}^{k-1}\rho \right\rangle_g, \quad \rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F), \tag{IV.40a}$$

$$\int_F \left\langle \mathsf{d}^k J^m_F\omega, \mathsf{d}^k\beta \right\rangle_g = \int_F \left\langle \mathsf{d}^k \operatorname{tr}^k_F \left(\omega - J^k_{\mathcal{W}}\omega - \sum_{k=l}^{m-1} J^k_l \omega \right), \mathsf{d}^k\beta \right\rangle_g, \quad \beta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F). \tag{IV.40b}$$

From Lemma IV.2.4 we find that $J^k_F\omega$ is well-defined. We then set

$$I^k_{\mathcal{P}} : C^\infty\Lambda^k(\mathcal{T}) \to \mathcal{P}\Lambda^k(\mathcal{T}), \quad \omega \mapsto J^k_{\mathcal{W}}\omega + J^k_k\omega + \dots + J^k_n\omega. \tag{IV.41}$$

We show that the operator $I^k_{\mathcal{P}}$ acts as the identity on $\mathcal{P}\Lambda^k(\mathcal{T})$, and its constituents J^k_F reproduce the geometric decomposition.

Lemma IV.4.1.
For each $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ we have $I^k_{\mathcal{P}}\omega = \omega$. Moreover, $J^k_{\mathcal{W}}\omega = \omega^{\mathcal{W}}$ and $J^k_F\omega = \mathring{\omega}_F$ for each $F \in \mathcal{T}$.

Proof. Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. We have $J^k_{\mathcal{W}}\omega = \omega^{\mathcal{W}}$ by definition. For $F \in \mathcal{T}^k$ we find $\operatorname{tr}^k_F\left(\omega - \omega^{\mathcal{W}}\right) \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$, and $J^k_F\omega = \mathring{\omega}_F$ follows easily. Next, let $m \in \{k, \dots, n-1\}$ and suppose that $J^k_F\omega = \mathring{\omega}_F$ for $F \in \mathcal{T}$ with $\dim F \leq m$. Let $F \in \mathcal{T}^{m+1}$. From definitions we conclude that

$$\operatorname{tr}^k_F\left(\omega - \omega^{\mathcal{W}} - \sum_{l=k}^{m-1} J^l\omega\right) \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F).$$

It follows that $J^k_F\omega = \mathring{\omega}_F$ and hence $J^k_m\omega = \omega^m$. An induction argument completes the proof. $\square$

Lemma IV.4.2.
Let $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$. If

$$\int_F \operatorname{tr}^k_F \omega = 0, \quad F \in \mathcal{T}^k, \tag{IV.42a}$$

$$\int_F \left\langle \operatorname{tr}^k_F \omega, \mathsf{d}^{k-1}\rho \right\rangle_g = 0, \quad \rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F), \quad F \in \mathcal{T}, \tag{IV.42b}$$

$$\int_F \left\langle \mathsf{d}^k \operatorname{tr}^k_F \omega, \mathsf{d}^k\beta \right\rangle_g = 0, \quad \beta \in \mathring{\underline{\mathcal{P}}}\Lambda^{k}(F) \quad F \in \mathcal{T}, \tag{IV.42c}$$

then $\omega = 0$.

Proof. If $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ such that (IV.42), then $J^k_{\mathcal{W}}\omega = 0$ and $J^k_k\omega = 0$. Rearranging the terms in (IV.39), an induction argument yields that $J^k_m\omega = 0$ for all $m \in \{k, \dots, n\}$. The claim is now a consequence of Lemma IV.4.1. $\square$

An auxiliary result yields an alternative characterization of $I^k_{\mathcal{P}}$.

Lemma IV.4.3.
Let $\omega \in C^\infty\Lambda^k(\mathcal{T})$ and $\omega' \in \mathcal{P}\Lambda^k(\mathcal{T})$. We have $\omega' = I^k_{\mathcal{P}}\omega$ if and only if

$$\int_F \operatorname{tr}^k_F \omega' = \int_F \operatorname{tr}^k_F \omega, \quad F \in \mathcal{T}^k, \tag{IV.43a}$$

$$\int_F \left\langle \operatorname{tr}^k_F \omega', \mathsf{d}^{k-1}\rho \right\rangle_g = \int_F \left\langle \operatorname{tr}^k_F \omega, \mathsf{d}^{k-1}\rho \right\rangle_g, \quad \rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F), \quad F \in \mathcal{T}, \tag{IV.43b}$$

$$\int_F \left\langle \mathsf{d}^k \operatorname{tr}^k_F \omega', \mathsf{d}^k\beta \right\rangle_g = \int_F \left\langle \mathsf{d}^k \operatorname{tr}^k_F \omega, \mathsf{d}^k\beta \right\rangle_g, \quad \beta \in \mathring{\underline{\mathcal{P}}}\Lambda^{k}(F) \quad F \in \mathcal{T}. \tag{IV.43c}$$

Proof. Let $\omega \in C^\infty\Lambda^k(\mathcal{T})$. We verify that $I^k_{\mathcal{P}}\omega$ satisfies (IV.43) by rearranging the terms in (IV.39) and the assumptions on the extension operators. If $\omega' \in \mathcal{P}\Lambda^k(\mathcal{T})$ is another solution to (IV.43), then we obtain $\omega' = I^k_{\mathcal{P}}\omega$ by applying Lemma IV.4.2 to $\omega' - I^k_{\mathcal{P}}\omega$. $\square$

Lemma IV.4.4.
Let $\omega \in C^\infty\Lambda^k(\mathcal{T})$ and $F \in \mathcal{T}$. If $\omega_F = 0$ then $\operatorname{tr}^k_F\left(I^k_{\mathcal{P}}\omega\right) = 0$.

Proof. Unfolding definitions we find

$$\begin{aligned}
\operatorname{tr}_F^k \left(I_{\mathcal{P}}^k \omega \right) &= \operatorname{tr}_F^k J_{\mathcal{W}}^k \omega + \sum_{m=k}^{n} \sum_{f \in \mathcal{T}^m} \operatorname{tr}_F^k \operatorname{Ext}_f^k J_f^k \omega \\
&= \sum_{f \in \Delta(F)^k} \operatorname{vol}(F)^{-1} \left(\int_f \operatorname{tr}_f^k \omega \right) \operatorname{ext}_{f,F}^k \operatorname{vol}_F + \sum_{f \in \Delta(F)} \operatorname{ext}_{f,F}^k J_f^k \omega.
\end{aligned}$$

If $\dim F = k$, then the claim follows from the direct decomposition (III.27) / (III.28). If $\dim F > k$, suppose that the claim has been proven for $f \in \Delta(F)$. Since $\omega_F = 0$ we have $\omega_f = 0$ for $f \in \Delta(F)$. Hence $\operatorname{tr}_F^k \left(I_{\mathcal{P}}^k \omega \right) = J_F^k \omega$, from which $\operatorname{tr}_F^k \left(I_{\mathcal{P}}^k \omega \right) = 0$ follows. An induction argument completes the proof. □

Lemma IV.4.5.
If $\omega \in C^\infty \Lambda^k(\mathcal{T}, \mathcal{U})$, then $I_{\mathcal{P}}^k \omega \in \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$.

Proof. This is an immediate consequence of Lemma IV.4.4 above. □

It remains to show that the interpolant commutes with the exterior derivative, so we have a commuting diagram

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\mathsf{d}^{k-1}} & C^\infty \Lambda^k(\mathcal{T}, \mathcal{U}) & \xrightarrow{\mathsf{d}^k} & C^\infty \Lambda^{k+1}(\mathcal{T}, \mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots \\
 & & \Big\downarrow{\scriptstyle I_{\mathcal{P}}^k} & & \Big\downarrow{\scriptstyle I_{\mathcal{P}}^{k+1}} & & \\
\dots & \xrightarrow{\mathsf{d}^{k-1}} & \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U}) & \xrightarrow{\mathsf{d}^k} & \mathcal{P}\Lambda^{k+1}(\mathcal{T}, \mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots
\end{array}$$

This the subject of the following lemma.

Lemma IV.4.6.
We have $\mathsf{d}^k I_{\mathcal{P}}^k \omega = I_{\mathcal{P}}^{k+1} \mathsf{d}^k \omega$ for $\omega \in C^\infty \Lambda^k(\mathcal{T})$.

Proof. Let $\omega \in C^\infty \Lambda^k(\mathcal{T}, \mathcal{U})$. For $F \in \mathcal{T}^{k+1}$ we observe

$$\begin{aligned}
\int_F \operatorname{tr}_F^{k+1} \mathsf{d}^k I_{\mathcal{P}}^k \omega &= \int_F \operatorname{tr}_F^{k+1} \mathsf{d}^k J_{\mathcal{W}}^k \omega = \int_F \mathsf{d}^k \operatorname{tr}_F^k J_{\mathcal{W}}^k \omega \\
&= \int_{\partial_{k+1} F} \operatorname{tr}_F^k J_{\mathcal{W}}^k \omega = \int_{\partial_{k+1} F} \operatorname{tr}_F^k \omega \\
&= \int_F \mathsf{d}^k \operatorname{tr}_F^k \omega \\
&= \int_F \operatorname{tr}_F^{k+1} \mathsf{d}^k \omega = \int_F \operatorname{tr}_F^{k+1} J_{\mathcal{W}}^{k+1} \mathsf{d}^k \omega = \int_F \operatorname{tr}_F^{k+1} I_{\mathcal{P}}^{k+1} \mathsf{d}^k \omega.
\end{aligned}$$

Let $F \in \mathcal{T}^m$ with $k \leq m \leq n$. For $\rho \in \mathring{\mathcal{P}}\Lambda^k(F)$ we find

$$\begin{aligned}
\int_F \left\langle I_{\mathcal{P}}^{k+1} \mathsf{d}^k \omega, \mathsf{d}^k \rho \right\rangle_g &= \int_F \left\langle \mathsf{d}^k \omega, \mathsf{d}^k \rho \right\rangle_g \\
&= \int_F \left\langle \mathsf{d}^k I_{\mathcal{P}}^k \omega, \mathsf{d}^k \rho \right\rangle_g = \int_F \left\langle \mathsf{d}^k I_{\mathcal{P}}^k \omega, \mathsf{d}^k \rho \right\rangle_g .
\end{aligned}$$

For $\beta \in \mathring{\mathcal{P}}\Lambda^{k+1}(F)$ we find

$$\begin{aligned}\int_F \left\langle \mathsf{d}^{k+1} I_{\mathcal{P}}^{k+1} \mathsf{d}^k \omega, \mathsf{d}^{k+1}\beta \right\rangle_g &= \int_F \left\langle \mathsf{d}^{k+1} \mathsf{d}^k \omega, \mathsf{d}^{k+1}\beta \right\rangle_g \\ &= \int_F \left\langle \mathsf{d}^{k+1} \mathsf{d}^k I_{\mathcal{P}}^{k} \omega, \mathsf{d}^{k+1}\beta \right\rangle_g = 0.\end{aligned}$$

In conjunction with Lemma IV.4.3, the desired result follows. □

Remark IV.4.7.
The definition of the interpolant and Lemma IV.4.3, implicitly use degrees of freedom associated with simplices of the triangulation. These functionals, however, employ an arbitrary Riemannian metric. When we restrict to finite element de Rham complexes of spaces of *uniform* polynomial order, then the degrees of freedom have canonical representations not involving a Riemannian metric (see Section 5 of [9]).

Remark IV.4.8.
In the sequel, we want to apply the commuting interpolant to differential forms that have well-defined traces on all subsimplices but do not necessarily have a classical (non-distributional) exterior derivative. Although some of the degrees of freedom in the definition of the commuting interpolant involve the exterior derivative of the differential form to be interpolated, this is of no further concern for our intended application. For $\omega \in C^\infty\Lambda^k(\mathcal{T},\mathcal{U})$, a simplex $F \in \mathcal{T}$ of dimension m, and $\beta \in \mathring{\mathcal{P}}\Lambda^k(F)$ we observe

$$\begin{aligned}&\int_F \left\langle \mathsf{d}^k \operatorname{tr}_F^k \omega, \mathsf{d}^k \beta \right\rangle_g = \int_F \mathsf{d}^k \operatorname{tr}_F^k \omega \wedge \star_g \mathsf{d}^k \beta \\ &= (-1)^{k+1} \int_F \operatorname{tr}_F^k \omega \wedge \mathsf{d}^{m-k-1} \star_g \mathsf{d}^k \beta + \sum_{f \in \Delta(F)^{m-1}} o(f,F) \int_f \operatorname{tr}_f^k \omega \wedge \operatorname{tr}_{F,f}^{m-k-1} \star_g \mathsf{d}^k \beta.\end{aligned}$$

Hence the presence of the exterior derivative may be traded in for taking traces on lower-dimensional simplices.

V. Differential Forms over Domains

The purpose of this chapter is to review fundamental results on the calculus of differential forms on domains. We pay particular attention to the differential forms with coefficients in L^p spaces and coordinate transformations with Lipschitz regularity. Finally, we discuss homogeneous boundary conditions in a setting of low regularity.

Our motivation for studying differential forms and coordinate transformations of low regularity lies in the construction of the smoothed projection later in this thesis. A component there are bi-Lipschitz coordinate transformations, which leave only the L^p classes of differential forms invariant. The pullback of a Lipschitz 0-form along a bi-Lipschitz mapping is again a Lipschitz 0-form, but this does not generalize to arbitrary k-forms. The reason is that the pullback of a form of positive degree involves coefficients of the Jacobian, which generally have no stronger regularity than being essentially bounded. Hence the pullback along mappings of low regularity is needed for this thesis.

In this context, the class of *flat differential forms* (see Example V.3.4) may be seen as the "smoothest" class of differential forms invariant under bi-Lipschitz mappings. Another important class of differential forms are the L^2 differential forms whose exterior derivative has L^2 coefficients. These constitute the L^2 de Rham complex, which we will pay further attention to in subsequent chapters. We generally address differential forms with L^p coefficients, including but not restricted to the important special cases $p \in \{1, 2, \infty\}$, in order to make these results available in the literature.

V.1. Elements of Lipschitz Analysis

We begin this chapter by establishing basic notions of Lipschitz analysis. The reader is referred to Luukkainen and Väisälä [134] for a general reference on Lipschitz analysis, but for specific results we also draw on Federer's monograph on geometric measure theory [88]. For the duration of this chapter, let $n \in \mathbb{N}$.

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ and let $\Phi : U \to V$ be a mapping. For a subset $A \subseteq U$, we let the *Lipschitz constant* $\operatorname{Lip}(\Phi, A) \in [0, \infty]$ of Φ over A be the minimum among those $L \in [0, \infty]$ that satisfy

$$\forall x, y \in A : \|\Phi(x) - \Phi(y)\| \leq L\|x - y\|.$$

We say that Φ is *Lipschitz* if $\operatorname{Lip}(\Phi, U) < \infty$. We may write $\operatorname{Lip}(\Phi) := \operatorname{Lip}(\Phi, U)$ if U is understood. More generally, we say that f is *locally Lipschitz* or *LIP* if for each $x \in U$ there exists a relatively open neighborhood $A \subseteq U$ of x such that $\Phi_{|A} : A \to Y$ is Lipschitz.

If Φ is invertible, then we call Φ *bi-Lipschitz* if both Φ and Φ^{-1} are Lipschitz, and we call Φ a *lipeomorphism* if both Φ and Φ^{-1} are locally Lipschitz. If $\Phi : U \to V$ is locally Lipschitz and injective such that $\Phi : U \to \Phi(U)$ is a lipeomorphism, then we call Φ a *LIP embedding*.

We recall some basic but useful facts. The composition of Lipschitz mappings is again Lipschitz, and the composition of locally Lipschitz mappings is again locally Lipschitz. If $\operatorname{Lip}(\Phi, U) < \infty$, then the continuous extension of Φ to $\overline{U}$ is Lipschitz with the same Lipschitz constant $\operatorname{Lip}(\Phi, U)$. For another observation we give a short proof.

Lemma V.1.1 (see [94, Lemma 2.3])**.**
Let $U \subset \mathbb{R}^n$ be compact and let $\Phi : U \to \mathbb{R}$ be locally Lipschitz. Then Φ is Lipschitz.

Proof. Using that Φ is locally Lipschitz and that U is compact, we infer the existence of a finite covering $U_1, \dots, U_N$ of U by relatively open subsets of U such that there exists $L \in [0, \infty)$ with $\operatorname{Lip}(\Phi, U_i) \leq L$ for each $1 \leq i \leq N$. By Lebesgue's number lemma, there exists $\gamma > 0$ such that for each $x \in U$ there exists $1 \leq i \leq N$ with $B_\gamma(x) \cap U \subseteq U_i$. Now let $x, y \in U$. If $\|x - y\| \leq \gamma$, then $|\Phi(x) - \Phi(y)| \leq L\|x - y\|$, since $x, y \in U_i$ for some $1 \leq i \leq N$. If instead $\|x - y\| > \gamma$, then we see

$$|\Phi(x) - \Phi(y)| \leq \frac{\Phi_{\max}(U) - \Phi_{\min}(U)}{\gamma} \|x - y\|,$$

using that Φ assumes a minimum and a maximum over U. The proof is complete. $\square$

We consider a special case of specific interest. Let $U, V \subseteq \mathbb{R}^n$ be open sets and let be $\Phi : U \to V$ be bi-Lipschitz. It follows from *Rademacher's theorem* [88, Theorem 3.1.6] that the Jacobians

$$\mathrm{D}\,\Phi : U \to \mathbb{R}^{n \times n}, \quad \mathrm{D}\,\Phi^{-1} : V \to \mathbb{R}^{n \times n}$$

exist almost everywhere and are essentially bounded. One can show that

$$\|\mathrm{D}\,\Phi\|_{L^\infty(U)} \leq \operatorname{Lip}(\Phi, U), \quad \|\mathrm{D}\,\Phi^{-1}\|_{L^\infty(V)} \leq \operatorname{Lip}(\Phi^{-1}, V). \tag{V.1}$$

According to [88, Lemma 3.2.8], the identities

$$\mathrm{D}\,\Phi^{-1}_{\Phi(x)} \cdot \mathrm{D}\,\Phi_x = \mathrm{Id}_U, \quad \mathrm{D}\,\Phi_{\Phi^{-1}(y)} \cdot \mathrm{D}\,\Phi^{-1}_y = \mathrm{Id}_V \tag{V.2}$$

hold for almost all $x \in U$ and $y \in V$, respectively. In particular, the Jacobians have full rank almost everywhere. Moreover, by [88, Corollary 4.1.26] the signs of the Jacobians are essentially locally constant. In particular, if U and V are connected, then there exists $o(\Phi) \in \{-1, 1\}$ such that

$$o(\Phi) = \operatorname{sgn} \det \mathrm{D}\,\Phi, \tag{V.3}$$

almost everywhere over U. One can show (see [88, Theorem 3.2.3]) that

$$\int_U u\,(\Phi(x))\,|\det \mathrm{D}\,\Phi_x|\,\mathrm{d}x = \int_V u(y)\,\mathrm{d}y \tag{V.4}$$

for every measurable $u : V \to \mathbb{R}$ if at least one of the integrals exists.

V.2. Differential Forms

The monographs by Lang [126], Lee [127], and by Agricola and Friedrich [92] introduce the calculus of differential forms with smooth coefficients. Differential forms with coefficients in L^p spaces have been subject of research for a long time (see, e.g., [100, 101, 112, 166]).

Let $U \subseteq \mathbb{R}^n$ be an open set. We let $M(U)$ denote the space of measurable functions over U. For $k \in \mathbb{Z}$ we let $M\Lambda^k(U)$ be the vector space of measurable differential k-forms over U. Note that $M(U) = M\Lambda^0(U)$. A specific subspace is the Banach space $C\Lambda^k(\overline{U})$ of bounded continuous differential k-forms over $\overline{U}$, equipped with the maximum norm. We let $C^\infty\Lambda^k(U)$ be the space of smooth differential forms over U, and we let $C^\infty\Lambda^k(\overline{U})$ denote the space of those smooth differential forms over U that are restrictions of members of $C^\infty\Lambda^k(\mathbb{R}^n)$. Lastly, we let $C^\infty_c\Lambda^k(U)$ be the space of smooth differential forms with compact support in U.

For $u \in M\Lambda^k(U)$ and $v \in M\Lambda^l(U)$ we let $u \wedge v \in M\Lambda^{k+l}(U)$ denote the *exterior product* of u and v, which satisfies $u \wedge v = (-1)^{kl} v \wedge u$.

We let $e_1, \dots, e_n$ be the canonical orthonormal basis of $\mathbb{R}^n$. The constant 1-forms $\mathsf{d}x^1, \dots, \mathsf{d}x^n \in M\Lambda^1(U)$ are uniquely defined by $\mathsf{d}x^i(e_j) = \delta_{ij}$, where $\delta_{ij} \in \{0,1\}$ denotes the Kronecker delta. More generally, the *basic k-alternators* are the exterior products

$$\mathsf{d}x^\sigma := \mathsf{d}x^{\sigma(1)} \wedge \cdots \wedge \mathsf{d}x^{\sigma(k)} \in M\Lambda^k(U), \quad \sigma \in \Sigma(1:k, 1:n),$$

and $\mathsf{d}x^\emptyset := 1$ in the case $k = 0$. The *volume form* over U is

$$\mathrm{vol}^n_U := \mathsf{d}x^1 \wedge \cdots \wedge \mathsf{d}x^n.$$

Every $u \in M\Lambda^k(U)$ can be written uniquely as

$$u = \sum_{\sigma \in \Sigma(1:k,1:n)} u_\sigma \mathsf{d}x^\sigma, \tag{V.5}$$

where $u_\sigma = u(e_{\sigma(1)}, \dots, e_{\sigma(k)}) \in M(U)$. In particular, every n-form $u \in M\Lambda^n(U)$ can be written as $u = u_{\mathrm{vol}} \mathrm{vol}^n_U$ for some unique $u_{\mathrm{vol}} \in M(U)$. Using this observation, we define the integral of an n-form $u \in M\Lambda^n(U)$ over U as

$$\int_U u := \int_U u_{\mathrm{vol}} \,\mathrm{d}x \tag{V.6}$$

whenever $u_{\mathrm{vol}} \in M(U)$ is integrable over U. Note that this definition presumes that $\mathbb{R}^n$ carries the canonical orientation.

The pointwise ℓ^2 product pairs up two measurable differential k-forms to get a measurable function,

$$\langle u, v \rangle := \sum_{\sigma \in \Sigma(1:k,1:n)} u_\sigma v_\sigma \in M(U), \quad u, v \in M\Lambda^k(U). \tag{V.7}$$

Accordingly, we define the pointwise ℓ^2 norm $|u| \in M(U)$ as

$$|u| := \sqrt{\langle u, u \rangle}, \quad u \in M\Lambda^k(U). \tag{V.8}$$

One can show that there exists a mapping

$$\star : M\Lambda^k(U) \to M\Lambda^{n-k}(U),$$

called the *Hodge star operator*, which is uniquely defined by the identity

$$u \wedge \star v = \langle u, v \rangle \operatorname{vol}^n_U, \quad u, v \in M\Lambda^k(U). \tag{V.9}$$

One can show that

$$\star \star u = (-1)^{k(n-k)} u, \quad u \in M\Lambda^k(U).$$

Furthermore, $|u| = |\star u|$ for all $u \in M\Lambda^k(U)$.

We let $L^p(U)$ denote the Lebesgue space with exponent $p \in [1, \infty]$, and let $L^p\Lambda^k(U)$ denote the Banach space of differential k-forms with coefficients (as in (V.5)) in $L^p(U)$. A compatible norm on $L^p\Lambda^k(U)$ is given by

$$\|u\|_{L^p\Lambda^k(U)} := \left\| \sqrt{\langle u, u \rangle} \right\|_{L^p(U)}, \quad u \in L^p\Lambda^k(U).$$

In the special case $p = 2$, this a Hilbert space with scalar product

$$\langle u, v \rangle_{L^2\Lambda^k(U)} := \int_U \langle u, v \rangle \mathrm{d}x, \quad u, v \in L^2\Lambda^k(U).$$

Remark V.2.1.
Our definition of pointwise ℓ^2 product (V.7) and the Hodge star (V.2) assume the choice of a canonical Riemannian metric over U. More generally, these structures can be defined for any choice of Riemannian metric over U. We do not explore this idea further in this chapter.

We conclude this section with the study of pullbacks of differential forms along bi-Lipschitz mappings. For the remainder of this section, we let $U, V \subseteq \mathbb{R}^n$ be open sets, and let $\Phi : U \to V$ be a bi-Lipschitz mapping.

The *pullback* $\Phi^* u \in M\Lambda^k(U)$ of $u \in M\Lambda^k(V)$ under Φ is defined as

$$\Phi^* u_x(\nu_1, \dots, \nu_k) := u_{\Phi(x)}(\mathrm{D}\,\Phi_x \cdot \nu_1, \dots, \mathrm{D}\,\Phi_x \cdot \nu_k), \quad \nu_1, \dots, \nu_k \in \mathbb{R}^n, \quad x \in U.$$

By the discussion at the beginning of Section 2 of [100], the algebraic identity

$$\Phi^*(u \wedge v) = \Phi^* u \wedge \Phi^* v$$

holds for $u \in M\Lambda^k(V)$ and $v \in M\Lambda^l(V)$. In particular, one can show that

$$(\Phi^* \operatorname{vol}^n_V) = \det(\mathrm{D}\,\Phi) \cdot \operatorname{vol}^n_U, \quad x \in U. \tag{V.10}$$

Next we show how the integral of n-forms transforms under pullback by bi-Lipschitz mappings.

Lemma V.2.2.
If $\Phi : U \to V$ is a bi-Lipschitz mapping between connected open subsets of $\mathbb{R}^n$, then

$$\int_U \Phi^*\left(u \operatorname{vol}^n_V\right) = o(\Phi) \int_V u \operatorname{vol}^n_V, \quad u \in M(V), \tag{V.11}$$

if any of the integrals exists.

Proof. Using (V.3), (V.4), and (V.10), we find

$$\begin{aligned}
\int_U \Phi^* \left(u \operatorname{vol}_V^n\right) &= \int_U (u \circ \Phi) \cdot \det(\mathrm{D}\,\Phi) \operatorname{vol}_U^n \\
&= \int_U (u \circ \Phi) \cdot \det(\mathrm{D}\,\Phi)\, \mathrm{d}x \\
&= \int_U u \circ \Phi \cdot \operatorname{sgn}\det \mathrm{D}\,\Phi \cdot |\det \mathrm{D}\,\Phi|\, \mathrm{d}x \\
&= o(\Phi) \int_U u \circ \Phi \cdot |\det \mathrm{D}\,\Phi|\, \mathrm{d}x = o(\Phi) \int_V u\, \mathrm{d}x = o(\Phi) \int_V u \operatorname{vol}_V^n .
\end{aligned}$$

This shows the desired identity. □

It can be shown that the pullback under bi-Lipschitz mappings preserves the L^p classes of differential forms over U (see Theorem 2.2. of [100]). For the purpose of this thesis, we prove that the pullback is an isomorphism of Banach spaces and we determine the operator norm of that isomorphism. Here and in the sequel, $n/\infty = 0$ for $n \in \mathbb{N}$.

Lemma V.2.3.
Let $\Phi : U \to V$ be a bi-Lipschitz mapping between open sets $U, V \subseteq \mathbb{R}^n$. For every $p \in [1, \infty]$ and $u \in L^p\Lambda^k(V)$ we have

$$\begin{aligned}
\|\Phi^* u\|_{L^p\Lambda^k(U)} &\leq \|\mathrm{D}\,\Phi\|_{L^\infty(U)}^k \left\|\det \mathrm{D}\,\Phi^{-1}\right\|_{L^\infty(V)}^{\frac{1}{p}} \|u\|_{L^p\Lambda^k(V)} \\
&\leq \|\mathrm{D}\,\Phi\|_{L^\infty(U)}^k \left\|\mathrm{D}\,\Phi^{-1}\right\|_{L^\infty(V)}^{\frac{n}{p}} \|u\|_{L^p\Lambda^k(V)}.
\end{aligned} \tag{V.12}$$

Proof. Let $\Phi : U \to V$ and $p \in [1, \infty]$ be as in the statement of the theorem, and let $u \in L^p\Lambda^k(U)$. For almost every $x \in U$ we observe

$$|\Phi^* u|_{|x} \leq \left\|\mathrm{D}\,\Phi_{|x}\right\|_{2,2}^k \sqrt{\sum_{\sigma \in \Sigma(1:k,1:n)} \left(u_{\sigma|\Phi(x)}\right)^2} = \left\|\mathrm{D}\,\Phi_{|x}\right\|_{2,2}^k |u|_{|\Phi(x)} .$$

From this we easily get

$$\|\Phi^* u\|_{L^p\Lambda^k(U)} \leq \|\,\mathrm{D}\,\Phi\|_{L^\infty(U)}^k \left\||u| \circ \Phi\right\|_{L^p\Lambda^k(U)}.$$

The desired statement follows trivially if $p = \infty$, and (V.4) gives

$$\begin{aligned}
\int_U |u|_{|\Phi(x)}^p \,\mathrm{d}x &\leq \left\|\det \mathrm{D}\,\Phi^{-1}\right\|_{L^\infty(V)} \int_U |u|_{|\Phi(x)}^p \left|\det \mathrm{D}\,\Phi_{|x}\right| \mathrm{d}x \\
&\leq \left\|\det \mathrm{D}\,\Phi^{-1}\right\|_{L^\infty(V)} \int_{\Phi(U)} |u|_x^p \,\mathrm{d}x
\end{aligned}$$

if $p \in [1, \infty)$. This shows the first estimate of (V.12). The second estimate in (V.12) follows by Hadamard's inequality, which estimates the determinant of a matrix by the product of the norms of its columns. □

V.3. The Exterior Derivative

We now address the exterior derivative in a setting of low regularity. To begin with, we define the exterior derivative in a weak sense over differential forms with locally integrable coefficients. We then turn our attention to the $W^{p,q}$ spaces of differential forms (see [97, 98, 121]). Eventually we consider a notion of homogeneous boundary condition.

The *exterior derivative* $\mathsf{d}^k : C^\infty\Lambda^k(U) \to C^\infty\Lambda^{k+1}(U)$ over smooth differential k-forms is defined by

$$\mathsf{d}^k u = \sum_{\sigma\in\Sigma(1:k,1:n)} \sum_{i=1}^{n} (\partial_i u_\sigma)\mathsf{d}x^i \wedge \mathsf{d}x^\sigma, \quad u \in C^\infty\Lambda^k(U). \tag{V.13}$$

One can show that d^k is a linear mapping satisfying the *differential property*

$$\mathsf{d}^{k+1}\mathsf{d}^k u = 0, \quad u \in C^\infty\Lambda^k(U),$$

and that it relates to the exterior product by

$$\mathsf{d}^{k+l}(u \wedge v) = \mathsf{d}^k u \wedge v + (-1)^k u \wedge \mathsf{d}^l v, \quad u \in C^\infty\Lambda^k(U), \quad v \in C^\infty\Lambda^l(U). \tag{V.14}$$

Moreover, $\mathsf{d}^k u \in C^\infty\Lambda^{k+1}(\overline{U})$ when $u \in C^\infty\Lambda^k(\overline{U})$.

We are interested in defining the exterior derivative in a weak sense over differential forms of low regularity. If $u \in M\Lambda^k(U)$ and $w \in M\Lambda^{k+1}(U)$ are locally integrable such that

$$\int_U w \wedge v = (-1)^{k+1} \int_U u \wedge \mathsf{d}^{n-k-1} v, \quad v \in C_c^\infty\Lambda^{n-k-1}(U), \tag{V.15}$$

then w is the only member of $M\Lambda^{k+1}(U)$ satisfying (V.15), up to equivalence almost everywhere, and we call $\mathsf{d}^k u := w$ the *weak exterior derivative* of u. Note that w is unique up to equivalence almost everywhere in U, and that $\mathsf{d}^k u$ has vanishing weak exterior derivative, since

$$\int_U \mathsf{d}^k u \wedge \mathsf{d}^{n-k-1} v = (-1)^k \int_U u \wedge \mathsf{d}^{n-k}\mathsf{d}^{n-k-1} v = 0, \quad v \in C_c^\infty\Lambda^{n-k-1}(U). \tag{V.16}$$

The weak exterior derivative of $u \in C^\infty\Lambda^k(\overline{U})$ agrees with the (strong) exterior derivative almost everywhere, and hence we call weak exterior derivatives simply *exterior derivatives* in the sequel. The product formula (V.14) generalizes in the obvious manner to the weak exterior derivative, provided all required weak exterior derivatives exist.

The Hodge star enters the definition of the *codifferential*, which is a differential operator given (in the strong sense) by

$$\delta^k : C^\infty\Lambda^k(U) \to C^\infty\Lambda^{k-1}(U), \quad u \mapsto (-1)^{k(n-k)+1} \star \mathsf{d}^{n-k} \star u. \tag{V.17}$$

A weak codifferential can be defined analogously to the weak exterior derivative.

Next we introduce a notion of Sobolev differential forms. For $p, q \in [1,\infty]$, we let $W^{p,q}\Lambda^k(U)$ be the space of those differential k-forms in $L^p\Lambda^k(U)$ that have a weak exterior derivative in $L^q\Lambda^{k+1}(U)$. The space $W^{p,q}\Lambda^k(U)$ is a Banach space with the norm

$$\|u\|_{W^{p,q}\Lambda^k(U)} = \|u\|_{L^p\Lambda^k(U)} + \left\|\mathsf{d}^k u\right\|_{L^q\Lambda^{k+1}(U)}. \tag{V.18}$$

It is obvious that $W^{p,q}\Lambda^k(U)$ is a Banach space. Since the exterior derivative of an exterior derivative is zero, even in the weak sense, we observe

$$\mathsf{d}^k W^{p,q}\Lambda^k(U) \subseteq W^{q,r}\Lambda^{k+1}(U), \quad p, q, r \in [1,\infty].$$

Hence one may study de Rham complexes of the form

$$\cdots \xrightarrow{\mathsf{d}^{k-1}} W^{p,q}\Lambda^k(U) \xrightarrow{\mathsf{d}^k} W^{q,r}\Lambda^{k+1}(U) \xrightarrow{\mathsf{d}^{k+1}} \cdots \tag{V.19}$$

Remark V.3.1.
The choice of the Lebesgue exponents determines analytical and algebraic properties of the de Rham complexes of the form (V.19). This is not subject of research in this thesis, but we refer to [102] for corresponding results over smooth manifolds without boundary. De Rham complexes of the above form with a Lebesgue exponent p fixed,

$$\cdots \xrightarrow{\mathsf{d}^{k-1}} W^{p,p}\Lambda^k(U) \xrightarrow{\mathsf{d}^k} W^{p,p}\Lambda^{k+1}(U) \xrightarrow{\mathsf{d}^{k+1}} \cdots \tag{V.20}$$

are known as L^p de Rham complexes (see [139]).

Example V.3.2.
The space $W^{1,1}\Lambda^k(U)$ contains all integrable differential k-forms over U with integrable weak exterior derivative. If U is bounded, then $W^{1,1}\Lambda^k(U)$ contains all the other spaces $W^{p,q}\Lambda^k(U)$ as embedded subspaces.

Example V.3.3.
The space $H\Lambda^k(U) := W^{2,2}\Lambda^k(U)$, consisting of those L^2 differential k-forms that have a weak exterior derivative with L^2 integrable coefficients, is a Hilbert space with the scalar product

$$\langle u, v\rangle_{H\Lambda^k(U)} = \langle u, v\rangle_{L^2\Lambda^k(U)} + \langle \mathsf{d}^k u, \mathsf{d}^k v\rangle_{L^2\Lambda^{k+1}(U)}, \quad u, v \in H\Lambda^k(U).$$

We write $\|\cdot\|_{H\Lambda^k(U)}$ for the corresponding norm. Note that $\|\cdot\|_{W^{2,2}\Lambda^k(U)}$ and $\|\cdot\|_{H\Lambda^k(U)}$ are equivalent but not identical norms on $H\Lambda^k(U)$. These spaces constitute the L^2 de Rham complex

$$\cdots \xrightarrow{\mathsf{d}^{k-1}} H\Lambda^k(U) \xrightarrow{\mathsf{d}^k} H\Lambda^{k+1}(U) \xrightarrow{\mathsf{d}^{k+1}} \cdots$$

which has received considerable attention in global and numerical analysis.

Example V.3.4.
The space $W^{\infty,\infty}\Lambda^k(U)$ of *flat differential forms* is spanned by those differential forms with essentially bounded coefficients that have a weak exterior derivative with essentially bounded coefficients. These spaces constitute the flat de Rham complex

$$\cdots \xrightarrow{\mathsf{d}^{k-1}} W^{\infty,\infty}\Lambda^k(U) \xrightarrow{\mathsf{d}^k} W^{\infty,\infty}\Lambda^{k+1}(U) \xrightarrow{\mathsf{d}^{k+1}} \cdots$$

Flat differential forms have been studied extensively in geometric integration theory [88, 180]; see in particular Theorem 1.5 of [100]. Furthermore, if U is bounded, then $W^{\infty,\infty}\Lambda^k(U)$ is a subspace of $W^{p,q}\Lambda^k(U)$ for all $p, q \in [1, \infty]$.

Differential forms with smooth coefficients are dense in $W^{p,q}\Lambda^k(U)$ for $p, q \in [1, \infty)$. This has been proven in [100, Lemma 1.3] using de Rham regularizers. We give a different proof, which uses standard techniques in functional analysis (see [86]). In Chapter VII, a generalization of this result will accommodate boundary conditions.

Lemma V.3.5.
Let $U \subseteq \mathbb{R}^n$ be open and let $p, q \in [1, \infty)$. Then $C^\infty\Lambda^k(U) \cap W^{p,q}\Lambda^k(U)$ is dense in $W^{p,q}\Lambda^k(U)$.

Proof. Let $u \in W^{p,q}\Lambda^k(U)$ and write $w := \mathsf{d}^k u$. Then $(u, w) \in L^p\Lambda^k(U) \times L^q\Lambda^{k+1}(U)$. We let $\tilde u \in L^p\Lambda^k(\mathbb{R}^n)$ and $\tilde w \in L^q\Lambda^{k+1}(\mathbb{R}^n)$ denote the extension by zero of u and w, respectively, onto $\mathbb{R}^n$. For $\epsilon > 0$ we let $\tilde u_\epsilon := \mu_\epsilon \star \tilde u$ and $\tilde w_\epsilon := \mu_\epsilon \star \tilde w$ denote the respective convolutions with the scaled mollifier. The differential forms $\tilde u_\epsilon$ and $\tilde w_\epsilon$ have smooth coefficients (see [27, Corollary 3.9.5]). The scaled mollifiers μ_ϵ have unit integral, and via Young's inequality (see [27, Theorem 3.9.4]) we thus find

$$\|\tilde u_\epsilon\|_{L^p\Lambda^k(U)} \leq \|u\|_{L^p\Lambda^k(U)}, \quad \|\tilde w_\epsilon\|_{L^q\Lambda^{k+1}(U)} \leq \|w\|_{L^q\Lambda^{k+1}(U)}.$$

Furthermore, we recall that $\tilde u_{\epsilon|U}$ converges to u in $L^p\Lambda^k(U)$ and that $\tilde w_{\epsilon|U}$ converges to w in $L^q\Lambda^{k+1}(U)$, as follows from well known results on the convolution with the standard mollifier (see [27, Theorem 4.2.4]). Let us assume that u has compact support in U. Then $\tilde u_\epsilon$ and $\tilde w_\epsilon$ have compact support in U for ϵ small enough. We then get $\mathsf{d}^k \tilde u_\epsilon = \tilde w_\epsilon$.

To treat the case of general u, we fix a countable locally finite covering $(U_i)_{i\in\mathbb{N}}$ of U by bounded open subsets and a countable smooth partition of unity $(\chi_i)_{i\in\mathbb{N}}$ over U such that $\operatorname{supp}\chi_i$ is compactly contained in U_i for each $i \in \mathbb{N}$.

Now $\chi_i u$ has compact support in U_i for each $i \in \mathbb{N}$, and hence it can be approximated by a smooth differential k-form compactly supported in U_i. Consequently, for every $\epsilon > 0$ we can fix $u_i \in C^\infty_c\Lambda^k(U_i)$ for every $i \in \mathbb{N}$ such that

$$\|\chi_i u - u_i\|_{L^p\Lambda^k(U)} + \left\|\mathsf{d}^k(\chi_i u) - \mathsf{d}^k u_i\right\|_{L^q\Lambda^{k+1}(U)} < \frac{\epsilon}{2^i}, \quad i \in \mathbb{N}, \quad \epsilon > 0.$$

Let us write $u_\star = \sum_{i\in\mathbb{N}} u_i$ and $w_\star = \sum_{i\in\mathbb{N}} \mathsf{d}^k u_i$. Since the covering $(U_i)_{i\in\mathbb{N}}$ is locally finite, we conclude that $u_\star \in C^\infty\Lambda^k(U)$ and $w_\star \in C^\infty\Lambda^{k+1}(U)$. Additionally, the triangle inequality gives

$$\|u - u_\star\|_{L^p\Lambda^k(U)} + \|w - w_\star\|_{L^q\Lambda^{k+1}(U)} \leq \sum_{i=1}^\infty \frac{\epsilon}{2^i} = \epsilon.$$

This completes the proof. □

We now study the behavior of the weak exterior derivative under bi-Lipschitz coordinate changes. Suppose that $U, V \subseteq \mathbb{R}^n$ are open sets, and let $\Phi : U \to V$

be a bi-Lipschitz mapping. If follows from Theorem 2.2 of [100] that whenever $u \in W^{p,q}\Lambda^k(V)$ with $p, q \in [1, \infty]$, then we also have $\Phi^* u \in W^{p,q}\Lambda^k(U)$ and

$$\mathsf{d}^k \Phi^* u = \Phi^* \mathsf{d}^k u. \tag{V.21}$$

In particular, the pullback along bi-Lipschitz mappings preserves the $W^{p,q}$ classes of differential forms.

In this thesis we are particularly interested in spaces of differential forms that satisfy homogeneous boundary conditions along a subset Γ of the boundary ∂U. We call these *partial boundary conditions*. We define homogeneous boundary conditions in the manner of Definition 3.3 of [99], which does not explicitly require assumptions on the regularity of ∂U. Thus we avoid the technicalities of generalized boundary traces.

Assume that $\Gamma \subseteq \partial U$ is a relatively open subset of ∂U. We define the space $W^{p,q}\Lambda^k(U, \Gamma)$ as the subspace of $W^{p,q}\Lambda^k(U)$ whose members adhere to the following condition: we have $u \in W^{p,q}\Lambda^k(U, \Gamma)$ if and only if for all $x \in \Gamma$ there exists $r > 0$ such that

$$\int_{U \cap B_r(x)} u \wedge \mathsf{d}^{n-k-1} v = (-1)^{k+1} \int_{U \cap B_r(x)} \mathsf{d}^k u \wedge v, \quad v \in C_c^\infty \Lambda^{n-k-1} \left(\mathring{B}_r(x) \right). \tag{V.22}$$

The definition implies that $W^{p,q}\Lambda^k(U, \Gamma)$ is a closed subspace of $W^{p,q}\Lambda^k(U)$, and hence a Banach space of its own. We also say that $u \in W^{p,q}\Lambda^k(U, \Gamma)$ satisfies partial boundary conditions along Γ.

Remark V.3.6.
The identity (V.22) resembles the integration by parts identity in the definition of the weak exterior derivative. Our definition of homogeneous boundary conditions is based on the idea that the trivial extension of any $u \in W^{p,q}\Lambda^k(U, \Gamma)$ outside of U should have a weak exterior derivative locally along Γ. For example, $W^{p,q}\Lambda^k(U, \partial U)$ is the subspace of $W^{p,q}\Lambda^k(U)$ whose member's extension to $\mathbb{R}^n$ by zero gives a member of $W^{p,q}\Lambda^k(\mathbb{R}^n)$. If the domain has a boundary of sufficient regularity, then an equivalent notion of homogeneous boundary conditions uses generalized trace operators [139, 177]. This thesis does not address inhomogeneous boundary conditions.

Another property of the $W^{p,q}$ classes of differential forms with homogeneous boundary conditions is that they are closed under taking the exterior derivative. Unfolding definitions we find

$$\mathsf{d}^k W^{p,q}\Lambda^k(U, \Gamma) \subseteq W^{q,r}\Lambda^{k+1}(U, \Gamma), \quad p, q, r \in [1, \infty]. \tag{V.23}$$

In other words, differential forms satisfying homogeneous boundary conditions along Γ have exterior derivatives satisfying homogeneous boundary conditions along Γ.

VI. Weakly Lipschitz Domains

The theoretical and numerical analysis of partial differential equations is affected by the properties of the geometric ambient. The aim of this chapter is to prepare this geometric ambient. We discuss the class of weakly Lipschitz domains and the geometry of boundary partitions.

A domain is *weakly Lipschitz* if its boundary can be flattened locally by a bi-Lipschitz coordinate transformation. The class of weakly Lipschitz domains can thus be regarded as a Lipschitz analogue to the class of smoothly bounded domains, whose boundaries can be flattened locally by a diffeomorphism. The terminology suggests that weakly Lipschitz domains are compared to the more common notion of *Lipschitz domains*, which are then also called *strongly Lipschitz domains*. A domain is (strongly) Lipschitz if its boundary is locally the graph of a Lipschitz function in an orthogonal coordinate system.

The notion of Lipschitz domain is standard in numerical analysis, but it is easy to see why weakly Lipschitz domains are worth being studied in the context of finite element methods. Every strongly Lipschitz domain is a weakly Lipschitz domain, but the converse is not true, and counterexamples include polyhedral domains in $\mathbb{R}^3$. For instance, the "crossed bricks domain" is not Lipschitz but weakly Lipschitz. We will prove that every polyhedral domain in $\mathbb{R}^3$ is weakly Lipschitz.

Even though the class of weakly Lipschitz domains is larger than the class of strongly Lipschitz domains, research has established that many analytical results known for more regular domains still remain true when considered on weakly Lipschitz domains [15, 38, 99, 106, 110, 151]. For example, one can show that the differential complex

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \tag{VI.1}$$

over a bounded three-dimensional weakly Lipschitz domain Ω satisfies Poincaré-Friedrichs inequalities and realizes the Betti numbers of the domain on cohomology. Furthermore, a vector field version of a Rellich-type compact embedding theorem is valid, and the scalar and vector Laplacians over Ω have a discrete spectrum. Recasting this in the calculus of differential forms, one establishes the analogous properties for the L^2 de Rham complex

$$H\Lambda^0(\Omega) \xrightarrow{\mathsf{d}^0} H\Lambda^1(\Omega) \xrightarrow{\mathsf{d}^1} \cdots \xrightarrow{\mathsf{d}^{n-1}} H\Lambda^n(\Omega) \tag{VI.2}$$

over a bounded weakly Lipschitz domain $\Omega \subset \mathbb{R}^n$.

Another concept of numerical analysis over strongly Lipschitz domains are collar neighborhoods. A collar neighborhood of a domain Ω is a neighborhood of its boundary $\partial\Omega$ that is homeomorphic to the topological space $\partial\Omega \times (-1,1)$ such that $\partial\Omega \times (-1,0)$ corresponds to the collar neighborhood's part inside the domain and $\partial\Omega \times (0,1)$ corresponds to the collar neighborhood's part outside the domain. For a strongly Lipschitz domain, such a collar neighborhood can be constructed using a transversal vector field along the $\partial\Omega$, and the corresponding homeomorphism can be chosen as bi-Lipschitz (see [56, 105, 160] for details). But this approach does not transfer to the case of weakly Lipschitz domains, the reason being that such a transversal vector field does not necessarily exist. Instead, we rely on the notion of Lipschitz collar from Lipschitz topology. We prove that the boundary of weakly Lipschitz domains allows for a bi-Lipschitz collar neighborhood. This collar neighborhood will be used later in the construction of an extension operator in Chapter VII.

Moreover, for the purpose of addressing partial differential equations with mixed boundary conditions later in this thesis, we discuss the geometric prerequisites of admissible boundary partitions and admissible boundary patches for weakly Lipschitz domains (based on [99]).

This chapter is structured in the following manner. In Section VI.1 we introduce weakly and strongly Lipschitz domains, and Lipschitz collars. In Section VI.2 we discuss admissible boundary partitions, and in Section VI.3 we prove that polyhedral domains in $\mathbb{R}^3$ are weakly Lipschitz.

VI.1. Classes of Domains

We commence this chapter with the classical notion of *Lipschitz domain*, which we also call *strongly Lipschitz domain* in this thesis. An open set $\Omega \subseteq \mathbb{R}^n$ is a *strongly Lipschitz domain* if for each $x \in \partial\Omega$ there exists a closed neighborhood $U_x \subseteq \mathbb{R}^n$ of x, positive numbers $\epsilon > 0$ and $h > 0$, an isometry $\zeta : U_x \to [-\epsilon,\epsilon]^{n-1} \times [-h,h]$ with $\zeta(x) = 0$, and a Lipschitz-continuous function $\gamma : [-\epsilon,\epsilon]^{n-1} \to (-h,h)$ such that

$$\zeta\left(\Omega \cap U_x\right) = \left\{ \, (y', y_n) \;\middle|\; y' \in [-\epsilon,\epsilon]^{n-1},\ y_n \in [-h,h],\ y_n < \gamma(y') \, \right\}, \tag{VI.3a}$$
$$\zeta\left(\partial\Omega \cap U_x\right) = \left\{ \, (y', y_n) \;\middle|\; y' \in [-\epsilon,\epsilon]^{n-1},\ y_n \in [-h,h],\ y_n = \gamma(y') \, \right\}, \tag{VI.3b}$$
$$\zeta\left(\Omega^c \cap U_x\right) = \left\{ \, (y', y_n) \;\middle|\; y' \in [-\epsilon,\epsilon]^{n-1},\ y_n \in [-h,h],\ y_n > \gamma(y') \, \right\}. \tag{VI.3c}$$

More generally, we call Ω a *weakly Lipschitz domain* if for all $x \in \partial\Omega$ there exist a closed neighborhood U_x of x in $\mathbb{R}^n$ and a bi-Lipschitz mapping $\varphi_x : U_x \to [-1,1]^n$ such that $\varphi_x(x) = 0$ and such that

$$\varphi_x(\Omega \cap U_x) = [-1,1]^{n-1} \times [-1,0), \tag{VI.4a}$$
$$\varphi_x(\partial\Omega \cap U_x) = [-1,1]^{n-1} \times \{0\}, \tag{VI.4b}$$
$$\varphi_x(\overline{\Omega}^c \cap U_x) = [-1,1]^{n-1} \times (0,1]. \tag{VI.4c}$$

In other words, a strongly Lipschitz domain is an open subset Ω of $\mathbb{R}^n$ whose boundary $\partial\Omega$ can be written locally as the graph of a Lipschitz function in some orthogonal coordinate system. A weakly Lipschitz domain is a domain whose boundary can be flattened locally by a bi-Lipschitz coordinate transformation. As the name already suggests, every strongly Lipschitz domain is also a weakly Lipschitz domain. The converse is generally false.

Lemma VI.1.1.
Let $\Omega \subseteq \mathbb{R}^n$ be a strongly Lipschitz domain. Then Ω is a weakly Lipschitz domain.

Proof. Let $x \in \partial\Omega$. There exist U_x, ϵ, h, ζ, and γ as in the definition of strongly Lipschitz domains. For $y \in (-h, h)$ we let $\varsigma_y : [-h, h] \to [-h, h]$ be the unique piecewise affine mapping with

$$\varsigma_y(-h) = -h, \quad \varsigma_y(0) = y, \quad \varsigma_y(h) = h.$$

We then have a bi-Lipschitz mapping $\varphi_\gamma : [-\epsilon, \epsilon]^{n-1} \times [-h, h] \to [-1, 1]^n$ defined by $\varphi_\gamma(y', y_n) := (y'/\epsilon, \varsigma_{\gamma(y')}(y_n)/h)$. Hence $\varphi_x := \varphi_\gamma \zeta_x$ is a bi-Lipschitz mapping from U_x to $[-1, 1]^n$ that satisfies the conditions (VI.4) in the definition of weakly Lipschitz domain. The proof is complete. □

Example VI.1.2.
The converse statement to Lemma VI.1.1 is generally false, and a counterexample is easily found. The *crossed bricks domain* Ω_{CB} (see Figure VI.1) is given by

$$\begin{aligned} \Omega_{CB} := (-1, 1) \times (0, 1) \times (0, -1) \cup (0, 1) \times (0, -1) \times (-1, 1) \\ \cup (0, 1) \times \{0\} \times (0, -1). \end{aligned} \tag{VI.5}$$

The domain Ω_{CB} is not a Lipschitz domain because at the origin it is not possible to write $\partial\Omega_{CB}$ as the graph of a Lipschitz function in any orthogonal coordinate system. If such a coordinate system existed, then the epigraph of the function describing the boundary would contain line segments in two opposite directions, which is not possible.

But Ω_{CB} is a weakly Lipschitz domain. This follows from Theorem VI.3.2 later in this chapter, but it is easy to verify in the particular example of Ω_{CB}. We first observe that near every non-zero $x \in \partial\Omega_{CB}$ we can write $\partial\Omega_{CB}$ as a Lipschitz graph, from which we can easily construct a suitable Lipschitz coordinate chart around x as we have done in the proof of Lemma VI.1.1. But this approach does not work at the origin. As a possible remedy, we deform Ω_{CB} into a strongly Lipschitz domain by a bi-Lipschitz mapping. Now it is easy to construct the desired bi-Lipschitz coordinate chart around the origin in which $\partial\Omega_{CB}$ is flattened.

A variant of the crossed bricks domain is displayed in the monograph of Monk [145, Figure 3.1, p.39], and another variant is discussed in [38]. For a generalization of this example, we refer to Example 2.2 in [12].

Remark VI.1.3.
A motivation for considering the class of weakly Lipschitz domains in finite element theory is the following observation: *every bounded domain $\Omega \subset \mathbb{R}^3$ with a finite triangulation is a weakly Lipschitz domain.* We prove that statement at the end of this chapter, using the results of Chapter II.

Remark VI.1.4.
A different access towards the idea originates from differential topology: a weakly

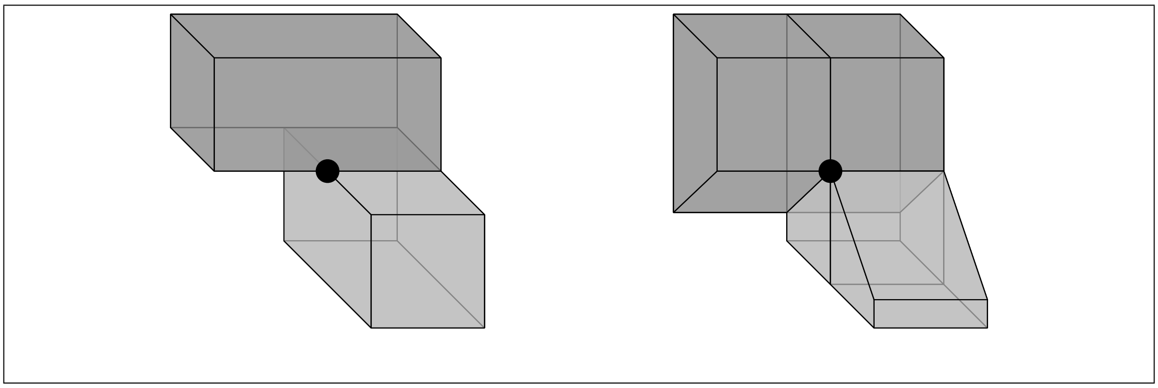

Figure VI.1: *Left*: polyhedral domain in 3D that is not the graph of a Lipschitz function at the marked point. *Right*: bi-Lipschitz transformation of that domain into a strongly Lipschitz domain.

Lipschitz domain is an n-dimensional *locally flat Lipschitz submanifold* of $\mathbb{R}^n$ in the sense of [134]. This idea has inspired the notion of weakly Lipschitz domains inside abstract Lipschitz manifolds [99].

At this point we gather several observations about weakly Lipschitz domains.

Lemma VI.1.5.
Let $\Omega \subset \mathbb{R}^n$ be open. Then Ω is a weakly Lipschitz domain if and only if $\overline{\Omega}^c$ is a weakly Lipschitz domain.

Proof. This can easily be seen from the definition of weakly Lipschitz domains. □

Lemma VI.1.6.
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain. Then there exists a finite family of closed sets $U_1, \dots, U_m \subseteq \overline{\Omega}$ that cover $\overline{\Omega}$, and a finite family $\varphi_1, \dots, \varphi_m$ of bi-Lipschitz mappings $\varphi_i : U_i \to [-1,1]^n$. Moreover, we can assume that the relative interiors of the U_i in $\overline{\Omega}$ constitute a finite covering of Ω.

Proof. The claim follows easily from definitions and compactness of $\overline{\Omega}$. □

Lemma VI.1.7.
Let $\Omega \subseteq \mathbb{R}^n$ be a weakly Lipschitz domain. Then for each $x \in \partial\Omega$ there exists a closed set $V_x \subseteq \partial\Omega$ and a bi-Lipschitz mapping $\theta_x : V_x \to [-1,1]^{n-1}$ with $\theta_x(x) = 0$ such that the relative interiors of V_x in $\partial\Omega$ constitute a covering of $\partial\Omega$.

Proof. From definitions we find that for each $x \in \partial\Omega$ there exists a closed set $U_x \subseteq \overline{\Omega}$ and a bi-Lipschitz mapping $\varphi_x : U_x \to [-1,1]^n$ such that $\varphi_x(x) = 0$ and (VI.4) holds. Then $(U_x \cap \partial\Omega)_{x\in\partial\Omega}$ is the desired covering of $\partial\Omega$ by closed sets whose relative interiors are again a covering. Furthermore, each $\varphi_{x|U_x\cap\partial\Omega}$ is a bi-Lipschitz mapping from $U_x \cap \partial\Omega$ onto $[-1,1]^{n-1}$. □

The above lemmas have been simple observations, but they prepare a much stronger result, namely the discussion of Lipschitz collars along the boundaries of weakly Lipschitz domains. A *Lipschitz collar* of a domain Ω is a LIP embedding

$$\Psi : \partial\Omega \times [-1,1] \to \mathbb{R}^n$$

such that $\Psi(z,0) = z$ for all $z \in \partial\Omega$, and such that Ψ maps $\partial\Omega \times [-1,0)$ into Ω and $\partial\Omega \times (0,1]$ into Ω^c. We show that every weakly Lipschitz domain allows for a Lipschitz collar.

Theorem VI.1.8.
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain. Then there exists a LIP embedding $\Psi : \partial\Omega \times [-1,1] \to \mathbb{R}^n$ such that $\Psi(x,0) = x$ for $x \in \partial\Omega$, and

$$\Psi\left(\partial\Omega \times [-1,0)\right) \subseteq \Omega, \quad \Psi\left(\partial\Omega \times (0,1]\right) \subseteq \overline{\Omega}^c. \tag{VI.6}$$

Moreover, we may assume that for every $t \in (0,1)$ the sets $\Omega \setminus \Psi\left(\partial\Omega, [-t,0)\right)$ and $\overline{\Omega} \cup \Psi\left(\partial\Omega, (0,t)\right)$ are weakly Lipschitz domains.

Proof. We first prove a one-sided version of the result. By Lemma VI.1.7 and the compactness of $\partial\Omega$ we obtain the existence of a collection $\{V_i\}_{i\in\mathbb{N}}$ of relatively open subsets of $\partial\Omega$ that constitute a covering of $\partial\Omega$, and a collection $\{\psi_i\}_{i\in\mathbb{N}}$ of LIP embeddings $\psi_i : V_i \times [0,1) \to \overline{\Omega}$ such that for each $i \in \mathbb{N}$ we have $\psi_i(x,0) = x$ for each $x \in \partial\Omega$. It follows that $\{(V_i, \psi_i)\}_{i\in\mathbb{N}}$ is a *local LIP collar* in the sense of Definition 7.2 in [134]. By Theorem 7.4 in [134], and a successive reparametrization, there exists a LIP embedding $\Psi^-(x,t) : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that $\Psi^-(x,0) = x$ for all $x \in \partial\Omega$.

We recall Lemma VI.1.5 to see that $\overline{\Omega}^c$ is a weakly Lipschitz domain. By the same arguments, there exists a LIP embedding $\Psi^+(x,t) : \partial\Omega \times [0,1] \to \Omega^c$ such that $\Psi^+(x,0) = x$ for all $x \in \partial\Omega$. We combine these two LIP embeddings. Let

$$\Psi : \partial\Omega \times [-1,1] \to \mathbb{R}^n, \quad (x,t) \mapsto \begin{cases} \Psi^-(x,-t) & \text{if } (x,t) \in \partial\Omega \times [-1,0), \\ x & \text{if } (x,t) \in \partial\Omega \times \{0\}, \\ \Psi^+(x,t) & \text{if } (x,t) \in \partial\Omega \times (0,1]. \end{cases}$$

Then Ψ is well-defined, bijective, and (VI.6) holds. Moreover, there exists $C \geq 1$ such that for all $x_1, x_2 \in \partial\Omega$, for all $t_1^-, t_2^- \in [-1,0]$, and for all $t_1^+, t_2^+ \in [0,1]$ we have

$$\begin{aligned} \frac{1}{C}\left(\|x_1 - x_2\| + \left|t_2^- - t_1^-\right|\right) &\leq \left\|\Psi(x_1,t_1^-) - \Psi(x_2,t_2^-)\right\| \\ &\leq C\left(\|x_1 - x_2\| + \left|t_2^- - t_1^-\right|\right), \end{aligned} \tag{VI.7}$$

$$\begin{aligned} \frac{1}{C}\left(\|x_1 - x_2\| + \left|t_2^+ - t_1^+\right|\right) &\leq \left\|\Psi(x_1,t_1^+) - \Psi(x_2,t_2^+)\right\| \\ &\leq C\left(\|x_1 - x_2\| + \left|t_2^+ - t_1^+\right|\right) \end{aligned} \tag{VI.8}$$

It remains to prove that Ψ is a LIP embedding, for which it suffices to show that

$$\begin{aligned} \frac{1}{C}\left(\|x_1 - x_2\| + |t_2 - t_1|\right) &\leq \|\Psi(x_1,t_1) - \Psi(x_2,t_2)\| \\ &\leq C\left(\|x_1 - x_2\| + |t_2 - t_1|\right) \end{aligned} \tag{VI.9}$$

for all $x_1, x_2 \in \partial\Omega$ and $t_1, t_2 \in [-1,1]$. If t_1 and t_2 are both non-negative or both non-positive, then (VI.7) or (VI.8) apply. To treat the case $t_1 < 0 < t_2$, we first observe that

$$\begin{aligned}
&\|\Psi(x_1,t_1)-\Psi(x_2,t_2)\| \\
&\quad \leq \|\Psi(x_1,t_1)-x_1\|+\|x_1-x_2\|+\|x_2-\Psi(x_2,t_2)\| \\
&\quad = \|\Psi(x_1,t_1)-\Psi(x_1,0)\|+\|x_1-x_2\|+\|\Psi(x_2,0)-\Psi(x_2,t_2)\| \\
&\quad \leq C|t_1|+C|t_2|+\|x_1-x_2\| \\
&\quad \leq C\,|t_1-t_2|+\|x_1-x_2\|.
\end{aligned}$$

On the other hand, we fix $z \in \partial\Omega$ on the straight line segment from $\Psi(x_1,t_1)$ to $\Psi(x_2,t_2)$ and find

$$\begin{aligned}
\|\Psi(x_1,t_1)-\Psi(x_2,t_2)\| &= \|\Psi(x_1,t_1)-z\|+\|z-\Psi(x_2,t_2)\| \\
&= \|\Psi(x_1,t_1)-\Psi(z,0)\|+\|\Psi(z,0)-\Psi(x_2,t_2)\| \\
&\geq \frac{1}{C}\left(|t_1|+|t_2|+\|x_1-z\|+\|z-x_2\|\right) \\
&= \frac{1}{C}\left(|t_1-t_2|+\|x_1-x_2\|\right).
\end{aligned}$$

Thus (VI.9) follows. Restricting and reparameterizing Ψ completes the proof. □

Remark VI.1.9.
Our Theorem VI.1.8 realizes the following idea from differential topology in a Lipschitz setting: if a surface is locally collared, then it is also globally collared. Such a result is well-known in the topological or smooth sense, but it seems to be only folklore in the Lipschitz sense. Notably, the result is mentioned in the unpublished preprint [94]. We have provided a proof for formal completeness.

VI.2. Admissible Boundary Partitions

One major topic of this thesis are mixed boundary conditions. To set up the geometric background, we discuss classes of boundary partitions in the context of weakly Lipschitz domains. Our main source is a publication by Gol'dshtein, Mitrea, and Mitrea [99].

Let $\Omega \subseteq \mathbb{R}^n$ be a weakly Lipschitz domain. An open set $\Gamma_T \subseteq \partial\Omega$ is called an *admissible boundary patch* if Γ_T is a topological submanifold of $\partial\Omega$ of dimension $n-1$ with boundary such that the following additional condition is satisfied: for each $x \in \partial\Gamma_T$ there exists a closed neighborhood V_x of x in $\partial\Omega$ and a bi-Lipschitz mapping $\theta_x : V_x \to [-1,1]^{n-1}$ such that

$$\theta_x(x) = 0, \tag{VI.10a}$$
$$\theta_x(\Gamma_T \cap V_x) = [-1,1]^{n-2} \times [-1,0), \tag{VI.10b}$$
$$\theta_x(\partial\Gamma_T \cap V_x) = [-1,1]^{n-2} \times \{0\}, \tag{VI.10c}$$
$$\theta_x((\partial\Omega \setminus \overline{\Gamma}_T) \cap V_x) = [-1,1]^{n-2} \times (0,1]. \tag{VI.10d}$$

If Γ_T is an admissible boundary patch, then $\Gamma_N := \partial\Omega \setminus \overline{\Gamma_T}$ is also an admissible boundary patch, called *complementary* to Γ_T. We can rewrite (VI.10d) as

$$\theta_x(\Gamma_N \cap V_x) = [-1,1]^{n-2} \times (0,1].$$

The admissible boundary patch Γ_T and its complementary boundary patch are topological manifolds with the same boundary Γ_I within $\partial\Omega$. Note that Γ_I is a topological submanifold of $\partial\Omega$ of dimension $n-2$ without boundary. We have $\Gamma_I = \partial\Gamma_T = \partial\Gamma_N$. We call $(\Gamma_T, \Gamma_I, \Gamma_N)$ an *admissible boundary partition.*

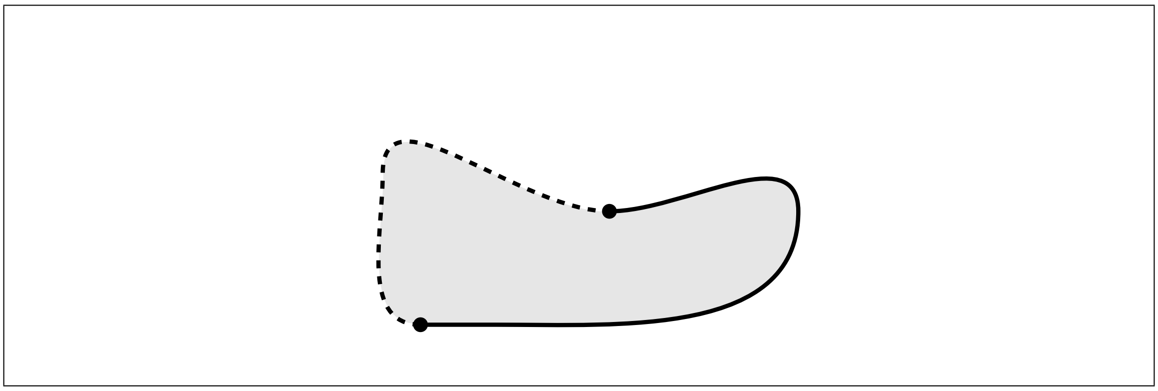

Figure VI.2: Schematic depiction of a domain Ω (shaded area), an admissible boundary patch Γ_T (thick line), its complementary boundary patch Γ_N (dashed line), and the interface Γ_I (dots) between the patches.

We provide an equivalent characterization of admissible boundary partitions. Assume that Γ_T, Γ_I, and Γ_N are subsets of $\partial\Omega$, such that the following condition is satisfied: for any $x \in \Gamma_I$, we can pick a closed neighborhood $U_x \subset \mathbb{R}^n$ of x and a bi-Lipschitz function $\varphi_x : U_x \to [-1,1]^n$ such that (VI.4) is satisfied and we additionally have

$$\varphi_x(\Gamma_T \cap U_x) = [-1,1]^{n-2} \times [-1,0) \times \{0\}, \tag{VI.11a}$$
$$\varphi_x(\Gamma_I \cap U_x) = [-1,1]^{n-2} \times \{0\} \times \{0\}, \tag{VI.11b}$$
$$\varphi_x(\Gamma_N \cap U_x) = [-1,1]^{n-2} \times (0,1] \times \{0\}. \tag{VI.11c}$$

Then Γ_T and Γ_N are mutually complementary admissible boundary patches with common boundary Γ_I, and the tuple $(\Gamma_T, \Gamma_I, \Gamma_N)$ is an admissible boundary partition.

Remark VI.2.1.
A weakly Lipschitz domain is a locally flat n-dimensional Lipschitz submanifold of $\mathbb{R}^n$ with boundary. In particular, $\partial\Omega$ is a locally flat Lipschitz submanifold of dimension $n-1$ without boundary. The tuple $(\Gamma_T, \Gamma_I, \Gamma_N)$ being an admissible boundary partition means that Γ_T and Γ_N are locally flat Lipschitz submanifolds of dimension $n-1$ of $\partial\Omega$ with common boundary $\Gamma_I := \partial\Gamma_T = \partial\Gamma_N$. In turn, Γ_I is a Lipschitz submanifold of dimension $n-2$ without boundary of $\partial\Omega$. Our definition is in accordance with Definition 3.7 of [99], when Remark 3.2 in that reference is taken into account. It also complies to the definition of Lipschitz manifolds in [134].

Weakly Lipschitz domains and admissible boundary partitions provide the geometric background to discuss the L^2 de Rham complex over a weakly Lipschitz domain. Even in our geometric setting of low regularity, this de Rham complex satisfies a Poincaré-Friedrichs inequality, a compact embedding result, and the harmonic spaces are isomorphic to the homology spaces of the domain (see Chapter VIII for details). It will be of interest, however, the discuss domains with additional regularity as a special case. This leads us back the class of strongly Lipschitz domains, for which a specialized class of boundary partitions is known.

Assume that Ω is a strongly Lipschitz domain and that $(\Gamma_T, \Gamma_I, \Gamma_N)$ is an admissible partition of $\partial\Omega$. We call the tuple $(\Omega, \Gamma_T, \Gamma_I, \Gamma_N)$ a *creased domain* if the following assumptions are satisfied: for every $x \in \Gamma_I$ there exists a closed neighborhood $U_x \subseteq \mathbb{R}^n$ of x, positive numbers $\epsilon > 0$ and $h > 0$, an isometry $\zeta : U_x \to [-\epsilon, \epsilon]^{n-1} \times [-h, h]$ with $\zeta(x) = 0$, and a Lipschitz-continuous function $\gamma : [-\epsilon, \epsilon]^{n-1} \to [-h, h]$ such that the conditions (VI.3) hold and additionally there exists a Lipschitz-continuous function $\chi : [-\epsilon, \epsilon]^{n-2} \to \mathbb{R}$ with $\chi(0) = 0$ and

$$\zeta\left(\Gamma_T \cap U_x\right) = \left\{ (y'', y_{n-1}, 0) \;\middle|\; \begin{array}{c} y'' \in [-\epsilon, \epsilon]^{n-2},\ y_{n-1} \in [-\epsilon, \epsilon], \\ y_{n-1} < \chi(y'') \end{array} \right\}, \tag{VI.12a}$$

$$\zeta\left(\partial\Omega \cap U_x\right) = \left\{ (y'', y_{n-1}, 0) \;\middle|\; \begin{array}{c} y'' \in [-\epsilon, \epsilon]^{n-2},\ y_{n-1} \in [-\epsilon, \epsilon], \\ y_{n-1} = \chi(y'') \end{array} \right\}, \tag{VI.12b}$$

$$\zeta\left(\Gamma_N \cap U_x\right) = \left\{ (y'', y_{n-1}, 0) \;\middle|\; \begin{array}{c} y'' \in [-\epsilon, \epsilon]^{n-2},\ y_{n-1} \in [-\epsilon, \epsilon], \\ y_{n-1} > \chi(y'') \end{array} \right\}, \tag{VI.12c}$$

and furthermore there exists a positive number $\kappa > 0$ such that

$$\begin{gathered} \forall (y'', y_{n-1}) \in (-\epsilon, \epsilon)^{n-2} \times (-\epsilon, \chi(y'')) : \partial_{n-1}\zeta(y'', y_{n-1}) \geq \kappa, \\ \forall (y'', y_{n-1}) \in (-\epsilon, \epsilon)^{n-2} \times (\chi(y''), \epsilon) : \partial_{n-1}\zeta(y'', y_{n-1}) \leq \kappa, \end{gathered}$$

where the derivative is taken almost everywhere.

Remark VI.2.2.
The notion of *creased domain* was introduced by Brown [41] in studying the well-posedness of the Poisson problem with mixed boundary conditions in spaces of higher regularity. It has been applied in context of differential forms for Jakab, Mitrea and Mitrea [113]. A creased domain is obviously not smoothly bounded.

VI.3. Polyhedral Domains

We close this chapter with a discussion of polyhedral domains and show that (in $\mathbb{R}^3$) they are weakly Lipschitz domains. The content of this section is not central to the remainder of this thesis but contributes additional motivation and context.

A *polyhedral domain* is an open set $\Omega \subseteq \mathbb{R}^n$ such that $\overline{\Omega}$ is a n-dimensional topological manifold with boundary such that Ω is the interior of that manifold, and such that there exists a simplicial complex $\mathcal{T}$ that triangulates $\overline{\Omega}$.

Remark VI.3.1.
The above definition rules out several pathological cases. For example, the slit domain $\Omega_S := (-1, 1)^2 \setminus [0, 1) \times \{0\}$ is not a polyhedral domain in our definition,

because Ω is not the interior of $\overline{\Omega}$. On the other hand, our definition of polyhedral domain captures many other domains of practical interest, such as the crossed bricks domain Ω_{CB} in Example VI.1.2.

Every polyhedral domain in $\mathbb{R}^2$ is a strongly Lipschitz domain, but this is no longer true in higher dimensions. Instead, a weaker statement holds: every polyhedral domain in $\mathbb{R}^3$ is a weakly Lipschitz domain.

Theorem VI.3.2.
Let $\Omega \subseteq \mathbb{R}^3$ be a bounded polyhedral domain. Then Ω is a weakly Lipschitz domain.

Proof. To prove the statement, we need to find for each $x \in \partial\Omega$ a compact neighborhood $U_x \subseteq \mathbb{R}^3$ of x and a bi-Lipschitz mapping $\varphi_x : U_x \to [-1,1]^3$ such that $\varphi_x(x) = 0$ and the conditions (VI.4) in the definition of weakly Lipschitz domains are satisfied. Since Ω is bounded and polyhedral, there exists a finite simplicial complex $\mathcal{T}$ that triangulates $\overline{\Omega}$.

Consider first the case that x is not a vertex of $\mathcal{T}$. Then x is either contained in the interior of a triangular boundary face of $\mathcal{T}$, or in the interior of an edge between two adjacent boundary triangles of $\mathcal{T}$. In both cases, we may choose $U_x := B_r(x)$ for $r > 0$ small enough, and $\varphi_x : U_x \to [-1,1]^3$ is easily constructed.

It remains to consider the case that x is a vertex. Let $r > 0$ be so small that $B_r(x)$ intersects $T \in \mathcal{T}^3$ if and only if $x \in T$. We observe that $\partial B_r(x) \cap \partial\Omega$ is a simple closed curve in $\partial B_r(x)$ composed of finitely many spherical arcs. Indeed, $\partial B_r(x) \cap \Omega$ has only one connected component and every point in $\partial B_r(x) \cap \partial\Omega$ is in the intersection of at most two triangular faces of $\mathcal{T}$. Hence $\partial B_r(x) \cap \partial\Omega$ is *locally flat* in the sense of [134, p.100]

By the Schoenflies theorem in the Lipschitz category (see Theorem 7.8 of [134]), there exists a bi-Lipschitz mapping

$$\varphi_x^0 : \partial B_r(x) \to \partial B_1(0) \subset \mathbb{R}^3$$

which maps $\partial B_r(x) \cap \partial\Omega$ onto $\partial B_1(0) \cap \{x \in \mathbb{R}^3 \mid x_3 = 0\}$. By radial continuation, we extend this to a bi-Lipschitz mapping

$$\varphi_x^I : B_r(x) \to B_1(0) \subset \mathbb{R}^3$$

which maps $B_r(x) \cap \Omega$ onto $B_1(0) \cap \{x \in \mathbb{R}^3 \mid x_3 < 0\}$ and which satisfies $\varphi_x^I(x) = 0$. Moreover, there exists a bi-Lipschitz mapping

$$\varphi^{II} : B_1(0) \to [-1,1]^3$$

which maps $B_1(0) \cap \{x \in \mathbb{R}^3 \mid x_3 < 0\}$ onto $[-1,1]^2 \times [-1,0)$ and satisfies $\varphi^{II}(0) = 0$. Specifically, we may set $\varphi^{II}(0) = 0$ and

$$\varphi^{II}(y) := \|y\|_{\ell^2}^{-1} \|y\|_{\ell^\infty} y, \quad y \in B_1(0) \setminus \{0\}.$$

Since all norms on $\mathbb{R}^3$ are equivalent, φ^{II} is a bi-Lipschitz mapping from the unit ball in the Euclidean norm to the unit ball in the ℓ^∞ norm. Eventually, we may pick

$$U_x := B_r(x), \quad \varphi_x := \varphi^{II} \varphi_x^I.$$

The proof is complete. $\square$

Remark VI.3.3.
One may conjecture that Theorem VI.3.2 can be generalized to higher-dimensional polyhedral domains, but we make no attempt at a proof here. The preceding proof critically relied on the generalized Schoenflies theorem in the Lipschitz category, which has been considered first in the topological category (see [40, 61, 182]). Generalizing this technique requires the discussion of spherical triangulations, which are beyond the scope of this thesis.

VII. Smoothed Projections

In previous chapters we have explored spaces of finite element differential forms over triangulations and Sobolev spaces of differential forms over domains. In this chapter we begin connecting these different fields of theory and develop a concept central to finite element exterior calculus: smoothed projections. These are projections from Sobolev de Rham complexes onto finite element de Rham complexes that commute with the exterior derivative and that satisfy bounds uniform in the discretization parameters.

The precise role of the smoothed projections in instantiating the abstract Galerkin theory of Hilbert complexes will be described in the next chapter; our main task in this chapter is their construction alone. We draw inspiration from earlier publications on smoothed projections ([9, 58]) but we also put considerable effort into extending the scope of applications.

One innovation of this thesis is that we address the Hodge Laplace equation over weakly Lipschitz domains. As a motivation, let us fix a bounded weakly Lipschitz domain $\Omega \subseteq \mathbb{R}^n$. When $f \in L^2(\Omega)$, then the weak formulation of the Poisson problem with right-hand side f is to find $u \in H^1(\Omega)$ such that

$$\int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x, \quad v \in H^1(\Omega).$$

The well-posedness of this problem, up to constant functions, is easily proven with the Poincaré inequality, and the Rellich-Kondrachov theorem follows with a partition of unity and locally flattening the boundary. As for the numerical analysis, there is no difficulty in showing the well-posedness of the primal finite element method for the Poisson equation. But proving the well-posedness of the mixed finite element method for the Poisson equation using a commuting projection (as in finite element exterior calculus) is not as trivial: after all, commuting projections have only been studied over strongly Lipschitz domains in the literature.

In order to generalize the existent literature on finite element exterior calculus we therefore need a smoothed projection over weakly Lipschitz domains. This is not entirely trivial because the original construction in [9] (see also [160]) for strongly Lipschitz domains utilized a transversal unit vector field along the boundary. Such a construction is not available for weakly Lipschitz domains, and hence different tools from Lipschitz topology are used in this chapter.

Another innovation of thesis is that we address the numerical analysis of the Hodge Laplace equation when *mixed boundary conditions* are imposed. Here, we speak of mixed boundary conditions when essential boundary conditions are imposed on one part of the boundary, while natural boundary conditions are imposed on the complementary boundary part. Special cases are the Poisson equation with mixed Dirichlet and Neumann boundary conditions [148] and the vector Laplace equation with mixed tangential and normal boundary conditions [89]. It is known in the theory of partial differential equations that the Hodge Laplace equation with mixed boundary conditions arises from Sobolev de Rham complexes with *partial boundary conditions* [99, 113]. These are composed of spaces of Sobolev differential forms in which boundary conditions are imposed only on a part of the boundary (corresponding to the essential boundary conditions).

Mixed boundary conditions for partial differential equations in vector analysis are a non-trivial topic, and even more so in numerical analysis. For an overview, we start with the Poisson problem with mixed boundary conditions. Suppose for simplicity that Ω is a bounded strongly Lipschitz domain with outward normal field $\vec{n}$ along $\partial\Omega$. We assume that $\Gamma_D \subseteq \partial\Omega$ is an admissible boundary patch and that Γ_N is its complementary boundary patch. Given a function f, the Poisson problem with mixed boundary conditions is finding the solution u of

$$-\Delta u = f, \quad u_{|\Gamma_D} = 0, \quad \nabla u_{|\Gamma_N} \cdot \vec{n} = 0. \tag{VII.1}$$

Here, we impose a homogeneous *Dirichlet boundary condition* along Γ_D and a homogeneous *Neumann boundary condition* along Γ_N. If $f \in L^2(\Omega)$, then a weak formulation characterizes the solution as the unique minimizer of the energy

$$\mathcal{J}(u) := \frac{1}{2}\int_\Omega |\operatorname{grad} u|^2 \,\mathrm{d}x - \int_\Omega fu \,\mathrm{d}x \tag{VII.2}$$

over $H^1(\Omega, \Gamma_D)$, the subspace of $H^1(\Omega)$ whose members satisfy the (essential) Dirichlet boundary condition along Γ_D. The well-posedness of this variational problem follows by a Friedrichs inequality with partial boundary conditions [148]. Moreover, the compactness of the embedding $H^1(\Omega, \Gamma_D) \to L^2(\Omega)$ is crucial in proving the compactness of the solution operator. A typical finite element method seeks a discrete approximation of u by minimizing $\mathcal{J}$ over a space of Lagrange elements in $H^1(\Omega, \Gamma_D)$. This Galerkin method is standard in the literature [32]. But still we cannot approach the Poisson problem with mixed boundary conditions by the current means of finite element exterior calculus due to the lack of a smoothed projection.

The natural generalization to vector-valued problems in three dimensions is given by the vector Laplace equation with mixed boundary conditions. This equation appears in electromagnetism or fluid dynamics. The analysis of this vector-valued partial differential equation, however, is considerably more complex. Given the vector field $\mathbf{f}$, we seek a vector field $\mathbf{u}$ that solves

$$\operatorname{curl}\operatorname{curl}\mathbf{u} - \operatorname{grad}\operatorname{div}\mathbf{u} = \mathbf{f} \tag{VII.3}$$

over the domain Ω. Moreover, we assume that Γ_T and Γ_N are mutually complementary admissible boundary patches of Ω. The boundary conditions on $\mathbf{u}$ are

$$\mathbf{u}_{|\Gamma_T} \times \vec{n} = 0, \quad (\operatorname{curl}\mathbf{u})_{|\Gamma_T} \cdot \vec{n} = 0, \quad \mathbf{u}_{|\Gamma_N} \cdot \vec{n} = 0, \quad (\operatorname{div}\mathbf{u})_{|\Gamma_N} = 0. \tag{VII.4}$$

Here we impose homogeneous *tangential boundary conditions* on $\mathbf{u}$ along a boundary part Γ_T, and homogeneous *normal boundary conditions* on $\mathbf{u}$ along the complementary boundary part Γ_N. When $\mathbf{f} \in L^2(\Omega, \mathbb{R}^3)$, then a variational formulation seeks the solution by minimizing the energy functional

$$\mathcal{J}(\mathbf{u}) := \frac{1}{2} \int_\Omega |\operatorname{div} \mathbf{u}|^2 + |\operatorname{curl} \mathbf{u}|^2 \, \mathrm{d}x - \int_\Omega \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \tag{VII.5}$$

over the space $\mathbf{H}(\operatorname{div}, \Omega, \Gamma_N) \cap \mathbf{H}(\operatorname{curl}, \Omega, \Gamma_T)$. Here $\mathbf{H}(\operatorname{div}, \Omega, \Gamma_N)$ is the subspace of $\mathbf{H}(\operatorname{div}, \Omega)$ satisfying normal boundary conditions along Γ_N, and $\mathbf{H}(\operatorname{curl}, \Omega, \Gamma_T)$ is the subspace of $\mathbf{H}(\operatorname{curl}, \Omega)$ satisfying tangential boundary conditions along Γ_T.

The additional complexity in comparison to the scalar-valued case begins with the correct definition of tangential and normal boundary conditions in a setting of low regularity [43, 44, 99, 176, 177]. When non-mixed boundary conditions are imposed, i.e. when either $\Gamma_T = \emptyset$ or $\Gamma_T = \partial\Omega$, then Rellich-type compact embeddings $\mathbf{H}(\operatorname{div}, \Omega, \Gamma_N) \cap \mathbf{H}(\operatorname{curl}, \Omega, \Gamma_T) \to \mathbf{L}^2(\Omega)$ and vector-valued Poincaré-Friedrichs inequalities have been known for a long time [64, 151, 175, 181]. Mixed boundary conditions in vector analysis, however, have only recently been addressed systematically [6, 15, 114, 115, 125].

Additional difficulties arise in numerical analysis. Minimizing (VII.5) over a finite element subspace of $\mathbf{H}(\operatorname{div}, \Omega, \Gamma_N) \cap \mathbf{H}(\operatorname{curl}, \Omega, \Gamma_T)$ generally does not lead to a consistent finite element method [11, 62]. But mixed finite element methods, which introduce either $\operatorname{div} \mathbf{u}$ or $\operatorname{curl} \mathbf{u}$ as auxiliary variables, have been studied with great success [30, 68, 109, 145]. Mixed boundary conditions for the vector Laplace equation, however, have not yet received much attention in numerical analysis (but see [105, 159]). In a mixed finite element method for the vector Laplace equation with mixed boundary conditions we may only incorporate the essential boundary conditions along Γ_T into the finite element space.

In Chapter VIII we attend particularly to a phenomenon that significantly affects the theoretical and numerical analysis of the vector Laplace equation but remains absent in the scalar-valued theory: the presence of non-trivial harmonic vector fields in $\mathbf{H}(\operatorname{div}, \Omega, \Gamma_N) \cap \mathbf{H}(\operatorname{curl}, \Omega, \Gamma_T)$. For a motivational example, let $\mathcal{H}(\Omega, \Gamma_T, \Gamma_N)$ be the subspace of $\mathbf{H}(\operatorname{div}, \Omega, \Gamma_N) \cap \mathbf{H}(\operatorname{curl}, \Omega, \Gamma_T)$ whose members have vanishing curl and vanishing divergence. This space has physical relevance; in fluid dynamics, for example, those vector fields describe the incompressible irrotational flows that satisfy given boundary conditions. In the case of non-mixed boundary conditions, their dimension corresponds to topological properties of the domain [138], and in particular that dimension is zero on contractible domains. But in the case of mixed boundary conditions, this dimension depends on the topology of both the domain Ω and the boundary part Γ_T. Thus $\mathcal{H}(\Omega, \Gamma_T, \Gamma_N)$ may have positive dimension if Γ_T has a sufficiently complicated topology even if Ω itself is contractible [99, 124]. This dimension can be calculated exactly from a given triangulation of Ω and Γ_T. In a finite element method, the subspace $\mathcal{H}(\Omega, \Gamma_T, \Gamma_N)$ must be approximated by discrete harmonic fields, i.e. the kernel of the finite element vector Laplacian.

It is instructive to study these partial differential equations in a unified manner using the calculus of differential forms. Both the Poisson problem and the vector

Laplace equation with mixed boundary conditions are special cases of the Hodge Laplace equation with mixed boundary conditions. The Hodge Laplace equation has been studied extensively over Sobolev spaces of differential forms [12, 42, 125, 139, 140, 141, 166, 176, 177]. The case of mixed boundary conditions has been a recent subject of research in the field of analysis on manifolds [99, 113]. As a theoretical basis, one studies de Rham complexes with partial boundary conditions,

$$\dots \xrightarrow{\mathsf{d}^{k-1}} H\Lambda^k(\Omega,\Gamma_T) \xrightarrow{\mathsf{d}^k} H\Lambda^{k+1}(\Omega,\Gamma_T) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{VII.6}$$

The choices $\Gamma_T = \emptyset$ and $\Gamma_T = \partial\Omega$ correspond to the widely studied special cases of either imposing no boundary conditions at all or boundary conditions along the entire boundary, respectively.

Moving towards the numerical analysis of mixed finite element methods for the Hodge Laplace equation with mixed boundary conditions, we adopt the framework of finite element exterior calculus. The calculus of differential forms has attracted interest as a unifying framework for mixed finite element methods [8, 9, 11, 72, 87, 109]. The numerical analysis of mixed finite element methods for the Hodge Laplace equation can be formulated in terms of finite element de Rham complexes, which mimic the differential complex (VII.6) on a discrete level. For an outline of the idea, we let $\mathcal{T}$ be a triangulation of Ω that also contains a triangulation $\mathcal{U}$ of Γ_T. The construction in Chapter IV provides a finite element de Rham complex

$$\dots \xrightarrow{\mathsf{d}^{k-1}} \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^k} \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{VII.7}$$

that features these essential boundary conditions along Γ_T. Indeed we have

$$\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) = \mathcal{P}\Lambda^k(\mathcal{T}) \cap H\Lambda^k(\Omega,\Gamma_T).$$

Smoothed projections from Sobolev de Rham complexes onto finite element de Rham complexes play a central role in the a priori error analysis within finite element exterior calculus. They are the main requirement to enable the abstract Galerkin theory of Hilbert complexes [11], which produces the stability and convergence of mixed finite element methods. Previous contributions [9, 58] provided the corresponding smoothed projections in the special cases of either fully essential, $\Gamma_T = \partial\Omega$, or fully natural boundary conditions, $\Gamma_T = \partial\Omega$, in finite element exterior calculus, but the general case of mixed boundary conditions has remained open. In order to overcome this limitation we need a smoothed projection that preserves partial boundary conditions. Already in the two special cases $\Gamma_T = \partial\Omega$ and $\Gamma_T = \partial\Omega$ one could observe that the smoothed projection depends on the boundary conditions, and hence we expect the same to be true in the treatment of general mixed boundary conditions.

What we specifically need is a projection $\pi^k : H\Lambda^k(\Omega,\Gamma_T) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ onto the finite element space that satisfies uniform L^2 bounds and that commutes with the differential operator. In particular, the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r, "\mathsf{d}^{k-1}"] & H\Lambda^k(\Omega,\Gamma_T) \arrow[r, "\mathsf{d}^k"] \arrow[d, "\pi^k"'] & H\Lambda^{k+1}(\Omega,\Gamma_T) \arrow[r, "\mathsf{d}^{k+1}"] \arrow[d, "\pi^{k+1}"'] & \dots \\
\dots \arrow[r, "\mathsf{d}^{k-1}"] & \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \arrow[r, "\mathsf{d}^k"] & \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \arrow[r, "\mathsf{d}^{k+1}"] & \dots
\end{tikzcd} \tag{VII.8}$$

Given such a projection, we obtain a priori convergence results for mixed finite element methods [11]. Smoothed projections have been developed in finite element exterior calculus [9, 58, 130] for non-mixed boundary conditions.

The agenda of this chapter is to construct such a smoothed projection. In particular, we prove the following main result.

Theorem.
Let $\Omega \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain, and let $\Gamma_T \subseteq \partial\Omega$ be an admissible boundary patch. Let $\mathcal{T}$ be a simplicial triangulation of Ω that contains a simplicial triangulation $\mathcal{U}$ of Γ_T, and let (VII.7) be a differential complex of finite element spaces of differential forms as in Chapter IV with essential boundary conditions along Γ_T. Then there exist bounded linear projections $\pi^k : L^2\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ such that the following diagram commutes:

$$\begin{array}{ccccccc} H\Lambda^0(\Omega,\Gamma_T) & \xrightarrow{\mathsf{d}^0} & H\Lambda^1(\Omega,\Gamma_T) & \xrightarrow{\mathsf{d}^1} & \cdots & \xrightarrow{\mathsf{d}^{n-1}} & H\Lambda^n(\Omega,\Gamma_T) \\ {\scriptstyle\pi^0}\big\downarrow & & {\scriptstyle\pi^1}\big\downarrow & & & & {\scriptstyle\pi^n}\big\downarrow \\ \mathcal{P}\Lambda^0(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^0} & \mathcal{P}\Lambda^1(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^1} & \cdots & \xrightarrow{\mathsf{d}^{n-1}} & \mathcal{P}\Lambda^n(\mathcal{T},\mathcal{U}). \end{array} \tag{VII.9}$$

Moreover, $\pi^k u = u$ for $u \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$. The L^2 operator norm of π^k is bounded uniformly in terms of the maximum polynomial order of (VII.7), the shape measure of the triangulation, and geometric properties of Ω and Γ_T. □

The smoothed projection enables the abstract Galerkin theory of Hilbert complexes for the numerical analysis of the Hodge Laplace equation with mixed boundary conditions over weakly Lipschitz domains. This applies to a large class of mixed finite element methods. As an immediate consequence, to be elaborated in the subsequent chapter, the a priori error estimates of finite element exterior calculus provide quasi-optimal convergence for mixed finite element methods for a large class of Hodge Laplace problems.

Constructing and analyzing such a smoothed projection requires significant technical effort. Even though we largely follow ideas in published literature [9, 58] we introduce significant technical modifications. The smoothed projection is constructed in several stages, which we give an outline of here. Suppose that $u \in L^2\Lambda^k(\Omega)$ is a square-integrable differential k-form over Ω.

First, an operator $E^k : L^2\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega^e)$ extends u over a neighborhood Ω^e of Ω. The basic idea is extending the differential form by reflection across the boundary. For strongly Lipschitz domains, such a parametrization can be constructed using the flow along a smooth vector field transversal to the boundary [9, 58], but for weakly Lipschitz domains such a transversal vector field does not necessarily exist. Instead we obtain the desired parametrized tubular neighborhood via a variant of the collaring theorem in Lipschitz topology [134]. Furthermore, in order to accommodate partial boundary conditions along Γ_T, we extend u by zero over a "bulge" attached to the domain along Γ_T. This is inspired by work of Gopalakrishnan and Qiu [105], who used a similar idea for strongly Lipschitz domains and boundary partitions with a piecewise C^1-interface. The resulting operator E^k commutes with the exterior derivative on $H\Lambda^k(\Omega, \Gamma_T)$.

Next, we construct a distortion $\mathfrak{D}_\varrho : \Omega^e \to \Omega^e$ which moves a neighborhood of the bulge into the latter but which is the identity outside of a small neighborhood of the bulge. We locally control the amount of distortion via a function ϱ. The pullback $\mathfrak{D}_\varrho^* E^k u$ of $E^k u$ along $\mathfrak{D}_\varrho$ vanishes in a neighborhood of Γ_T and commutes with the exterior derivative.

Subsequently, a mollification operator $R_\varrho^k : L^2\Lambda^k(\Omega^e) \to C^\infty\Lambda^k(\overline{\Omega})$ smooths the differential form $\mathfrak{D}_\varrho^* E^k u$ to a smooth differential form over Ω that vanishes in a neighborhood of Γ_T. This is based on the idea of taking the convolution with a smooth bump function. In order to guarantee uniform bounds for shape-regular families of meshes, the mollification radius is locally controlled by a function ϱ. This is similar to [58], but we elaborate the details of the construction and make a minor correction; see also Remark VII.8.12. We find that the mollified differential form has well-defined degrees of freedom.

The regularized differential form has well-defined degrees of freedom, and thus the interpolant $I_\mathcal{P}^k : C^\infty\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T})$ can be applied. Since the regularized differential form vanishes near Γ_T, the interpolation gives an element of $\mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$. In combination, this yields a *smoothed interpolant* $Q^k : L^2\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$ that commutes with the exterior derivative on $H\Lambda^k(\Omega, \Gamma_T)$. In making the mollification radius depending on the local mesh size we can prove uniform bounds for the smoothed interpolant. But Q^k is generally not idempotent. We can, however, control the interpolation error over the finite element space. To enforce idempotence, we prove a bound on the interpolation error over the finite element space and apply the "Schöberl trick" [159]. If the smoothed interpolant is sufficiently close to the identity over the finite element space, then a commuting and uniformly bounded discrete inverse exists. The composition of this discrete inverse with the smoothed interpolant gives the desired smoothed projection.

In order to derive the aforementioned interpolation error estimate over the finite element space, we call on geometric measure theory [88, 180]. The principle motivation in utilizing geometric measure theory is the low regularity of the boundary, which requires new techniques in finite element theory. A key observation, which seems to be of independent interest, is the identification of the degrees of freedom as *flat chains* in the sense of geometric measure theory. The desired estimate of the interpolation error over the finite element space is proven eventually with distortion estimates on flat chains. To the author's best understanding, our results close a gap in some proofs in the literature; see also Remark VII.8.9.

Most of the literature on commuting projections focuses on the L^2 theory (but see also [56] or [82]). We consider differential forms with coefficients in general L^p spaces, following [100], which includes the $W^{p,q}$ classes of differential forms in particular.

Commuting projections are a standard tool in finite element analysis, and the calculus of differential forms has been promoted as a unifying language for finite element methods for problems in vector analysis [109]. A bounded projection operator that commutes with the exterior derivative up to a controllable error was derived in

[52]. A bounded commuting projection operator for the de Rham complex without boundary conditions has been derived in [9] in the case of quasi-uniform triangulations, which was subsequently generalized in [58] to shape-uniform triangulations and de Rham complexes with full boundary conditions. The existence of a smoothed projection that respects partial boundary conditions has been an unproven assumption in [28]. Commuting projections have been derived in [56] and [82] with different methods. We mention the local bounded interpolant in [160], given in the language of classical vector analysis, as one of the first contributions to research on commuting interpolants and projections. This interpolant was only later generalized to differential forms in [72], and a variant of that preserves partial boundary condition was given in [105]. A commuting projection for spaces with weighted norms that arise in the numerical analysis of axisymmetric Maxwell's equation was given in [104]. A local commuting projection was given in [87].

In addition to this research in numerical analysis, we address a topic that is of purely analytical interest. Specifically, we prove that smooth differential forms over a weakly Lipschitz domain Ω which vanish near an admissible boundary patch Γ_T are dense in $W^{p,q}\Lambda^k(\Omega, \Gamma_T)$ for $p, q \in [1, \infty)$. When Ω is a (strongly) Lipschitz domain and $\Gamma_D \subseteq \partial\Omega$ is a suitable boundary patch, then the density of $C^\infty(\overline{\Omega}) \cap H^1(\Omega, \Gamma_D)$ in $H^1(\Omega, \Gamma_D)$ (see [76, 77]) and analogous density result for differential forms with partial boundary conditions over strongly Lipschitz domains (see [113]) have been available in the literature before. In this chapter we generalize these results to Sobolev spaces of differential forms over weakly Lipschitz domains. Specifically, we prove the following theorem.

Theorem.
Let Ω be a bounded weakly Lipschitz domain and let Γ_T be an admissible boundary patch. Then the smooth differential k-forms in $C^\infty\Lambda^k(\overline{\Omega})$ that vanish near Γ_T constitute a dense subset of $W^{p,q}\Lambda^k(\Omega, \Gamma_T)$ for all $p, q \in [1, \infty)$.

VII.1. Extension Operators

For the duration of this entire chapter, we let $\Omega \subset \mathbb{R}^n$ be a bounded weakly Lipschitz domain and $n \geq 2$. Additionally, we assume that $\Gamma_T \subseteq \partial\Omega$ is a fixed admissible boundary patch. The reader may assume $\Gamma_T = \emptyset$ in a simplified reading. We let $\Gamma_N := \partial\Omega \setminus \overline{\Gamma_T}$ denote the complementary boundary patch and we let $\Gamma_I = \overline{\Gamma_T} \cap \overline{\Gamma_N}$ be the interface between Γ_T and Γ_N. In other words, $(\Gamma_T, \Gamma_I, \Gamma_N)$ is assumed to be an admissible boundary partition.

In this section we make use of Lipschitz collars in the construction of extension operators. The basic idea is to reflect a differential form over Ω along the boundary onto the exterior of Ω. But to accommodate partial boundary conditions, we first extend the differential form by zero onto a "bulge" attached at Γ_T. The original idea of reflection is then applied to the domain with the bulge attached. The resulting extension operator commutes with the exterior derivative.

By an application of Theorem VI.1.8, the bounded weakly Lipschitz domain Ω admits a Lipschitz collar Ψ_0. We recall that this is a LIP embedding

$$\Psi_0 : \partial\Omega \times [-1,1] \to \mathbb{R}^n$$

such that $\Psi_0(x,0) = x$ for $x \in \partial\Omega$ and such that

$$\Psi_0\left(\partial\Omega \times [-1,0)\right) \subset \Omega, \quad \Psi_0\left(\partial\Omega \times (0,1]\right) \subset \overline{\Omega}^c.$$

Based on this, we define the auxiliary domains

$$\Upsilon := \Psi_0\left(\Gamma_T \times (0, {}^1\!/\!_2)\right), \quad \Omega^b := \Omega \cup \Gamma_T \cup \Upsilon. \tag{VII.10}$$

We think of Υ as a *bulge* attached to the domain Ω along Γ_T, which results in the combined domain Ω^b. The following lemma is easily verified.

Lemma VII.1.1.
The domains Υ and Ω^b as defined above are bounded weakly Lipschitz domains.

Proof. The proof is not very difficult but we give the technical details. We want to construct the coordinate charts that flatten the boundaries of Υ and Ω^b as in the definition of weakly Lipschitz domains.

Consider $x \in \overline{\Gamma}_T$. By Lemma VI.1.7 there exists a neighborhood V_x of x in $\partial\Omega$ and a bi-Lipschitz mapping $\theta_x : V_x \to [-1,1]^{n-1}$ with $\theta_x(x) = 0$. If $x \in \Gamma_T$, then we assume without loss of generality that V_x is contained in Γ_T. If instead $x \in \Gamma_I$, then we assume additionally that V_x and θ_x satisfy the properties (VI.10) stated in the definition of admissible boundary patches.

For $t_0 \in (-1,1)$ and $\epsilon > 0$ small enough we let

$$U_{x,t_0} := \Phi_0\Big(V_x \times [t_0 - \epsilon, t_0 + \epsilon]\Big)$$

and define a bi-Lipschitz mapping $\varphi_{x,t_0} : U_{x,t_0} \to [-1,1]^n$ by setting

$$\varphi_{x,t_0}\Big(\Phi_0(\theta_x(z), t)\Big) := \Big(z, (t - t_0)/\epsilon\Big).$$

If $x \in \Gamma_T$, then this easily produces the required flattenings of the boundary for $t = \frac{1}{2}$ and $t = 0$. If $x \in \Gamma_I$ and $t \in (-\frac{1}{2}, \frac{1}{2})$, then the required coordinate chart is found similarly. In the special case that $x \in \Gamma_I$ and $t \in \{-\frac{1}{2}, \frac{1}{2}\}$ we obtain the desired coordinate chart with another bi-Lipschitz transformation φ applied after φ_{x,t_0}, where φ depends only on t and whether we consider Υ or Ω^b. This completes the construction. $\square$

We define the extension operator

$$E_0^k : M\Lambda^k(\Omega) \to M\Lambda^k(\Omega^b) \quad u \mapsto \begin{cases} u & \text{over } \Omega, \\ 0 & \text{over } \Upsilon. \end{cases} \tag{VII.11}$$

The properties of E_0^k are stated in the following lemma.

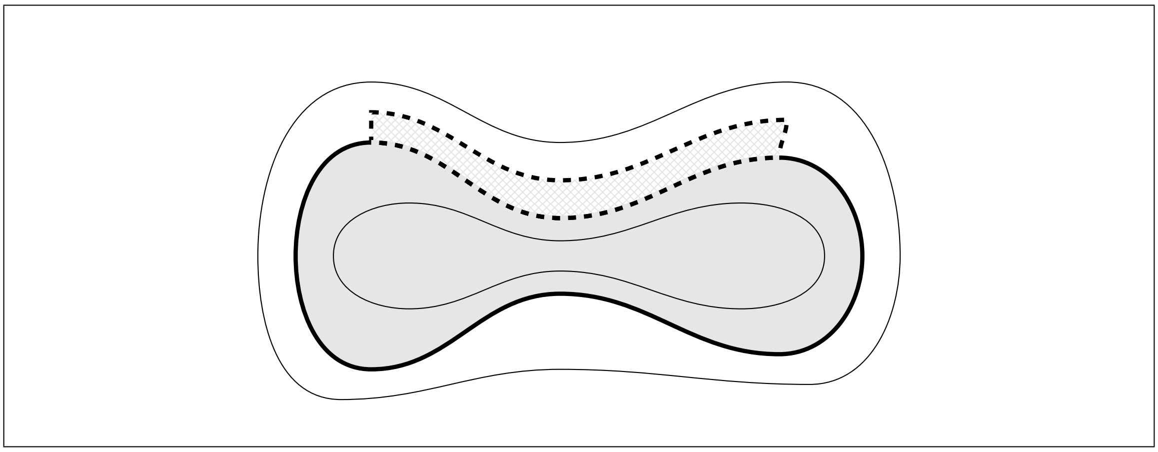

Figure VII.1: Domain Ω (gray) with a bulge Υ (shaded) attached. The thick line is the boundary part Γ_N of the original domain, and the contact line between Ω and Υ is the boundary part Γ_T. The thinner lines inside and outside of the domain indicate the inner and outer boundaries, respectively, of a Lipschitz collar.

Theorem VII.1.2.
We have a bounded linear operator

$$E_0^k : L^p\Lambda^k(\Omega) \to L^p\Lambda^k(\Omega^b), \quad p \in [1, \infty].$$

Moreover, for $p, q \in [1, \infty]$ we have

$$u \in W^{p,q}\Lambda^k(\Omega, \Gamma_T) \implies E_0^k u \in W^{p,q}\Lambda^k(\Omega^b)$$

and

$$\mathsf{d}^k E_0^k u = E_0^{k+1}\mathsf{d}^k u, \quad u \in W^{p,q}\Lambda^k(\Omega, \Gamma_T).$$

Proof. It is clear that E_0^k is bounded. That E_0^k maps $W^{p,q}\Lambda^k(\Omega, \Gamma_T)$ into $W^{p,q}\Lambda^k(\Omega^b)$ and commutes with the exterior derivative is easily seen from the definition of $W^{p,q}\Lambda^k(\Omega, \Gamma_T)$. □

Remark VII.1.3.
It is evident that $E_0^k u$ is generally not a member of $W^{p,q}\Lambda^k(\Omega^b)$ for arbitrary $u \in W^{p,q}\Lambda^k(\Omega)$ unless $\Gamma_T = \emptyset$.

We return to the original idea of extending a differential form by reflection along the boundary. Since Ω^b is a bounded weakly Lipschitz domain, we may apply Theorem VI.1.8 again to obtain a LIP embedding

$$\Psi_b : \partial\Omega^b \times [-1, 1] \to \mathbb{R}^n$$

such that $\Psi_b(x, 0) = x$ for $x \in \partial\Omega^b$ and such that

$$\Psi_b\left(\partial\Omega^b \times [-1, 0)\right) \subset \Omega^b, \quad \Psi_b\left(\partial\Omega^b \times (0, 1]\right) \subset \mathbb{R}^n \setminus \overline{\Omega^b}.$$

The Lipschitz collar of Ω^b allows us to reflect points across the boundary. We write

$$\mathcal{C}\Omega^b := \Psi_b\left(\partial\Omega^b \times (-1,1)\right), \tag{VII.12}$$

$$\mathcal{C}^-\Omega^b := \Psi_b\left(\partial\Omega^b \times (-1,0)\right), \quad \mathcal{C}^+\Omega^b := \Psi_b\left(\partial\Omega^b \times (0,1)\right), \tag{VII.13}$$

$$\Omega^e := \overline{\Omega^b} \cup \mathcal{C}^+\Omega^b. \tag{VII.14}$$

The domain $\mathcal{C}\Omega^b$ is an open neighborhood of $\partial\Omega$. The domains $\mathcal{C}^-\Omega^b$ and $\mathcal{C}^+\Omega^b$ represent the interior and exterior parts of the collar neighborhood $\mathcal{C}\Omega^b$, respectively. Finally, Ω^e is an extension of the domain Ω^b. In particular, Ω is compactly contained in Ω^e. Furthermore, in accordance with Theorem VI.1.8 we can assume without loss of generality that Ω^e is a weakly Lipschitz domain.

For every $x \in \mathcal{C}\Omega^b$ there exist unique $x_0 \in \partial\Omega^b$ and $t \in [-1,1]$ such that $x = \Psi_b(x_0,t)$. Hence we may define a bi-Lipschitz mapping

$$\mathcal{R} : \mathcal{C}^+\Omega^b \to \mathcal{C}^-\Omega^b, \quad \Psi_b(x,t) \to \Psi_b(x,-t). \tag{VII.15}$$

This formalizes the idea of reflecting a point across the boundary. Using the pullback along $\mathcal{R}$, we introduce the extension operator

$$E_r^k : M\Lambda^k(\Omega^b) \to M\Lambda^k(\Omega^e), \quad u \mapsto \begin{cases} u & \text{over } \Omega^b, \\ \mathcal{R}^* u & \text{over } \mathcal{C}^+\Omega^b. \end{cases} \tag{VII.16}$$

We gather some properties of the extension operator E_r^k. For future use, we define

$$C_b := \max\left\{ \operatorname{Lip}\left(\mathcal{R}, \mathcal{C}^+\Omega^b\right), \operatorname{Lip}\left(\mathcal{R}^{-1}, \mathcal{C}^-\Omega^b\right) \right\}.$$

It is easily seen that $C_b \geq 1$. We first show that E_r^k satisfies local estimates:

Lemma VII.1.4.
Let $p \in [1,\infty]$. We have a bounded linear operator

$$E_r^k : L^p\Lambda^k(\Omega^b) \to L^p\Lambda^k(\Omega^e), \quad u \mapsto E_r^k u.$$

Moreover, for $0 \leq s \leq t \leq 1$ and $A \subseteq \partial\Omega^b$ closed we have

$$\|E_r^k u\|_{L^p\Lambda^k\Psi_b(A\times[s,t])} \leq C_b^{k+\frac{n}{p}} \|u\|_{L^p\Lambda^k\Psi_b(A\times[-t,-s])}, \quad u \in L^p\Lambda^k(\Omega^b). \tag{VII.17}$$

Proof. Let $p \in [1,\infty]$, let $A \subseteq \partial\Omega^b$ be closed, let $0 \leq s \leq t \leq 1$, and let $u \in L^p\Lambda^k(\Omega^b)$. We apply Lemma V.2.3 to find

$$\begin{aligned} \|E_r^k u\|_{L^p\Lambda^k\Psi_b(A\times[s,t])} &= \|\mathcal{R}^* u\|_{L^p\Lambda^k\Psi_b(A\times[s,t])} \\ &\leq \|\operatorname{D}\mathcal{R}\|^k_{L^\infty(\mathcal{C}^+\Omega^b)} \|\operatorname{D}\mathcal{R}^{-1}\|^{\frac{n}{p}}_{L^\infty(\mathcal{C}^-\Omega^b)} \|u\|_{L^p\Lambda^k\Psi_b(A\times[-t,-s])}. \end{aligned}$$

Hence (VII.17) holds. Setting $A \times [s,t] = \partial\Omega^b \times [0,1]$, we find

$$\begin{aligned} \|E_r^k u\|_{L^p\Lambda^k(\Omega^e)} &\leq \|u\|_{L^p\Lambda^k(\Omega^b)} + \|E_r^k u\|_{L^p\Lambda^k(\mathcal{C}^+\Omega^b)} \\ &\leq \|u\|_{L^p\Lambda^k(\Omega^b)} + C_b^{k+\frac{n}{p}} \|u\|_{L^p\Lambda^k(\mathcal{C}^-\Omega^b)} \\ &\leq \left(1 + C_b^{k+\frac{n}{p}}\right) \|u\|_{L^p\Lambda^k(\Omega^b)}. \end{aligned}$$

Hence E_r^k is bounded from $L^p\Lambda^k(\Omega^b)$ to $L^p\Lambda^k(\Omega^e)$. $\square$

We show that E_r^k commutes with the exterior derivative.

Lemma VII.1.5.
Let $p, q \in [1, \infty]$ and let $u \in W^{p,q}\Lambda^k(\Omega^b)$. Then $E_r^k u \in W^{p,q}\Lambda^k(\Omega^e)$ with $E_r^{k+1}\mathsf{d}^k u = \mathsf{d}^k E_r^k u$.

Proof. Because Ω is bounded, it suffices to consider the case $p = q = 1$. Let us assume that $u \in W^{1,1}\Lambda^k(\Omega^b)$. We have $E_r^k u \in L^1\Lambda^k(\Omega^e)$ and $E_r^{k+1}\mathsf{d}^k u \in L^1\Lambda^{k+1}(\Omega^e)$ by Lemma VII.1.4. To prove that $E_r^k u \in W^{1,1}\Lambda^k(\Omega^e)$ with $E_r^{k+1}\mathsf{d}^k u = \mathsf{d}^k E_r^k u$, it suffices to show that there exists a covering $(U_i)_{i \in \mathbb{N}}$ of Ω^e by open subsets $U_i \subseteq \Omega^e$ such that $E_r^k u_{|U_i} \in W^{1,1}\Lambda^k(U_i)$ and $E_r^{k+1}\mathsf{d}^k u = \mathsf{d}^k E_r^k u$ over U_i.

From the definition of weakly Lipschitz domains we easily see that there exists a family $(\theta_i)_{i\in\mathbb{N}}$ of LIP embeddings $\theta_i : (-1,1)^{n-1} \to \partial\Omega^b$ whose images cover $\partial\Omega^b$. We define

$$\varphi_i : (-1,1)^n \to \mathcal{C}\Omega^b, \quad (y,t) \mapsto \Psi_b\left(\theta_i(y), t\right).$$

These are a family of LIP embeddings whose images $U_i := \varphi_i\left((-1,1)^n\right)$ cover $\mathcal{C}\Omega^b$. Together with Ω^b we thus have a finite covering of Ω^e.

We recall that $E_r^k u_{|\Omega^b} \in W^{1,1}\Lambda^k(\Omega^b)$ with $E_r^{k+1}\mathsf{d}^k u = \mathsf{d}^k E_r^k u$ over Ω^b. It remains to show that $E_r^k u_{|U_i} \in W^{1,1}\Lambda^k(U_i)$ and $E_r^{k+1}\mathsf{d}^k u = \mathsf{d}^k E_r^k u$ over U_i for $i \in \mathbb{N}$. We define

$$u_i := \varphi_i^*\left(E_r^k u_{|U_i}\right), \quad w_i := \varphi_i^*\left(E_r^{k+1}\mathsf{d}^k u_{|U_i}\right).$$

It suffices to show $u_i \in W^{1,1}\Lambda^k((-1,1)^n)$ and $\mathsf{d}^k u_i = w_i$ over $(-1,1)^n$. We let

$$\mathscr{R} : (-1,1)^{n-1} \times (0,1) \to (-1,1)^{n-1} \times (-1,0)$$

be the reflection by the n-th coordinate. It is evident that

$$\begin{aligned} u_{i|(-1,1)^{n-1}\times(0,1)} &= \mathscr{R}^* u_{i|(-1,1)^{n-1}\times(-1,0)} \\ w_{i|(-1,1)^{n-1}\times(0,1)} &= \mathscr{R}^* w_{i|(-1,1)^{n-1}\times(-1,0)} = \mathscr{R}^*\mathsf{d}^k u_{i|(-1,1)^{n-1}\times(-1,0)}. \end{aligned}$$

By Lemma V.3.5 there exists a sequence $(u_i^j)_{j\in\mathbb{N}}$ of smooth differential k-forms over $(-1,1)^{n-1} \times (-1,0)$ that converge to u_i over $(-1,1)^{n-1} \times (-1,0)$ in the $W^{1,1}\Lambda^k$ norm for $j \to \infty$. We let $\overset{\shortparallel}{u}{}_i^j$ be the extension of u_i^j from $(-1,1)^{n-1} \times (-1,0)$ to $(-1,1)^n$ by pullback along $\mathscr{R}$. Then $\overset{\shortparallel}{u}{}_i^j$ is a locally integrable differential k-form over $(-1,1)^n$ with locally integrable weak exterior derivative. It is easy to observe that $\overset{\shortparallel}{u}{}_i^j$ converges to u_i in $L^1\Lambda^k\left((-1,1)^n\right)$ and $\mathsf{d}^k\overset{\shortparallel}{u}{}_i^j$ converges to w_i in $L^1\Lambda^{k+1}\left((-1,1)^n\right)$ for $j \to \infty$. Hence $u_i \in W^{1,1}\Lambda^k\left((-1,1)^n\right)$ with $\mathsf{d}^k u_i = w_i$. The proof is complete. $\square$

We can now verify that the extension operator E_r^k has the following properties.

Theorem VII.1.6.
We have bounded linear operators

$$E_r^k : L^p\Lambda^k(\Omega^b) \to L^p\Lambda^k(\Omega^e), \quad p \in [1, \infty].$$

For $p, q \in [1, \infty]$ we have

$$E_r^k \left(W^{p,q}\Lambda^k(\Omega^b)\right) \subseteq W^{p,q}\Lambda^k(\Omega^e),$$

with

$$\mathsf{d}^k E_r^k u = E_r^{k+1} \mathsf{d}^k u, \quad u \in W^{p,q}\Lambda^k(\Omega^b).$$

Proof. This is a combination of Lemma VII.1.4 and Lemma VII.1.5. □

Combining these two operators, we introduce

$$E^k : M\Lambda^k(\Omega) \to M\Lambda^k(\Omega^e), \quad u \mapsto E_r^k E_0^k u. \tag{VII.18}$$

The following theorem summarizes the above observations.

Theorem VII.1.7.
We have a bounded operator

$$E^k : L^p\Lambda^k(\Omega) \to L^p\Lambda^k(\Omega^e), \quad p \in [1, \infty].$$

Moreover, for $u \in L^p\Lambda^k(\Omega)$ with $p \in [1, \infty]$ we have

$$\operatorname{supp} E^k u \cap \Upsilon = \emptyset.$$

For $p, q \in [1, \infty]$ we have a bounded operator

$$E^k : W^{p,q}\Lambda^k(\Omega, \Gamma_T) \to W^{p,q}\Lambda^k(\Omega^e), \quad p, q \in [1, \infty],$$

such that

$$\mathsf{d}^k E^k u = E^{k+1} \mathsf{d}^k u, \quad u \in W^{p,q}\Lambda^k(\Omega, \Gamma_T).$$

Proof. This is a combination of Theorem VII.1.2 and Theorem VII.1.6. □

In the sequel, we will require several local bounds of these extension operators. This is accomplished with the following lemma.

Lemma VII.1.8.
There exists $L_E \geq 1$, depending only on Ψ_b, such that for all $p \in [1, \infty]$, for all $\delta > 0$, and all closed sets $A \subset \overline{\Omega}$ we have

$$\|E^k u\|_{L^p\Lambda^k(B_\delta(A)\cap\Omega^e)} \leq \left(1 + C_b^{k+\frac{n}{p}}\right) \|u\|_{L^p\Lambda^k\left(B_{\delta L_E}(A)\cap\Omega\right)}, \quad u \in L^p\Lambda^k(\Omega^b). \tag{VII.19}$$

Proof. Let $\delta \in \mathbb{R}_0^+$, let $p \in [1, \infty]$, and let $A \subset \overline{\Omega}$ be closed. Then

$$\begin{aligned}
\|E^k u\|_{L^p\Lambda^k(B_\delta(A)\cap\Omega^e)} &= \|E_r^k E_0^k u\|_{L^p\Lambda^k(B_\delta(A)\cap\Omega^e)} \\
&\leq \|E_0^k u\|_{L^p\Lambda^k\left(B_\delta(A)\cap\Omega^b\right)} + \|E_r^k E_0^k u\|_{L^p\Lambda^k\left(B_\delta(A)\cap\mathcal{C}^+\Omega^b\right)} \\
&= \|u\|_{L^p\Lambda^k(B_\delta(A)\cap\Omega)} + \|E_r^k E_0^k u\|_{L^p\Lambda^k\left(B_\delta(A)\cap\mathcal{C}^+\Omega^b\right)}.
\end{aligned}$$

We set $G^+ := B_\delta(A) \cap \mathcal{C}^+\Omega^b$ and $G^- = \mathcal{R}(G^+)$. Using Lemma VII.1.4, we find

$$\|E_r^k E_0^k u\|_{L^p\Lambda^k(G^+)} \leq C_b^{k+\frac{n}{p}} \|E_0^k u\|_{L^p\Lambda^k(G^-)} = C_b^{k+\frac{n}{p}} \|u\|_{L^p\Lambda^k(G^-\cap\Omega)}.$$

Let $x \in B_\delta(A) \cap \mathcal{C}^+\Omega^b$ be fixed but arbitrary. There exist $z \in A$ with $\|z - x\| \leq \delta$, and $y \in \partial\Omega^b$ on the straight line segment between x and z. Since $x \in \mathcal{C}^+\Omega^b$, there exist $x_0 \in \partial\Omega^b$ and $t \in [0, 1]$ with $x = \Psi_b(x_0, t)$. It is easily seen that

$$\|\mathcal{R}(x) - x\| = \|\Psi_b(x, -t) - \Psi_b(x, t)\| \leq 2\,\mathrm{Lip}(\Psi_b) \cdot t$$

and

$$|t| \leq \sqrt{\|x_0 - y\|^2 + |t|^2} \leq \mathrm{Lip}(\Psi_b^{-1})\,\|\Psi(x_0, t) - \Psi(y, 0)\| = \mathrm{Lip}(\Psi_b^{-1})\,\|x - y\|\,.$$

We then find that

$$\begin{aligned}
\|\mathcal{R}(x) - z\| &\leq \|\mathcal{R}(x) - x\| + \|x - z\| \\
&\leq 2\,\mathrm{Lip}(\Psi_b)\,\mathrm{Lip}(\Psi_b^{-1})\|x - y\| + \|x - z\| \\
&\leq \left(1 + 2 \cdot \mathrm{Lip}(\Psi_b)\,\mathrm{Lip}(\Psi_b^{-1})\right)\delta.
\end{aligned}$$

We choose $L_E := \left(1 + 2\,\mathrm{Lip}(\Psi_b)\,\mathrm{Lip}(\Psi_b^{-1})\right)$. Hence $G^- \cap \Omega \subseteq B_{L_E\delta}(A) \cap \Omega$. This completes the proof. □

Remark VII.1.9.
In the special case that $\Gamma_T = \emptyset$, we have $\Upsilon = \emptyset$ and $\Omega = \Omega^b$. The operator E_0^k does not enter the construction then. On the other hand, in the special case $\Gamma_T = \partial\Omega$, the set Υ wraps around the whole of Ω. The construction of E^k can then be simplified as follows: We set $\Omega^e = \mathbb{R}^n$ and pick E_r^k as the trivial extension onto $\mathbb{R}^n$.

Remark VII.1.10.
Whenever we say in the sequel that a quantity depends only on Ω, then the quantity may depend also on the arbitrary choice of Lipschitz collars at several points. This use of terminology will simplify the exposition in the sequel.

VII.2. A Distortion Theorem

In this section we discuss a geometric result that enters the construction of the smoothed projection and is also of independent interest. The basic idea is as follows: given a domain $\Upsilon \subseteq \mathbb{R}^n$, we search for a homeomorphism of $\mathbb{R}^n$ that moves $\partial\Upsilon$ into Υ and that is the identity outside of a neighborhood of $\partial\Upsilon$. Moreover, we want to take the metric structure of $\mathbb{R}^n$ into account: the homeomorphism should be a lipeomorphism, and we want to control locally how far the homeomorphism moves the boundary into the domain. Specifically, we prove the following result.

Theorem VII.2.1.
Let $\Upsilon \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain. Then there exist constants $\delta_D > 0$ and $L_D \geq 1$, depending only on Υ, with the following significance. For any non-negative function $\varrho : \mathbb{R}^n \to \mathbb{R}_0^+$ satisfying

$$\mathrm{Lip}(\varrho, \mathbb{R}^n) < \delta_D, \quad \varrho_{\max}(\mathbb{R}^n) < \delta_D, \tag{VII.20}$$

there exists a bi-Lipschitz mapping $\mathfrak{D}_\varrho : \mathbb{R}^n \to \mathbb{R}^n$ with the following properties. We have

$$\operatorname{Lip}(\mathfrak{D}_\varrho) \leq L_D (1 + \operatorname{Lip}(\varrho)), \quad \operatorname{Lip}\left(\mathfrak{D}_\varrho^{-1}\right) \leq L_D (1 + \operatorname{Lip}(\varrho)). \tag{VII.21a}$$

We have

$$\mathfrak{D}_\varrho(\Upsilon) \subseteq \Upsilon. \tag{VII.21b}$$

For all $x \in \mathbb{R}^n$ we have

$$\|x - \mathfrak{D}_\varrho(x)\| \leq L_D \varrho(x). \tag{VII.21c}$$

For every $x \in \mathbb{R}^n$ we have $x = \mathfrak{D}_\varrho(x)$ if

$$\operatorname{dist}(x, \partial\Upsilon) \geq L_D \varrho(x). \tag{VII.21d}$$

For all $x \in \partial\Upsilon$ we have

$$\mathfrak{D}_\varrho\left(B_{\varrho(x)/L_D}(x)\right) \subseteq \Upsilon. \tag{VII.21e}$$

Remark VII.2.2.
We discuss the meaning and application of Theorem VII.2.1 before we give the proof. The mapping $\mathfrak{D}_\varrho$ is a distortion of $\mathbb{R}^n$ which moves Υ into itself. The function ϱ controls the amount of distortion near Υ. The distortion $\mathfrak{D}_\varrho$ contracts a neighborhood of Υ into the domain.

Specifically, we interpret the properties (VII.21) in the following manner. Property (VII.21b) formalizes that the distortion moves $\partial\Upsilon$ into Υ; in particular, Υ is mapped into itself. Property (VII.21d) formalizes that the homeomorphism is the identity outside of a neighborhood of $\partial\Upsilon$. By Property (VII.21c) the amount of distortion is locally bounded by ϱ, and Property (VII.21e) formalizes that the distortion is proportional to ϱ near the boundary.

Remark VII.2.3.
In our application, Υ will be the bulge attached to the domain Ω, as introduced in the previous section. Moreover we will set $\varrho := \epsilon\rho$, where $\rho : \mathbb{R}^n \to \mathbb{R}_0^+$ will be a non-negative Lipschitz continuous function that indicates the local mesh size of a triangulation of Ω, and where $\epsilon > 0$ is a parameter to be chosen so small that the conditions of Theorem VII.2.1 are satisfied. If a differential form vanishes over Υ, then the pullback along $\mathfrak{D}_\varrho$ will vanish in a neighborhood of Υ.

Proof of Theorem VII.2.1. Since $\Upsilon \subseteq \mathbb{R}^n$ is a bounded weakly Lipschitz domain, we may apply Theorem VI.1.8 to deduce the existence of a LIP embedding

$$\Xi : \partial\Upsilon \times [-1, 1] \to \mathbb{R}^n$$

such that $\Xi(x, 0) = x$ for $x \in \partial\Upsilon$ and such that $\Xi(\partial\Upsilon, [0, 1]) \subset \overline{\Upsilon}$. In particular, there exist constants $c_\Xi, C_\Xi > 0$ such that

$$\|\Xi(x_1, t_1) - \Xi(x_2, t_2)\| \leq C_\Xi \sqrt{\|x_1 - x_2\|^2 + |t_1 - t_2|^2},$$
$$\sqrt{\|x_1 - x_2\|^2 + |t_1 - t_2|^2} \leq c_\Xi \|\Xi(x_1, t_1) - \Xi(x_2, t_2)\|$$

for $x_1, x_2 \in \partial\Upsilon$ and $t_1, t_2 \in [-1,1]$. We note in particular that

$$c_\Xi^{-1} \operatorname{Lip}(\varrho) \leq \operatorname{Lip}(\varrho\Xi) \leq C_\Xi \operatorname{Lip}(\varrho).$$

For $\alpha \in [0, 1/5]$ we consider the parametrized mappings

$$\begin{aligned} \zeta_\alpha &: [-1,1] \to [-1,1], \quad t \mapsto \int_{-1}^{t} 1 + \chi_{[-2\alpha,\alpha]} - \frac{2}{3}\chi_{[\alpha,3\alpha]} \,\mathrm{d}\lambda - 1, \\ \zeta_\alpha^{-1} &: [-1,1] \to [-1,1], \quad t \mapsto \int_{-1}^{s} 1 - \frac{2}{3}\chi_{[-2\alpha,\alpha]} + 2\chi_{[3\alpha,4\alpha]} \,\mathrm{d}\lambda - 1, \end{aligned}$$

where χ_I denotes the indicator function of the interval $I \subseteq [-1,1]$. As the notation already suggests, these two mappings are mutually inverse for α fixed. We easily see that they are strictly monotonically increasing, and that their Lipschitz constants are uniformly bounded for $\alpha \in [0, 1/5]$. In particular ζ_α and ζ_α^{-1} are bi-Lipschitz. Moreover, for $\alpha \in [0, 1/5]$ we observe that

$$\zeta_\alpha(t) = \zeta_\alpha^{-1}(t) = t, \quad t \notin [-2\alpha, 4\alpha], \tag{VII.22}$$

$$\zeta_\alpha([-\alpha,\alpha]) = [\alpha, 3\alpha]. \tag{VII.23}$$

We now write $\zeta(t;\alpha) = \zeta_\alpha(t)$ and $\zeta^{-1}(t;\alpha) = \zeta_\alpha^{-1}(t)$ for $(t,\alpha) \in [-1,1] \times [0, 1/5]$. Assume from now on that

$$\varrho_{\max}(\mathbb{R}^n) < 1/5, \quad \operatorname{Lip}(\varrho, \mathbb{R}^n) < \min\left\{1, \operatorname{Lip}(\Xi)^{-1}\right\}. \tag{VII.24}$$

The latter implies that $\operatorname{Lip}(\varrho\Xi) < 1$. We define homeomorphisms

$$\mathfrak{D}_\varrho : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathfrak{D}_\varrho^{-1} : \mathbb{R}^n \to \mathbb{R}^n,$$

in the following manner. Assume that $x \in \mathbb{R}^n$. If there exist $x_0 \in \partial\Upsilon$ and $t \in [-1,1]$ such that $x = \Xi(x_0, t)$, then we set

$$\mathfrak{D}_\varrho(x) := \Xi(x_0, t'), \quad t' := \zeta\left(t; \frac{\varrho(x_0)}{8}\right), \tag{VII.25}$$

$$\mathfrak{D}_\varrho^{-1}(x) := \Xi(x_0, t''), \quad t'' := \zeta^{-1}\left(t; \frac{\varrho(x_0)}{8}\right). \tag{VII.26}$$

Otherwise, we set $\mathfrak{D}_\varrho(x) := x$. It follows from the construction that $\mathfrak{D}_\varrho$ and $\mathfrak{D}_\varrho^{-1}$ are bi-Lipschitz and mutually inverse. In particular, (VII.21a) is implied by

$$\begin{aligned} \operatorname{Lip}(\mathfrak{D}_\varrho) &\leq 1 + \frac{C_\Xi}{c_\xi}\left(1 + \operatorname{Lip}(\zeta)\operatorname{Lip}(\varrho)\right), \\ \operatorname{Lip}\left(\mathfrak{D}_\varrho^{-1}\right) &\leq 1 + \frac{C_\Xi}{c_\xi}\left(1 + \operatorname{Lip}\left(\zeta^{-1}\right)\operatorname{Lip}(\varrho)\right). \end{aligned}$$

From the construction we immediately see that (VII.21b) holds, since $\mathfrak{D}_\varrho$ maps $\Xi(\partial\Upsilon, [0,1])$ into itself. Moreover, $\mathfrak{D}_\varrho$ and $\mathfrak{D}_\varrho^{-1}$ act like the identity outside of $\Xi(\partial\Upsilon, [-1,1])$.

Let us assume for the remainder of this proof that $x = \Xi(x_0, t)$ for $x_0 \in \partial\Upsilon$ and $t \in [-1, 1]$. Using (VII.22) and (VII.25), we see that $\mathfrak{D}_\varrho(x) \neq x$ implies $|t| \leq \varrho(x_0)/2$. Thus

$$|\varrho(x_0) - \varrho(x)| = |\varrho(\Xi(x_0, 0)) - \varrho(\Xi(x_0, t))| \leq \frac{\operatorname{Lip}(\varrho\Xi)\varrho(x_0)}{2} \leq \frac{\varrho(x_0)}{2}.$$

This shows that $\varrho(x_0) \leq 2\varrho(x)$. By the definition of ζ and $\mathfrak{D}_\varrho$ we then see

$$\|x - \mathfrak{D}_\varrho(x)\| \leq \operatorname{Lip}(\Xi) \left| t - \zeta\left(t; \frac{\varrho(x_0)}{8}\right) \right| \leq \frac{6}{8} \operatorname{Lip}(\Xi)\varrho(x_0) \leq \frac{3}{2} \operatorname{Lip}(\Xi)\varrho(x),$$

proving (VII.21c). Furthermore, using (VII.22) we note that $x \neq \mathfrak{D}_\varrho(x)$ implies

$$\operatorname{dist}(x, \partial\Upsilon) \leq \|x - x_0\| \leq \operatorname{Lip}(\Xi)|t| \leq \frac{\operatorname{Lip}(\Xi)}{2}\varrho(x_0) \leq \operatorname{Lip}(\Xi)\varrho(x).$$

Conversely, this means that $x = \mathfrak{D}_\varrho(x)$ is implied by

$$\operatorname{dist}(x, \partial\Upsilon) \geq \operatorname{Lip}(\Xi)\varrho(x),$$

which proves (VII.21d).

It remains to prove (VII.21e). Let $x_0 \in \partial\Upsilon$ and define $A \subseteq \partial\Upsilon \times [-1, 1]$ by

$$A := \left(B_{\varrho(x_0)/8}(x_0) \cap \partial\Upsilon\right) \times \left(-\frac{7}{64}\varrho(x_0), \frac{7}{64}\varrho(x_0)\right).$$

If $y \in \partial\Upsilon$ with $|x_0 - y| \leq \varrho(x_0)/8$, then $|\varrho(x_0) - \varrho(y)| \leq \varrho(x_0)/8$ since we assume $\operatorname{Lip}(\varrho) < 1$. In particular, $\varrho(y) \geq 7\varrho(x_0)/8$ follows. Via (VII.22) we thus find $\mathfrak{D}_\varrho(\Xi(A)) \subseteq \Upsilon$. Furthermore, we observe that A contains a ball around x_0 of radius $7\varrho(x_0)/64$ in $\partial\Upsilon \times [-1, 1]$. Hence $\Xi(A)$ contains a ball around x_0 of radius $7c_\Xi^{-1}\varrho(x_0)/64$. This shows (VII.21e), and completes the proof. □

VII.3. Smoothing Operators

In this section we construct a smoothing operator for differential forms on weakly Lipschitz domains. We define the smoothed differential form at each point by averaging the original differential form in a small neighborhood of that point. A technical difference to the classical mollification operator is that we let the mollification radius vary across the domain.

We let $\varrho : \Omega^e \to \mathbb{R}^+$ be a non-negative smooth function that assumes a positive minimum over $\overline{\Omega}$. We introduce the mapping

$$\Phi_\varrho : \overline{\Omega} \times B_1(0) \to \mathbb{R}^n, \quad (x, y) \mapsto x + \varrho(x)y. \tag{VII.27}$$

When we regard the second variable as a parameter, then we have a family of mappings

$$\Phi_{\varrho,y} : \overline{\Omega} \to \mathbb{R}^n, \quad x \mapsto \Phi_\varrho(x, y).$$

Here and in the sequel, we let $B_\varrho(A)$ for any $A \subseteq \Omega^e$ be defined as the union of the balls $B_{\varrho(x)}(x)$ for $x \in A$.

We study some properties of $\Phi_{\varrho,y}$. When $y \in B_1(0)$ and $x_1, x_2 \in \Omega$, then

$$\|\Phi_{\varrho,y}(x_1) - \Phi_{\varrho,y}(x_2)\| \leq (1 + \operatorname{Lip}(\varrho)) \|x_1 - x_2\| . \tag{VII.28}$$

Moreover, for any $y \in B_1(0)$ and $x \in \Omega$ we have

$$\|\Phi_{\varrho,y}(x) - x\| \leq \varrho(x). \tag{VII.29}$$

The latter inequality implies that for ϱ small enough we have

$$\Phi_{\varrho,y}\left(\overline{\Omega}\right) \subseteq \Omega^e. \tag{VII.30}$$

Under the additional condition that $\operatorname{Lip}(\varrho) < 1/2$, we observe for $y \in B_1(0)$ and $x_1, x_2 \in \overline{\Omega}$ that

$$\begin{aligned} \|\Phi_{\varrho,y}(x_1) - \Phi_{\varrho,y}(x_1)\| &= \|x_1 - x_2 + (\varrho(x_1) - \varrho(x_2))\, y\| \\ &\geq \Big| \|x_1 - x_2\| - \operatorname{Lip}(\varrho) \|x_1 - x_2\| \Big| \geq \frac{1}{2} \|x_1 - x_2\| . \end{aligned} \tag{VII.31}$$

We conclude that for ϱ and $\operatorname{Lip}(\varrho)$ small enough, the mapping

$$\Phi_{\varrho,y} : \overline{\Omega} \to \Omega^e$$

is a LIP embedding for every $y \in B_1(0)$.

Remark VII.3.1.
Similarly as in Remark VII.2.3, we have $\varrho = \epsilon\rho$ in applications, where ρ is a fixed smooth function with positive minimum over $\overline{\Omega}$ and $\epsilon > 0$ is a parameter to be chosen small enough.

The smoothing operator in this section uses the standard mollifier μ as a building block and can be seen as a generalization of the classical smoothing by convolution. For every $u \in L^1\Lambda^k(\Omega^e)$ we define

$$R^k_\varrho u_{|x} := \int_{\mathbb{R}^n} \mu(y)(\Phi^*_{\varrho,y} u)_{|x} \mathrm{d}y, \quad x \in \overline{\Omega}. \tag{VII.32}$$

We first show that R^k_ϱ maps into $C^\infty\Lambda^k(\overline{\Omega})$ and satisfies a local bound. In particular, it is a bounded mapping into $C\Lambda^k(\overline{\Omega})$ with respect to the maximum norm.

Lemma VII.3.2.
Suppose that we have LIP embeddings $\Phi_{\varrho,y} : \overline{\Omega} \to \Omega^e$ for all $y \in B_1(0)$. The operator

$$R^k_\varrho : L^p\Lambda^k(\Omega^e) \to C^\infty\Lambda^k(\overline{\Omega}), \quad p \in [1, \infty],$$

is well-defined and linear. Moreover, for every $p \in [1, \infty]$, $u \in L^p\Lambda^k(\Omega^e)$, and measurable set $A \subseteq \overline{\Omega}$ we have

$$\|R^k_\varrho u\|_{C\Lambda^k(A)} \leq \operatorname{vol}^n(B_1(0)) \frac{(1 + \operatorname{Lip}(\varrho))^k}{\varrho_{\inf}(A)^{\frac{n}{p}}} \|u\|_{L^p\Lambda^k(\Phi_\varrho(A, B_1))}. \tag{VII.33}$$

Proof. Let $p \in [1,\infty]$ and let $u \in L^p\Lambda^k(\Omega^e)$. Under the assumptions on ϱ we have a LIP embedding $\Phi_{\varrho,y} : \overline{\Omega} \to \Omega^e$ for every $y \in B_1(0)$. Hence $\mu(y)(\Phi^*_{\varrho,y}u)_{|x}$ is measurable in y for every $x \in \overline{\Omega}$, and the integral (VII.32) is well-defined. Using elementary results, we find for every $x \in \overline{\Omega}$ that

$$\begin{aligned}
\left|R^k_\varrho u\right|_{|x} &\leq \int_{\mathbb{R}^n} \mu(y) \left\|\mathrm{D}_x \Phi_{\varrho,y}\right\|^k_{2,2|x} |u|_{\Phi_{\varrho,y}(x)} \\
&\leq \operatorname{Lip}(\Phi_{\varrho,y}, \overline{\Omega})^k \int_{\mathbb{R}^n} \mu(y) |u|_{\Phi_{\varrho,y}(x)} \\
&\leq (1 + \operatorname{Lip}(\varrho))^k \int_{\mathbb{R}^n} \mu(y) |u|_{\Phi_{\varrho,y}(x)}.
\end{aligned}$$

A substitution of variables and Hölder's inequality give

$$\int_{\mathbb{R}^n} \mu(y)|u|_{x+\varrho(x)y} \,\mathrm{d}y = \int_{\mathbb{R}^n} \frac{\mu(\varrho(x)^{-1}y)}{\varrho(x)^n} \left|u\left(x+y\right)\right| \mathrm{d}y \leq \frac{\operatorname{vol}^n(B_1(0))}{\varrho(x)^{\frac{n}{p}}} \|u\|_{L^p\left(B_{\varrho(x)}(x)\right)}.$$

These estimates in combination yield (VII.33). In order to prove the smoothness of $R^k_\varrho u$ over Ω, we first change the form of the integral. By a substitution of variables we find for $x \in \Omega^e$ that

$$\begin{aligned}
R^k_\varrho u_{|x} &= \sum_{\sigma \in \Sigma(1:k,0:n)} \int_{\mathbb{R}^n} \mu(y) u_\sigma\left(x + \varrho(x)y\right) \left(\Phi^*_{\varrho,y} \mathsf{d}x^\sigma\right)_{|x} \mathrm{d}y \\
&= \sum_{\sigma \in \Sigma(1:k,0:n)} \int_{\mathbb{R}^n} \mu_{\varrho(x)}\left(y - x\right) u_\sigma(y) \left(\Phi^*_{\varrho,\varrho(x)^{-1}(y-x)} \mathsf{d}x^\sigma\right)_{|x} \mathrm{d}y.
\end{aligned}$$

We know that $u_\sigma \in L^1(\Omega)$, that ϱ and Φ_ϱ are smooth, and that $\overline{\Omega}$ is compact. The desired smoothness of $R^k_\varrho u$ over Ω is now a simple consequence of the dominated convergence theorem. Furthermore, $R^k_\varrho u \in C^\infty\Lambda^k(\overline{\Omega})$, as can easily be seen when picking x in a sufficiently small open neighborhood of $\overline{\Omega}$. □

Lemma VII.3.3.
Suppose that $\Phi_{\varrho,y} : \overline{\Omega} \to \Omega^e$ is a LIP embedding for all $y \in B_1(0)$. We then have

$$\mathsf{d}^k R^k_\varrho u = R^{k+1}_\varrho \mathsf{d}^k u, \quad u \in W^{p,q}\Lambda^k(\Omega^e), \quad p,q \in [1,\infty].$$

Proof. Let $v \in C^\infty_c\Lambda^{n-k-1}(\Omega)$. By Fubini's theorem we have

$$\begin{aligned}
\int_\Omega R^k_\varrho u \wedge \mathsf{d}^{n-k-1} v &= \int_\Omega \int_{\mathbb{R}^n} \mu(y) \Phi^*_{\varrho,y} u \,\mathrm{d}y \wedge \mathsf{d}^{n-k-1} v \\
&= \int_\Omega \int_{\mathbb{R}^n} \mu(y) \Phi^*_{\varrho,y} u \wedge \mathsf{d}^{n-k-1} v \,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \mu(y) \int_\Omega \Phi^*_{\varrho,y} u \wedge \mathsf{d}^{n-k-1} v \,\mathrm{d}y,
\end{aligned}$$

and similarly

$$\int_\Omega R^k_\varrho \mathsf{d}^k u \wedge v = \int_\Omega \int_{\mathbb{R}^n} \mu(y) \Phi^*_{\varrho,y} \mathsf{d}^k u \,\mathrm{d}y \wedge v = \int_{\mathbb{R}^n} \mu(y) \int_\Omega \Phi^*_{\varrho,y} \mathsf{d}^k u \wedge v \,\mathrm{d}y.$$

Since $\Phi_{\varrho,y} : \Omega \to \Omega^e$ is a LIP embedding for every $y \in B_1(0)$, we obtain

$$\int_\Omega \Phi^*_{\varrho,y} u \wedge \mathsf{d}^{n-k-1} v = (-1)^{k+1} \int_\Omega \mathsf{d}^k \Phi^*_{\varrho,y} u \wedge v = (-1)^{k+1} \int_\Omega \Phi^*_{\varrho,y} \mathsf{d}^k u \wedge v.$$

By definition, $\mathsf{d}^k R^k_\varrho u = R^{k+1}_\varrho \mathsf{d}^k u$. The proof is complete. □

VII.4. Commuting Approximation

We are now in the position to combine the extension operator of Section VII.1, the distortion operator of Section VII.2, and smoothing operator of the preceding section. We let $\varrho \in C^\infty(\Omega^e)$ be a smooth function whose exact properties we will stipulate soon and let $\delta > 0$ be a small parameter to be determined below. We then define

$$M^k_\varrho : L^p\Lambda^k(\Omega) \to C^\infty\Lambda^k(\Omega), \quad u \mapsto R^k_{\delta\varrho} \mathfrak{D}^*_\varrho E^k u, \quad p \in [1,\infty]. \tag{VII.34}$$

The properties of M^k_ϱ are summarized as follows.

Theorem VII.4.1.
Assume that ϱ satisfies the conditions of Lemma VII.3.2 and Theorem VII.2.1 applied to Υ, and that $\mathfrak{D}_\varrho$ maps Ω^e into itself. Assume also that $\delta \in (0,1)$ with $\delta^{-1} > 2L_D$. Then M^k_ϱ is well-defined. Moreover, there exist $C^M_{n,k,p} > 0$ and $L_M > 0$, not depending on ϱ, such that for all measurable $A \subseteq \overline{\Omega}$ we have

$$\|M^k_\varrho u\|_{C\Lambda^k(A)} \leq C^M_{n,k,p} \frac{(1+\operatorname{Lip}(\varrho))^{k+\frac{n}{p}}}{\varrho_{\inf}(A)^{\frac{n}{p}}} \|u\|_{L^p\Lambda^k(B_{L_M(1+\operatorname{Lip}(\varrho))\varrho_{\sup}(A)}(A)\cap\Omega)}. \tag{VII.35}$$

Additionally, if $p,q \in [1,\infty]$ and $u \in W^{p,q}\Lambda^k(\Omega,\Gamma_T)$ then

$$M^{k+1}_\varrho \mathsf{d}^k u = \mathsf{d}^k M^k_\varrho u, \tag{VII.36}$$

and $M^k_\varrho u$ vanishes in a neighborhood of Γ_T.

Proof. We combine Theorem VII.1.7, Theorem VII.2.1 together with (V.21), and Theorem VII.3.2. We then have a linear mapping

$$R^k_{\delta\varrho} \mathfrak{D}^*_\varrho E^k : L^p\Lambda^k(\Omega) \to C^\infty\Lambda^k(\overline{\Omega}), \quad p \in [1,\infty].$$

By the same token we immediately deduce (VII.36).

Next we prove (VII.35). Assume that $u \in L^p\Lambda^k(\Omega)$ and that $A \subseteq \Omega$ is measurable. Via Theorem VII.3.2 we find

$$\|R^k_{\delta\varrho} \mathfrak{D}^*_\varrho E^k u\|_{C\Lambda^k(A)} \leq \operatorname{vol}^n(B_1(0)) \frac{(1+\operatorname{Lip}(\varrho))^k}{(\delta\varrho_{\inf}(A))^{\frac{n}{p}}} \|\mathfrak{D}^*_\varrho E^k u\|_{L^p\Lambda^k\left(\Phi_{\delta\varrho}(A,B_1)\right)},$$

By Theorem VII.2.1 and Lemma V.2.3 we have

$$\|\mathfrak{D}^*_\varrho E^k u\|_{L^p\Lambda^k\left(\Phi_{\delta\varrho}(A,B_1)\right)} \leq L_D^{k+\frac{n}{p}} (1+\operatorname{Lip}(\varrho))^{k+\frac{n}{p}} \|E^k u\|_{L^p\Lambda^k\left(\mathfrak{D}_\varrho\Phi_{\delta\varrho}(A,B_1)\right)}.$$

For $x \in A$ and $y \in B_{\delta\varrho(x)}(x)$ we find

$$|\varrho(y) - \varrho(x)| \leq \operatorname{Lip}(\varrho)\|y - x\| \leq \delta \operatorname{Lip}(\varrho)\varrho(x).$$

By (VII.21c) we then observe

$$\|\mathfrak{D}_\varrho(y) - y\| \leq L_D \varrho(y) \leq L_D \left(\varrho(x) + \delta \operatorname{Lip}(\varrho)\varrho(x)\right).$$

Consequently,

$$\mathfrak{D}_\varrho \Phi_{\delta\varrho}(A, B_1) \subseteq \Phi_{L_D(1+\delta \operatorname{Lip}(\varrho))\varrho}(A, B_1) \cap \Omega^e \subseteq B_{L_D(1+\delta \operatorname{Lip}(\varrho))\varrho_{\sup}(A)}(A) \cap \Omega^e.$$

Applying Lemma VII.1.8, we thus obtain

$$\|E^k u\|_{L^p\Lambda^k\left(\mathfrak{D}_\varrho \Phi_{\delta\varrho}(A,B_1)\right)} \leq \left(1 + C_b^{k+\frac{n}{p}}\right) \|u\|_{L^p\Lambda^k\left(B_{L_E L_D(1+\delta \operatorname{Lip}(\varrho))\varrho_{\sup}(A)}(A)\cap\Omega\right)}.$$

This provides the desired local estimate.

We consider the special case $A = B_{\operatorname{Lip}(\varrho)^{-1}\varrho(x)}(x) \cap \overline{\Omega}$ for $x \in \Gamma_T$. We have

$$\delta\varrho(y) \leq \delta\varrho(x) + \delta \operatorname{Lip}(\varrho)\|y - x\| \leq 2\delta\varrho(x)$$

for all $y \in A$, and thus

$$B_{\delta\varrho_{\sup}(A)}(A) \subseteq B_{2\delta\varrho(x)}(x).$$

In particular

$$\|R^k_{\delta\varrho}\mathfrak{D}^*_\varrho E^k u\|_{C\Lambda^k(A)} \leq \operatorname{vol}^n(B_1(0)) \frac{(1 + \delta \operatorname{Lip}(\varrho))^k}{(\delta\varrho_{\inf}(A))^{\frac{n}{p}}} \|\mathfrak{D}^*_\varrho E^k u\|_{L^p\Lambda^k\left(B_{2\delta\varrho(x)}(x)\right)}$$

and

$$\|\mathfrak{D}^*_\varrho E^k u\|_{L^p\Lambda^k\left(B_{2\delta\varrho(x)}(x)\right)} \leq L_D^{k+\frac{n}{p}} (1 + \delta \operatorname{Lip}(\varrho))^{k+\frac{n}{p}} \|E^k u\|_{L^p\Lambda^k\left(\mathfrak{D}_\varrho B_{2\delta\varrho(x)}(x)\right)}.$$

We now assume $2\delta < 1/L_D$. In combination with Theorem VII.2.1 we conclude

$$\mathfrak{D}_\varrho B_{2\delta\varrho(x)}(x) \subseteq \Upsilon.$$

Hence $M^k_\varrho u$ vanishes in an open neighborhood of Γ_T in $\overline{\Omega}$. The proof is complete. □

Remark VII.4.2.
The proof of Theorem VII.4.1 shows that $L_M \leq L_D L_E$ and that

$$C^M_{n,k,p} \leq \operatorname{vol}^n(B_1(0))\delta^{-\frac{n}{p}} L_D^{k+\frac{n}{p}} \left(1 + C_b^{k+\frac{n}{p}}\right).$$

In the remainder of this chapter, we instantiate M^k_ϱ with a specific choice of ϱ for an application in finite element exterior calculus, where ϱ relates to the local mesh size of a triangulation. But another specific choice of ϱ, namely choosing it constant near Ω, enables a new density result for Sobolev spaces of differential forms over weakly Lipschitz domains, which is of general interest for functional analysis.

Theorem VII.4.3.
Let Ω be a bounded weakly Lipschitz domain and let Γ_T be an admissible boundary patch. Then the smooth differential k-forms in $C^\infty\Lambda^k(\overline{\Omega})$ that vanish near Γ_T constitute a dense subset of $W^{p,q}\Lambda^k(\Omega,\Gamma_T)$ for all $p,q\in[1,\infty)$.

Proof. Let $p,q\in[1,\infty)$ and $u\in W^{p,q}\Lambda^k(\Omega,\Gamma_T)$. We let $\chi:\Omega^e\to\mathbb{R}$ be a non-negative smooth function with compact support that equals 1 in an open neighborhood of Ω in Ω^e. For $\epsilon>0$ small enough, Theorem VII.4.1 provides an operator $M^k_{\epsilon\chi}:L^p\Lambda^k(\Omega)\to C^\infty\Lambda^k(\Omega)$. We define

$$Y_\epsilon := \Omega\cap B_{L_D\epsilon}(\partial\Omega), \quad Z_\epsilon := \Omega\setminus Y_\epsilon.$$

On the one hand,

$$\left\|u-M^k_{\epsilon\chi}u\right\|_{L^p\Lambda^k(Y_\epsilon)} \leq \|u\|_{L^p\Lambda^k(Y_\epsilon)} + \left\|M^k_{\epsilon\chi}u\right\|_{L^p\Lambda^k(Y_\epsilon)}.$$

For ϵ small enough, we use Young's inequality for convolutions (see [27, Theorem 3.9.4] again), Lemma V.2.3, and Lemma VII.1.8 to see

$$\begin{aligned}\left\|M^k_{\epsilon\chi}u\right\|_{L^p\Lambda^k(Y_\epsilon)} &\leq \left\|\mathfrak{D}^*_{\epsilon\chi}E^k u\right\|_{L^p\Lambda^k(Y_{2\epsilon})}\\ &\leq 2L_D^{k+\frac{n}{p}}\left\|E^k u\right\|_{L^p\Lambda^k(Y_{3\epsilon})} \leq 2L_D^{k+\frac{n}{p}}\left(1+C_b^{k+\frac{n}{p}}\right)\|u\|_{L^p\Lambda^k(Y_{L_E3\epsilon}\cap\Omega)}.\end{aligned}$$

This is again a bound in terms of an integral over a neighborhood of $\partial\Omega$. There exists $C>0$ such that for ϵ small enough

$$Y_{L_E3\epsilon}\cap\Omega\subset\Psi_0\left(\partial\Omega\times[-CL_E3\epsilon,0]\right).$$

The volume of the latter set is bounded by

$$\operatorname{vol}^n\Big(\Psi_0\left(\partial\Omega\times[-CL_E3\epsilon,0]\right)\Big)\leq CL_E\operatorname{Lip}(\Psi_0)^n\operatorname{vol}^{n-1}(\partial\Omega)\cdot 3\epsilon$$

and thus converges to zero as ϵ converges to zero. We conclude

$$\lim_{\epsilon\to 0}\left\|u-M^k_{\epsilon\chi}u\right\|_{W^{p,q}\Lambda^k(Y_\epsilon)}=0.$$

On the other hand, we have

$$\left\|u-M^k_{\epsilon\chi}u\right\|_{L^p\Lambda^k(Z_\epsilon)} = \|u-\mu_{\delta\epsilon}\star u\|_{L^p\Lambda^k(Z_\epsilon)} \leq \|u-\mu_{\delta\epsilon}\star u\|_{L^p\Lambda^k(\Omega)}.$$

By basic results on mollifications, the last expression converges to zero as ϵ converges to zero. Since $\Omega=Y_\epsilon\cup Z_\epsilon$ for all $\epsilon>0$, and since $M^{k+1}_{\epsilon\chi}\mathsf{d}^k u=\mathsf{d}^k M^k_{\epsilon\chi}u$, the combination of both observations provides

$$\lim_{\epsilon\to 0}\left\|u-M^k_{\epsilon\chi}u\right\|_{W^{p,q}\Lambda^k(\Omega)}=0.$$

We observe that $M^k_{\epsilon\chi}u\in C^\infty\Lambda^k(\overline{\Omega})$ with support having a positive distance from Γ_T for all $\epsilon>0$. The proof is complete. □

Remark VII.4.4.
The preceding approximation theorem generalizes Theorem V.3.5 in the case of weakly Lipschitz domains. The smooth differential k-forms over Ω that are contained in $W^{p,q}\Lambda^k(\Omega)$ and vanish in an open neighborhood of Γ_T are a dense subset of the space $W^{p,q}\Lambda^k(\Omega,\Gamma_T)$. This result apparently has not been available in the literature before. The idea of proving a commuting mollification operator, however, is inspired by previous works in global analysis [100].

VII.5. Elements of Geometric Measure Theory

Before we continue to develop the smoothed projection, we need to prepare definitions and results in geometric measure theory. This is non-standard material in the context of numerical analysis. Our exposition in this section is specifically targeted towards applications later in this chapter.

Whitney's monograph [180] is our main reference. Our motivation for studying geometric measure theory lies in proving Theorem VII.8.7 later in this chapter. The key observation is that finite element differential forms are flat differential forms, and that the degrees of freedom are flat chains (see Lemma VII.5.2). This allows us to estimate Lipschitz deformations of degrees of freedom (Lemma VII.5.4), which is of critical importance in the construction of the smoothed projection.

We begin with basic notions of chains and cochains in geometric measure theory, which can be found in Sections 1-3 of Chapter V in [180]. Throughout this section, we fix for each simplex $S \subseteq \mathbb{R}^n$ an orientation. We may identify each positively oriented simplex S with the indicator function $\chi_S : \mathbb{R}^n \to \mathbb{R}$. Let $k \in \mathbb{Z}$ be arbitrary. To each finite formal sum $\sum_i a_i S_i$ of (oriented) k-simplices with real coefficients we may associate the function $\sum_i a_i \chi_{S_i}$. We call two such finite formal sums $\sum_i a_i S_i$ and $\sum_j b_j T_j$ *equivalent*, and write $\sum_i a_i S_i \sim \sum_j b_j T_j$, if the associated functions $\sum_i a_i \chi_{S_i}$ and $\sum_j b_j \chi_{T_j}$ agree almost everywhere with respect to the k-dimensional Hausdorff measure.

The boundary $\partial_k S$ of a positively oriented k-simplex $S \subseteq \mathbb{R}^n$ is defined as

$$\partial_k S = \sum_{F \in \Delta(S)^{k-1}} o(F,S) F. \tag{VII.37}$$

By linear extension, $\partial_k \sum_i a_i S_i = \sum_i a_i \partial_k S_i$, which defines a linear operator on the finite formal sums of positively oriented k-simplices. Furthermore, it is apparent that this operation preserves the equivalence relation.

The space $\mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ of polyhedral k-chains in $\mathbb{R}^n$ is the vector space of finite real formal sums of positively oriented k-simplices with the equivalence relation factored out. If $S \in \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$, then we write $S \sim \sum_i a_i S_i$ if the latter formal sum represents S. We may identify a polyhedral k-chain $S \sim \sum_i a_i S_i$ in $\mathbb{R}^n$ by the function $\chi_S = \sum_i a_i \chi_{S_i}$ whenever convenient. The boundary operator (VII.37) gives rise to a linear mapping $\partial_k : \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n) \to \mathcal{C}_{k-1}^{\mathrm{pol}}(\mathbb{R}^n)$.

The *mass* $|S|_k$ of a polyhedral k-chain S in $\mathbb{R}^n$ is defined as the L^1 norm of the associated function χ_S with respect to the k-dimensional Hausdorff measure.[i] Hence, if $S \sim \sum_i a_i S_i$ with the simplices S_i being essentially disjoint with respect to the k-dimensional Hausdorff measure, then

$$|S|_k = \sum_i |a_i| \operatorname{vol}^k(S_i).$$

It is easy to see that $|\cdot|_k$ is a norm on the polyhedral chains, called *mass norm*. We write $\mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ for the Banach space that results by taking the completion of the polyhedral chains with respect to the mass norm.

[i]We assume the convention that the k-dimensional Hausdorff volume of a k-simplex S equals its k-dimensional volume $\operatorname{vol}^k(S)$.

The *flat norm* $\|S\|_{k,\flat}$ of a polyhedral k-chain $S \in \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ is defined as

$$\|S\|_{k,\flat} := \inf_{Q \in \mathcal{C}_{k+1}^{\mathrm{pol}}(\mathbb{R}^n)} \Big(|S - \partial_k Q|_k + |Q|_{k+1} \Big). \tag{VII.38}$$

As the name already suggest, one can show that $\|\cdot\|_{k,\flat}$ is a norm on the polyhedral chains. The Banach space $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$ is defined as the completion of $\mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ with respect to the flat norm. It is apparent from the definition that

$$\|S\|_{k,\flat} \leq |S|_k, \quad S \in \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n).$$

In particular, $\mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ is densely embedded in $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$.

We show that the boundary operator is bounded with respect to the flat norm. To see this, let $S \in \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ let $\epsilon > 0$, and let $Q \in \mathcal{C}_{k+1}^{\mathrm{pol}}(\mathbb{R}^n)$ such that $|S - \partial_{k+1} Q|_k + |Q|_{k+1} \leq \|S\|_{k,\flat} + \epsilon$. We then observe that

$$\begin{aligned} \|\partial_k S\|_{k-1,\flat} &\leq |\partial_k S - \partial_k (S - \partial_{k+1} Q)|_{k-1} + |S - \partial_{k+1} Q|_k \\ &\leq |S - \partial_{k+1} Q|_k \\ &\leq \|S\|_{k,\flat} + \epsilon. \end{aligned}$$

By taking ϵ to zero in the limit, we have $\|\partial_k S\|_{k-1,\flat} \leq \|S\|_{k,\flat}$. Using the density of $\mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ in $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$, we find

$$\|\partial_k \alpha\|_{k-1,\flat} \leq \|\alpha\|_{k,\flat}, \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^n).$$

We remark that the boundary operator is generally not bounded with respect to the mass norm. This can be seen by shrinking a single simplex: the surface measure scales differently than the volume.

Remark VII.5.1.
The space $\mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ is a subspace of the Banach space of functions over $\mathbb{R}^n$ integrable with respect to the k-dimensional Hausdorff measure. The members of $\mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ play a similar role as the simple functions in the theory of the Lebesgue measure. The Banach space $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$ can be motivated by the following example: for $r > 0$ small, consider the two opposing longer sides of the rectangle $[0, r] \times [0, 1]$. The mass norm of these two edges is 2 regardless of $r > 0$. But in the flat norm for r small enough, their norm is r, corresponding to area of the original rectangle. In this sense, the flat norm takes into account the distance between simplices.

The chains in the space $\mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ are the most important ones in this chapter. We discuss the space $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$ to utilize some technical tools in geometric measure theory that are stated for flat chains in the literature.

The Banach space $\mathcal{C}_k^{\flat}(\mathbb{R}^n)$ of flat chains has a dual space, which is called the Banach space of *flat cochains*. The space of flat cochains can be represented by a class of differential forms: to every cochain we associate a differential form such that evaluating the cochain on a simplex is equal to integrating the associated differential form over that simplex. This is another instance of a recurrent idea throughout differential geometry. Specifically, the space of flat cochains can be represented by the space of flat differential forms. Flat forms were studied in Whitney's book [180], there mainly as representations of flat cochains, and in functional analysis (see [100]). For the following facts, we refer to Section 2 of [100] and Chapters IX and X of Whitney's book [180].

Flat differential forms have well-defined traces on simplices. More precisely, for each m-simplex $S \subset \mathbb{R}^n$ there exists a bounded linear mapping

$$\operatorname{tr}_S^k : W^{\infty,\infty}\Lambda^k(\mathbb{R}^n) \to W^{\infty,\infty}\Lambda^k(S),$$

which extends the trace of smooth forms. In particular, for $u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$ the trace $\operatorname{tr}_S^k u$ depends only on the values of u near S. We write

$$\int_S u := \int_S \operatorname{tr}_S^k u$$

for the integral of $u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$ over a k-simplex S. This induces a bilinear pairing between $\mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$ and $W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$. We have

$$\left| \int_S u \right| \leq |S|_k \|u\|_{W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)}, \quad S \in \mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n), \quad u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n). \tag{VII.39}$$

This pairing furthermore extends to flat chains. We have

$$\left| \int_\alpha u \right| \leq \|\alpha\|_{k,\flat} \|u\|_{W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)}, \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^n), \quad u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n). \tag{VII.40}$$

The exterior derivative between spaces of flat forms is dual to the boundary operator between spaces of flat chains. We have

$$\int_{\partial_k \alpha} u = \int_\alpha \mathsf{d}^k u, \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^n), \quad u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n), \tag{VII.41}$$

as a generalized Stokes' theorem.

Many results in geometric measure theory are invariant under Lipschitz mappings. We recall some basic facts about pushforwards of chains and pullbacks of differential forms along Lipschitz mappings. Here we refer to Paragraph 7 in Chapter X of Whitney's book [180].

Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz mapping. Then there exists a mapping

$$\varphi_* : \mathcal{C}_k^{\flat}(\mathbb{R}^m) \to \mathcal{C}_k^{\flat}(\mathbb{R}^n), \tag{VII.42}$$

called the *pushforward* along φ, which commutes with the boundary operator,

$$\partial_k \varphi_* \alpha = \varphi_* \partial_k \alpha, \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^m), \tag{VII.43}$$

and which satisfies the norm estimates

$$\|\varphi_* \alpha\|_{k,\flat} \leq \max\left\{ \operatorname{Lip}(\varphi, \mathbb{R}^m)^k, \operatorname{Lip}(\varphi, \mathbb{R}^m)^{k+1} \right\} \|\alpha\|_{k,\flat}, \quad \alpha \in \mathcal{C}_k^{\flat}(\mathbb{R}^m), \tag{VII.44}$$

$$|\varphi_* S|_k \leq \operatorname{Lip}(\varphi, \mathbb{R}^m)^k |S|_k, \quad S \in \mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^m). \tag{VII.45}$$

The pushforward of chains is dual to the pullback of differential forms. We recall that this is a mapping

$$\varphi^* : W^{\infty,\infty}\Lambda^k(\mathbb{R}^n) \to W^{\infty,\infty}\Lambda^k(\mathbb{R}^m) \tag{VII.46}$$

which commutes with the exterior derivative,

$$\mathsf{d}^k \varphi^* u = \varphi^* \mathsf{d}^k u, \quad u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n), \tag{VII.47}$$

and satisfies the norm estimate

$$\|\varphi^* u\|_{L^\infty\Lambda^k(\mathbb{R}^m)} \leq \operatorname{Lip}(\varphi, \mathbb{R}^n)^k \|u\|_{L^\infty\Lambda^k(\mathbb{R}^n)}, \quad u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n). \tag{VII.48}$$

The pushforward and the pullback are related by the identity

$$\int_{\varphi_*\alpha} u = \int_\alpha \varphi^* u, \quad u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n), \quad \alpha \in \mathcal{C}^\flat_k(\mathbb{R}^m). \tag{VII.49}$$

Lastly, if $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ and $\psi : \mathbb{R}^l \to \mathbb{R}^m$ are Lipschitz mappings, then $\varphi\psi : \mathbb{R}^l \to \mathbb{R}^n$ is a Lipschitz mapping, and we have $(\varphi\psi)_* = \varphi_*\psi_*$ and $(\varphi\psi)^* = \psi^*\varphi^*$ over the spaces of chains and differential forms, respectively.

Having outlined basic concepts of geometric measure theory, we provide a new result which makes these notions interesting for finite element theory: the degrees of freedom in finite element exterior calculus are flat chains.

Lemma VII.5.2.
Let $F \subset \mathbb{R}^n$ be a closed oriented m-simplex and let $\eta \in C^\infty\Lambda^{m-k}(F)$. Then there exists a flat chain $\alpha(F,\eta) \in C^\flat_k(\mathbb{R}^n)$ such that for all $u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$ we have

$$\int_F \operatorname{tr}^k_F u \wedge \eta = \int_{\alpha(F,\eta)} u. \tag{VII.50}$$

Moreover, $\alpha(F,\eta) \in \mathcal{C}^{\text{mass}}_k(\mathbb{R}^n)$ and $\partial_k \alpha(F,\eta) \in \mathcal{C}^{\text{mass}}_{k-1}(\mathbb{R}^n)$.

Proof. We first assume that $\dim F = n$, and that F is positively oriented. We use Theorem 15A of [180, Chapter IX] to deduce the existence of $\alpha(F,\eta) \in C^\flat_k(\mathbb{R}^n)$ such that

$$\int_F \operatorname{tr}^k_F u \wedge \eta = \int_{\alpha(F,\eta)} u, \quad u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n),$$

and such that

$$|\alpha(F,\eta)|_k = \|\eta\|_{L^1\Lambda^{m-k}(F)}.$$

In particular, we even have $\alpha(F,\eta) \in \mathcal{C}^{\text{mass}}_k(\mathbb{R}^n)$.

Now assume that $\dim F = m < n$. There exists a simplex $F_0 \subseteq \mathbb{R}^m$ and an isometric inclusion $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ which maps F_0 onto F. Recall that the pullback of a flat form along a Lipschitz mapping is well-defined. We have

$$\begin{aligned}\int_F \operatorname{tr}^k_F u \wedge \eta = \int_{\varphi_* F_0} \operatorname{tr}^k_F u \wedge \eta &= \int_{F_0} \varphi^* \operatorname{tr}^k_F u \wedge \varphi^*\eta \\ &= \int_{\alpha(F_0,\varphi^*\eta)} \varphi^* \operatorname{tr}^k_F u = \int_{\varphi_*\alpha(F_0,\varphi^*\eta)} u\end{aligned}$$

for $u \in W^{\infty,\infty}\Lambda^k(\mathbb{R}^n)$. Thus we may choose $\alpha(F,\eta) = \varphi_* \alpha(F_0, \varphi^* \eta) \in \mathcal{C}_k^{\mathrm{mass}}(\mathbb{R}^n)$. It remains to show that $\partial_{k-1}\alpha(F,\eta) \in \mathcal{C}_{k-1}^{\mathrm{mass}}(\mathbb{R}^n)$. For $u \in W^{\infty,\infty}\Lambda^{k-1}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\int_{\partial_k \alpha(F,\eta)} u \\
&= \int_{\alpha(F,\eta)} \mathsf{d}^{k-1} u = \int_F \eta \wedge \operatorname{tr}_F^k \mathsf{d}^{k-1} u \\
&= (-1)^{m-k+1} \int_F \mathsf{d}^{m-k}\eta \wedge \operatorname{tr}_F^{k-1} u + (-1)^{m-k} \sum_{f \in \Delta(F)^{m-1}} o(f,F) \int_f \operatorname{tr}_{F,f}^{m-k} \eta \wedge \operatorname{tr}_f^{k-1} u \\
&= (-1)^{m-k+1} \int_{\alpha(F,\mathsf{d}^{m-k}\eta)} u + (-1)^{m-k} \sum_{f \in \Delta(F)^{m-1}} o(f,F) \int_{\alpha(f,\operatorname{tr}_{F,f}^{m-k}\eta)} u.
\end{aligned}$$

In particular, $\partial_k \alpha \in \mathcal{C}_{k-1}^{\mathrm{mass}}(\mathbb{R}^n)$. The proof is complete. □

Remark VII.5.3.
The degrees of freedom in finite element exterior calculus can be described in terms of integrals over simplices weighted against polynomial differential forms (see, e.g., Chapter IV). Hence Lemma VII.5.2 can be applied to identify the degrees of freedom with flat chains.

We finish this section with an estimate on the deformation of flat chains by Lipschitz mappings. This result is applied later in this chapter and constitutes the rationale for considering geometric measure theory.

Lemma VII.5.4.
Let $F \subseteq \mathbb{R}^n$ be an m-simplex and let $\eta \in C^\infty\Lambda^{m-k}(F)$. Let $\alpha(F,\eta) \in C_k^\flat(\mathbb{R}^n)$ be the associated flat chain in the manner of Lemma VII.5.2. Let $r > 0$ be fixed and let $\varphi : B_{2r}(F) \to B_{3r}(F)$ be a Lipschitz mapping that maps $B_r(F)$ into $B_{2r}(F)$. Then

$$\|\varphi_*\alpha - \alpha\|_{k,\flat} \leq \|\varphi - \operatorname{Id}\|_{L^\infty(B_{2r}(F),\mathbb{R}^n)} \left(\mathfrak{L}^k |\alpha|_k + \mathfrak{L}^{k-1} |\partial_k \alpha|_{k-1} \right), \qquad \text{(VII.51)}$$

where $\mathfrak{L} := \sup\{\operatorname{Lip}(\varphi, B_{2r}(F)), 1\}$.

Proof. To prove this result, we gather several additional notions of Whitney's monograph. For any open set $U \subseteq \mathbb{R}^n$, a polyhedral chain $S \sim \sum_i a_i S_i \in \mathcal{C}_k^{\mathrm{pol}}(\mathbb{R}^n)$ is *in* U if all S_i are contained in U, and S is *of* U if there exists an open set $V \subseteq \mathbb{R}^n$ compactly contained in U such that S is a chain in V (see [180]).

The support of a flat chain $\alpha \in \mathcal{C}_k^\flat(\mathbb{R}^n)$ is the set of all points $x \in \mathbb{R}^n$ such that for all $\epsilon > 0$ there exists $u \in C^\infty\Lambda^k(\mathbb{R}^n)$ with support in $B_\epsilon(x)$ such that $\int_\alpha S \neq 0$. It follows from Definition (1) in Section I.13 of [180, p.52] and the discussion in Section V.10 of [180] up to Theorem V.10A that our definition of support agrees with the definition of support in [180, Section VII.3]

Having established these additional notions, the claim is now an application of Theorem 13A in Chapter X in [180] together with Equation VIII.1.(7) in [180, p.233]. □

VII.6. Review of Triangulations

Up to now, we have only addressed topics of purely analytical interest in this chapter. Towards our goal of smoothed projections onto finite element spaces, we briefly discuss triangulations. This builds upon the concepts of Chapter II.

Let $\mathcal{T}$ be a triangulation of the bounded weakly Lipschitz domain Ω, and let $\mathcal{U}$ be a simplicial subcomplex of $\mathcal{T}$ that triangulates Γ_T. We equip the simplices of $\mathcal{T}$ with fixed but arbitrary orientations, except for n-simplices, which we assume to be equipped with the Euclidean orientation of $\mathbb{R}^n$. We have defined the geometric shape measure of $\mathcal{T}$ by

$$\mu(\mathcal{T}) = \max_{T \in \mathcal{T}} \frac{\operatorname{diam}(T)^n}{\operatorname{vol}^n(T)}.$$

Several other relevant quantities can be bounded in terms of $\mu(\mathcal{T})$ and the ambient dimension, as has been demonstrated in Chapter II. This includes the constant $\mu_{\mathrm{N}}(\mathcal{T})$, which bounds the numbers of simplices adjacent to a given simplex, and the constant $\mu_{\mathrm{lqu}}(\mathcal{T})$, which measures how the (generalized) diameters of adjacent simplices compare. We also recall the constant $\mu_{\mathrm{r}}(\mathcal{T})$, but for the purpose of this chapter, we will use $\mu_{\mathrm{r}}(\mathcal{T})$ only in the definition of another quantity. We define

$$\mu_{\mathrm{b}}(\mathcal{T}) := \sup_{T \in \mathcal{T}} \sup \left\{ \epsilon > 0 \mid B_{\epsilon h_T}(T) \subseteq [\mathcal{T}(T)] \right\}. \tag{VII.52}$$

Lemma VII.6.1.
There exists a lower bound for $\mu_{\mathrm{b}}(\mathcal{T})$ that depends only on $\mu_{\mathrm{r}}(\mathcal{T})$ and Ω.

Proof. Since Ω is a weakly Lipschitz domain, there exists a finite covering $U_1, \dots, U_N$ of $\overline{\Omega}$ by closed subsets together with a family $\varphi_1, \dots, \varphi_N$ of bi-Lipschitz mappings $\varphi_i : U_i \to [-1,1]^n$. By Lebesgue's number lemma, there exists $\gamma > 0$ such that for all $x \in \overline{\Omega}$ there exists $1 \leq i \leq N$ such that $B_\gamma(x) \cap \overline{\Omega} \subseteq U_i$.

Let $x \in \overline{\Omega}$ and let $y \in B_\gamma(x) \cap \overline{\Omega}$. Let $1 \leq i \leq N$ such that $B_\gamma(x) \cap \overline{\Omega} \subseteq U_i$. A path from x to y is a continuous mapping $p : [0,1] \to \overline{\Omega}$ with $p(0) = x$ and $p(1) = y$. Let $\mathtt{T}$ be the set of all finite ordered subsets $t_0, \dots, t_M$ of $[0,1]$ that contain 0 and 1. We define the length $\mathtt{L}(p)$ of a path p from x to y as

$$\mathtt{L}(p) := \sup_{\{t_0, \dots, t_M\} \in \mathtt{T}} \sum_{i=1}^{M} \| p(t_i) - p(t_{i-1}) \| .$$

It is obvious that $\|x - y\| \leq \mathtt{L}(p)$. Furthermore, there exists a path p from x to y such that $\mathtt{L}(p) \leq \operatorname{Lip}(\varphi_i) \operatorname{Lip}(\varphi_i^{-1}) \|x - y\|$, namely the image of the straight line segment from $\varphi_i(x)$ to $\varphi_i(y)$ under φ_i^{-1}.

For $r > 0$ we let $\mathtt{B}_r(x)$ be the set of all points in $\overline{\Omega}$ such that a path from x to y of length r is contained in $\overline{\Omega}$. We also see that $\mathtt{B}_r(x) \subseteq B_r(x)$. Hence, if $r < \gamma$, then

$$B_{\frac{r}{\operatorname{Lip}(\varphi_i) \operatorname{Lip}(\varphi_i^{-1})}}(x) \subseteq \mathtt{B}_r(x).$$

Now, let $T \in \mathcal{T}$ with $x \in T$. We write

$$C = \min_{1 \leq i \leq N} \operatorname{Lip}(\varphi_i)^{-1} \operatorname{Lip}(\varphi_i^{-1})^{-1}.$$

Let $\epsilon > 0$. The definition of $\mu_{\mathrm{r}}(\mathcal{T})$ implies that $\mathsf{B}_{\epsilon h_T}(x) \subset [\mathcal{T}(T)]$ for $\epsilon < \mu_{\mathrm{r}}(\mathcal{T})$. If additionally $\epsilon \operatorname{diam}(\Omega) < \gamma$, then $B_{C\epsilon h_T}(x) \subset [\mathcal{T}(T)]$. This completes the proof. □

Remark VII.6.2.
The underlying principle of the proof is that the *inner path metric* is equivalent to the Euclidean metric over $\overline{\Omega}$.

Additionally, we introduce the constant $\epsilon_\Omega > 0$ as the supremum

$$\epsilon_\Omega := \sup \{ \, \epsilon > 0 \mid \forall T \in \mathcal{T} : B_{2\epsilon h_T}(T) \subseteq \Omega^e \, \} . \tag{VII.53}$$

We have defined ϵ_Ω such that the $h_T \epsilon_\Omega$-neighborhood of every $T \in \mathcal{T}$ is compactly contained in Ω^e. Since $h_T \leq \operatorname{diam}(\Omega)$, there exists a lower bound for ϵ_Ω that is independent of the triangulation $\mathcal{T}$.

Finally, for each n-simplex $T \in \mathcal{T}^n$ of the triangulation, we fix an affine transformation $\varphi_T(x) = M_T x + b_T$ where $b_T \in \mathbb{R}^n$ and $M_T \in \mathbb{R}^{n \times n}$ are such that $\varphi_T(\Delta_n) = T$. Each matrix M_T is invertible, and

$$\|M_T\|_{2,2} \leq c_M h_T, \quad \|M_T^{-1}\|_{2,2} \leq C_M h_T^{-1} \tag{VII.54}$$

for constants $c_M, C_M > 0$ that depend only on $\mu(\mathcal{T})$ and n.

VII.7. Review of Interpolants

We define the finite element spaces on the background of Chapter IV. To every simplex $F \in \mathcal{T}$ of the triangulation we associate an admissible sequence type $\mathcal{P}_T \in \mathscr{A}$. Moreover, we assume that the hierarchy condition holds, i.e., we have

$$\forall T \in \mathcal{T} : \forall F \in \Delta(T) : \mathcal{P}_F \leq \mathcal{P}_T.$$

This family of admissible sequence types describes a finite element de Rham complex

$$\dots \xrightarrow{\mathsf{d}^{k-1}} \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^k} \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k+1}} \dots$$

In Chapter IV we have also introduced the commuting interpolant $I_{\mathcal{P}}^k$, which maps from the space $C^\infty\Lambda^k(\mathcal{T},\mathcal{U})$ onto the space $\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$. In particular, we have a commuting diagram

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\mathsf{d}^{k-1}} & C^\infty\Lambda^k(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^k} & C^\infty\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots \\
 & & \Big\downarrow{\scriptstyle I_{\mathcal{P}}^k} & & \Big\downarrow{\scriptstyle I_{\mathcal{P}}^{k+1}} & & \\
\dots & \xrightarrow{\mathsf{d}^{k-1}} & \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^k} & \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots
\end{array}$$

We will now combine these ideas with results in geometric measure theory that we have presented in the preceding section. We begin with the following basic observation.

Lemma VII.7.1.
We have $C^{\infty}\Lambda^{k}(\mathcal{T},\mathcal{U}) \subseteq W^{\infty,\infty}\Lambda^{k}(\mathcal{T},\mathcal{U})$.

Proof. Let $u \in C^{\infty}\Lambda^{k}(\mathcal{T},\mathcal{U})$. As described in Section IV.1, we may identify u with a differential form over Ω. Since the triangulation is finite and $\operatorname{tr}^{k}_{T} u$ is the restriction of a member of $C^{\infty}\Lambda^{k}(\mathbb{R}^{n})$ for each $T \in \mathcal{T}^{n}$, we conclude that $u \in L^{\infty}\Lambda^{k}(\Omega)$ and $\mathsf{d}^{k} u \in L^{\infty}\Lambda^{k+1}(\Omega)$, where the exterior derivative is taken in the sense of Section IV.1.

Next, let $v \in C^{\infty}_{c}\Lambda^{n-k-1}(\Omega)$. We calculate

$$\begin{aligned}
&\int_{\Omega} u \wedge \mathsf{d}^{n-k-1} v \\
&= \sum_{T \in \mathcal{T}^{n}} \int_{T} u_{T} \wedge \mathsf{d}^{n-k-1} v \\
&= (-1)^{k} \sum_{\substack{T \in \mathcal{T}^{n} \\ F \in \Delta(T)^{n-1}}} o(F,T) \int_{F} u_{F} \wedge \operatorname{tr}^{n-k-1}_{F} v + (-1)^{k+1} \sum_{T \in \mathcal{T}^{n}} \int_{T} \mathsf{d}^{k} u_{T} \wedge v \\
&= (-1)^{k+1} \sum_{T \in \mathcal{T}^{n}} \int_{T} \mathsf{d}^{k} u_{T} \wedge v \\
&= (-1)^{k+1} \int_{\Omega} \mathsf{d}^{k} u \wedge v
\end{aligned}$$

with integration by parts and using that u has single-valued traces. This shows that $u \in W^{\infty,\infty}\Lambda^{k}(\Omega)$.

Finally, suppose that $x \in \Gamma_{T}$ and let $r > 0$ be so small that $B_{r}(x)$ intersects $\partial\Omega$ only along Γ_{T}. Let $v \in C^{\infty}_{c}\Lambda^{n-k-1}(\mathbb{R}^{n})$ with support compactly contained in $B_{r}(x)$. Then $u \in W^{\infty,\infty}\Lambda^{k}(\Omega, \Gamma_{T})$ follows as above via integration by parts, using that u has single-valued traces on subsimplices and vanishing traces over simplices in $\mathcal{U}$. The proof is complete. □

We have seen in Chapter IV that the degrees of freedom of the finite element de Rham complexes, which enter the definition of the finite element interpolant, are defined in terms of the integrals over simplices of $\mathcal{T}$ against polynomial differential forms over those simplices. By the results of Section VII.5 these are flat chains of finite mass. Specifically, to each simplex $F \in \mathcal{T}$ we associate a finite-dimensional vector space $\mathcal{PC}^{F}_{k}$ of linear functionals over $C^{\infty}\Lambda^{k}(\mathcal{T})$. After fixing an arbitrary smooth Riemannian metric g over F, the space $\mathcal{PC}^{F}_{k}$ can be written as the span of the following three types of functionals:

- For each simplex $F \in \mathcal{T}$, the space $\mathcal{PC}^{F}_{k}$ includes the functional
$$u \mapsto \int_{F} u.$$
- For each simplex $F \in \mathcal{T}$ and $\rho \in \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F)$, the space $\mathcal{PC}^{F}_{k}$ includes the functional
$$u \mapsto \int_{F} u \wedge \star_{g} \mathsf{d}^{k-1} \rho.$$

- For each simplex $F \in \mathcal{T}$ and $\beta \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$, the space $\mathcal{PC}_k^F$ includes the functional

$$u \mapsto \int_F \mathsf{d}^k u \wedge \star_g \mathsf{d}^k \beta.$$

It follows from Theorem VII.5.2 that $\mathcal{PC}_k^F$ is a finite-dimensional space of flat chains with finite mass whose boundaries have finite mass too. In the case of the third class of functionals, we take Remark IV.4.8 into account in order to see that.

Correspondingly, we define the spaces of flat chains

$$\mathcal{PC}_k(\mathcal{T}) := \bigoplus_{F \in \mathcal{T}} \mathcal{PC}_k^F.$$

These observations allow us to extend the finite element interpolant to a continuous mapping over flat differential forms. We have a linear operator

$$I_{\mathcal{P}}^k : W^{\infty,\infty}\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T})$$

that is uniquely defined by setting

$$\int_S I_{\mathcal{P}}^k u = \int_S u, \quad u \in W^{\infty,\infty}\Lambda^k(\Omega), \quad S \in \mathcal{PC}_k(\mathcal{T}).$$

For $u \in W^{\infty,\infty}\Lambda^k(\Omega)$ and $S \in \mathcal{PC}_k(\mathcal{T})$ we additionally observe that

$$\int_S I_{\mathcal{P}}^{k+1} \mathsf{d}^k u = \int_S \mathsf{d}^k u = \int_{\partial_k S} u = \int_{\partial_k S} I_{\mathcal{P}}^k u = \int_S \mathsf{d}^k I_{\mathcal{P}}^k u. \qquad \text{(VII.55)}$$

This implies that

$$I_{\mathcal{P}}^{k+1} \mathsf{d}^k u = \mathsf{d}^k I_{\mathcal{P}}^k u, \quad u \in W^{\infty,\infty}\Lambda^k(\Omega).$$

Furthermore, we observe that for all $F \in \mathcal{T}$ we have $\operatorname{tr}_F^k I_{\mathcal{P}}^k u = 0$ if $\operatorname{tr}_F^k u = 0$. As a consequence, the mapping $I_{\mathcal{P}}^k$ maps $W^{\infty,\infty}\Lambda^k(\Omega, \Gamma_T)$ into $\mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$. In particular, we have a commuting diagram

$$\begin{array}{ccccccc}
\dots \xrightarrow{\mathsf{d}^{k-1}} & W^{\infty,\infty}\Lambda^k(\Omega, \Gamma_T) & \xrightarrow{\mathsf{d}^k} & W^{\infty,\infty}\Lambda^{k+1}(\Omega, \Gamma_T) & \xrightarrow{\mathsf{d}^{k+1}} \dots \\
& {\scriptstyle I_{\mathcal{P}}^k}\downarrow & & {\scriptstyle I_{\mathcal{P}}^{k+1}}\downarrow & \\
\dots \xrightarrow{\mathsf{d}^{k-1}} & \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U}) & \xrightarrow{\mathsf{d}^k} & \mathcal{P}\Lambda^{k+1}(\mathcal{T}, \mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} \dots
\end{array}$$

from the differential complex of flat differential forms over Ω with partial boundary conditions along Γ_T onto the finite element de Rham complex over $\mathcal{T}$ relative to $\mathcal{U}$.

In addition to that, $I_{\mathcal{P}}^k$ can be extended to a bounded operator over the space $C\Lambda^k(\overline{\Omega})$ of differential k-forms over $\overline{\Omega}$ with continuous coefficients. We have a bounded linear mapping

$$I_{\mathcal{P}}^k : C\Lambda^k(\overline{\Omega}) \to \mathcal{P}\Lambda^k(\mathcal{T})$$

uniquely defined

$$\int_S I^k_{\mathcal{P}} u = \int_S u, \quad u \in C\Lambda^k(\overline{\Omega}), \quad S \in \mathcal{PC}_k(\mathcal{T}).$$

Similar as above, we observe that for all $F \in \mathcal{T}$ we have $\operatorname{tr}^k_F I^k_{\mathcal{P}} u = 0$ if $\operatorname{tr}^k_F u = 0$. Hence $I^k_{\mathcal{P}}$ maps members of $C\Lambda^k(\overline{\Omega})$ whose trace on simplices in $\mathcal{U}$ vanish into the space $\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$.

We finish this section with the discussion of several *inverse inequalities*. These follow easily from scaling arguments and the equivalence of norms on finite-dimensional vector spaces. They build upon the fact that the finite element spaces over each triangle are contained within the pullback of a finite-dimensional vector space over a reference triangle.

By construction, the pullbacks $\varphi^*_T u_{|T}$ lie in a common finite-dimensional vector space as $u \in \mathcal{P}\Lambda^k(T)$ and $T \in \mathcal{T}^n$ vary. For example, this can be a fixed space of differential forms with polynomial coefficients of sufficiently high order. Hence for each $p \in [1,\infty]$ there exists a constant $C^\flat_{n,k,p} > 0$ such that

$$\|\varphi^*_T u\|_{W^{\infty,\infty}\Lambda^k(\Delta_n)} \leq C^\flat_{n,k,p} \|\varphi^*_T u\|_{L^p\Lambda^k(\Delta_n)}, \quad u \in \mathcal{P}\Lambda^k(\mathcal{T}), \quad T \in \mathcal{T}^n. \tag{VII.56}$$

The constant $C^\flat_{n,k,p}$ depends only on p, n, and the maximal polynomial order in the finite element de Rham complex.

Another inverse inequality applies to the degrees of freedom. By Lemma VII.5.2, each degree of freedom can be identified with a flat chain of finite mass whose boundary is again a flat chain of finite mass. In general, the boundary operator is an unbounded operator as a mapping between spaces of polyhedral chains with respect to the mass norm. But in the present setting, the pushforward of the degrees of freedom onto the reference simplex takes values in a finite-dimensional vector space. We conclude that there exists $C_\partial > 0$ such that

$$|\varphi^{-1}_{T*} \partial_k S|_{k-1} \leq C_\partial |\varphi^{-1}_{T*} S|_k, \quad S \in \mathcal{PC}^F_k, \quad F \in \Delta(T), \quad T \in \mathcal{T}^n. \tag{VII.57}$$

Again, the constant C_∂ depends only on n and the maximal polynomial order in the finite element de Rham complex.

Finally, we have a local bound for the interpolant. We observe that there exists a constant $C_I > 0$ such that for $T \in \mathcal{T}^n$ and $u \in C\Lambda^k(\overline{\Omega})$ we have

$$\|\varphi^*_T I^k_{\mathcal{P}} u\|_{L^\infty\Lambda^k(\Delta_n)} \leq C_I \sup_{\substack{F \in \Delta(T) \\ S \in \mathcal{PC}^F_k}} |\varphi_{T*} S|^{-1}_k \int_{\varphi^{-1}_{T*} S} \varphi^*_T u. \tag{VII.58}$$

Similar as above, C_I depends only on n and the maximal polynomial order in the finite element de Rham complex. Note that this inequality immediately implies

$$\|\varphi^*_T I^k_{\mathcal{P}} u\|_{L^\infty\Lambda^k(\Delta_n)} \leq C_I \|\varphi^*_T u\|_{C\Lambda^k(\Delta_n)}, \quad u \in C\Lambda^k(\overline{\Omega}). \tag{VII.59}$$

Remark VII.7.2.
The existence of constants $C^\flat_{n,k,p}$, C_∂, and C_I as above follows trivially if the triangulation $\mathcal{T}$ and the maximal polynomial order of the finite element spaces are

fixed. But in applications we consider families of triangulations with associated finite element de Rham complexes. We then demand uniform bounds for the three constants. Such uniform bounds hold if the triangulations are shape-regular and the finite element spaces have uniformly bounded polynomial order. This thesis does not address estimates that are uniform in the polynomial order, as would be relevant for p- and hp-methods.

VII.8. Construction of the Smoothed Projection

In this section we complete the agenda of this chapter and devise the smoothed projection from a Sobolev de Rham complex with partial boundary conditions onto a conforming finite element de Rham complex.

In order to instantiate the smoothing operator of Section VII.4, we need to specify a function controlling the smoothing radius. For our particular application, that function should indicate the local mesh size. First we prove the existence of a mesh size function $\mathtt{H}$ with Lipschitz regularity and then the existence of a mesh size function $\mathtt{h}$ that is smooth.

Lemma VII.8.1.
There exists $L_\Omega > 0$, depending only on Ω, and a Lipschitz continuous function $\mathtt{H} : \overline{\Omega} \to \mathbb{R}_0^+$ such that

$$\forall F \in \mathcal{T} : \mu_{\mathrm{lqu}}(\mathcal{T})^{-1} h_F \leq \mathtt{H}_{|F} \leq \mu_{\mathrm{lqu}}(\mathcal{T}) h_F, \tag{VII.60}$$

$$\operatorname{Lip}(\mathtt{H}, \overline{\Omega}) \leq \mu_{\mathrm{lqu}}(\mathcal{T}) L_\Omega. \tag{VII.61}$$

Proof. We define $\mathtt{H} : \overline{\Omega} \to \mathbb{R}_0^+$ as follows. If $V \in \mathcal{T}^0$, then we set $\mathtt{H}(V) = h_V$. We then extend $\mathtt{H}$ to each $T \in \mathcal{T}$ by affine interpolation between the vertices of T. With this definition, $\mathtt{H}$ is continuous, and (VII.60) follows from definitions. It remains to prove (VII.61). Obviously, $\operatorname{Lip}(\mathtt{H}, T) \leq \mu_{\mathrm{lqu}}(\mathcal{T})$ for $T \in \mathcal{T}^n$.

Since Ω is a bounded weakly Lipschitz domain, there exists be a finite family $(U_i)_{1 \leq i \leq N}$ of open sets $U_i \subseteq \Omega$ such that the union of all U_i equals Ω, and such that there exist $\varphi_i : U_i \to (-1,1)^n$ bi-Lipschitz for each $1 \leq i \leq N$ (see Lemma VI.1.6). By Lebesgue's number lemma, and the precompactness of Ω, we may pick $\gamma > 0$ so small that for each $x \in \Omega$ there exists $1 \leq i \leq N$ such that $B_\gamma(x) \cap \Omega \subseteq U_i$.

First assume that $x, y \in \Omega$ with $0 < \|x - y\| \leq \gamma$. Then there exists $1 \leq i \leq N$ with $x, y \in U_i$. For $M \in \mathbb{N}$, consider a partition of the line segment in $(-1,1)^n$ from $\varphi(x)$ to $\varphi(y)$ into M subsegments of equal length with points $\varphi_i(x) = z_0, z_1, \dots, z_M = \varphi_i(x)$. Let $x_m := \varphi_i^{-1}(z_m) \in U_i$. For M large enough, the straight line segment between x_{m-1} and x_m is contained in U_i for all $1 \leq m \leq M$. After a further subpartitioning, not necessarily equidistant, we may assume to have a sequence $x = w_0, \dots, w_{M'} = y$ for some $M' \in \mathbb{N}$ such that for all $1 \leq m \leq M'$ the points w_{m-1} and w_m are connected by a straight line segment in U_i and such that there exists $F_m \in \mathcal{T}$ with $w_{m-1}, w_m \in F_m$. We observe

$$\begin{aligned} |\mathtt{H}(y) - \mathtt{H}(x)| &\leq \sum_{m=1}^{M'} |\mathtt{H}(w_m) - \mathtt{H}(w_{m-1})| \\ &\leq \mu_{\mathrm{lqu}}(\mathcal{T}) \sum_{m=1}^{M'} \|w_m - w_{m-1}\| \\ &= \mu_{\mathrm{lqu}}(\mathcal{T}) \sum_{m=1}^{M} \|x_m - x_{m-1}\| \\ &\leq \mu_{\mathrm{lqu}}(\mathcal{T}) \operatorname{Lip}(\varphi_i^{-1}) \sum_{m=1}^{M} \|\varphi_i(x_m) - \varphi_i(x_{m-1})\| \\ &\leq \mu_{\mathrm{lqu}}(\mathcal{T}) \operatorname{Lip}(\varphi_i^{-1}) \cdot \|\varphi_i(y) - \varphi_i(x)\| \\ &\leq \mu_{\mathrm{lqu}}(\mathcal{T}) \operatorname{Lip}(\varphi_i^{-1}) \operatorname{Lip}(\varphi_i) \cdot \|y - x\|. \end{aligned}$$

If we instead assume that $x, y \in \Omega$ with $\|x - y\| \geq \gamma$, then

$$|\mathtt{H}(y) - \mathtt{H}(x)| \leq \operatorname{diam}(\Omega) \leq \frac{\operatorname{diam}(\Omega)}{\gamma} \|y - x\|.$$

Hence $\operatorname{Lip}(\mathtt{H}, \Omega) \leq \mu_{\mathrm{lqu}}(\mathcal{T}) L_\Omega$ with

$$L_\Omega := \sup \left\{ \gamma^{-1} \operatorname{diam}(\Omega),\ \operatorname{Lip}(\varphi_1^{-1}) \operatorname{Lip}(\varphi_1), \dots,\ \operatorname{Lip}(\varphi_N^{-1}) \operatorname{Lip}(\varphi_N) \right\}.$$

Thus $\operatorname{Lip}(\mathtt{H}, \overline{\Omega}) \leq \mu_{\mathrm{lqu}}(\mathcal{T}) L_\Omega$ because any Lipschitz continuous function is Lipschitz continuous over the closure of its domain with the same Lipschitz constant. □

Lemma VII.8.2.
There exist a compactly supported smooth function $\mathtt{h} : \Omega^e \to \mathbb{R}_0^+$ and constants $C_h > 0$ and $L_h > 0$, depending only on Ω and $\mu_{\mathrm{lqu}}(\mathcal{T})$, such that

$$\operatorname{Lip}(\mathtt{h}, \Omega^e) \leq L_h, \tag{VII.62}$$

and such that for all $F \in \mathcal{T}$ and $x \in F$ we have

$$C_h^{-1} h_F \leq \mathtt{h}(x) \leq C_h h_F. \tag{VII.63}$$

Moreover, $\operatorname{supp} \mathtt{h}$ depends only on Ω.

Proof. Let $\mathtt{H} : \overline{\Omega} \to \mathbb{R}_0^+$ as in the previous lemma. Consider the Lipschitz collar $\Psi_0 : \partial\Omega \times [-1, 1] \to \mathbb{R}^n$ introduced in Section VII.1, and write $G := \Psi_0(\partial\Omega, (0, 1))$. For $x \in \overline{\Omega} \cup G$ we define

$$\mathtt{H}^e(x) := \begin{cases} \mathtt{H}(x) & \text{if } x \in \overline{\Omega}, \\ \mathtt{H}(\Psi_0(x_0, -t)) & \text{if } x = \Psi(x_0, t), \quad (x_0, t) \in \partial\Omega \times (0, 1). \end{cases} \tag{VII.64}$$

It is easy to see that $\operatorname{Lip}(\mathtt{H}^e, \overline{\Omega} \cup G) \leq (1 + C_0) \operatorname{Lip}(\mathtt{H}^e, \overline{\Omega})$ for a constant C_0 that depends only on Ψ_0. Note that $\mathtt{H}^e$ is just the extension by reflection of $\mathtt{H}$ along the Lipschitz collar. We extend $\mathtt{H}^e$ trivially to a function over $\mathbb{R}^n$.

Next we let $\chi : \Omega^e \to \mathbb{R}$ be a smooth non-negative function that assumes its maximum 1 over an open neighborhood of Ω and has support compactly contained in Ω^e. Then

$$(\chi \mathtt{H}^e)_{|\Omega} = \mathtt{H}_{|\Omega}, \quad (\chi \mathtt{H}^e)_{\max}(\Omega^e) \leq \mathtt{H}_{\max}(\overline{\Omega}),$$
$$\operatorname{Lip}(\chi \mathtt{H}^e, \Omega^e) \leq \mu_{\mathrm{lqu}}(\mathcal{T}) L_\Omega + \mathtt{H}_{\max}(\overline{\Omega}) \operatorname{Lip}(\chi).$$

For $r > 0$ yet to be determined, we let $\mathtt{h} := \mu_r \star \chi \mathtt{H}^e$. Then $\mathtt{h}$ is smooth. For $r > 0$ small enough, $\operatorname{supp} \mathtt{h}$ is compactly contained in Ω^e and

$$\operatorname{Lip}(\mathtt{h}, \Omega^e) \leq \mu_{\mathrm{lqu}}(\mathcal{T}) L_\Omega + \mathtt{H}_{\max}(\overline{\Omega}) \operatorname{Lip}(\chi),$$

which gives the constant in (VII.62). Note that $\mathtt{h}(x)$ is contained in the convex hull of all values of $\chi \mathtt{H}^e$ in $B_r(x)$. We find for all $x \in \Omega$ that

$$\mathtt{h}(x) = \int_{B_r(x) \cap \Omega} \mu_r(y) \mathtt{H}(x+y) \,\mathrm{d}y + \int_{B_r(x) \setminus \Omega} \mu_r(y) \mathtt{H}^e(x+y) \,\mathrm{d}y.$$

Hence for $r > 0$ small enough there exists $C > 0$ such that for all $x \in \overline{\Omega}$ the value $\mathtt{h}(x)$ lies in the convex combination of the values of $\mathtt{H}$ over $B_{Cr}(x)$. In particular, if $r < \mu_{\mathrm{b}}(\mathcal{T}) h_{\min}$, where $h_{\min}$ is the shortest edge length in $\mathcal{T}$, then (VII.63) holds for $F \in \mathcal{T}$ and $x \in F$. The proof is complete. $\square$

Remark VII.8.3.
The existence of Lipschitz-continuous mesh size functions was used before in the literature [58]. We mention that the existence of a smooth mesh size function is also used in [56].

We have given close attention to estimating the Lipschitz constant $\operatorname{Lip}(\mathtt{H})$. An interesting observation in the light of Lemma VII.8.1 is that $\operatorname{Lip}(\mathtt{H})$ is the product of $\mu_{\mathrm{lqu}}(\mathcal{T})$, which depends only on the shape of the simplices, and L_Ω, which depends only the geometry. Conceptually, L_Ω compares the *inner path metric* of $\overline{\Omega}$ to the Euclidean metric over $\overline{\Omega}$. The equivalence of these two metrics is non-trivial in general but holds for bounded weakly Lipschitz domains.

We will use the smooth mesh size function $\mathtt{h}$, but we will generally need to rescale it by a fixed parameter $\epsilon > 0$ in the sequel. Our goal is to choose $\epsilon > 0$ so small that the conditions of Theorem VII.4.1 are satisfied by $\varrho = \epsilon \mathtt{h}$. This enables us to work with the smoothing operator $M_{\epsilon \mathtt{h}}$ in this section.

Lemma VII.8.4.
There exists $\epsilon_0 > 0$, depending only on Ω and $\mu(\mathcal{T})$, such that for all $\epsilon \in (0, \epsilon_0)$ the function $\varrho = \epsilon \mathtt{h}$ satisfies the conditions of Theorem VII.4.1.

Proof. First, we let $\epsilon > 0$ be so small that for each $T \in \mathcal{T}$ we have $B_{\epsilon h_T / C_h}(T) \subseteq \Omega^e$. It suffices that $\epsilon / C_h < \epsilon_\Omega$. Under that condition, $\Psi_{\epsilon \mathtt{h}}$, defined as in Section VII.3, maps Ω into Ω^e. If additionally $\epsilon L_h < 1/2$, then the conditions of Lemma VII.3.2 are satisfied. Second, we choose ϵ so small that $\epsilon L_h < \delta_D$ and $\epsilon \mathtt{h}_{\max}(\mathbb{R}^n) < \delta_D$. For the latter it suffices that $\epsilon \operatorname{diam}(\Omega) < \delta_D$. Then the conditions of Theorem VII.2.1 are satisfied. Lastly, we write $r := \operatorname{diam}(\Omega) L_D$ and choose ϵ so small that the ϵr-neighborhood of $\operatorname{supp} \mathtt{h}$ is contained in Ω^e, which depends only on Ω. It follows via

(VII.21c) that $\mathfrak{D}_{\epsilon\mathtt{h}}$ maps Ω^e bijectively into itself. Under these assumptions $\varrho := \epsilon\mathtt{h}$ satisfies the required properties, and the proof is complete. $\square$

In the sequel, we call a quantity *uniformly bounded* if it can be bounded in terms of the geometry, the mesh regularity, and the maximal polynomial degree of the finite element space. We remind the reader that bounds uniform in the polynomial degree are not subject of this thesis.

Towards the definition of the smoothed projection, we first define a *smoothed interpolant*. Let $\epsilon > 0$ be small enough; we assume in particular $\epsilon < \epsilon_0$. We define

$$Q_\epsilon^k : L^p\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \subseteq L^p\Lambda^k(\Omega), \quad u \mapsto I_\mathcal{P}^k M_{\epsilon\mathtt{h}}^k u, \quad p \in [1,\infty]. \tag{VII.65}$$

We show that Q_ϵ^k satisfies uniform local bounds and commutes with the exterior derivative:

Theorem VII.8.5.
Let $\epsilon > 0$ be small enough. We have a bounded linear operator

$$Q_\epsilon^k : L^p\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \subseteq L^p\Lambda^k(\Omega), \quad p \in [1,\infty].$$

For each $p \in [1,\infty]$ there exists uniformly bounded $C_{Q,p} > 0$ such that

$$\|Q_\epsilon^k u\|_{L^p\Lambda^k(T)} \leq C_{Q,p}\epsilon^{-\frac{n}{p}}\|u\|_{L^p\Lambda^k(\mathcal{T}(T))}, \quad u \in L^p\Lambda^k(\Omega), \quad T \in \mathcal{T}^n, \tag{VII.66}$$

and

$$\|Q_\epsilon^k u\|_{L^p\Lambda^k(\Omega)} \leq C_N^{\frac{1}{p}} C_{Q,p}\epsilon^{-\frac{n}{p}}\|u\|_{L^p\Lambda^k(\Omega)}, \quad u \in L^p\Lambda^k(\Omega). \tag{VII.67}$$

Moreover, we have

$$\mathsf{d}^k Q_\epsilon^k u = Q_\epsilon^{k+1}\mathsf{d}^k u, \quad u \in W^{p,q}\Lambda^k(\Omega), \quad p,q \in [1,\infty]. \tag{VII.68}$$

Proof. Let $u \in L^p\Lambda^k(\Omega)$ and $T \in \mathcal{T}^n$. Then

$$\begin{aligned}\|Q_\epsilon^k u\|_{L^p\Lambda^k(T)} &= \|I_\mathcal{P}^k M_{\epsilon\mathtt{h}}^k u\|_{L^p\Lambda^k(T)} \\ &\leq \operatorname{vol}^n(T)^{\frac{1}{p}}\|I_\mathcal{P}^k M_{\epsilon\mathtt{h}}^k u\|_{L^\infty\Lambda^k(T)} \leq h_T^{\frac{n}{p}}\|I_\mathcal{P}^k M_{\epsilon\mathtt{h}}^k u\|_{L^\infty\Lambda^k(T)}.\end{aligned}$$

Estimate (VII.59) gives

$$\begin{aligned}\|I_\mathcal{P}^k M_{\epsilon\mathtt{h}}^k u\|_{L^\infty\Lambda^k(T)} &= \|\varphi_T^{-*}\varphi_T^* I_\mathcal{P}^k M_{\epsilon\mathtt{h}}^k u\|_{L^\infty\Lambda^k(T)} \\ &\leq C_M^k h_T^{-k}\|\varphi_T^* I_\mathcal{P}^k M_{\epsilon\mathtt{h}}^k u\|_{L^\infty\Lambda^k(\Delta_n)} \\ &\leq C_I C_M^k h_T^{-k}\|\varphi_T^* M_{\epsilon\mathtt{h}}^k u\|_{L^\infty\Lambda^k(\Delta_n)} \\ &\leq C_I c_M^k C_M^k\|M_{\epsilon\mathtt{h}}^k u\|_{C\Lambda^k(T)}.\end{aligned}$$

For $\epsilon > 0$ small enough, we may apply Theorem VII.4.1 with $\varrho = \epsilon\mathtt{h}$ to find

$$\begin{aligned}\|M_{\epsilon\mathtt{h}}^k u\|_{C\Lambda^k(T)} &\leq C_{n,k,p}^M \frac{(1+\epsilon\operatorname{Lip}(\mathtt{h}))^{k+\frac{n}{p}}}{(\epsilon\mathtt{h}_{\min}(T))^{\frac{n}{p}}}\|u\|_{L^p\Lambda^k(B_{\epsilon L_M(1+\epsilon\operatorname{Lip}(\mathtt{h}))\mathtt{h}_{\max}(T)}(T)\cap\Omega)} \\ &\leq C_{n,k,p}^M \frac{C_h^{\frac{n}{p}}(1+\epsilon L_h)^{k+\frac{n}{p}}}{\epsilon^{\frac{n}{p}} h_T^{\frac{n}{p}}}\|u\|_{L^p\Lambda^k(B_{\epsilon C_h L_M(1+\epsilon L_h)h_T}(T)\cap\Omega)}\end{aligned}$$

Under the condition that ϵ is so small that $\epsilon C_h L_M (1+\epsilon L_h) < \mu_{\mathrm{b}}(\mathcal{T})$ we observe

$$B_{\epsilon C_h L_M (1+\epsilon L_h) h_T}(T) \cap \Omega \subseteq \mathcal{T}(T).$$

Thus the local bound (VII.66) follows. The global bound (VII.67) is obtained via

$$\begin{aligned}
\|Q_\epsilon^k u\|_{L^p\Lambda^k(\Omega)}^p &= \sum_{T\in\mathcal{T}^n} \|Q_\epsilon^k u\|_{L^p\Lambda^k(T)}^p \leq C_{Q,p}^p \sum_{T\in\mathcal{T}^n} \|u\|_{L^p\Lambda^k(\mathcal{T}(T))}^p \\
&\leq C_{Q,p}^p \mu_{\mathrm{N}}(\mathcal{T}) \sum_{T\in\mathcal{T}^n} \|u\|_{L^p\Lambda^k(T)}^p \leq C_{Q,p}^p \mu_{\mathrm{N}}(\mathcal{T}) \|u\|_{L^p\Lambda^k(\Omega)}^p
\end{aligned}$$

for $p \in [1,\infty)$, and for $p=\infty$ similarly.

Next, if $F \in \mathcal{T}$ with $F \subseteq \Gamma_T$, then $M_{\epsilon\mathfrak{h}}^k u$ vanishes near F. By the properties of $I_{\mathcal{P}}^k$ we conclude that $Q_\epsilon^k u \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$. Finally, the commutativity with the exterior derivative (VII.68) follows from Theorem VII.4.1 and the commutativity of the finite element interpolant on flat differential forms. □

Remark VII.8.6.
For the preceding lemma, it suffices that $\epsilon > 0$ is so small that $\epsilon L C_h (1+\epsilon L_h) < \mu_{\mathrm{b}}(\mathcal{T})$ and Lemma VII.8.4 applies. We may assume $C_{Q,p} \leq C_{n,k,p}^M C_h^{n/p} (1+L_h)^{k+\frac{n}{p}}$.

The smoothed interpolant Q_ϵ^k is local and satisfies uniform bounds. Although Q_ϵ^k generally does not reduce to the identity over $\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$, we can show that, for $\epsilon > 0$ small enough, it is close to the identity and satisfies a local error estimate.

Theorem VII.8.7.
Let $\epsilon > 0$ be small enough. Then for every $p \in [1,\infty]$ there exists a uniformly bounded constant $C_{e,p} > 0$ such that

$$\|u - Q_\epsilon^k u\|_{L^p\Lambda^k(T)} \leq \epsilon C_{e,p} \|u\|_{L^p\Lambda^k(\mathcal{T}(T))}, \quad u \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}), \quad T \in \mathcal{T}^n.$$

Proof. We prove the statement by a series of inequalities. Let $u \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ and let $T \in \mathcal{T}^n$. Then

$$\begin{aligned}
\|u - Q_\epsilon^k u\|_{L^p\Lambda^k(T)} &\leq \operatorname{vol}^n(T)^{\frac{1}{p}} \|u - Q_\epsilon^k u\|_{L^\infty\Lambda^k(T)} \\
&\leq h_T^{\frac{n}{p}} \|u - Q_\epsilon^k u\|_{L^\infty\Lambda^k(T)} \\
&\leq h_T^{\frac{n}{p}} \|E^k u - Q_\epsilon^k u\|_{L^\infty\Lambda^k(T)} \\
&\leq C_M^k h_T^{\frac{n}{p}-k} \|\varphi_T^* I_{\mathcal{P}}^k (E^k u - M_{\epsilon\mathfrak{h}}^k u)\|_{L^\infty\Lambda^k(\varphi_T^{-1}T)}.
\end{aligned}$$

By (VII.58) and (VII.49), we have

$$\|\varphi_T^* I_{\mathcal{P}}^k (E^k u - R_{\epsilon\mathfrak{h}}^k E^k u)\|_{L^\infty\Lambda^k(\varphi_T^{-1}T)} \leq C_I \sup_{\substack{F\in\Delta(T)\\ S\in\mathcal{PC}_k^F}} |\varphi_{T*}^{-1} S|_k^{-1} \int_S E^k u - R_{\delta\epsilon\mathfrak{h}}^k \mathfrak{D}_{\epsilon\mathfrak{h}}^* E^k.$$

We need to bound the last expression. Fix $F \in \Delta(T)$ and $S \in \mathcal{PC}_k^F$. We see that

$$\int_S E^k u - R_{\delta\epsilon\mathfrak{h}}^k \mathfrak{D}_{\epsilon\mathfrak{h}}^* E^k u = \int_S \int_{\mathbb{R}^n} \mu(y) \left(E^k u - \Phi_{\delta\epsilon\mathfrak{h},y}^* \mathfrak{D}_{\epsilon\mathfrak{h}}^* E^k u \right) \mathrm{d}y.$$

We want to change the order of integration between those two integrals. As a technical tool, we use Theorem VI.7A of [180], which implies that integrable continuous differential k-forms over $\mathbb{R}^n$ are densely embedded in the space of flat chains over $\mathbb{R}^n$, such that the pairing of the induced flat chain with a flat differential form is the usual scalar product between k-forms. Consider a sequence of continuous integrable differential k-forms $(S_i)_{i\in\mathbb{N}}$ such that $S_i \to S$ in $\mathcal{C}^\flat_k(\mathbb{R}^n)$. We then find with Fubini's theorem and theorem of dominated convergence that

$$\begin{aligned}
&\int_S \int_{\mathbb{R}^n} \mu(y) \Phi^*_{\delta\epsilon\mathtt{h},y} \mathfrak{D}^*_{\epsilon\mathtt{h}} E^k u \,\mathrm{d}y \\
&\quad = \lim_{i\to\infty} \int_{\mathbb{R}^n} \left\langle S_i, \int_{\mathbb{R}^n} \mu(y) \Phi^*_{\delta\epsilon\mathtt{h},y} \mathfrak{D}^*_{\epsilon\mathtt{h}} E^k u \,\mathrm{d}y \right\rangle \,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n} \mu(y) \lim_{i\to\infty} \int_{\mathbb{R}^n} \left\langle S_i, \Phi^*_{\delta\epsilon\mathtt{h},y} \mathfrak{D}^*_{\epsilon\mathtt{h}} E^k u \right\rangle \,\mathrm{d}x \,\mathrm{d}y \\
&\quad = \int_{\mathbb{R}^n} \mu(y) \int_S \Phi^*_{\delta\epsilon\mathtt{h},y} \mathfrak{D}^*_{\epsilon\mathtt{h}} E^k u \,\mathrm{d}y.
\end{aligned}$$

Using these observations and (VII.49) again, we have

$$\int_{\mathbb{R}^n} \mu(y) \int_S E^k u - \Phi^*_{\delta\epsilon\mathtt{h},y} \mathfrak{D}^*_{\epsilon\mathtt{h}} E^k u \mathrm{d}y = \int_{\mathbb{R}^n} \mu(y) \int_{\varphi^{-1}_{T*} S - \varphi^{-1}_{T*} \mathfrak{D}_{\epsilon\mathtt{h}*} \Phi_{\delta\epsilon\mathtt{h},y*} S} \varphi^*_T E^k u \,\mathrm{d}y.$$

Before we proceed with bounding this term, we gather some auxiliary estimates. For $\lambda > 0$ we find that

$$\begin{aligned}
&\sup_{\hat{x}\in B_\lambda(\varphi^{-1}_T F)} \sup_{y\in B_1(0)} \left\| \hat{x} - \varphi^{-1}_T \mathfrak{D}_{\epsilon\mathtt{h}} \Phi_{\delta\epsilon\mathtt{h},y}(\varphi_T \hat{x}) \right\| \\
&\leq \sup_{\hat{x}\in B_\lambda(\varphi^{-1}_T F)} \sup_{y\in B_1(0)} C_M h_T^{-1} \left\| \varphi_T(\hat{x}) - \mathfrak{D}_{\epsilon\mathtt{h}} \Phi_{\delta\epsilon\mathtt{h},y}(\varphi_T \hat{x}) \right\| \\
&\leq \sup_{\hat{x}\in B_\lambda(\varphi^{-1}_T F)} \sup_{y\in B_1(0)} C_M h_T^{-1} \left\| \varphi_T(\hat{x}) - \mathfrak{D}_{\epsilon\mathtt{h}}(\varphi_T \hat{x}) \right\| \\
&\qquad + C_M h_T^{-1} \left\| \mathfrak{D}_{\epsilon\mathtt{h}}(\varphi_T \hat{x}) - \mathfrak{D}_{\epsilon\mathtt{h}} \Phi_{\delta\epsilon\mathtt{h},y}(\varphi_T \hat{x}) \right\| \\
&\leq \sup_{\hat{x}\in B_\lambda(\varphi^{-1}_T F)} \sup_{y\in B_1(0)} C_M h_T^{-1} L_D \epsilon \mathtt{h}(\varphi_T \hat{x}) + C_M h_T^{-1} L_D \left(1 + \epsilon L_h\right) \delta\epsilon \mathtt{h}(\varphi_T \hat{x}) \\
&\leq \sup_{\hat{x}\in B_\lambda(\varphi^{-1}_T F)} \sup_{y\in B_1(0)} C_M h_T^{-1} L_D \left(2 + \epsilon L_h\right) \epsilon \mathtt{h}(\varphi_T \hat{x}).
\end{aligned}$$

Since $\hat{x} \in B_\lambda(\varphi^{-1}_T F)$, we have $\varphi_T(\hat{x}) \in B_{h_T c_M \lambda}(F)$. Assuming $c_M \lambda < \mu_{\mathrm{b}}(\mathcal{T})$, we moreover have $\varphi_T \hat{x} \in \mathcal{T}(F)$, and hence $\mathtt{h}(\varphi_T \hat{x}) \leq \mu_{\mathrm{lqu}}(\mathcal{T}) C_h h_T$. For $c_M \lambda < \mu_{\mathrm{b}}(\mathcal{T})$ we thus conclude

$$\sup_{\hat{x}\in B_\lambda(\varphi^{-1}_T F)} \sup_{y\in B_1(0)} \left\| \hat{x} - \varphi^{-1}_T \mathfrak{D}_{\epsilon\mathtt{h}} \Phi_{\delta\epsilon\mathtt{h},y}(\varphi_T \hat{x}) \right\| \leq C_M L_D \left(2 + \epsilon L_h\right) \epsilon \mu_{\mathrm{lqu}}(\mathcal{T}) C_h.$$

We observe that

$$\sup_{y\in B_1(0)} \operatorname{Lip}\left(\varphi^{-1}_T \mathfrak{D}_{\epsilon\mathtt{h}} \Phi_{\delta\epsilon\mathtt{h},y} \varphi_T, B_\lambda(\varphi^{-1}_T F) \right) \leq c_M C_M L_D \left(1 + \epsilon L_h\right)^2.$$

We introduce the constants

$$\mathfrak{K} := c_M C_M L_D \left(1 + L_h\right)^2, \quad \mathfrak{L} := C_M L_D \left(2 + L_h\right) \mu_{\mathrm{lqu}}(\mathcal{T}) C_h$$

and henceforth assume that $\epsilon < 1$.

We proceed with the main part of the proof. With (VII.40), it follows that

$$\begin{aligned}
&\int_{\varphi_{T*}^{-1} S - \varphi_{T*}^{-1} \mathfrak{D}_{\epsilon\mathfrak{h}*} \Phi_{\delta\epsilon\mathfrak{h},y*} S} \varphi_T^* E^k u \,\mathrm{d}s \mathrm{d}y \\
&\qquad \leq \sup_{y \in B_1(0)} \| \varphi_{T*}^{-1} S - \varphi_{T*}^{-1} \mathfrak{D}_{\epsilon\mathfrak{h}*} \Phi_{\delta\epsilon\mathfrak{h},y*} S \|_{k,\flat} \cdot \| \varphi_T^* E^k u \|_{W^{\infty,\infty}\Lambda^k(B_{\mathfrak{L}\epsilon}(\Delta_n))},
\end{aligned}$$

We need to bound this product.

We begin with the second factor. For ϵ small enough we observe

$$\| \varphi_T^* E^k u \|_{W^{\infty,\infty}\Lambda^k(B_{\mathfrak{L}\epsilon}(\Delta_n))} \leq \left(1 + C_b^{k+1+\frac{n}{p}}\right) c_M^{k+1} C_M^{k+1} \| \varphi_T^* u \|_{W^{\infty,\infty}\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))}.$$

To see this, assume that $L_E \mathfrak{L} \epsilon < \mu_{\mathrm{b}}(\mathcal{T})$. Applying Lemma VII.1.8 gives

$$\begin{aligned}
\| \varphi_T^* E^k u \|_{L^\infty\Lambda^k(B_{\mathfrak{L}\epsilon}(\Delta_n))} &\leq c_M^k h_T^k \| E^k u \|_{L^\infty\Lambda^k(B_{\mathfrak{L}\epsilon h_T}(T))} \\
&\leq \left(1 + C_b^{k+\frac{n}{p}}\right) c_M^k h_T^k \| u \|_{L^\infty\Lambda^k(B_{L_E \mathfrak{L}\epsilon h_T}(T) \cap \overline{\Omega})} \\
&\leq \left(1 + C_b^{k+\frac{n}{p}}\right) c_M^k h_T^k \| u \|_{L^\infty\Lambda^k(\mathcal{T}(T))} \\
&\leq \left(1 + C_b^{k+\frac{n}{p}}\right) c_M^k C_M^k \| \varphi_T^* u \|_{L^\infty\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))}
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\| \varphi_T^* E^{k+1} \mathsf{d}^k u \|_{L^\infty\Lambda^{k+1}(B_{\mathfrak{L}\epsilon}(\Delta_n))} &\leq c_M^{k+1} h_T^{k+1} \| E^{k+1} \mathsf{d}^k u \|_{L^\infty\Lambda^{k+1}(B_{\mathfrak{L}\epsilon h_T}(T))} \\
&\leq \left(1 + C_b^{k+1+\frac{n}{p}}\right) c_M^{k+1} h_T^{k+1} \| \mathsf{d}^k u \|_{L^\infty\Lambda^{k+1}(B_{L_E \mathfrak{L}\epsilon h_T}(T) \cap \overline{\Omega})} \\
&\leq \left(1 + C_b^{k+1+\frac{n}{p}}\right) c_M^{k+1} h_T^{k+1} \| \mathsf{d}^k u \|_{L^\infty\Lambda^{k+1}(\mathcal{T}(T))} \\
&\leq \left(1 + C_b^{k+1+\frac{n}{p}}\right) c_M^{k+1} C_M^{k+1} \| \varphi_T^* \mathsf{d}^k u \|_{L^\infty\Lambda^{k+1}(\varphi_T^{-1}\mathcal{T}(T))}.
\end{aligned}$$

The inverse inequality (VII.56) gives

$$\| \varphi_T^* u \|_{W^{\infty,\infty}\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))} \leq C_{n,k,p}^\flat \| \varphi_T^* u \|_{L^p\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))}.$$

Another pullback estimate then provides

$$\| \varphi_T^* u \|_{L^p\Lambda^k(\varphi_T^{-1}\mathcal{T}(T))} \leq c_M^k C_M^{\frac{n}{p}} h_T^{k-\frac{n}{p}} \| u \|_{L^p\Lambda^k(\mathcal{T}(T))}.$$

On the other hand, we apply Lemma VII.5.4 to bound the remaining factor. Let $\lambda > 0$ as above. By applying Lemma VII.5.4 with $r = \lambda/3$, we then estimate

$$\begin{aligned}
&\sup_{y \in B_1(0)} \| \varphi_{T*}^{-1} S - \varphi_{T*}^{-1} \mathfrak{D}_{\epsilon\mathfrak{h}*} \Phi_{\delta\epsilon\mathfrak{h},y*} S \|_{k,\flat} \\
&\qquad = \sup_{y \in B_1(0)} \| \varphi_{T*}^{-1} S - \varphi_{T*}^{-1} \mathfrak{D}_{\epsilon\mathfrak{h}*} \Phi_{\delta\epsilon\mathfrak{h},y*} \varphi_{T*} \varphi_{T*}^{-1} S \|_{k,\flat} \\
&\qquad \leq \epsilon \cdot \mathfrak{L} \cdot \max(1, \mathfrak{K})^k \cdot \left(|\varphi_{T*}^{-1} S|_k + |\partial_k \varphi_{T*}^{-1} S|_{k-1} \right).
\end{aligned}$$

The inverse inequality (VII.57) gives

$$|\partial_k \varphi_{T*}^{-1} S|_{k-1} \leq C_\partial |\varphi_{T*}^{-1} S|_k.$$

This completes the proof. □

Remark VII.8.8.
With the notation as in the proof of Theorem VII.8.7, we have

$$C_{e,p} := C_M^k C_I \left(1 + C_b^{k+1+\frac{n}{p}}\right) c_M^{k+1} C_M^{k+1} C_{n,k,p}^\flat c_M^k C_M^{\frac{n}{p}} (1 + C_\partial) \mathfrak{L} \cdot \max(1, \mathfrak{K})^k.$$

It suffices that $L_E \mathfrak{L} \epsilon < \mu_{\mathrm{b}}(\mathcal{T})$ and $\epsilon < 1$.

Remark VII.8.9.
Our Theorem VII.8.7 resembles Lemma 5.5 in [9] and Lemma 4.2 in [58]. We give a brief motivation why our method of proof differs from theirs. In order to obtain the interpolation error estimate over simplices $T \in \mathcal{T}$, the authors of the aforementioned references suppose that finite element differential forms are piecewise Lipschitz near T. This holds if T is an interior simplex but not if T touches the boundary of Ω. In what appears to be a gap in the proof, it is not clear how their method applies for such T. The reason is that their extension operator, like ours, involves a pullback along a bi-Lipschitz mapping, so the extended finite element differential form is not necessarily Lipschitz continuous anywhere outside of Ω. The extended differential form, however, is still a flat form, and this motivates our utilization of geometric measure theory to prove the desired estimate for the interpolation error. A particular merit of our solution is that no modification to original construction in [9, 58] is necessary.

We mention that interpolation error estimates (similar to Theorem VII.8.7) were used earlier in [159], which in turn refers to a technical report for the details of the proof. This technical report, however, has not been published as of the time of this writing, and so comparing our proof of the interpolation error estimate, though desirable, is currently not possible. On the other hand, a uniformly bounded commuting projection is constructed in [56] with different techniques.

We are now in the position to prove the main result of this chapter. For $\epsilon > 0$ small enough, the mapping $Q_\epsilon^k : \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ is close enough to the identity operator to be invertible. This leads to the smoothed projection.

Theorem VII.8.10.
Let $\epsilon > 0$ be small enough. There exists a bounded linear operator

$$\pi^k : L^p\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \subseteq L^p\Lambda^k(\Omega), \quad p \in [1,\infty],$$

such that

$$\pi^k u = u, \quad u \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}),$$

such that

$$\mathsf{d}^k \pi^k u = \pi^{k+1} \mathsf{d}^k u, \quad u \in W^{p,q}\Lambda^k(\Omega, \Gamma_T), \quad p, q \in [1,\infty],$$

and such that for all $p \in [1,\infty]$ there exist uniformly bounded $C_{\pi,p} > 0$ with

$$\|\pi^k u\|_{L^p\Lambda^k(\Omega)} \leq C_{\pi,p}\epsilon^{-\frac{n}{p}}\|u\|_{L^p\Lambda^k(\Omega)}, \quad u \in L^p\Lambda^k(\Omega).$$

Proof. If $\epsilon > 0$ is small enough and $p \in [1,\infty]$, then Theorem VII.8.7 implies that

$$\|u - Q^k_\epsilon u\|_{L^p\Lambda^k(\Omega)} \leq \frac{1}{2}\|u\|_{L^p\Lambda^k(\Omega)}, \quad u \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}).$$

By standard results, the linear mapping $Q^k_\epsilon : \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ is invertible. Let $J^k_\epsilon : \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ be its inverse. J^k_ϵ does not depend on p, since Q^k_ϵ does not depend on p. The construction of J^k_ϵ via a Neumann series reveals that

$$\|J^k_\epsilon u\|_{L^p\Lambda^k(\Omega)} \leq 2\|u\|_{L^p\Lambda^k(\Omega)}, \quad u \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}).$$

So J^k_ϵ is bounded. Moreover, J^k_ϵ commutes with the exterior derivative because

$$\mathsf{d}^k J^k_\epsilon u = J^{k+1}_\epsilon Q^{k+1}_\epsilon \mathsf{d}^k J^k_\epsilon u = J^{k+1}_\epsilon \mathsf{d}^k Q^k_\epsilon J^k_\epsilon u = J^{k+1}_\epsilon \mathsf{d}^k u, \quad u \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}).$$

The theorem follows with $\pi^k := J^k_\epsilon Q^k_\epsilon$. □

Remark VII.8.11.
Specifically, it suffices for Theorem VII.8.10 that $\epsilon > 0$ is so small that Theorem VII.8.5 and Theorem VII.8.7 apply and that $C_{e,p}\epsilon < 2$. We may assume $C_{\pi,p} \leq 2C_{Q,p}\mu_{\mathrm{N}}(\mathcal{T})^{\frac{1}{p}}$.

Remark VII.8.12.
We compare our construction of the smoothed projection with previous constructions in the literature, with particular focus on the role of the mesh size function.

The smoothed projection constructed in [9] applies to *quasi-uniform* families of triangulations. In that case, a classical mollification operator can be used instead of our $R^k_{\epsilon\mathrm{h}}$. That result was expanded in [58] to include *shape-uniform* families of triangulations. The Lipschitz continuous mesh size function of Lemma VII.8.1 is specifically inspired by the construction in [58]. But simple examples show that, contrarily to the statement in [58, p.821], a regularization operator with that mesh size function does not yield a continuous differential form. This is due to the differential of the mesh size function being discontinuous in general. The discontinuity of the differential thwarts the global continuity of the regularized differential forms in [58]. This is our motivation to employ a smooth mesh size function as a remedy.

But it is insightful to inspect the situation in more detail. The Lipschitz continuous mesh size function in Lemma VII.8.1 is the limit of the smoothed mesh size function in Lemma VII.8.2 for decreasing smoothing radius. It is natural to ask how this limit process is reflected in the regularization operator. The gradient of the original mesh size function features tangential continuity. Using this additional property, one can show that the regularization operator of [58] does yield differential forms that are piecewise continuous with respect to the triangulation and that are single-valued along simplex boundaries. Consequently, the regularized differential form, though not continuous, still has well-defined degrees of freedom, and the canonical interpolant can be applied as intended. We emphasize that the main result of [58] remains unchanged.

Remark VII.8.13.
Several estimates in this section depend on a Lebesgue exponent $p \in [1, \infty]$. We carefully observe that it suffices to consider only the case $p = 1$: a sufficiently small choice of $\epsilon > 0$ enables Theorem VII.8.10 for all $p \in [1, \infty]$ simultaneously.

Remark VII.8.14.
Throughout this chapter, we have provided explicit formulas for the admissible ranges of ϵ, and we have derived explicit estimates for several constants. In general, these quantities are effectively computable. The only exception are constructions that involve Lipschitz collars. It seems a reasonable assumption that explicit constructions of Lipschitz collars are feasible, at least in principle, for polyhedral domains. Provided such results, all constants in this chapter become effectively computable. Further research on this topic could reveal dependencies on the geometric properties of the domain, such as the boundary curvature.

VIII. A Priori Error Estimates

Commuting projections are of general relevance in the theory of mixed finite element methods. Arnold, Falk, and Winther [11] have embedded mixed finite element methods in the theoretical framework of Hilbert complexes (see Brüning and Lesch [42]), and have developed what can be called a *Galerkin theory of Hilbert complexes*. Here, abstract projection operators take a central role: they are assumed to commute with the differential operators and to satisfy uniform bounds. Given such operators, one can relate algebraic and analytical properties of discrete subcomplexes to the original complex, prove the stability of approximate discrete problems, and derive abstract a priori error estimates for Galerkin approximations.

The major application of this abstract theory is mixed finite element methods for the Hodge Laplace equation, which requires stably bounded smoothed projections from the L^2 de Rham complex over a bounded Lipschitz domain onto finite element de Rham complexes. This was initially accomplished for L^2 de Rham complexes without boundary conditions and finite element spaces over quasi-uniform families of triangulations [9]. Subsequently this was extended to shape-uniform families of triangulations and full homogeneous boundary conditions [58].

In the preceding chapters, we have developed a smoothed projection over the L^2 de Rham complex over weakly Lipschitz domains and moreover considered partial boundary conditions. Furthermore, we have extended the class of finite element spaces to the case of non-uniform polynomial order. Our smoothed projection enables the abstract Galerkin theory of Hilbert complexes.

The aim of this chapter is to elaborate on this application. In particular, we identify the *mixed boundary conditions* of the Hodge Laplace equation associated to the L^2 de Rham complex with *partial boundary conditions*. We give special attention to *harmonic forms with mixed boundary conditions*. The harmonic forms span the kernel of the Hodge Laplace operator and play a singular role in the convergence theory of finite element exterior calculus. When we consider the de Rham complex with either no and full homogeneous boundary conditions, then the dimension of the space of harmonic forms reflects topological properties of the domain; but when we consider the de Rham complex with partial boundary conditions, then the situation is more complicated and new qualitative properties are present: the dimension of the space of harmonic forms depends not only on the topology of the domain but also on the topology of the boundary patch along which the boundary conditions are imposed. Even if the domain itself is topologically simple, the space of harmonic forms satisfying mixed boundary conditions may have a large dimension. This qualitative difference apparently has not been discussed in the literature yet.

To begin with, we give a brief review of the theory of Hilbert complexes in Section VIII.1. We then describe the Hodge Laplace equation with mixed boundary conditions in Section VIII.2, for which we primarily use results of Gol'dshtein, Mitrea and Mitrea [99]. Finite element de Rham complexes are briefly recapitulated in Section VIII.3, where we prove discrete Poincaré-Friedrichs inequalities and show the stability of discrete Hodge Laplace equations. The a priori error estimates in Section VIII.4 are an example application of the abstract Galerkin theory.

This chapter addresses a priori error estimates for the mixed formulation of the Hodge Laplacian equation, which is also known as the *source problem*. The abstract Galerkin theory of Hilbert complexes also gives a priori error estimates for the corresponding eigenvalue problem (see [11]) among other applications, but this is not addressed here.

VIII.1. Notions of Hilbert Complexes

We review basic notions of Hilbert complexes. A thorough discussion in functional analysis has been provided by Brüning and Lesch [42], and a Galerkin theory of Hilbert complexes has been initiated by Arnold, Falk, and Winther [11].

We first recall some notions of linear operators over Hilbert spaces. For every Hilbert space W we let $\langle \cdot, \cdot \rangle_W$ be the associated scalar product and $\| \cdot \|_W$ be the associated norm. We may leave out the subscript and simply write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, if there is no danger of confusion. If $A \subseteq W$ is a linear subspace of W, then we let $A^{\perp_W}$ denote the orthogonal complement of A in W. We also write $A^\perp = A^{\perp_W}$ if the ambient Hilbert space W is known from context.

Suppose that W and $\widetilde{W}$ are Hilbert spaces and that $d : \operatorname{dom}(d) \subseteq W \to \widetilde{W}$ is an unbounded linear operator with domain $\operatorname{dom}(d)$. We let $\ker d = \ker(d)$ denote the kernel of d and let $\operatorname{ran} d = \operatorname{ran}(d)$ denote the *range* of d. We say that d is a closed operator if the *graph*

$$\operatorname{graph}(d) := \left\{ (x, dx) \in W \times \widetilde{W} \;\middle|\; x \in \operatorname{dom}(d) \right\}$$

is a closed subset of $W \times \widetilde{W}$. We say that d is *densely-defined* if $\operatorname{dom}(d)$ is dense in W. The mapping d is called *bounded* if there exists a constant $C > 0$ such that

$$\|dx\| \leq C\|x\|, \quad x \in \operatorname{dom}(d). \tag{VIII.1}$$

We say that d has *closed range* if the range of d is closed; one can show [179, Lemma IV.5.2] that this is equivalent to the existence of $c > 0$ such that

$$\forall y \in \operatorname{ran}(d) : \exists x \in \operatorname{dom}(d) : dx = y \textit{ and } \|x\| \leq c\|y\|, \tag{VIII.2}$$

or equivalently,

$$\forall x \in \operatorname{dom}(d) : \exists x_0 \in \ker(d) : \|x - x_0\| \leq c\|dx\|.$$

The smallest C satisfying (VIII.1) is called the *norm* of d. The smallest c satisfying (VIII.2) is called the *Poincaré-Friedrichs constant* of d. Furthermore, any densely-defined bounded operator can be extended uniquely to the whole of W, which makes it a bounded operator in the classical sense.

Whenever $d : \operatorname{dom}(d) \subseteq W \to \widetilde{W}$ is a densely-defined closed linear operator, then we write $d^* : \operatorname{dom}(d^*) \subseteq \widetilde{W} \to W$ for the adjoint operator. The adjoint is a densely-defined closed linear operator too, and we have $d^{**} = d$. One can show that d is bounded if and only if d^* is bounded, in which case the norms agree. Similarly, one can show that d has closed range if and only if d^* has closed range, in which case the Poincaré-Friedrichs constants agree.

We call d *self-adjoint* if $d = d^*$. Moreover, we make extensive use the concept of pseudoinverse of a bounded linear operator with closed range. We refer to Beutler [25] and Desoer and Whalen [73] for further information on this subject. The *pseudoinverse* of a densely-defined closed linear operator d with closed range is the unique bounded linear operator $d^\dagger : \widetilde{W} \to W$ that is defined by

$$d^\dagger y := \operatorname*{argmin}_{\substack{x \in \operatorname{dom}(x) \\ y = dx}} \|x\|. \tag{VIII.3}$$

One can show that $d^{*\dagger} = d^{\dagger *}$. We remark that the pseudoinverse is the solution operator to the (possibly inconsistent) least-squares problem $dx = y$ with unknown x and data y. The pseudoinverse gives the $x \in \operatorname{dom}(d)$ with minimal norm among the minimizers of $\|y - dx\|$. A particularly important property is

$$\forall x \in \operatorname{dom}(d) : dd^\dagger dx = dx.$$

The norm of $d^\dagger$ is precisely the reciprocal of the Poincaré-Friedrichs constant of d.

A *Hilbert complex* (W, d) consists of a sequence of Hilbert spaces $W = (W^k)_{k\in\mathbb{Z}}$ together with a sequence $d = (d^k)_{k\in\mathbb{Z}}$ of densely-defined closed unbounded operators $d^k : \operatorname{dom}(d^k) \subseteq W^k \to W^{k+1}$ that satisfy the *differential property*

$$\operatorname{ran} d^k \subseteq \ker d^{k+1}. \tag{VIII.4}$$

A Hilbert complex can be visualized as a diagram:

$$\dots \xrightarrow{d^{k-1}} W^k \xrightarrow{d^k} W^{k+1} \xrightarrow{d^{k+1}} \dots \tag{VIII.5}$$

We call a Hilbert complex *bounded* if all differentials $(d^k)_{k\in\mathbb{Z}}$ are bounded operators. We call a Hilbert complex *closed* if all differentials $(d^k)_{k\in\mathbb{Z}}$ have closed range.

If (W, d) is a Hilbert complex, then the *adjoint Hilbert complex* $(W, d)^*$ is the Hilbert complex (W, d^*) where $W = (W^k)_{k\in\mathbb{Z}}$ is the same family of Hilbert spaces and $d^* = (d^*_k)_{k\in\mathbb{Z}}$ is the sequence of adjoint operators. We have the differential property $d^*_{k-1} d^*_k = 0$. We visualize $(W, d)^*$ as the diagram

$$\dots \xleftarrow{d^*_{k-1}} W^k \xleftarrow{d^*_k} W^{k+1} \xleftarrow{d^*_{k+1}} \dots \tag{VIII.6}$$

We note that $(W, d)^{**} = (W, d)$. Moreover, $(W, d)^*$ is bounded if and only if (W, d) is bounded, and $(W, d)^*$ is closed if and only if (W, d) is closed.

Remark VIII.1.1.
The adjoint of a Hilbert complex conforms to a different indexing convention than proper Hilbert complexes. Hence, the adjoint Hilbert complexes are technically a different category of objects. It is convenient, however, to ignore this technical matter, since the differences are only an index convention.

Let (W, d) be an arbitrary but fixed Hilbert complex. We assume that (W, d) is closed. A fundamental concept in the category of Hilbert complexes are *Poincaré-Friedrichs inequalities.* There exists a constant $C_{\mathrm{PF}} > 0$ such that

$$\|x\| \leq C_{\mathrm{PF}} \|d^k x\|, \quad x \in W^k \cap \left(\ker(d^k)\right)^{\perp}. \tag{VIII.7}$$

We call (VIII.7) a *Poincaré-Friedrichs inequality* and C_{PF} the *Poincaré-Friedrichs constant* of (W, d).

Another fundamental concept in the category of Hilbert complexes are harmonic spaces. The *k-th harmonic space* of (W, d) is

$$\mathfrak{H}^k = \ker d^k \cap \ker d^*_{k-1}. \tag{VIII.8}$$

This is a closed subspace of W^k because the kernel of closed unbounded operators is closed and the intersection of closed sets is closed again. By basic facts on Hilbert spaces we have

$$\ker d^k = (\operatorname{ran} d^*_{k-1})^{\perp}, \quad (\operatorname{ran} d^{k-1})^{\perp} = \ker d^*_{k-1}.$$

Hence $\mathfrak{H}^k$ satisfies several identities, such as

$$\mathfrak{H}^k = \ker d^k \cap (\operatorname{ran} d^{k-1})^{\perp} = \ker d^*_{k-1} \cap (\operatorname{ran} d^*_k)^{\perp}. \tag{VIII.9}$$

Recall that the differentials are assumed to have closed ranges. The *abstract Hodge decomposition* of W^k is the orthogonal decomposition

$$W^k = \operatorname{ran} d^{k-1} \oplus \mathfrak{H}^k \oplus \operatorname{ran} d^*_k. \tag{VIII.10}$$

Again, there several equivalent ways to write that decomposition, such as

$$W^k = \operatorname{ran} d^{k-1} \oplus \mathfrak{H}^k \oplus \left(\ker d^k\right)^{\perp}. \tag{VIII.11}$$

The principle relevance of the harmonic spaces is that they appear as "defects" in partial differential equations, as we explore in this thesis.

Remark VIII.1.2.
The homology theory of Hilbert complexes is not a mere specialization of homological algebra. The reason is that differential complexes in homological algebra are always constructed from a category of objects; for example, differential complexes of vector spaces are constructed from the category of vector spaces with linear mappings. But Hilbert spaces with closed densely-defined unbounded mappings do not constitute a category. For example, the product of closed unbounded operators is not closed in general. This forecloses an immediate application of homological algebra. Still many ideas of homological algebra have analogues for Hilbert complexes.

We continue to assume that (W, d) is a fixed closed Hilbert complex. We introduce the sets

$$\operatorname{dom}\left(D^k\right) := \operatorname{dom}(d^k) \cap \operatorname{dom}(d^*_{k-1}),$$
$$\operatorname{dom}\left(\boldsymbol{\Delta}_k\right) := \left\{ x \in \operatorname{dom}\left(D^k\right) \;\middle|\; d^*_{k-1}x \in \operatorname{dom}(d^{k-1}),\ d^k x \in \operatorname{dom}(d^*_k) \right\}.$$

The k-th *Dirac operator* associated to (W, d) is the unbounded operator

$$D^k : \operatorname{dom}\left(D^k\right) \subseteq W^k \to W^{k-1} \oplus W^{k+1}, \quad x \mapsto \left(d^*_{k-1}x, d^k x\right). \tag{VIII.12}$$

We let D^*_k denote the adjoint of D^k. By the Hodge decomposition, it is easy to verify that D^k is densely-defined, closed, and has closed range with the Poincaré-Friedrichs constant being the one of (W, d).

The k-th *Hodge Laplacian* or *Hodge Laplace operator* associated to (W, d) is the unbounded operator

$$\boldsymbol{\Delta}_k : \operatorname{dom}\left(\boldsymbol{\Delta}_k\right) \subseteq W \to W, \quad x \mapsto d^*_k d^k x + d^{k-1} d^*_{k-1} x. \tag{VIII.13}$$

Note that in the sense of unbounded operators

$$\boldsymbol{\Delta}_k = D^*_k D^k = d^*_k d^k + d^{k-1} d^*_{k-1}.$$

The constructions immediately imply that

$$\mathfrak{H}^k = \ker \boldsymbol{\Delta}_k = \ker D^k. \tag{VIII.14}$$

It is easily seen that D^k is densely-defined and closed, and that D^k has closed range if (W, d) is closed. The corresponding properties of $\boldsymbol{\Delta}_k$ require more work.

Theorem VIII.1.3.
The Hodge-Laplacian $\boldsymbol{\Delta}_k$ is densely-defined and closed. Moreover, $\boldsymbol{\Delta}_k$ is self-adjoint and positive semi-definite. We have $\ker \boldsymbol{\Delta}_k = \ker d^k \cap \ker d^*_{k-1}$.

$$\ker \boldsymbol{\Delta}_k = \ker d^k \cap \ker d^*_{k-1} = \left(\operatorname{ran} \boldsymbol{\Delta}_k\right)^{\perp}. \tag{VIII.15}$$

Proof. We show that $\boldsymbol{\Delta}_k$ is densely-defined. The Lax-Milgram theorem (see [86, p.315]) implies that for every $F \in \operatorname{dom}(D^k)'$ there exists $x_F \in \operatorname{dom}(D)$ such that

$$\langle x_F, y\rangle_{W^k} + \langle d^k x_F, d^k y\rangle_{W^{k+1}} + \langle d^*_{k-1} x_F, d^*_{k-1} y\rangle_{W^{k-1}} = F(y), \quad y \in \operatorname{dom}(D^k). \tag{VIII.16}$$

Provided that there exists $f \in W$ such that $F(y) = \langle f, y\rangle_{W^k}$ for all $y \in \operatorname{dom}(D^k)$, then $x_F \in \operatorname{dom}(D^* D)$ follows by definitions. In particular, for every $f \in W$ there exists $x_f \in \operatorname{dom}(D^* D)$ satisfying (VIII.16) with $F(y) = \langle f, y\rangle_{W^k}$. Let us suppose that $x^0 \in \operatorname{dom}(D^k)$ is orthogonal to $\operatorname{dom}(\boldsymbol{\Delta}_k)$ in the Hilbert space $\operatorname{dom}(D^k)$ with the graph scalar product. Then for all $f \in W^k$ we have

$$\langle f, x^0\rangle = \langle x_f, x^0\rangle_{W^k} + \langle d^k x_f, d^k x^0\rangle_{W^{k+1}} + \langle d^*_{k-1} x_f, d^*_{k-1} x^0\rangle_{W^{k-1}} = 0.$$

We conclude $x^0 = 0$. Hence $\operatorname{dom}(\boldsymbol{\Delta}_k)$ is dense in $\operatorname{dom}(D^k)$, and so it is dense in W^k. In particular, $\boldsymbol{\Delta}_k$ is densely-defined.

Next we show that $\mathbf{\Delta}_k$ is self-adjoint, which also implies that $\mathbf{\Delta}_k$ is closed. First, we observe from definitions that $\operatorname{dom}(\mathbf{\Delta}_k) \subseteq \operatorname{dom}(\mathbf{\Delta}_k^*)$, where $\mathbf{\Delta}_k^*$ denotes the adjoint of $\mathbf{\Delta}_k$. Let $x \in \operatorname{dom}(\mathbf{\Delta}_k^*)$, so there exists $C_x > 0$ such that $\langle x, \mathbf{\Delta}_k z\rangle \leq C_x\|x\| \cdot \|z\|$ for all $z \in \mathbf{\Delta}_k$. We have $x \in \operatorname{dom}(D^k)$ if there exists $C_x' > 0$ such that $\langle x, D_k^* y\rangle \leq C_x'\|x\| \cdot \|y\|$ for all $y \in \operatorname{dom}(D_k^*)$. To show this condition, it suffices to consider $y \in \operatorname{dom}(D_k^*)$ with $y \perp \ker(D_k^*)$, which is equivalent to $y \in \operatorname{ran}(D^k)$. But since the Hilbert complex is assumed to be closed, there exists $c > 0$ such that for $y \in \operatorname{dom}(D_k^*) \cap \operatorname{ran}(D^k)$ there exists $z \in \operatorname{dom}(D^k)$ with $D^k z = y$ and $\|z\| \leq c\|y\|$. But then

$$\langle x, D_k^* y\rangle = \langle x, D_k^* D z\rangle \leq C_x\|x\| \cdot \|z\| \leq C_x\|x\| \cdot c\|y\|.$$

Hence $x \in \operatorname{dom}(D^k)$. Next, we have $D^k x \in \operatorname{dom}(D_k^*)$ if $\langle D^k x, D^k \cdot\rangle$ is a bounded linear functional over W^k, which is already the case if the functional is bounded over the dense subset $\operatorname{dom}(\mathbf{\Delta}_k)$. This last condition, however, is satisfied, since $x \in \operatorname{dom}(\mathbf{\Delta}_k^*)$. We conclude that $\mathbf{\Delta}_k = D_k^* D^k$ is self-adjoint.

That $\mathbf{\Delta}^k$ is positive semi-definite is easily verified too. Finally, (VIII.15) follows from definitions. □

We are particularly interested in the Hodge Laplace equation $\mathbf{\Delta}_k u = f$. Since the self-adjoint operator $\mathbf{\Delta}_k$ generally has a non-trivial kernel, we may use a Lagrange multiplier. In applications, the kernel is finite-dimensional.

The *k-th Hodge Laplace problem* associated to the Hilbert complex (W, d) is to find $u \in \operatorname{dom}(\mathbf{\Delta}_k)$ and $p \in \mathfrak{H}^k$ such that

$$\mathbf{\Delta}_k u + p = f, \quad u \perp \mathfrak{H}^k, \tag{VIII.17}$$

for given $f \in W^k$. We call this the strong formulation of the Hodge Laplace problem. One can show that there exists a bounded operator $\mathbf{\Delta}_k^\dagger$, the pseudoinverse of $\mathbf{\Delta}_k$, such that for every $f \in W$ the solution of (VIII.17) is given by $u = \mathbf{\Delta}_k^\dagger f$ and $p = f - \mathbf{\Delta}_k u$. This means that the strong formulation is well-posed.

Similar to the case of the Poisson problem, a weak formulation of the Hodge Laplace problem is of interest. We equip the space $\operatorname{dom}(D^k)$ with the graph scalar product of D^k, which makes $\operatorname{dom}(D^k)$ into a Hilbert space. We let $\operatorname{dom}(D^k)'$ denote the dual Hilbert space. The weak k-th Hodge Laplace problem associated to (W, d) is to find $u \in \operatorname{dom}(D^k)$ and $p \in \mathfrak{H}^k$ such that

$$\langle d^k u, d^k v\rangle + \langle d_k^* u, d_k^* v\rangle + \langle p, v\rangle = F(v), \quad v \in \operatorname{dom}(D^k), \tag{VIII.18a}$$

$$\langle u, q\rangle = 0, \quad q \in \mathfrak{H}^k, \tag{VIII.18b}$$

for given functional $F \in \operatorname{dom}(D^k)'$. The existence and uniqueness of a solution is now an elementary consequence of the Lax-Milgram theorem together with a small modification to account for the Lagrange multiplier. Moreover, if $F(v) = \langle f, v\rangle$, then (u, p) is a solution of the weak formulation if and only if it is a solution of the strong formulation.

It is intuitive that the weak formulation of the Hodge Laplace problem is more amenable for Galerkin methods. But in practice, the weak formulation is still too strong. A conforming Galerkin method will approximate the solution in $\operatorname{dom}(D^k)$, which still poses too strong requirements on the approximation space.

Instead, we will focus on a third formulation of the Hodge Laplace problem. In accordance with the notation in [11], we abbreviate $V^k = \operatorname{dom}(d^k)$ for the domains of the differentials equipped with the graph scalar product of d. The corresponding norm on the Hilbert space V^k will be written $\|\cdot\|_{V^k}$ in the sequel, and induced norm on the dual Hilbert space V' will be written $\|\cdot\|_{V'}$. The mixed formulation of the Hodge Laplace problem is now to find the unknowns

$$(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k \tag{VIII.19}$$

such that for given right-hand side

$$(G, F, r) \in \left(V^{k-1}\right)' \times \left(V^k\right)' \times \mathfrak{H}^k \tag{VIII.20}$$

we have

$$\langle \sigma, \tau \rangle - \langle u, d^{k-1}\tau \rangle = G(\tau), \quad \tau \in V^{k-1}, \tag{VIII.21a}$$

$$\langle d^{k-1}\sigma, v \rangle + \langle d^k u, d^k v \rangle + \langle p, v \rangle = F(v), \quad v \in V^k, \tag{VIII.21b}$$

$$\langle u, q \rangle = \langle r, q \rangle, \quad q \in \mathfrak{H}^k. \tag{VIII.21c}$$

If $(u, p) \in \operatorname{dom}(\boldsymbol{\Delta}_k) \times \mathfrak{H}^k$ is the strong solution of the Hodge Laplace problem with right-hand side f, then (σ, u, p), where $\sigma = d_k^* u$, solves the mixed problem with $G = 0$ and $q = 0$ and $F(v) = \langle f, v \rangle$. On the other hand, if $G = 0$ and $r = 0$ and $F(v) = \langle f, v \rangle$, then the solution of the mixed from (σ, u, p) satisfies $\sigma = d_k^* u$ and (u, p) is the solution of the strong formulation. Proving the stability of the mixed formulation is more complex than for the mixed formulation. The stability constant depends on the Poincaré-Friedrichs constant of (W, d). We recall the following result.

Theorem VIII.1.4.
There exists a constant $C > 0$, depending only the Poincaré-Friedrichs constant C_{PF}, such that for every right-hand side $(G, F, q) \in \left(V^{k-1}\right)' \times \left(V^k\right)' \times \mathfrak{H}^k$ there exists a unique solution $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ of (VIII.21) such that

$$\left(\|\sigma\|_V + \|u\|_V + \|p\|_W\right) \leq C \left(\|G\|_{V'} + \|F\|_{V'} + \|r\|_W\right).$$

Proof. This follows by Theorem 3.1, Theorem 3.2 and the subsequent discussion in [11]. The constant C depends only on the constant γ in the statement of Theorem 3.2 in [11], which in turn only depends on the Poincaré-Friedrichs constant. □

VIII.2. L^2 de Rham Complex over Domains

The framework of Hilbert complexes is applied in several instances throughout this thesis. Here we consider the most central application: the L^2 de Rham complex over a weakly Lipschitz domain. More precisely, we consider the case of partial boundary conditions.

Throughout this chapter we let $\Omega \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain. In addition, we let $(\Gamma_T, \Gamma_I, \Gamma_N)$ be an admissible boundary partition. We focus on spaces of the form $L^{2,2}\Lambda^k(\Omega, \Gamma_T)$ for $k \in \mathbb{Z}$. Following [99], we write

$$H_T\Lambda^k(\Omega) := W^{2,2}\Lambda^k(\Omega, \Gamma_T), \tag{VIII.22}$$

$$H^\star_N\Lambda^k(\Omega) := \star W^{2,2}\Lambda^{n-k}(\Omega, \Gamma_N), \tag{VIII.23}$$

and consider the unbounded linear operators

$$\mathsf{d}^k : H_T\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \to H_T\Lambda^{k+1}(\Omega), \tag{VIII.24}$$

$$\delta^k : H^\star_N\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \to H^\star_N\Lambda^{k-1}(\Omega). \tag{VIII.25}$$

These linear operators are closed and densely-defined, since $H_T\Lambda^k(\Omega)$ and $H^\star_N\Lambda^k(\Omega)$ are Hilbert spaces, and additionally they are mutually adjoint, which means

$$\int_\Omega \mathsf{d}^k u \wedge \star v = \int_\Omega u \wedge \star\delta^{k+1} v, \quad u \in H_T\Lambda^k(\Omega), \quad v \in H^\star_N\Lambda^{k+1}(\Omega). \tag{VIII.26}$$

The identity (VIII.26) is an easy consequence of approximation by smooth differential forms (see Lemma V.3.5). We may assemble the L^2 de Rham complex with tangential boundary conditions along Γ_T,

$$\dots \xrightarrow{\mathsf{d}^{k-1}} H_T\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \xrightarrow{\mathsf{d}^k} H_T\Lambda^{k+1}(\Omega) \subseteq L^2\Lambda^{k+1}(\Omega) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{VIII.27}$$

and its adjoint L^2 de Rham complex with normal boundary conditions along Γ_N,

$$\dots \xleftarrow{\delta^k} H^*_N\Lambda^k(\Omega) \subseteq L^2\Lambda^k(\Omega) \xleftarrow{\delta^{k+1}} H^*_N\Lambda^{k+1}(\Omega) \subseteq L^2\Lambda^{k+1}(\Omega) \xleftarrow{\delta^{k+2}} \dots \tag{VIII.28}$$

It is evident that the Hilbert complexes (VIII.27) and (VIII.28) are mutually adjoint. That (VIII.24) and (VIII.25) have closed range follows by [99, Proposition 4.3(i)]. In particular, the Hilbert complexes (VIII.27) and (VIII.28) are closed. One implication is a Poincaré-Friedrichs inequality. There exists $C_{\mathrm{PF}} > 0$ such that

$$\forall u \in H_T\Lambda^k(\Omega) : \exists u_0 \in H_T\Lambda^k(\Omega) \cap \ker \mathsf{d}^k : \|u - u_0\| \leq C_{\mathrm{PF}}\|\mathsf{d}^k u\|.$$

The space of *k-th harmonic forms with mixed boundary conditions* is defined as

$$\mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N) := \left\{ p \in H_T\Lambda^k(\Omega) \cap H^*_N\Lambda^k(\Omega) \;\middle|\; \mathsf{d}^k p = 0,\ \delta^k p = 0 \right\}.$$

It can be shown that $\mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N)$ has finite dimension [99, Proposition 4.3]. Putting all this together, we have the L^2 orthogonal *Hodge decomposition*

$$L^2\Lambda^k(\Omega) := \mathsf{d}^{k-1} H\Lambda^{k-1}_T(\Omega) \oplus \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N) \oplus \delta^{k+1} H^\star\Lambda^{k+1}_N(\Omega). \tag{VIII.29}$$

The dimension of $\mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N)$ depends only on the topology of Ω and Γ_T, and is of independent interest. More specifically, from Theorem 5.3 in [99] we find that $\dim \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N)$ equals the k-th topological Betti number $b_k(\overline{\Omega}, \Gamma_T)$ of $\overline{\Omega}$ relative to Γ_T, and equals $(n-k)$-th topological Betti number $b_{n-k}(\overline{\Omega}, \Gamma_N)$ of $\overline{\Omega}$ relative to Γ_N. Thus we have

$$\dim \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N) = b_k(\overline{\Omega}, \Gamma_T) = b_{n-k}(\overline{\Omega}, \Gamma_N), \quad k \in \mathbb{Z}. \tag{VIII.30}$$

The special cases $\Gamma_T = \emptyset$ and $\Gamma_T = \partial\Omega$ have received the most attention in the literature. In that case, the Betti numbers correspond to the topological properties of the domain, such as the number of holes of a certain dimension. In the presence of mixed boundary conditions, the Betti numbers depend also on the topology of the boundary patch Γ_T.

Example VIII.2.1.
For every weakly Lipschitz domain, the Betti number $b_0(\overline{\Omega}, \Gamma_T)$ is the number of connected components that do not touch Γ_T. The space $\mathfrak{H}^0(\Omega, \Gamma_T, \Gamma_N)$ corresponds to the span of the indicator functions of those components of Ω.

We consider an example over a domain with a very simple topology, where the harmonic forms with mixed boundary conditions may still have a very non-trivial kernel, depending on the topology of the boundary patch. Consider the example $\Omega = (-1,1)^2$ and let $\Gamma_T \subseteq \partial\Omega$ be the union of $m \in \mathbb{N}$ relatively open subsets of $\partial\Omega$ that pairwise have positive distance. In that case one can show that

$$b_2(\overline{\Omega}, \Gamma_T) = 0, \quad b_1(\overline{\Omega}, \Gamma_T) = m-1, \quad b_0(\overline{\Omega}, \Gamma_T) = 0.$$

In particular, $\dim \mathfrak{H}^1(\Omega, \Gamma_T, \Gamma_N) = m-1$.

We are also interested in an analogue of the Rellich embedding for differential forms. The intersection $H_T\Lambda^k(\Omega) \cap H^\star_N\Lambda^k(\Omega)$ is a Hilbert space that can be equipped with the compatible norm that is uniquely defined by

$$\|u\|^2_{H_T\Lambda^k(\Omega)\cap H^\star_N\Lambda^k(\Omega)} := \|u\|^2_{H_T\Lambda^k(\Omega)} + \|u\|^2_{H^\star_N\Lambda^k(\Omega)}, \quad u \in H_T\Lambda^k(\Omega) \cap H^\star_N\Lambda^k(\Omega).$$

From Proposition 4.4 of [99] we know that the embedding

$$H_T\Lambda^k(\Omega) \cap H^\star_N\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega), \tag{VIII.31}$$

known as *Rellich embedding*, is compact. Stronger conditions on the domain and the boundary patch imply refined versions of the Rellich embedding. We use this later in this chapter after having introduced fractional Sobolev spaces.

The L^2 de Rham complex with tangential boundary conditions along Γ_T gives rise to the Hodge Laplace operator, which we have introduced abstractly in the previous section. Let

$$\operatorname{dom}(\mathbf{\Delta}_k) := \left\{ u \in H\Lambda^k_T(\Omega) \cap H^\star\Lambda^k_N(\Omega) \;\middle|\; \begin{array}{c} \mathsf{d}^{k-1}u \in H^\star\Lambda^{k+1}_N(\Omega), \\ \delta^{k+1}u \in H\Lambda^{k-1}_T(\Omega) \end{array} \right\}$$

The k-th *Hodge Laplacian* with respect to these boundary conditions is the unbounded operator

$$\mathbf{\Delta}_k : \operatorname{dom}(\mathbf{\Delta}_k) \subseteq L^2\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega), \quad u \mapsto \delta^{k+1}\mathsf{d}^k u + \mathsf{d}^{k-1}\delta^k u.$$

As a consequence of Theorem VIII.1.3, the operator $\mathbf{\Delta}_k$ is densely-defined, closed, self-adjoint, and has closed range. Furthermore,

$$\mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N) = \ker \mathbf{\Delta}_k, \quad \operatorname{ran} \mathbf{\Delta}_k = \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N)^\perp.$$

Since $\mathbf{\Delta}_k$ has closed range, it has a bounded pseudoinverse. We let

$$\mathscr{G}_k : L^2\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega)$$

denote the pseudoinverse of the Hodge Laplacian in the sense of [73]. We have

$$\mathscr{G}_k f \in \operatorname{dom}(\mathbf{\Delta}_k), \quad \mathscr{G}_k f \perp \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N), \quad f - \mathbf{\Delta}_k \mathscr{G}_k f \in \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N).$$

for all $f \in L^2\Lambda^k(\Omega)$. One can show that the operator norm of $\mathscr{G}_k$ is bounded by C_{PF}^2. Additionally, $\mathscr{G}_k$ takes values in the intersection $H_T\Lambda^k(\Omega) \cap H_N^\star\Lambda^k(\Omega)$ and is bounded as an operator from $L^2\Lambda^k(\Omega)$ to $H_T\Lambda^k(\Omega) \cap H_N^\star\Lambda^k(\Omega)$. The compactness of $\mathscr{G}_k$ follows from the compactness of the Rellich embedding.

The strong formulation of the k-th Hodge Laplace equation with mixed boundary conditions asks for $u \in \operatorname{dom}(\mathbf{\Delta}_k)$ and $p \in \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N)$ such that

$$\mathbf{\Delta}_k u = f - p, \tag{VIII.32}$$

for given data $f \in L^2\Lambda^k(\Omega)$. The solution to this system is given by

$$u = \mathscr{G}_k f, \quad p = f - \mathbf{\Delta}_k \mathscr{G}_k f$$

The strong formulation should not be regarded as amenable for a finite element method. For example, it is difficult to construct shape functions in $\operatorname{dom}(\mathbf{\Delta}_k)$. The weak formulation of the Hodge Laplace equation is not of much interest in this thesis either: even though one can construct piecewise polynomial differential forms in $H_T\Lambda^k(\Omega) \cap H_N^*\Lambda^k(\Omega)$, such differential forms are generally not dense in $H_T\Lambda^k(\Omega) \cap H_N^*\Lambda^k(\Omega)$, which makes the weak formulation a generally inconsistent method; see also Remark VIII.2.3 below.

We consider the mixed formulation of the Hodge Laplace equation. We search for the unknown

$$(\sigma, u, p) \in H_T\Lambda^{k-1}(\Omega) \times H_T\Lambda^k(\Omega) \times \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N) \tag{VIII.33}$$

which for given right-hand side

$$(G, F, r) \in H_T\Lambda^{k-1}(\Omega)' \times H_T\Lambda^k(\Omega)' \times \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N) \tag{VIII.34}$$

solves the problem

$$\langle \sigma, \tau \rangle - \langle u, \mathsf{d}^{k-1}\tau \rangle = G(\tau), \quad \tau \in H_T\Lambda^{k-1}(\Omega), \tag{VIII.35a}$$

$$\langle \mathsf{d}^{k-1}\sigma, v \rangle + \langle \mathsf{d}^k u, \mathsf{d}^k v \rangle + \langle p, v \rangle = F(v), \quad v \in H_T\Lambda^k(\Omega), \tag{VIII.35b}$$

$$\langle u, q \rangle = \langle r, q \rangle, \quad q \in \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N). \tag{VIII.35c}$$

The well-posedness of this problem is a direct consequence of Theorem VIII.1.4.

Corollary VIII.2.2.
There exists a constant $C > 0$, depending only on the Poincaré-Friedrichs constant of (VIII.27), such that for every

$$(G, F, q) \in H_T\Lambda^{k-1}(\Omega)' \times H_T\Lambda^k(\Omega)' \times \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N)$$

there exists a unique solution

$$(\sigma, u, p) \in H_T\Lambda^{k-1}(\Omega) \times H_T\Lambda^k(\Omega) \times \mathfrak{H}^k(\Omega, \Gamma_T, \Gamma_N)$$

of the mixed formulation such that

$$(\|\sigma\|_V + \|u\|_V + \|p\|_W) \leq C\left(\|G\|_{V'} + \|F\|_{V'} + \|r\|_W\right).$$

Remark VIII.2.3.
The weak formulation of the Hodge Laplace equation leads to a semi-elliptic variational formulation over the intersection space $H_T\Lambda^k(\Omega) \cap H_N\Lambda^k(\Omega)$. It is not trivial how to use this as the base of a Galerkin method (but see, e.g., [63] for approaches in that direction). The reason is that every simplexwise polynomial subspace of $H_T\Lambda^k(\Omega) \cap H_N\Lambda^k(\Omega)$ is necessarily a space of differential k-forms with coefficients in the Sobolev space $H^1(\Omega)$. But such a space generally has infinite codimension (see Costabel [62]). By contrast, a Galerkin method based on the mixed formulation of the Hodge Laplace equation requires only a conforming discretization of the spaces $H_T\Lambda^k(\Omega)$. Indeed, we will develop a convergent and stable mixed finite element method in this chapter.

As mentioned above, there exist refinements of the Rellich embedding for special domains and boundary conditions. To enable the discussion of this, it will be helpful to discuss differential forms with coefficients in Sobolev-Slobodeckij spaces of higher, possibly non-integer order. We give only a few definitions, and refer to specialized literature (e.g., [7, 45, 74, 158]) for further information.

We begin with the scalar-valued case. For every $s \in \mathbb{N}_0$ we let $W^s(\Omega)$ denote the *Sobolev-Slobodeckij space* of order s. This is a Hilbert space with scalar product

$$\langle u, v\rangle_{W^s(\Omega)} := \sum_{r=0}^{s} \sum_{\alpha \in A(r,n)} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2(\Omega)}, \quad u, v \in W^s(\Omega), \tag{VIII.36}$$

where the partial derivatives are taken in the weak sense. We write $\|\cdot\|_{W^s(\Omega)}$ for the associated norm. Note that $L^2(\Omega) = W^0(\Omega)$. In order to treat fractional Sobolev spaces, we introduce for every $\theta \in (0,1)$ the (possibly infinite) quantities

$$[u, v]_{W^\theta(\Omega)} := \int_\Omega \int_\Omega \frac{\langle u(x) - u(y), v(x) - v(y)\rangle}{|x-y|^{n+2\theta}} \,\mathrm{d}x\mathrm{d}y, \quad u, v \in L^2(\Omega). \tag{VIII.37}$$

The members of $u \in L^2(\Omega)$ with $[u,u]_{W^\theta(\Omega)} < \infty$ constitute the Hilbert space $W^\theta(\Omega)$, the Sobolev-Slobodeckij space of order θ. This is a Hilbert space with scalar product

$$\langle u, v\rangle_{W^\theta(\Omega)} := \langle u, v\rangle_{L^2(\Omega)} + [u, v]_{W^\theta(\Omega)}. \tag{VIII.38}$$

We write $\|\cdot\|_{W^\theta(\Omega)}$ for the corresponding norm.

Combining these definitions, we develop for $s = r + \theta$ with $r \in \mathbb{N}_0$ and $\theta \in (0,1)$ the Sobolev-Slobodeckij space $W^s(\Omega)$. This is a Hilbert space continuously embedded in $L^2(\Omega)$, and its scalar product is

$$\langle u, v\rangle_{W^s(\Omega)} = \langle u, v\rangle_{W^r(\Omega)} + \sum_{\alpha \in A(r,n)} [\partial^\alpha u, \partial^\alpha v]_{W^\theta(\Omega)}, \quad u, v \in W^s(\Omega).$$

We write $\|\cdot\|_{W^s(\Omega)}$ for the corresponding norm. We avoid further details of fractional Sobolev spaces at this point. We remark that $W^t(\Omega)$ is continuously embedded in $W^s(\Omega)$ for $t > s$ as can be verified easily from definitions.

For $k \in \mathbb{Z}$ and $s \in \mathbb{R}_0^+$ we let $W^s\Lambda^k(\Omega)$ denote the subspace of $L^2\Lambda^k(\Omega)$ that is spanned by differential k-forms with coefficients in $W^s(\Omega)$. This space is isometric to a direct sum of several copies of $W^s(\Omega)$ and satisfies completely analogous properties as the scalar-valued instances. We let $\|\cdot\|_{W^s\Lambda^k(\Omega)}$ denote the corresponding norm.

We now provide the stronger versions of the Rellich embedding, which in this case are known as Gaffney inequalities. We say that a *Gaffney inequality* with exponent $s \in \mathbb{R}^+$ holds if there exists $s \in \mathbb{R}_0^+$ and $C > 0$ such that

$$\|u\|_{W^s\Lambda^k(\Omega)} \leq C\|u\|_{H_T\Lambda^k(\Omega)\cap H_N^\star\Lambda^k(\Omega)}, \quad u \in H_T\Lambda^k(\Omega) \cap H_N^\star\Lambda^k(\Omega). \tag{VIII.39}$$

Such inequalities have been investigated under different conditions. We mention that a Gaffney inequality with $s = 1$ holds if Ω is a strongly convex Lipschitz domain and $\Gamma_T \in \{\emptyset, \partial\Omega\}$ (see [140]). In general, a Gaffney inequality with $s = \frac{1}{2}$ holds for Ω being a weakly Lipschitz domain and Γ_T being an admissible boundary patch. Several Gaffney inequalities with $s \in [\frac{1}{2}, 1]$ can be found in the literature with various conditions on the domains and the boundary conditions, and we refer to Subsection 7.7 of [9] for further information.

VIII.3. Conforming Discretizations

Having reviewed facts on the L^2 de Rham complex, we now investigate its relation to finite element de Rham complexes. Let $\mathcal{T}$ be a triangulation of Ω and let $\mathcal{U} \subseteq \mathcal{T}$ be a triangulation of Γ_T. Furthermore, let $\mathcal{P} : \mathcal{T} \to \mathscr{A}$ be a hierarchical association of admissible sequence types to simplices. We may consider the finite element de Rham complex with partial boundary conditions.

$$0 \to \mathcal{P}\Lambda^0(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \mathcal{P}\Lambda^n(\mathcal{T},\mathcal{U}) \to 0 \tag{VIII.40}$$

By the results of the previous chapter, there exist bounded operators

$$\pi^k : L^2\Lambda^k(\Omega) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \subset L^2\Lambda^k(\Omega), \quad k \in \mathbb{Z},$$

that act as the identity on the finite element spaces and for which the diagram

$$\begin{tikzcd}
\dots \arrow[r, "\mathsf{d}^{k-1}"] & H\Lambda^k(\Omega,\Gamma_T) \arrow[r, "\mathsf{d}^k"] \arrow[d, "\pi^k"'] & H\Lambda^{k+1}(\Omega,\Gamma_T) \arrow[r, "\mathsf{d}^{k+1}"] \arrow[d, "\pi^{k+1}"'] & \dots \\
\dots \arrow[r, "\mathsf{d}^{k-1}"] & \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \arrow[r, "\mathsf{d}^k"] & \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \arrow[r, "\mathsf{d}^{k+1}"] & \dots
\end{tikzcd}$$

commutes. Furthermore, the L^2 operator norms of smoothed projections π^k can be bounded in terms of properties of Ω, the geometric shape constant, and the maximal polynomial order indicated by $\mathcal{P}$. In the sequel, we largely keep the dependence on $\mathcal{P}$ implicit in the notation.

The finite element de Rham complex is a Hilbert complex with respect to the L^2 scalar product. Consequently, the theory of Hilbert complexes can be instantiated similarly as for the L^2 de Rham complex with tangential boundary conditions.

Since the finite element spaces are finite-dimensional, the finite element de Rham complex is bounded and closed. The *k-th discrete harmonic space* is defined as

$$\begin{aligned}\mathfrak{H}^k_{\mathcal{P}}(\mathcal{T},\mathcal{U}) := \ker\left(\mathsf{d}^k : \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U})\right)\\ \cap\left(\mathsf{d}^{k-1}\mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U})\right)^{\perp}.\end{aligned} \tag{VIII.41}$$

This corresponds to the identity (VIII.9) satisfied by harmonic spaces of a Hilbert complex. The identity (VIII.8) is less helpful to define $\mathfrak{H}^k_{\mathcal{P}}(\mathcal{T},\mathcal{U})$ because the adjoint of the discrete exterior derivative does not have a neat description.

We have the L^2-orthogonal decomposition

$$\begin{aligned}\mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) = \mathsf{d}^{k-1}\mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \oplus \mathfrak{H}^k_{\mathcal{P}}(\mathcal{T},\mathcal{U})\\ \oplus \ker\left(\mathsf{d}^k : \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U})\right)^{\perp},\end{aligned} \tag{VIII.42}$$

known as *discrete Hodge decomposition*. Moreover, there exists a constant $C_{\mathrm{PF},\mathcal{P}} > 0$ such that a discrete Poincaré-Friedrichs inequality holds:

$$\forall u \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) : \exists u_0 \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \cap \ker \mathsf{d}^k : \|u - u_0\|_{L^2\Lambda^k} \leq C_{\mathrm{PF},\mathcal{P}} \|\mathsf{d}^k u\|_{L^2\Lambda^{k+1}}.$$

The presence of the smoothed projections is an additional structure which allows us to relate concepts of (VIII.40) to concepts of (VIII.24). This pertains to the Poincaré-Friedrichs inequality. A consequence of Theorem 3.6 of [11], and its proof, is the estimate

$$C_{\mathrm{PF},\mathcal{P}} \leq \|\pi^{k-1}\| \cdot C_{\mathrm{PF}}. \tag{VIII.43}$$

There are different techniques to determine the dimensions of the discrete harmonic spaces. We do not engage in this topic deeper here; in subsequent chapters we will determine the dimensions of the harmonic spaces with different methods. We show that the dimension of $\mathfrak{H}^k_{\mathcal{P}}(\mathcal{T},\mathcal{U})$ is the Betti number $b_k(\overline{\Omega}, \Gamma_T)$ of $\overline{\Omega}$ relative to Γ_T.

Finally, the smoothed projection can be used to provide a priori error estimates in a Galerkin setting. We consider the mixed Hodge Laplace equation for the finite element de Rham complex with a right-hand side $f \in L^2\Lambda^k(\Omega)$. We search for

$$(\sigma_h, u_h, p_h) \in \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \times \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \times \mathfrak{H}^k_{\mathcal{P}}(\mathcal{T},\mathcal{U})$$

solving the discrete system

$$\langle \sigma_h, \tau_h \rangle - \langle u_h, \mathsf{d}^{k-1} \tau_h \rangle = 0, \quad \tau_h \in \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U}), \tag{VIII.44a}$$

$$\langle \mathsf{d}^{k-1} \sigma_h, v_h \rangle + \langle \mathsf{d}^k u_h, \mathsf{d}^k v_h \rangle + \langle p_h, v_h \rangle = \langle f, v_h \rangle, \quad v_h \in \mathcal{P}\Lambda^{k}(\mathcal{T},\mathcal{U}), \tag{VIII.44b}$$

$$\langle u_h, q_h \rangle = 0, \quad q_h \in \mathfrak{H}^k_{\mathcal{P}}(\mathcal{T},\mathcal{U}). \tag{VIII.44c}$$

Analogously to the mixed formulation of the original Hodge Laplace problem, the problem on the finite element spaces is well-posed. For every $f \in L^2\Lambda^k(\Omega)$ there exists a unique solution (σ_h, u_h, p_h) of (VIII.44), and we have

$$\|u_h\|_{H\Lambda^k} + \|\sigma_h\|_{H\Lambda^{k-1}} + \|p_h\|_{L^2\Lambda^k(\Omega)} \leq C \|f\|_{L^2\Lambda^k(\Omega)},$$

where $C > 0$ depends only on $C_{\mathrm{PF},\mathcal{P}}$. This discrete system can serve to compute approximate solutions of the original system. We note that the variables u_h and σ_h are approximations to u and σ, respectively, in conforming finite element spaces. By contrast, the discrete harmonic form p_h is an approximation to p in a generally non-conforming space of discrete harmonic forms. If the space of discrete harmonic forms is non-trivial, then a basis should be computed in the first place.

VIII.4. Finite Element Approximation

The smoothed projection allows us to derive a priori error estimates, relating the respective solutions of (VIII.21) and (VIII.44) in a priori error estimates. We define

$$\begin{aligned} E(v) &:= \inf_{v_h \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})} \|v - v_h\|_{L^2\Lambda^k(\Omega)}, \quad v \in L^2\Lambda^k(\Omega), \\ E_{\mathsf{d}}(v) &:= \inf_{v_h \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})} \|v - v_h\|_{H\Lambda^k(\Omega)}, \quad v \in H\Lambda^k(\Omega), \\ \mathfrak{a} &:= \sup_{\substack{p \in \mathfrak{H}^k(\Omega,\Gamma_T,\Gamma_N) \\ \|p\|_{L^2\Lambda^k(\Omega)}=1}} \|p - \pi^k p\|_{L^2\Lambda^k(\Omega)}. \end{aligned}$$

Note that the quantity $\mathfrak{a}$ measures the approximation of the harmonic forms by the finite element space.

Let (σ, u, p) solve the original mixed Hodge Laplace system (VIII.21) and let (σ_h, u_h, p_h) solve the discrete Hodge Laplace system (VIII.44), then Theorem 3.9 of [11] implies

$$\begin{aligned} &\|\sigma - \sigma_h\|_{H\Lambda^{k-1}(\Omega)} + \|u - u_h\|_{H\Lambda^k(\Omega)} + \|p - p_h\|_{L^2\Lambda^k(\Omega)} \\ &\leq C\left(E^{k-1}_{\mathsf{d}}(\sigma) + E^k_{\mathsf{d}}(u) + E^k(p) + \mathfrak{a} E^k_{\mathsf{d}}(u_{\mathsf{d}}) \right). \end{aligned}$$

Here and below, u_{d} is the $L^2\Lambda^k(\Omega)$-orthogonal projection of u onto $\mathsf{d}^k H_T\Lambda^k(\Omega)$.

This priori error estimate shows the convergence of the method. Improved error estimates reflect that in typical applications different parts of the error display different convergence behaviors. We utilize the solution operator $\mathscr{G}_k$ in the case of the Hodge Laplace operator of the L^2 de Rham complex with partial boundary conditions along Γ_T. In addition to the quantity $\mathfrak{a}$, we introduce the quantities

$$\begin{aligned}
\mathfrak{b} &:= \sup_{\substack{f \in L^2\Lambda^k(\Omega) \\ f \neq 0}} \frac{\|(1-\pi^k)\mathscr{G}_k f\|_{L^2\Lambda^k(\Omega)}}{\|f\|_{L^2\Lambda^k(\Omega)}}, \\
\mathfrak{c}_{\mathsf{d}}^k &:= \sup_{\substack{f \in L^2\Lambda^k(\Omega) \\ f \neq 0}} \frac{\|(1-\pi^{k+1})\mathsf{d}^k\mathscr{G}_k f\|_{L^2\Lambda^{k+1}(\Omega)}}{\|f\|_{L^2\Lambda^k(\Omega)}}, \\
\mathfrak{c}_{\delta}^k &:= \sup_{\substack{f \in L^2\Lambda^k(\Omega) \\ f \neq 0}} \frac{\|(1-\pi^{k-1})\delta^k\mathscr{G}_k f\|_{L^2\Lambda^{k-1}(\Omega)}}{\|f\|_{L^2\Lambda^k(\Omega)}}, \\
\mathfrak{c} &:= \sup\{\mathfrak{c}_{\mathsf{d}}^k, \mathfrak{c}_{\delta}^k, \mathfrak{c}_{\mathsf{d}}^{k-1}, \mathfrak{c}_{\delta}^{k+1}\}.
\end{aligned}$$

The mapping properties of $\mathscr{G}_k$ imply that these quantities are well-defined and bounded in terms of C_{PF} and the L^2 norm of the smoothed projection. Via Theorem 3.11 of [11] we conclude the existence of $C > 0$, depending only on C_{PF} and the L^2 operator norm of the smoothed projection, such that

$$\begin{aligned}
\|\mathsf{d}^{k-1}(\sigma - \sigma_h)\|_{L^2\Lambda^k(\Omega)} &\leq C E^k(\mathsf{d}^{k-1}\sigma), \\
\|\sigma - \sigma_h\|_{L^2\Lambda^{k-1}(\Omega)} &\leq C\left(E^{k-1}(\sigma) + \mathfrak{c}E^k(\mathsf{d}^{k-1}\sigma)\right), \\
\|p - p_h\|_{L^2\Lambda^k(\Omega)} &\leq C\left(E^k(p) + \mathfrak{a}E^k(\mathsf{d}^{k-1}\sigma)\right), \\
\|\mathsf{d}^k(u - u_h)\|_{L^2\Lambda^{k+1}(\Omega)} &\leq C\left(E^{k+1}(\mathsf{d}^k u) + \mathfrak{c}E^k(\mathsf{d}^{k-1}\sigma) + \mathfrak{c}E^k(p)\right), \\
\|u - u_h\|_{L^2\Lambda^k(\Omega)} &\leq C\begin{pmatrix} E^k(u) + \mathfrak{c}E^{k+1}(\mathsf{d}^k u) + \mathfrak{c}E^{k-1}(\sigma) \\ +(\mathfrak{c}^2 + \mathfrak{b})\left(E^k(\mathsf{d}^{k-1}\sigma) + E^k(p)\right) + \mathfrak{a}E^k(u_{\mathsf{d}}) \end{pmatrix}.
\end{aligned}$$

Remark VIII.4.1.
The relevance of these estimates is that the quantities $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ often display additional convergence behavior, which facilitates improved error estimates. In applications, we have uniformly bounded sequences of finite element de Rham complexes with uniformly bounded smoothed projections, such that the projections converge pointwise to the identity. One can then show, via compactness of the Rellich embedding, that the quantities $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ converge to zero. If a Gaffney inequality such as (VIII.39) is valid, then their convergence order can often be quantified, e.g. in terms of the mesh size using the Bramble-Hilbert lemma. We refer to [11] for a discussion and further details.

IX. Discrete Distributional Differential Forms

In this and the next chapter we devote ourselves to the interaction of finite element exterior calculus and topics in a posteriori error estimation. Our main reference is an important publication by Braess and Schöberl [34], who provide multiple new ideas in the area of vector-valued finite element methods.

One of their contributions have been distributional finite element sequences in the language of vector calculus. The agenda of this chapter is to integrate this interesting concept into finite element exterior calculus and generalize it substantially. This leads to the notion of *discrete distributional differential form*, from which we assemble *discrete distributional de Rham complexes*. Apart from exploring the technical definitions, we study the homology theory and the Poincaré-Friedrichs inequalities of such de Rham complexes.

Distributional finite element spaces appear throughout numerical analysis, in areas such as a posteriori error estimation and of discontinuous Galerkin methods. The research on discrete distributional differential forms, however, produces new results that return to conforming finite element spaces again: for example, this research has provided the first computation of the homology spaces and Poincaré-Friedrichs inequalities of (conforming) finite element de Rham complexes with *partial* boundary conditions.

Braess and Schöberl have introduced distributional finite element sequences as theoretical background for research on equilibrated a posteriori error estimation in computational electromagnetism. Their seminal publication studies these sequences in the language of vector calculus and with lowest polynomial order over local element patches.

We have skimmed over several examples of distributional finite element sequences already in the introduction of this thesis. For further motivation we revisit these objects, which are derived from the work of Braess and Schöberl. Here we employ the formalism of vector calculus, close to [34], but the remainder of this chapter employs the calculus of differential forms. Let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain with a triangulation $\mathcal{T}$.

Our starting point is the lowest-order finite element complex defined with respect to this triangulation and satisfying boundary conditions:

$$\mathcal{P}_0^1(\mathcal{T}) \xrightarrow{\text{grad}} \mathbf{Nd}_0^0(\mathcal{T}) \xrightarrow{\text{curl}} \mathbf{RT}_0^0(\mathcal{T}) \xrightarrow{\text{div}} \mathcal{P}_{-1}^0(\mathcal{T}). \tag{IX.1}$$

This differential complex is assembled from the piecewise affine Lagrange elements $\mathcal{P}_0^1(\mathcal{T})$ with Dirichlet boundary conditions, the lowest-order Nédélec space $\mathbf{Nd}_0^0(\mathcal{T})$ with tangential boundary conditions, the lowest-order Raviart-Thomas space $\mathbf{RT}_0^0(\mathcal{T})$

with normal boundary conditions, and the space of piecewise constant functions $\mathcal{P}^0_{-1}(\mathcal{T})$.

The subindex of the last space indicates that no continuity or boundary conditions are imposed. Similarly, we let $\mathbf{RT}^0_{-1}(\mathcal{T})$ denote the space of piecewise lowest-order Raviart-Thomas vector fields, without imposing normal continuity or boundary conditions. The divergence of such a vector field exists in the sense of distributions: if $\mathbf{u} \in \mathbf{RT}^0_{-1}(\mathcal{T})$ with support over a tetrahedron $T \in \mathcal{T}^3$, then

$$-\int_T \mathbf{u} \cdot \operatorname{grad} \phi \, \mathrm{d}x = \int_T (\operatorname{div} \mathbf{u}) \phi \, \mathrm{d}x - \sum_{F \in \Delta(T)^2} \int_F \phi (\mathbf{u} \cdot \vec{n}_{T,F}) \, \mathrm{d}s, \quad \phi \in C^\infty(\overline{\Omega}), \tag{IX.2}$$

gives an explicit formula for the distributional divergence. Here $\vec{n}_{T,F}$ denotes the outward unit normal of T along F. The face integrals of ϕ are taken against constant functions; we let $\mathbf{RT}^0_{-1}(\mathcal{T}^2)$ denote the space of distributions over $C^\infty(\overline{\Omega})$ spanned by integral functionals over faces of $\mathcal{T}$. We then write $\mathcal{P}^0_{-1}(\mathcal{T}^3) := \mathcal{P}^0_{-1}(\mathcal{T})$ and define the space of distributions $\mathcal{P}^0_{-2}(\mathcal{T}) := \mathcal{P}^0_{-1}(\mathcal{T}^3) \oplus \mathbf{RT}^0_{-1}(\mathcal{T}^2)$ as a direct sum. This leads to another differential complex

$$\mathcal{P}^1_0(\mathcal{T}) \xrightarrow{\operatorname{grad}} \mathbf{Nd}^0_0(\mathcal{T}) \xrightarrow{\operatorname{curl}} \mathbf{RT}^0_{-1}(\mathcal{T}) \xrightarrow{\operatorname{div}} \mathcal{P}^0_{-2}(\mathcal{T}). \tag{IX.3}$$

We repeat this construction. Let $\mathbf{Nd}^0_{-1}(\mathcal{T})$ denote the space of vector fields that are piecewise in the Nédélec space of lowest order but which do not necessarily satisfy tangential continuity or boundary conditions. If $\mathbf{u} \in \mathbf{Nd}^0_{-1}(\mathcal{T})$ is supported over a tetrahedron $T \in \mathcal{T}^3$, then it is observed for $\boldsymbol{\phi} \in \mathbf{C}^\infty(\overline{\Omega})$ that

$$\int_T \mathbf{u} \cdot \operatorname{curl} \boldsymbol{\phi} \, \mathrm{d}x = \int_T \operatorname{curl} \mathbf{u} \cdot \boldsymbol{\phi} \, \mathrm{d}x - \sum_{F \in \Delta(T)^2} \int_F \mathbf{u} \cdot (\boldsymbol{\phi} \times \vec{n}_{T,F}) \, \mathrm{d}s. \tag{IX.4}$$

This defines the distributional curl over $\mathbf{Nd}^0_{-1}(\mathcal{T})$. The face terms integrate the tangential trace of $\boldsymbol{\phi}$ against a tangential lowest-order Nédélec vector field over faces, and we let $\mathbf{Nd}^0_{-1}(\mathcal{T}^2)$ denote the space of such functionals over vector fields; we write accordingly $\mathbf{RT}^0_{-1}(\mathcal{T}^3) := \mathbf{RT}^0_{-1}(\mathcal{T})$. With $\mathbf{RT}^0_{-2}(\mathcal{T}) := \mathbf{RT}^0_{-1}(\mathcal{T}^3) \oplus \mathbf{Nd}^0_{-1}(\mathcal{T}^2)$ we define a space of distributions over $\mathbf{C}^\infty(\overline{\Omega})$ and thus get a well-defined mapping $\operatorname{curl} : \mathbf{Nd}^0_{-1}(\mathcal{T}) \to \mathbf{RT}^0_{-2}(\mathcal{T})$. In order to arrange a complete differential complex, we want to determine the divergence over the space $\mathbf{RT}^0_{-2}(\mathcal{T})$. Letting $\mathbf{Nd}^0_{-1}(\mathcal{T}^1)$ denote the space of distributions that are spanned by the integration over edges of $\mathcal{T}$ and defining $\mathcal{P}^0_{-3}(\mathcal{T}) := \mathcal{P}^0_{-2}(\mathcal{T}) \oplus \mathbf{Nd}^0_{-1}(\mathcal{T}^1)$, we are in the position to consider the differential complex

$$\mathcal{P}^1_0(\mathcal{T}) \xrightarrow{\operatorname{grad}} \mathbf{Nd}^0_{-1}(\mathcal{T}) \xrightarrow{\operatorname{curl}} \mathbf{RT}^0_{-2}(\mathcal{T}) \xrightarrow{\operatorname{div}} \mathcal{P}^0_{-3}(\mathcal{T}). \tag{IX.5}$$

Finally, we let $\mathcal{P}^1_{-1}(\mathcal{T})$ denote the space of piecewise affine functions. If $T \in \mathcal{T}^3$ and $u \in \mathcal{P}^1_{-1}(\mathcal{T})$ is supported over T, then the distributional gradient of u is defined by

$$-\int_T (\operatorname{div} \boldsymbol{\phi}) u \, \mathrm{d}x = \int_T \boldsymbol{\phi} \cdot \operatorname{grad} u \, \mathrm{d}x - \sum_{F \in \Delta(T)^2} \int_F (\boldsymbol{\phi} \cdot \vec{n}_{T,F}) u \, \mathrm{d}s, \quad \boldsymbol{\phi} \in \mathbf{C}^\infty(\overline{\Omega}). \tag{IX.6}$$

More generally, we let $\mathcal{P}^1_{-1}(\mathcal{T}^3)$, $\mathcal{P}^1_{-1}(\mathcal{T}^2)$, $\mathcal{P}^1_{-1}(\mathcal{T}^1)$, and $\mathcal{P}^1_{-1}(\mathcal{T}^0)$ denote the spaces of functionals over $C^\infty(\overline{\Omega})$ spanned by integrals of functions against affine functions over tetrahedra, by integrals of vector fields against affine normal fields along faces, by integrals of vector fields against affine tangential fields along edges, and by point evaluations, respectively. With a similar construction as above, we introduce the differential complex

$$\mathcal{P}^0_{-1}(\mathcal{T}) \xrightarrow{\text{grad}} \mathbf{Nd}^0_{-2}(\mathcal{T}) \xrightarrow{\text{curl}} \mathbf{RT}^0_{-3}(\mathcal{T}) \xrightarrow{\text{div}} \mathcal{P}^0_{-4}(\mathcal{T}), \tag{IX.7}$$

consisting of distributional finite element spaces. These and similar differential complexes have been discussed by Braess and Schöberl [34, Equations (3.3), (3.5), (3.7), (3.16-3.18)] albeit only over local patches.

We can make several interesting observations at this point. The conforming de Rham complex (IX.1) is a starting point for a succession of differential complexes, finishing with (IX.7). On the other hand, consider the differential complex

$$\mathcal{P}^0_{-1}(\mathcal{T}^3) \xrightarrow{\text{grad}} \mathcal{P}^0_{-1}(\mathcal{T}^2) \xrightarrow{\text{curl}} \mathcal{P}^0_{-1}(\mathcal{T}^1) \xrightarrow{\text{div}} \mathcal{P}^0_{-1}(\mathcal{T}^0), \tag{IX.8}$$

consisting of spaces of distributions spanned by taking volume averages, normal averages along faces, tangential averages along lines, and point evaluations. This is another subcomplex of (IX.7) and it is not difficult to see that this complex is isomorphic to the simplicial chain complex

$$\mathcal{C}_3(\mathcal{T}) \xrightarrow{\partial_3} \mathcal{C}_2(\mathcal{T}) \xrightarrow{\partial_2} \mathcal{C}_1(\mathcal{T}) \xrightarrow{\partial_1} \mathcal{C}_0(\mathcal{T}). \tag{IX.9}$$

In this sense, the simplicial chain complex of $\mathcal{T}$ is a subcomplex of (IX.7), and taking jump terms corresponds to applying the simplicial boundary operator. This observation will be of fundamental importance throughout this chapter.

We can view the right-hand sides of the integration by parts formulas (IX.2), (IX.4), and (IX.6) as composed of two different classes of operators: a piecewise differential operator on the one hand, and an operator corresponding to the "jump terms" on the other hand. Whereas the jump terms play no role in the conforming finite element complex (IX.1), the situation is exactly reversed for the differential complex (IX.8). This indicative of the fact that "taking jump terms" constitutes differential complexes on its own. We employ the calculus of differential forms to treat this in a uniform manner.

Furthermore, we are interested in understanding boundary conditions of distributional de Rham complexes. It is natural to define such boundary conditions indirectly by imposing boundary conditions on the test spaces. For example, if the test functions are assumed to compactly supported, then the aforementioned jump terms do not involve integrals associated to simplices of $\mathcal{T}$ included in the boundary $\partial\Omega$ of the domain. This suggests that the boundary conditions are partially encoded in the differential operators of the differential complex, which is a phenomenon very different from the case of conforming finite element de Rham complexes.

The notion of *double complex* in homological algebra puts these observations into a broader context. The Čech de Rham complex in differential topology is the most prominent example [178]. We remark that Falk and Winther [87] have recently introduced a finite element Čech de Rham complex to finite element theory, albeit not for questions of homological nature. In our case we start with the following diagram:

$$
\begin{tikzcd}
\mathcal{P}^1_{-1}(\mathcal{T}^3) \arrow[r, "\operatorname{grad}_\mathcal{T}"] \arrow[d] & \mathbf{Nd}^0_{-1}(\mathcal{T}^3) \arrow[r, "\operatorname{curl}_\mathcal{T}"] \arrow[d] & \mathbf{RT}^0_{-1}(\mathcal{T}^3) \arrow[r, "\operatorname{div}_\mathcal{T}"] \arrow[d] & \mathcal{P}^0_{-1}(\mathcal{T}^3) \\
\mathcal{P}^1_{-1}(\mathcal{T}^2) \arrow[r, "-\operatorname{grad}_\mathcal{T}"] \arrow[d] & \mathbf{Nd}^0_{-1}(\mathcal{T}^2) \arrow[r, "-\operatorname{curl}_\mathcal{T}"] \arrow[d] & \mathbf{RT}^0_{-1}(\mathcal{T}^2) & \\
\mathcal{P}^1_{-1}(\mathcal{T}^1) \arrow[r, "\operatorname{grad}_\mathcal{T}"] \arrow[d] & \mathbf{Nd}^0_{-1}(\mathcal{T}^1) & & \\
\mathcal{P}^1_{-1}(\mathcal{T}^0) & & &
\end{tikzcd}
\tag{IX.10}
$$

The spaces in this diagram have been introduced above. The horizontal mappings are piecewise differential operators, thus the rows of the diagram are differential complexes by themselves. Conversely, the vertical mappings correspond to the boundary terms in partial integration formulas as we have used above. It is an original observation of this research that these "jump-terms" are operators in their own right and that the columns of the above diagram are differential complexes.

The diagram (IX.10) is a *double complex* in the sense of homological algebra [93]. The distributional finite element sequence (IX.7) corresponds to the sequence of diagonals, also called *total complex*, of the double complex. Furthermore, our earlier observations transfer: the simplicial chain complex is included in the left-most column, whereas the conforming finite element complex with essential boundary conditions is included in the top-most row.

Finite element exterior calculus has provided a unified framework for conforming finite element de Rham complexes. The major contribution of the present chapter is the extension of finite element exterior calculus to distributional finite element de Rham complexes and their incorporation into the larger picture. On the one hand, the machinery of exterior calculus may improves understanding of the original Braess-Schöberl sequences. On the other hand, integrating distributional de Rham complexes into finite element exterior calculus provides new tools that are of independent interest.

The original distributional finite element sequences have been studied only for the lowest-order case in two and three dimensions over local element patches. We switch from classical vector calculus to the calculus of differential forms, and we study distributional finite element de Rham complexes over triangulations in arbitrary dimensions and arbitrary topology, with general partial boundary conditions, and without restrictions on the polynomial degree.

Again we give a rough outline of the theoretical framework. Let $\Omega \subset \mathbb{R}^n$ be a bounded weakly Lipschitz domain. Moreover we let $\mathcal{T}$ be a triangulation of Ω and let $\mathcal{V} \subseteq \mathcal{T}$ be the simplicial subcomplex that triangulates $\partial\Omega$. We consider a conforming finite element de Rham complex

$$\dots \xrightarrow{\mathsf{d}^{k-1}} \mathcal{P}\Lambda^{k}(\mathcal{T},\mathcal{V}) \xrightarrow{\mathsf{d}^{k}} \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{V}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{IX.11}$$

of the form discussed previously in this thesis (see Chapter IV). This differential complex can be redirected at index k into a distributional de Rham complex if we omit the inter-element continuity and the boundary conditions along $\mathcal{V}$, leading to differential complexes of the form

$$\dots \xrightarrow{\mathsf{d}^{k-2}} \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{V}) \xrightarrow{\mathsf{d}^{k-1}} \mathcal{P}\Lambda^{k}_{-1}(\mathcal{T}) \xrightarrow{\mathsf{d}^{k}} \mathcal{P}\Lambda^{k+1}_{-2}(\mathcal{T}) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{IX.12}$$

Here, the subindex -1 indicates omitting the aforementioned continuity and boundary conditions, and subsequent spaces are spanned by distributions containing functionals associated to lower dimensional simplices. Via a succession of generalized finite element complexes we eventually arrive at the differential complex

$$0 \to \mathcal{P}\Lambda^{0}_{-1}(\mathcal{T}) \xrightarrow{\mathsf{d}^{0}} \mathcal{P}\Lambda^{1}_{-2}(\mathcal{T}) \xrightarrow{\mathsf{d}^{1}} \dots \xrightarrow{\mathsf{d}^{n-1}} \mathcal{P}\Lambda^{n}_{-n-1}(\mathcal{T}) \to 0. \tag{IX.13}$$

Similarly as in the preceding outline in the language of vector calculus, the simplicial chain complex of $\mathcal{T}$ is isomorphic to a subcomplex of (IX.13). In the course of this chapter we illuminate how properties of that subcomplex, such as the homology theory or Poincaré-Friedrichs inequalities, can be related to the corresponding properties of other discrete distributional de Rham complexes.

Distributional differential forms appear in different areas of mathematics. Originally, de Rham [66] introduced the term "currents" for continuous linear functionals on a class of locally convex spaces of smooth differential forms. Geometric integration theory [123] knows simplicial chain complexes as a specific example of currents, which is also rediscovered in this work. Christiansen [54] has studied distributional finite element complexes in Regge calculus.

Given these differential complexes, how can we relate their homology spaces? The answer adapts methods of homological algebra. We construct isomorphisms between the homology spaces of the triangulation, the discrete harmonic forms of the standard finite element complex, and discrete distributional harmonic forms of distributional finite element complexes such as (IX.7). In particular, the homology of these complexes reflects topological properties of the domain. To the author's best knowledge, this is the first derivation in the literature of the homology theory of finite element de Rham complexes with partial boundary conditions.

When a differential complex is equipped with a Hilbert space structure, then it is natural to ask for Poincaré-Friedrichs inequalities. In the case of discrete distributional de Rham complexes we want to prove that the Poincaré-Friedrichs constants with respect to mesh dependent norms can be bounded in terms of only the geometry, the mesh regularity, and the polynomial order.

We recall that the conforming finite element de Rham complexes over triangulations of weakly Lipschitz domains satisfy such uniform inequalities; this follows easily from the existence of uniformly bounded commuting projections. In particular the discrete Poincaré-Friedrichs constant $C_{\mathrm{PF},\mathcal{P}}$ satisfies

$$\forall \omega \in \mathsf{d}^k \mathcal{P}\Lambda^k(\mathcal{T}) : \exists \rho \in \mathcal{P}\Lambda^k(\mathcal{T}) : \begin{cases} \|\rho\|_{L^2\Lambda^k(\Omega)} \leq C_{\mathrm{PF},\mathcal{P}} \|\omega\|_{L^2\Lambda^{k+1}(\Omega)}, \\ \mathsf{d}^k \rho = \omega. \end{cases} \tag{IX.14}$$

In other words, the Poincaré-Friedrichs constant measures the norm of the generalized solution operator for the flux equation $\mathsf{d}^k \rho = \omega$. As we have seen in Chapter VIII, the stability of finite element methods for the Hodge Laplace equation can be evaluated solely in terms of discrete Poincaré-Friedrichs constants.

In this chapter we establish analogous Poincaré-Friedrichs inequalities for complexes of discrete distributional differential forms with respect to mesh-dependent scalar products. Our analysis bounds the Poincaré-Friedrichs constants of discrete distributional de Rham complexes in terms of the Poincaré-Friedrichs constant of the complex of Whitney forms.

Specifically, we reduce the construction of a solution to the discrete distributional flux equation $\mathsf{d}^k \rho = \omega$ between spaces of discrete distributional differential forms to the solution of a flux equation between spaces of simplicial chains. Solving that problem has stability and complexity comparable to the flux equation between Whitney forms as we demonstrate using the duality between the simplicial chain complex and the complex of Whitney forms. The reduction to this simplified problem employs only local computations and is the only part of the estimate that depends on the polynomial order of the finite element spaces. Here, Poincaré-Friedrichs and inverse inequalities for the horizontal and vertical differential complexes, i.e., the rows and columns of the diagram (IX.10), are instrumental and easy to prove via scaling arguments.

Thus we bound, for example, the Poincaré-Friedrichs constant of (IX.13) in terms of the Poincaré-Friedrichs constant of the complex of Whitney forms, up to terms which are influenced only by the polynomial order and the mesh quality. We have thus reduced a global problem on high-order finite element spaces to a global problem on lowest-order case.

A side product of this research pertains to the flux equation $\mathsf{d}^k \rho = \omega$ between conforming finite element spaces. Solving this equation can be reduced via local operations to solving the analogous equation between spaces of simplicial chains. Thus, algorithmically solving $\mathsf{d}^k \rho = \omega$ requires a global computation only as difficult as solving $\mathsf{d}^k \rho = \omega$ between lowest-order Whitney forms, and additional local operations whose stability and complexity may be polynomial order dependent.

IX.1. Basic Definitions

Throughout this entire chapter we let $\mathcal{T}$ be a finite simplicial complex and we let $\mathcal{U} \subseteq \mathcal{T}$ be a simplicial subcomplex. Moreover we assume that for each simplex $C \in \mathcal{T}$ we have fixed a differential complex

$$\dots \xrightarrow{\mathsf{d}_C^{k-1}} \Lambda^k(C) \xrightarrow{\mathsf{d}_C^k} \Lambda^{k+1}(C) \xrightarrow{\mathsf{d}_C^{k+1}} \dots \tag{IX.15}$$

where $\Lambda^k(C)$, $k \in \mathbb{Z}$, is a finite-dimensional subspace of $C^\infty\Lambda^k(C)$, and where we have

$$\operatorname{tr}^k_{C,F} \Lambda^k(C) = \Lambda^k(F), \quad k \in \mathbb{Z}, \quad C \in \mathcal{T}, \quad F \in \Delta(C). \tag{IX.16}$$

In particular we have a commuting diagram

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\mathsf{d}^{k-1}_C} & \Lambda^k(C) & \xrightarrow{\mathsf{d}^k_C} & \Lambda^{k+1}(C) & \xrightarrow{\mathsf{d}^{k+1}_C} & \dots \\
 & & \downarrow{\scriptstyle \operatorname{tr}^k_{C,F}} & & \downarrow{\scriptstyle \operatorname{tr}^{k+1}_{C,F}} & & \\
\dots & \xrightarrow{\mathsf{d}^{k-1}_F} & \Lambda^k(F) & \xrightarrow{\mathsf{d}^k_F} & \Lambda^{k+1}(F) & \xrightarrow{\mathsf{d}^{k+1}_F} & \dots
\end{array}$$

Remark IX.1.1.
In many applications, $\mathcal{T}$ is the triangulation of an n-dimensional topological manifold with boundary, often even a polyhedral domain in $\mathbb{R}^n$. The subcomplex $\mathcal{U}$ is the triangulation of an admissible boundary patch. In this chapter, however, such assumptions are not necessary as such. Additional assumptions on the triangulations will be required later for stronger results.

Example IX.1.2.
An example for the differential complexes (IX.15) is given in Chapter IV. Let $\mathcal{P} : \mathcal{T} \to \mathscr{A}$ be a family of admissible sequence types associated to simplices that satisfy the hierarchy condition. We then define

$$\Lambda^k(C) := \operatorname{tr}^k_C \mathcal{P}\Lambda^k(\mathcal{T}), \quad C \in \mathcal{T}.$$

By construction, we have a surjective mapping $\operatorname{tr}^k_{C,F} : \Lambda^k(C) \to \Lambda^k(F)$ for all $C \in \mathcal{T}$ and $F \in \Delta(C)$. This is the most important example for the differential complexes (IX.15) in this chapter.

We introduce the direct sums

$$\Lambda^k_{-1}(\mathcal{T}^m, \mathcal{U}) := \bigoplus_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} \Lambda^k(C), \quad k, m \in \mathbb{Z}. \tag{IX.17}$$

We also introduce the alternative notation

$$\Gamma^k_{-1}(\mathcal{T}^m, \mathcal{U}) := \Lambda^k_{-1}(\mathcal{T}^m, \mathcal{U}), \quad k, m \in \mathbb{Z}, \tag{IX.18}$$

to be motivated soon in this chapter. We write ω_C for the component of $\omega \in \Lambda^k_{-1}(\mathcal{T}^m, \mathcal{U})$ associated to the simplex $C \in \mathcal{T}^m \setminus \mathcal{U}^m$.

We now define two operators which feature the differential property and which are central objects of investigation in this chapter. Let $k, m \in \mathbb{Z}$. We first define the *horizontal differential operator*

$$\mathsf{D}^m_k : \Lambda^k_{-1}(\mathcal{T}^m, \mathcal{U}) \longrightarrow \Lambda^{k+1}_{-1}(\mathcal{T}^m, \mathcal{U}) \tag{IX.19}$$

by applying the exterior derivative on each simplex, which means that

$$\mathsf{D}^m_k \omega := \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} \mathsf{d}^k_C \omega_C, \quad \omega \in \Lambda^k(\mathcal{T}^m, \mathcal{U}). \tag{IX.20}$$

It is obvious that

$$\mathsf{D}_{k+1}^{m}\mathsf{D}_{k}^{m}\omega = 0, \quad \omega \in \Lambda^{k}(\mathcal{T}^{m},\mathcal{U}). \tag{IX.21}$$

The simplicial chain complex of $\mathcal{T}$ relative to $\mathcal{U}$ is an additional structure in this context, and it is not quite as obvious that this gives rise to another differential operator on the spaces $\Lambda_{-1}^{k}(\mathcal{T}^{m},\mathcal{U})$. We define the *vertical differential operator*

$$\mathsf{T}_{k}^{m} : \Gamma_{-1}^{k}(\mathcal{T}^{m},\mathcal{U}) \longrightarrow \Gamma_{-1}^{k}(\mathcal{T}^{m-1},\mathcal{U}) \tag{IX.22}$$

by setting

$$\mathsf{T}_{k}^{m}\omega := \sum_{\substack{C\in\mathcal{T}^{m}\setminus\mathcal{U}^{m}\\ F\in\mathcal{T}^{m-1}\setminus\mathcal{U}^{m-1}\\ F\in\Delta(C)}} o(F,C)\,\mathrm{tr}_{C,F}^{k}\,\omega_{C}, \quad \omega \in \Gamma^{k}(\mathcal{T}^{m},\mathcal{U}). \tag{IX.23}$$

It is easy to see that

$$\mathsf{T}_{k}^{m-1}\mathsf{T}_{k}^{m}\omega = 0, \quad \omega \in \Gamma_{-1}^{k}(\mathcal{T}^{m},\mathcal{U}), \tag{IX.24}$$

as follows by checking that for all $\omega \in \Gamma_{-1}^{k}(\mathcal{T}^{m},\mathcal{U})$ we have

$$\mathsf{T}_{k}^{m-1}\mathsf{T}_{k}^{m}\omega = \sum_{\substack{C\in\mathcal{T}^{m}\setminus\mathcal{U}^{m}\\ F\in\Delta(C)^{m-1}\setminus\mathcal{U}^{m-1}\\ f\in\Delta(F)^{m-2}\setminus\mathcal{U}^{m-2}}} o(F,C)o(f,F)\,\mathrm{tr}_{C,f}^{k}\,\omega_{C} = 0.$$

The differential property of the vertical differential is completely analogous to the differential property of the simplicial boundary operator (II.42).

It is obvious from the construction that

$$\mathsf{D}_{k}^{m-1}\mathsf{T}_{k}^{m}\omega = \mathsf{T}_{k+1}^{m}\mathsf{D}_{k}^{m}\omega, \quad \omega \in \Lambda_{-1}^{k}(\mathcal{T}^{m},\mathcal{U}). \tag{IX.25}$$

Lastly, it will be of interest to study the kernels of the operators D_{k}^{m} and T_{k}^{m} in their own right. We define

$$\Lambda^{k}(\mathcal{T}^{m},\mathcal{U}) := \left\{\, \omega \in \Lambda_{-1}^{k}(\mathcal{T}^{m},\mathcal{U}) \;\middle|\; \mathsf{T}_{k}^{m}\omega = 0 \,\right\}, \tag{IX.26}$$

$$\Gamma^{k}(\mathcal{T}^{m},\mathcal{U}) := \left\{\, \omega \in \Gamma_{-1}^{k}(\mathcal{T}^{m},\mathcal{U}) \;\middle|\; \mathsf{D}_{k}^{m}\omega = 0 \,\right\}. \tag{IX.27}$$

Remark IX.1.3.
The horizontal and the vertical differential operator have many analogous properties. As the reader may already tell, the spaces with symbol Λ will be used when discussing the horizontal differential operator, and the spaces with symbol Γ will be used when discussing the vertical differential operator. Using such notation will be helpful in later parts of this chapter.

Due to the differential properties (IX.21) and (IX.24), we can introduce several differential complexes. For each $m \in \mathbb{Z}$ fixed, we may consider the differential complex

$$\dots \xrightarrow{\mathsf{D}_{k-1}^{m}} \Lambda_{-1}^{k}(\mathcal{T}^{m},\mathcal{U}) \xrightarrow{\mathsf{D}_{k}^{m}} \Lambda_{-1}^{k+1}(\mathcal{T}^{m},\mathcal{U}) \xrightarrow{\mathsf{D}_{k+1}^{m}} \dots \tag{IX.28}$$

and, because of (IX.25), we may also consider the subcomplex

$$\dots \xrightarrow{\mathsf{D}^m_{k-1}} \Lambda^k(\mathcal{T}^m,\mathcal{U}) \xrightarrow{\mathsf{D}^m_k} \Lambda^{k+1}(\mathcal{T}^m,\mathcal{U}) \xrightarrow{\mathsf{D}^m_{k+1}} \dots \tag{IX.29}$$

Analogously, for each $k \in \mathbb{Z}$ fixed, we have the differential complex

$$\dots \xrightarrow{\mathsf{T}^{m+1}_k} \Gamma^k_{-1}(\mathcal{T}^m,\mathcal{U}) \xrightarrow{\mathsf{T}^m_k} \Gamma^k_{-1}(\mathcal{T}^{m-1},\mathcal{U}) \xrightarrow{\mathsf{T}^{m-1}_k} \dots \tag{IX.30}$$

and, because of (IX.25), a subcomplex is given by

$$\dots \xrightarrow{\mathsf{T}^{m+1}_k} \Gamma^k(\mathcal{T}^m,\mathcal{U}) \xrightarrow{\mathsf{T}^m_k} \Gamma^k(\mathcal{T}^{m-1},\mathcal{U}) \xrightarrow{\mathsf{T}^{m-1}_k} \dots \tag{IX.31}$$

The homology spaces of these complexes will be determined in the course of this chapter.

We also consider scalar products on the spaces $\Lambda^k_{-1}(\mathcal{T}^m,\mathcal{U}) = \Gamma^k_{-1}(\mathcal{T}^m,\mathcal{U})$. These allow us to utilize the framework of Hilbert complexes in this chapter. For a particularly relevant family of scalar products, suppose that we have chosen a family $\alpha : \mathbb{Z} \to \mathbb{R}$ of real numbers, and suppose that for each $C \in \mathcal{T}$ and $k \in \mathbb{Z}$ we have fixed a scalar product

$$\langle \cdot, \cdot \rangle_{L^2\Lambda^k(C)} : \Lambda^k(C) \times \Lambda^k(C) \to \mathbb{R}.$$

We then define the scalar product

$$\langle \omega, \eta \rangle_\alpha = \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{\alpha(m)} \langle \omega_C, \eta_C \rangle_{L^2\Lambda^k(C)}, \quad \omega, \eta \in \Lambda^k_{-1}(\mathcal{T}^m,\mathcal{U}).$$

Example IX.1.4.
Suppose that $\mathcal{T}$ is n-dimensional and let $k, m \in \mathbb{Z}$. Generalizing a scalar product used by Braess and Schöberl [34, Subsection 3.4], we may consider

$$\langle \omega, \eta \rangle_h := \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{n-m} \langle \omega_C, \eta_C \rangle_{L^2\Lambda^k(C)}, \quad \omega, \eta \in \Lambda^k_{-1}(\mathcal{T}^m,\mathcal{U}). \tag{IX.32}$$

Another scalar product takes the form

$$\langle \omega, \eta \rangle_{-h} := \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{m-n} \langle \omega_C, \eta_C \rangle_{L^2\Lambda^k(C)}, \quad \omega, \eta \in \Lambda^k_{-1}(\mathcal{T}^m,\mathcal{U}). \tag{IX.33}$$

Later in this chapter we prove Poincaré-Friedrichs inequalities with respect to the scalar product (IX.33).

In order to motivate these definitions and the terminology, in particular the term *discrete distributional differential form*, we consider the following example in detail.

Example IX.1.5.
Assume that $\Omega \subseteq \mathbb{R}^n$ is a weakly Lipschitz domain and that Γ_T is an admissible boundary patch. Additionally we assume that $\mathcal{T}$ is a triangulation of Ω, and that $\mathcal{U}$ is a simplicial subcomplex of $\mathcal{T}$ that triangulates Γ_T. We let Γ_N be the boundary

patch complementary to Γ_T. There exists a simplicial subcomplex $\mathcal{V}$ of $\mathcal{T}$ that triangulates Γ_N. For every $k \in \mathbb{Z}$ we let $C^\infty\Lambda^k(\overline{\Omega}, \Gamma_T)$ be the space of smooth differential k-forms over Ω that are restrictions of smooth differential k-forms over $\mathbb{R}^n$ and whose boundary trace vanishes over Γ_T.

We consider a simplex $C \in \mathcal{T}^m \setminus \mathcal{U}^m$ and a differential k-form $\omega_C \in \Lambda^k(C)$ over C. For every smooth differential form $\phi \in C^\infty\Lambda^{n-m+k}(\overline{\Omega}, \Gamma_T)$ we define

$$\langle \omega_C, \phi \rangle := \int_C \omega_C \wedge \operatorname{tr}_C^{m-k} \star\phi.$$

Since $\star_\Omega\phi \in C^\infty\Lambda^{m-k}(\overline{\Omega}, \Gamma_T)$, we easily verify that this integral is well-defined. We also note that this pairing generalizes the L^2 scalar product of differential forms. Thus ω_C acts as a functional on $C^\infty\Lambda^{m-k}(\overline{\Omega}, \Gamma_T)$, and in this sense, ω_C is a *distributional differential form.* Since ω_C will be a member of a finite element space of polynomial differential forms over C, we call it a *discrete distributional differential form.* Consequently, $\Lambda^k_{-1}(\mathcal{T}^m, \mathcal{U})$ is a finite-dimensional space of functionals over $C^\infty\Lambda^{n-m+k}(\overline{\Omega}, \Gamma_T)$.

To give a motivation for the horizontal and vertical differential operators, suppose that $\psi \in C^\infty\Lambda^{n-m+k+1}(\overline{\Omega}, \Gamma_T)$. By the definition of the codifferential (V.17) we find

$$\begin{aligned}
\langle \omega_C, \delta^{n-m+k+1}\psi \rangle &= (-1)^{n-m+k+1} \int_C \omega_C \wedge \operatorname{tr}_C^{m-k} \star\star^{-1} \mathsf{d}^{m-k-1} \star\psi \\
&= (-1)^{n-m+k+1} \int_C \omega_C \wedge \operatorname{tr}_C^{m-k} \mathsf{d}^{m-k-1} \star\psi \\
&= (-1)^{n-m+k+1} \int_C \omega_C \wedge \mathsf{d}_C^{m-k-1} \operatorname{tr}_C^{m-k-1} \star\psi.
\end{aligned}$$

By Stokes' theorem over simplices (III.3), we have

$$\begin{aligned}
\int_C \omega_C \wedge \mathsf{d}_C^{m-k-1} \operatorname{tr}_C^{m-k-1} \star\psi &= (-1)^{k+1} \int_C \mathsf{d}_C^k\omega_C \wedge \operatorname{tr}_C^{m-k-1} \star\psi \\
&\quad + (-1)^k \sum_{\substack{F \in \Delta(C)^{m-1} \\ F \notin \mathcal{U}}} o(F, C) \int_F \operatorname{tr}_{C,F}^k \omega_C \wedge \operatorname{tr}_F^{m-k-1} \star\psi.
\end{aligned}$$

In combination, this means that

$$\begin{aligned}
\langle \omega_C, \delta^{n-m+k+1}\psi \rangle &= (-1)^{n-m} \int_C \mathsf{d}_C^k\omega_C \wedge \operatorname{tr}_{\Omega,C}^{m-k-1} \star_\Omega\psi \\
&\quad + (-1)^{n-m+1} \sum_{\substack{F \in \Delta(C)^{m-1} \\ F \notin \mathcal{U}}} o(F, C) \int_F \operatorname{tr}_{C,F}^k \omega_C \wedge \operatorname{tr}_F^{m-k-1} \star\psi,
\end{aligned}$$

which we restate as

$$\langle \omega_C, \delta^{n-m+k+1}\psi \rangle = (-1)^{n-m} \langle \mathsf{D}_k^m\omega_C, \psi \rangle + (-1)^{n-m+1} \langle \mathsf{T}_k^m\omega_C, \psi \rangle.$$

This motivates the introduction of the horizontal and vertical differential operators.

Now we motivate the spaces $\Lambda^k(\mathcal{T}^m, \mathcal{U})$ and $\Gamma^k(\mathcal{T}^m, \mathcal{U})$. Consider the special case $m = n$ and suppose that $\omega \in \Lambda^k(\mathcal{T}^n, \mathcal{U})$. The condition $\mathsf{T}^n_k \omega = 0$ then means that for all distinct n-simplices $T, T' \in \mathcal{T}^n$ sharing a common face $F \in \Delta(T)^{n-1} \cap \Delta(T')^{n-1}$ we have

$$o(F,T) \operatorname{tr}^k_{T,F} \omega_T + o(F,T') \operatorname{tr}^k_{T',F} \omega_{T'} = 0.$$

We conclude that $\operatorname{tr}^k_{T,F} \omega_T = \operatorname{tr}^k_{T',F} \omega_{T'}$ since T and T' induce opposing orientations on F. Moreover, if $T \in \mathcal{T}^n$ and $F \in \Delta(T)^{n-1}$ with $F \in \mathcal{V}$, then $\mathsf{T}^n_k \omega = 0$ implies that

$$\operatorname{tr}^k_{T,F} \omega_T = 0.$$

In summary, this means that each $\omega \in \Lambda^k(\mathcal{T}^n, \mathcal{U})$ has single-valued traces and satisfies homogeneous boundary conditions along Γ_N. In particular, if additionally the assumptions of Example IX.1.2 hold, then

$$\Lambda^k(\mathcal{T}^n, \mathcal{U}) = \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{V}).$$

Thus we see that when boundary conditions are imposed on the test function space along Γ_T, triangulated by $\mathcal{U}$, then boundary conditions are imposed on the conforming finite element spaces along Γ_N, triangulated by $\mathcal{V}$.

As a motivation for introducing $\Gamma^k(\mathcal{T}^m, \mathcal{U})$ we consider the special case $k = 0$. Suppose that $\Lambda^k(C)$ contains the constant function 1_C for each $C \in \mathcal{T}^m$. Then $\Gamma^0(\mathcal{T}^m, \mathcal{U})$ is just the space of constant functions associated to m-simplices in $\mathcal{T} \setminus \mathcal{U}$. Consequently, $\Gamma^0(\mathcal{T}^m, \mathcal{U})$ is isomorphic to $\mathcal{C}_m(\mathcal{T}, \mathcal{U})$, and the vertical differential operator corresponds, up to signs, to the simplicial boundary operator.

Remark IX.1.6.
Our notion of discrete distributional differential form is similar but different from the notion of currents [66] introduced by de Rham. Currents over an n-dimensional smooth manifold in the sense of de Rham are $(n-k)$-forms with distributional coefficients, which act as functionals on compactly supported smooth k-forms. This extends the canonical pairing of $(n-k)$-forms and k-forms. The notion of current only employs the differentiable structure on manifolds. By contrast, our notion extends the L^2 pairing of differential forms of the same degree and thus requires a Riemannian metric.

IX.2. Homology of Horizontal Complexes

In this section we study the homology spaces of the horizontal differential complexes (IX.28). These homology spaces are isomorphic to the direct sum of the homology spaces of the differential complexes on simplices (IX.15). This is an easy observation that we make explicit because its analogue regarding the vertical differential complexes will not be as obvious.

It is instrumental that the horizontal complexes can be localized in the sense of the following lemma.

Lemma IX.2.1.
Let $m \in \mathbb{Z}$. Then the differential complex

$$\dots \xrightarrow{\mathsf{D}^m_{k-1}} \Lambda^k_{-1}(\mathcal{T}^m,\mathcal{U}) \xrightarrow{\mathsf{D}^m_k} \Lambda^{k+1}_{-1}(\mathcal{T}^m,\mathcal{U}) \xrightarrow{\mathsf{D}^m_{k+1}} \dots \tag{IX.34}$$

is the direct sum of the differential complexes

$$\dots \xrightarrow{\mathsf{d}^{k-1}_C} \Lambda^k(C) \xrightarrow{\mathsf{d}^k_C} \Lambda^{k+1}(C) \xrightarrow{\mathsf{d}^{k+1}_C} \dots \tag{IX.35}$$

over $C \in \mathcal{T}^m \setminus \mathcal{U}^m$.

Proof. This is evident from the definitions. □

In order to control the homology of the horizontal complexes, we introduce a new condition. We say that the *local exactness condition* holds if for each $C \in \mathcal{T} \setminus \mathcal{U}$, the sequence (IX.34) is exact at every non-zero index and furthermore $\ker \mathsf{d}^0_C$ is spanned by the constant functions over C. This implies that the differential complex

$$0 \to \ker \mathsf{d}^0_C \longrightarrow \Lambda^0(C) \xrightarrow{\mathsf{d}^0_C} \Lambda^1(C) \xrightarrow{\mathsf{d}^1_C} \dots \tag{IX.36}$$

is exact for each $C \in \mathcal{T}^m \setminus \mathcal{U}^m$, and that $\ker \mathsf{d}^0_C = \operatorname{span}\{1_C\}$. The following result is easily verified.

Lemma IX.2.2.
Assume that the local exactness condition holds. Let $m \in \mathbb{Z}$. Then

$$\ker\left(\mathsf{D}^m_k : \Lambda^k_{-1}(\mathcal{T}^m,\mathcal{U}) \to \Lambda^{k+1}_{-1}(\mathcal{T}^m,\mathcal{U})\right) = \operatorname{ran}\left(\mathsf{D}^m_{k-1} : \Lambda^{k-1}_{-1}(\mathcal{T}^m,\mathcal{U}) \to \Lambda^k_{-1}(\mathcal{T}^m,\mathcal{U})\right)$$

for $k \in \mathbb{Z} \setminus \{0\}$, and

$$\ker\left(\mathsf{D}^m_0 : \Lambda^0_{-1}(\mathcal{T}^m,\mathcal{U}) \to \Lambda^1_{-1}(\mathcal{T}^m,\mathcal{U})\right) = \bigoplus_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} \operatorname{span}\{1_C\}.$$

Example IX.2.3.
We recall the setting of Example IX.1.2. Suppose we have fixed an admissible sequence type $\mathcal{P}_C \in \mathscr{A}$ for each $C \in \mathcal{T}$ such that $\mathcal{P}_F \leq \mathcal{P}_C$ for $F \in \Delta(C)$ and $C \in \mathcal{T}$. Then the sequences

$$\dots \xrightarrow{\mathsf{d}^{k-1}_C} \mathcal{P}_C\Lambda^k(C) \xrightarrow{\mathsf{d}^k_C} \mathcal{P}_C\Lambda^{k+1}(C) \xrightarrow{\mathsf{d}^{k+1}_C} \dots$$

realize the absolute cohomology of the simplex C, and hence the local exactness condition holds. To see this, we first recall that $1_C \in \mathcal{P}_C\Lambda^0(C)$ by construction. On the other hand, suppose that $k > 0$ and that $\omega \in \mathcal{P}_C\Lambda^k(C)$ with $\mathsf{d}^k_C\omega = 0$. Without loss of generality, we assume that C is full-dimensional. Because the L^2 de Rham complex over contractible domains realizes the absolute cohomology, there exists $\xi \in H\Lambda^{k-1}(C)$ such that $\mathsf{d}^{k-1}\xi = \omega$. We invoke the L^2 bounded commuting projection π^{k-1} of Chapter VII and check that $\pi^{k-1}\xi \in \mathcal{P}\Lambda^{k-1}(C)$ with $\mathsf{d}^{k-1}\pi^{k-1}\xi = \omega$. This yields the property required for this example.

IX.3. Homology of Vertical Complexes

We study the homology spaces of the vertical complexes in this section, which parallels the study of the homology spaces of the horizontal complexes of the previous section in many regards. The crucial observation is the local decomposition of the vertical complexes.

Let $m \in \mathbb{Z}$ and consider the differential complex

$$\dots \xrightarrow{\mathsf{T}_k^{m+1}} \Gamma_{-1}^k(\mathcal{T}^m, \mathcal{U}) \xrightarrow{\mathsf{T}_k^m} \Gamma_{-1}^k(\mathcal{T}^{m-1}, \mathcal{U}) \xrightarrow{\mathsf{T}_k^{m-1}} \dots \tag{IX.37}$$

In order to decompose this complex into local contributions, some additional assumptions need to be made.

First we introduce the notation

$$\mathring{\Gamma}^k(C) := \left\{ \omega \in \Gamma^k(C) \;\middle|\; \forall F \in \Delta(C) \setminus \{C\} : \operatorname{tr}_{C,F}^k \omega = 0 \right\} \tag{IX.38}$$

for the subspace of $\Gamma^k(C)$ whose members have vanishing trace on the proper subsimplices of C. We say that the *geometric decomposition condition* holds if we have linear extension operators

$$\operatorname{ext}_{F,C}^k : \mathring{\Gamma}^k(F) \to \Gamma^k(C),$$

for every $C \in \mathcal{T}$ and $F \in \Delta(C)$ such that

(i) for all $F \in \mathcal{T}$ we have

$$\operatorname{ext}_{F,F}^k \omega = \omega, \quad \omega \in \mathring{\Gamma}^k(F), \tag{IX.39a}$$

(ii) for all $C \in \mathcal{T}$ with $F \in \Delta(C)$ and $f \in \Delta(F)$ we have

$$\operatorname{tr}_{C,F}^k \operatorname{ext}_{f,C}^k = \operatorname{ext}_{f,F}^k, \tag{IX.39b}$$

(iii) and for all $C \in \mathcal{T}$ and $F, G \in \Delta(C)$ with $F \notin \Delta(G)$ we have

$$\operatorname{tr}_{C,G}^k \operatorname{ext}_{F,C}^k = 0. \tag{IX.39c}$$

Note that the extension operators $\operatorname{ext}_{F,C}^k$ are largely analogous to the assumptions on the local extension operators in Chapter IV. Under these conditions we obtain a representation of $\Gamma_{-1}^k(\mathcal{T}^m, \mathcal{U})$ as a direct sum similar to the geometric decomposition in Chapter IV.

Lemma IX.3.1.
Let $k, m \in \mathbb{Z}$. Then

$$\Gamma_{-1}^k(\mathcal{T}^m, \mathcal{U}) = \bigoplus_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} \bigoplus_{F \in \Delta(C)} \operatorname{ext}_{F,C}^k \mathring{\Gamma}^k(F). \tag{IX.40}$$

Proof. Let $\omega \in \Gamma^k_{-1}(\mathcal{T}^m, \mathcal{U})$ and let $C \in \mathcal{T}^m \setminus \mathcal{U}^m$. We define recursively

$$\mathring{\omega}_C^{(F)} := \operatorname{tr}^k_{C,F} \left(\omega - \sum_{\substack{f \in \Delta(C) \\ \dim f < \dim F}} \operatorname{ext}^k_{f,C} \mathring{\omega}_C^{(f)} \right).$$

It is easy to see that $\mathring{\omega}_C^{(V)} \in \mathring{\Gamma}^k(V)$ for $V \in \Delta(C)^0$. Next, suppose that $F \in \Delta(C)$ and that $\mathring{\omega}_C^{(f)} \in \mathring{\Gamma}^k(f)$ for $f \in \Delta(C)$ with $\dim f < \dim F$. Then we easily see that $\mathring{\omega}_C^{(F)}$ is well-defined and a member of $\mathring{\Gamma}^k(F)$. An induction argument gives

$$\omega_C = \bigoplus_{F \in \Delta(C)} \operatorname{ext}^k_{F,C} \mathring{\omega}_C^{(F)}. \tag{IX.41}$$

The desired claim follows. □

Assuming that the geometric decomposition assumption holds, we can now construct the local vertical complexes. We define

$$\Gamma^m_k(F) := \bigoplus_{\substack{C \in \mathcal{T}^m \setminus \mathcal{U}^m \\ F \in \Delta(C)}} \operatorname{ext}^k_{F,C} \mathring{\Gamma}^k(F), \quad F \in \mathcal{T}, \quad m \in \mathbb{Z}. \tag{IX.42}$$

The next two lemmas formalize that these spaces enable a local decomposition of the vertical differential complexes.

Lemma IX.3.2.
Let $m, k \in \mathbb{Z}$. Then

$$\Gamma^k_{-1}(\mathcal{T}^m, \mathcal{U}) = \bigoplus_{F \in \mathcal{T}} \Gamma^m_k(F).$$

Proof. Using Definition (IX.42) we observe

$$\begin{aligned}
\Gamma^k_{-1}(\mathcal{T}^m, \mathcal{U}) &= \bigoplus_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} \bigoplus_{F \in \Delta(C)} \operatorname{ext}^k_{F,C} \mathring{\Gamma}^k(F) \\
&= \bigoplus_{\substack{F \in \mathcal{T} \\ C \in \mathcal{T}^m \setminus \mathcal{U}^m \\ F \in \Delta(C)}} \operatorname{ext}^k_{F,C} \mathring{\Gamma}^k(F) \\
&= \bigoplus_{F \in \mathcal{T}} \bigoplus_{\substack{C \in \mathcal{T}^m \setminus \mathcal{U}^m \\ F \in \Delta(C)}} \operatorname{ext}^k_{F,C} \mathring{\Gamma}^k(F) = \bigoplus_{F \in \mathcal{T}} \Gamma^m_k(F),
\end{aligned}$$

which is the desired result. □

Lemma IX.3.3.
Let $m, k \in \mathbb{Z}$ and $F \in \mathcal{T}$. Then $\mathsf{T}^m_k \Gamma^m_k(F) \subseteq \Gamma^{m-1}_k(F)$.

Proof. Let $C \in \mathcal{T}^m \setminus \mathcal{U}^m$ with $F \in \Delta(C)$, and let $\omega_C^F = \operatorname{ext}_{F,C}^k \mathring{\omega}_C^F$ for some $\mathring{\omega}_C^F \in \mathring{\Gamma}^k(F)$. Then $\omega_C^F \in \Gamma_k^m(F)$, and we calculate

$$\begin{aligned}
\mathsf{T}_k^m \omega_C^F &= \sum_{\substack{G \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1} \\ G \in \Delta(C)}} o(G,C) \operatorname{tr}_{C,G}^k \omega_C^F \\
&= \sum_{\substack{G \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1} \\ G \in \Delta(C)}} o(G,C) \operatorname{tr}_{C,G}^k \operatorname{ext}_{F,C}^k \mathring{\omega}_C^F \\
&= \sum_{\substack{G \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1} \\ G \in \Delta(C)}} o(G,C) \operatorname{ext}_{F,G}^k \mathring{\omega}_C^F,
\end{aligned}$$

where we used Lemma IX.3.1. The final expression is an element of $\Gamma_k^m(F)$, which implies the desired result. □

In combination, these observations imply the following decomposition of vertical complexes.

Lemma IX.3.4.
Assume the geometric decomposition condition holds. Let $k \in \mathbb{Z}$. Then the differential complex

$$\dots \xrightarrow{\mathsf{T}_k^{m+1}} \Gamma_{-1}^k(\mathcal{T}^m, \mathcal{U}) \xrightarrow{\mathsf{T}_k^m} \Gamma_{-1}^k(\mathcal{T}^{m-1}, \mathcal{U}) \xrightarrow{\mathsf{T}_k^{m-1}} \dots \tag{IX.43}$$

is the direct sum of the differential complexes

$$\dots \xrightarrow{\mathsf{T}_k^{m+1}} \Gamma_k^m(F) \xrightarrow{\mathsf{T}_k^m} \Gamma_k^{m-1}(F) \xrightarrow{\mathsf{T}_k^{m-1}} \dots \tag{IX.44}$$

over all $F \in \mathcal{T} \setminus \mathcal{U}$.

We have decomposed the vertical differential complexes into local differential complexes associated to simplices of $\mathcal{T}$. The next step is analyze the homology spaces of the local vertical complexes. At this point we refer to material from Section II.1 in Chapter II, in particular the definition of the micropatch $\mathcal{M}(\mathcal{T}, F)$ and the micropatch boundary $\mathcal{N}(\mathcal{T}, \mathcal{U}, F)$. We prove the following algebraic result.

Lemma IX.3.5.
Let $F \in \mathcal{T}$ and $k \in \mathbb{Z}$. Then the differential complex

$$\dots \xrightarrow{\mathsf{T}_k^{m+1}} \Gamma_k^m(F) \xrightarrow{\mathsf{T}_k^m} \Gamma_k^{m-1}(F) \xrightarrow{\mathsf{T}_k^{m-1}} \dots \tag{IX.45}$$

is isomorphic to the differential complex

$$\dots \xrightarrow{\partial_{m+1} \otimes \mathrm{Id}} \mathcal{C}_m^F(\mathcal{T}, \mathcal{U}) \otimes \mathring{\Gamma}^k(F) \xrightarrow{\partial_m \otimes \mathrm{Id}} \mathcal{C}_{m-1}^F(\mathcal{T}, \mathcal{U}) \otimes \mathring{\Gamma}^k(F) \xrightarrow{\partial_{m-1} \otimes \mathrm{Id}} \dots \tag{IX.46}$$

Proof. For the duration of this proof, we introduce linear mappings

$$\Theta_m : \mathcal{C}_m^F(\mathcal{T}, \mathcal{U}) \otimes \mathring{\Gamma}^k(F) \to \Gamma_k^m(F)$$

for $m \in \mathbb{Z}$ that are defined by setting

$$\Theta_m \left(C \otimes \mathring{\omega}^{(F)} \right) := \operatorname{ext}^k_{F,C} \mathring{\omega}^{(F)}$$

for $C \in \mathcal{M}(\mathcal{T}, F)^m \setminus \mathcal{N}(\mathcal{T}, \mathcal{U}, F)^m$ and $\mathring{\omega}^{(F)} \in \mathring{\Gamma}^k(F)$. Each of these mappings is invertible, and we observe

$$\begin{aligned}
\Theta_m \left(\partial_m C \otimes \mathring{\omega}^{(F)} \right) &= \sum_{\substack{G \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1} \\ G \in \Delta(C)}} o(G, C) \Theta_m \left(G \otimes \mathring{\omega}^{(F)} \right) \\
&= \sum_{\substack{G \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1} \\ G \in \Delta(C)}} o(G, C) \operatorname{ext}^k_{F,G} \mathring{\omega}^{(F)} \\
&= \sum_{\substack{G \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1} \\ G \in \Delta(C)}} o(G, C) \operatorname{tr}^k_{C,G} \operatorname{ext}^k_{F,C} \mathring{\omega}^{(F)} \\
&= \mathsf{T}^m_k \operatorname{ext}^k_{F,C} \mathring{\omega}^{(F)} \\
&= \mathsf{T}^m_k \Theta_m \left(C \otimes \mathring{\omega}^{(F)} \right).
\end{aligned}$$

This means that the isomorphisms Θ_m constitute a isomorphism of the differential complexes (IX.45) and (IX.46). This completes the proof. □

The simplicial homology spaces of the micropatches, which we have discussed in Chapter II, determine the homology spaces of the vertical complexes provided that the geometric decomposition condition holds. We are particularly interested in the following special case. We say that the *local patch condition* holds if

$$b^F_m(\mathcal{T}, \mathcal{U}) = 0, \quad m \in \mathbb{Z} \setminus \{n\}, \quad F \in \mathcal{T}.$$

We can then characterize the homology spaces of the vertical differential complex (IX.37) in the following manner.

Lemma IX.3.6.
Suppose that the geometric decomposition condition and the local patch condition are satisfied. Then we have

$$\begin{aligned}
&\ker \left(\mathsf{T}^m_k : \Gamma^k_{-1}(\mathcal{T}^m, \mathcal{U}) \to \Gamma^k_{-1}(\mathcal{T}^{m-1}, \mathcal{U}) \right) \\
&\quad = \operatorname{ran} \left(\mathsf{T}^{m+1}_k : \Gamma^k_{-1}(\mathcal{T}^{m+1}, \mathcal{U}) \to \Gamma^k_{-1}(\mathcal{T}^m, \mathcal{U}) \right)
\end{aligned}$$

for $m \in \mathbb{Z} \setminus \{n\}$ and $k \in \mathbb{Z}$.

It is not straight-forward to give an intuitive characterization of the spaces $\Gamma^k(\mathcal{T}^m, \mathcal{U})$ in the general case. If $\mathcal{T}$ triangulates a manifold, then additional structure is available, and we obtain the following result as a consequence of Lemma II.7.1.

Lemma IX.3.7.
Suppose $\mathcal{T}$ triangulates an n-dimensional topological manifold M with boundary, and that $\mathcal{U}$ triangulates a topological submanifold Γ of ∂M of dimension $n-1$

with boundary. There exists a simplicial subcomplex $\mathcal{V} \subset \mathcal{T}$ that triangulates the closure of the complement of Γ in $\partial\Omega$. Assume that the geometric decomposition assumption holds true, and let $k \in \mathbb{Z}$. Then Lemma IX.3.6 applies, and for every $\omega \in \Gamma^k(\mathcal{T}^n,\mathcal{U})$ we have

$$\omega = \sum_{F \in \mathcal{T} \setminus \mathcal{V}} \sum_{\substack{T \in \mathcal{T}^n \\ F \in \Delta(T)}} \operatorname{ext}^k_{F,T} \mathring{\omega}^{(F)},$$

where $\mathring{\omega}^{(F)} \in \mathring{\Gamma}^k(F)$ for $F \in \mathcal{T} \setminus \mathcal{V}$.

IX.4. A First Application

We have introduced practically feasible conditions under which the horizontal and vertical differential complexes have homology spaces that are easy to describe. At this point, we can already provide a result on the homology theory of conforming finite element complexes. We consider the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Lambda^0(\mathcal{T}^n,\mathcal{U}) \arrow[r, "\mathsf{D}^n_0"] \arrow[d] & \Lambda^1(\mathcal{T}^n,\mathcal{U}) \arrow[r, "\mathsf{D}^n_1"] \arrow[d] & \dots \\
\Gamma^0(\mathcal{T}^n,\mathcal{U}) \arrow[r] \arrow[d, "-\mathsf{T}^n_0"'] & \Lambda^0_{-1}(\mathcal{T}^n,\mathcal{U}) \arrow[r, "\mathsf{D}^n_0"] \arrow[d, "-\mathsf{T}^n_0"'] & \Lambda^1_{-1}(\mathcal{T}^n,\mathcal{U}) \arrow[r, "\mathsf{D}^n_1"] \arrow[d, "-\mathsf{T}^n_1"'] & \dots \\
\Gamma^0(\mathcal{T}^{n-1},\mathcal{U}) \arrow[r] \arrow[d, "\mathsf{T}^n_0"'] & \Lambda^0_{-1}(\mathcal{T}^{n-1}v) \arrow[r, "-\mathsf{D}^{n-1}_0"] \arrow[d, "\mathsf{T}^{n-1}_0"'] & \Lambda^1_{-1}(\mathcal{T}^{n-1},\mathcal{U}) \arrow[r, "-\mathsf{D}^{n-1}_1"] \arrow[d, "\mathsf{T}^{n-1}_1"'] & \dots \\
\dots & \dots & \dots &
\end{tikzcd} \tag{IX.47}$$

The left-most horizontal and the top-most vertical arrows denote the respective inclusion mappings. The choice of signs on the arrows in this diagram is motivated by our observations in Example IX.1.5. We recall the identities

$$\mathsf{D}^m_{k+1}\mathsf{D}^m_k = 0, \quad \mathsf{T}^{m-1}_k\mathsf{T}^m_k = 0, \quad \mathsf{T}^m_{k+1}\mathsf{D}^m_k = \mathsf{D}^{m-1}_k\mathsf{T}^m_k.$$

This implies that (IX.47) constitutes a double complex in the sense of [93, Chapter 1, § 3.5]. This allows us to utilize known results from homological algebra in our setting. Specifically, in this section we will derive a result on the homology spaces of the differential complexes (IX.29) and (IX.31). We notate the homology spaces of the differential complexes (IX.29) and (IX.31) by

$$\begin{aligned}
\mathcal{H}\Lambda^k(\mathcal{T}^n,\mathcal{U}) &:= \frac{\ker\left(\mathsf{D}^n_k : \Lambda^k(\mathcal{T}^n,\mathcal{U}) \to \Lambda^{k+1}(\mathcal{T}^n,\mathcal{U})\right)}{\operatorname{ran}\left(\mathsf{D}^n_{k-1} : \Lambda^{k-1}(\mathcal{T}^n,\mathcal{U}) \to \Lambda^k(\mathcal{T}^n,\mathcal{U})\right)}, \\
\mathcal{H}\Gamma^0(\mathcal{T}^m,\mathcal{U}) &:= \frac{\ker\left(\mathsf{T}^m_0 : \Gamma^0(\mathcal{T}^m,\mathcal{U}) \to \Gamma^0(\mathcal{T}^{m-1},\mathcal{U})\right)}{\operatorname{ran}\left(\mathsf{T}^{m+1}_0 : \Gamma^0(\mathcal{T}^{m+1},\mathcal{U}) \to \Gamma^0(\mathcal{T}^m,\mathcal{U})\right)}.
\end{aligned}$$

With a result from homological algebra, we obtain the following fact.

Theorem IX.4.1.
Assume that the local exactness condition, the geometric decomposition condition, and the local patch condition hold. Then the rows and columns of (IX.47) are exact, with the possible exception of the top-most row and the left-most column. Moreover,

$$\Gamma^0(\mathcal{T}^{n-k},\mathcal{U}) \simeq \mathcal{C}_0(\mathcal{T}^{n-k},\mathcal{U}). \tag{IX.48}$$

and we have isomorphisms between homology spaces

$$\mathcal{H}_{n-k}(\mathcal{T},\mathcal{U}) \simeq \mathcal{H}\Gamma^0(\mathcal{T}^{n-k},\mathcal{U}) \simeq \mathcal{H}\Lambda^k(\mathcal{T}^n,\mathcal{U}) \tag{IX.49}$$

for $0 \leq k \leq n$.

Proof. Under the assumptions of the theorem, the rows and columns of (IX.47) are exact, with the possible exception of the top-most row and left-most column. This follows from Lemma IX.2.2 and Lemma IX.3.6. The local exactness condition implies in particular that $\Gamma^0(\mathcal{T}^m,\mathcal{U}) \simeq \mathcal{C}_0(\mathcal{T}^m,\mathcal{U})$ for all $m \in \mathbb{Z}$, and that the following diagram commutes:

$$\begin{array}{ccc}
\Gamma^0(\mathcal{T}^m,\mathcal{U}) & \xrightarrow{\simeq} & \mathcal{C}_0(\mathcal{T}^m,\mathcal{U}) \\
{\scriptstyle \mathsf{T}_0^m}\downarrow & & {\scriptstyle \partial_m}\downarrow \\
\Gamma^0(\mathcal{T}^{m-1},\mathcal{U}) & \xrightarrow{\simeq} & \mathcal{C}_0(\mathcal{T}^{m-1},\mathcal{U})
\end{array}$$

Hence (IX.48) and the first isomorphism in (IX.49) follow. Finally, the second isomorphism in (IX.49) follows via a standard result in homological algebra on double complexes; see for example Proposition 3.11 of [150], Chapter 9.2 of [31], or Corollary 6.4 of [14]. □

Example IX.4.2.
We continue our example application. The differential complex

$$0 \longrightarrow \Lambda^0(\mathcal{T}^n,\mathcal{U}) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda^n(\mathcal{T}^n,\mathcal{U}) \longrightarrow 0, \tag{IX.50}$$

is composed of spaces of finite element differential forms whose traces on simplices of $\mathcal{V}$ vanish. This is a conforming discretization of the L^2 de Rham complex with partial tangential boundary conditions along Γ_T. The dimension of the homology spaces $\mathcal{H}\Lambda^k(\mathcal{T}^n,\mathcal{U})$ of the finite element complex are related to the Betti numbers. We have

$$\begin{aligned}
\dim \mathcal{H}\Lambda^k(\mathcal{T}^n,\mathcal{U}) &= b_{n-k}(\overline{\Omega},\Gamma_N) \\
&= \dim \mathfrak{H}^{n-k}(\Omega,\Gamma_N,\Gamma_T) \\
&= \dim \mathfrak{H}^{k}(\Omega,\Gamma_T,\Gamma_N) = b_k(\overline{\Omega},\Gamma_T)
\end{aligned}$$

for $0 \leq k \leq n$. This includes the special cases $\Gamma_T = \emptyset$ and $\Gamma_T = \Gamma$, which have been treated earlier in the literature [58].

Remark IX.4.3.
Christiansen has analyzed the finite element de Rham cohomology on triangulated manifolds without boundary [52]. Arnold, Falk and Winther have derived the finite element de Rham cohomology without boundary conditions over domains from the L^2 de Rham complex [9], which Christiansen and Winther have successively extended to the case of essential boundary conditions [58]. Christiansen et al. have also derived the finite element de Rham cohomology without boundary conditions within the framework of element systems via de Rham mappings [56]. With different techniques, we have derived the finite element de Rham cohomology without reference to the L^2 de Rham complex.

IX.5. Discrete Distributional de Rham Complexes

In this section we introduce discrete distributional de Rham complexes. We continue to assume that $\mathcal{T}$ is a finite n-dimensional simplicial complex, and that $\mathcal{U}$ is a simplicial subcomplex.

For $k, m, b \in \mathbb{Z}$ we introduce the direct sums

$$\Lambda^k_{-b}(\mathcal{T}^m, \mathcal{U}) := \sum_{i=0}^{b-1} \Lambda^{k-i}_{-1}(\mathcal{T}^{m-i}, \mathcal{U}), \tag{IX.51a}$$

$$\Gamma^k_{-b}(\mathcal{T}^m, \mathcal{U}) := \sum_{i=0}^{b-1} \Gamma^{k+i}_{-1}(\mathcal{T}^{m+i}, \mathcal{U}). \tag{IX.51b}$$

From now on we may also write

$$\Lambda^k_0(\mathcal{T}^m, \mathcal{U}) = \Lambda^k(\mathcal{T}^m, \mathcal{U}), \quad \Gamma^k_0(\mathcal{T}^m, \mathcal{U}) = \Gamma^k(\mathcal{T}^m, \mathcal{U})$$

whenever convenient. We remark that

$$\Lambda^k_{-b}(\mathcal{T}^m, \mathcal{U}) \subseteq \Lambda^k_{-b-1}(\mathcal{T}^m, \mathcal{U}), \quad \Gamma^k_{-b}(\mathcal{T}^m, \mathcal{U}) \subseteq \Gamma^k_{-b-1}(\mathcal{T}^m, \mathcal{U}),$$

and that for $b \geq 1$ we have

$$\Lambda^k_{-b}(\mathcal{T}^m, \mathcal{U}) = \Gamma^{k-b+1}_{-b}(\mathcal{T}^{m-b+1}, \mathcal{U}),$$

as is easily verified from definitions.

We introduce the discrete distributional exterior derivative

$$\mathsf{d}^{n-m+k} : \Lambda^k_{-1}(\mathcal{T}^m, \mathcal{U}) \to \Lambda^{k+1}_{-1}(\mathcal{T}^m, \mathcal{U}) \oplus \Lambda^k_{-1}(\mathcal{T}^{m-1}, \mathcal{U})$$

by setting

$$\mathsf{d}^{n-m+k}\omega := (-1)^{n-m}\mathsf{D}^m_k\omega + (-1)^{n-m+1}\mathsf{T}^m_k\omega, \quad \omega \in \Lambda^k_{-1}(\mathcal{T}^m, \mathcal{U}).$$

This operator is inspired by the observations in Example IX.1.5. Taking direct sums, we obtain operators

$$\mathsf{d}^{n-m+k} : \Lambda^k_{-b}(\mathcal{T}^m, \mathcal{U}) \to \Lambda^{k+1}_{-b-1}(\mathcal{T}^m, \mathcal{U}),$$
$$\mathsf{d}^{n-m+k} : \Gamma^k_{-b}(\mathcal{T}^m, \mathcal{U}) \to \Gamma^k_{-b-1}(\mathcal{T}^{m-1}, \mathcal{U}).$$

Via (IX.21), (IX.24) and (IX.25), we verify the differential property

$$\mathsf{d}^{n-m+k+1}\mathsf{d}^{n-m+k}\omega = 0, \quad \omega \in \Lambda^k_{-b}(\mathcal{T}^m, \mathcal{U}).$$

This differential property motivates us to consider differential complexes of discrete distributional differential forms assembled from spaces of the type $\Lambda^k_{-b}(\mathcal{T}^m, \mathcal{U})$ and $\Gamma^k_{-b}(\mathcal{T}^m, \mathcal{U})$, which leads us to the notion of discrete distributional de Rham complex. Consider again the differential complex

$$0 \to \Lambda^0(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda^n(\mathcal{T}^n) \to 0, \tag{IX.52}$$

which resembles the standard finite element complex. This complex may be "redirected" at any index k in the following sense: we replace $\Lambda^k(\mathcal{T}^n)$ with $\Lambda^k_{-1}(\mathcal{T}^n)$, and continue with the spaces $\Lambda^{k+1}_{-2}(\mathcal{T}^n)$, $\Lambda^{k+2}_{-3}(\mathcal{T}^n)$, and so forth, where the exterior derivative is to be understood in the generalized sense. We thus have a differential complex

$$\dots \xrightarrow{\mathsf{d}^{k-2}} \Lambda^{k-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^{k-1}} \Lambda^k_{-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^k} \Lambda^{k+1}_{-2}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^{k+1}} \dots \tag{IX.53}$$

We see that the original complex is already trivially redirected at the n-forms, noting $\Lambda^n(\mathcal{T}^n) = \Lambda^n_{-1}(\mathcal{T}^n)$. The original finite element complex is a subcomplex of the differential complex redirected at the $(n-1)$-forms. In turn, the latter is a subcomplex of the differential complex redirected at the $n-2$ forms. More generally, the differential complex redirected at the k-forms is a subcomplex of the differential complex redirected at the $(k-1)$-forms.

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\mathsf{d}^{k-2}} & \Lambda^{k-1}_{-1}(\mathcal{T}^n) & \xrightarrow{\mathsf{d}^{k-1}} & \Lambda^k_{-2}(\mathcal{T}^n) & \xrightarrow{\mathsf{d}^k} & \Lambda^{k+1}_{-3}(\mathcal{T}^n) & \xrightarrow{\mathsf{d}^{k+1}} & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \xrightarrow{\mathsf{d}^{k-2}} & \Lambda^{k-1}(\mathcal{T}^n) & \xrightarrow{\mathsf{d}^{k-1}} & \Lambda^k_{-1}(\mathcal{T}^n) & \xrightarrow{\mathsf{d}^k} & \Lambda^{k+1}_{-2}(\mathcal{T}^n) & \xrightarrow{\mathsf{d}^{k+1}} & \dots
\end{array} \tag{IX.54}$$

We thus have a succession of (inclusions of) differential complexes of discrete distributional differential forms. Proceeding in this manner, we eventual obtain a "maximal" differential complex that is redirected already at the 0-forms:

$$0 \to \Lambda^0_{-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda^n_{-n-1}(\mathcal{T}^n) \to 0. \tag{IX.55}$$

We have constructed a family of differential complexes starting from the original finite element complex (IX.52) and completely analogously we start a similar construction with the differential complex of simplicial chains. Consider the differential complex

$$0 \to \Gamma^0(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Gamma^0(\mathcal{T}^0) \to 0. \tag{IX.56}$$

Let us fix an index $m \in \mathbb{Z}$. The differential complex (IX.56) at index m looks like

$$\dots \xrightarrow{\mathsf{d}^{n-m-2}} \Gamma^0(\mathcal{T}^{m+1}) \xrightarrow{\mathsf{d}^{n-m-1}} \Gamma^0(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{n-m}} \Gamma^0(\mathcal{T}^{m-1}) \xrightarrow{\mathsf{d}^{n-m+1}} \dots \tag{IX.57}$$

We redirect this complex at index m to obtain

$$\dots \xrightarrow{\mathsf{d}^{n-m-2}} \Gamma^0(\mathcal{T}^{m+1}) \xrightarrow{\mathsf{d}^{n-m-1}} \Gamma^0_{-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{n-m}} \Gamma^0_{-2}(\mathcal{T}^{m-1}) \xrightarrow{\mathsf{d}^{n-m+1}} \dots \tag{IX.58}$$

Again we observe a sequence of discrete distributional de Rham complexes. The original differential complex (IX.56) is already redirected at index 0, because $\Gamma^0_{-1}(\mathcal{T}^n) = \Gamma^0(\mathcal{T}^n)$. This complex is a subcomplex of the differential complex redirected at index $m = 1$, which is a subcomplex of the differential complex redirected at index $m = 2$, and so forth. In general, the differential complex redirected at index m is included in the differential complex redirected at index $m - 1$.

$$\begin{tikzcd}
\dots \arrow[r, "\mathsf{d}^{n-m-2}"] & \Gamma^0_{-1}(\mathcal{T}^{m+1}) \arrow[r, "\mathsf{d}^{n-m-1}"] & \Gamma^0_{-2}(\mathcal{T}^m) \arrow[r, "\mathsf{d}^{n-m}"] & \Gamma^0_{-3}(\mathcal{T}^{m-1}) \arrow[r, "\mathsf{d}^{n-m+1}"] & \dots \\
\dots \arrow[r, "\mathsf{d}^{n-m-2}"] & \Gamma^0(\mathcal{T}^{m+1}) \arrow[r, "\mathsf{d}^{n-m-1}"] \arrow[u] & \Gamma^0_{-1}(\mathcal{T}^m) \arrow[r, "\mathsf{d}^{n-m}"] \arrow[u] & \Gamma^0_{-2}(\mathcal{T}^{m-1}) \arrow[r, "\mathsf{d}^{n-m+1}"] \arrow[u] & \dots
\end{tikzcd} \tag{IX.59}$$

As before, we have a succession of (inclusions of) differential complexes. The maximal example of this second family of differential complexes is obtained by redirecting (IX.56) already at index $m = n$,

$$0 \to \Gamma^0_{-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Gamma^0_{-n-1}(\mathcal{T}^0) \to 0. \tag{IX.60}$$

Unfolding definitions we find that this differential complex is, in fact, identical to (IX.55). Hence we have two families of differential complexes, one starting at (IX.52) and the other starting at (IX.56), that lead to the same discrete distributional de Rham complex.

We have encountered differential complexes that generalize the finite element complex (IX.52) and are formulated on the lower dimensional skeletons $\mathcal{T}^{[m]}$ of $\mathcal{T}$, for $0 \leq m \leq n$.

$$0 \to \Lambda^0(\mathcal{T}^m) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{m-1}} \Lambda^m(\mathcal{T}^m) \to 0, \tag{IX.61}$$

Similar as above, we may redirect this complex at any index k, and obtain differential complexes

$$\dots \xrightarrow{\mathsf{d}^{k+n-m-2}} \Lambda^{k-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m-1}} \Lambda^k_{-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m}} \Lambda^{k+1}_{-2}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m+1}} \dots \tag{IX.62}$$

We have interpreted the simplicial chain complex of $\mathcal{T}$ relative to $\mathcal{U}$ as a differential complex of (constant) 0-forms on simplices. We generalize this differential complex as a complex of k-forms on simplices whose piecewise exterior derivative vanishes for $0 \leq k \leq n$.

$$0 \to \Gamma^k(\mathcal{T}^n) \xrightarrow{\mathsf{d}^k} \dots \xrightarrow{\mathsf{d}^{n-1}} \Gamma^k(\mathcal{T}^k) \to 0. \tag{IX.63}$$

We may redirect this complex at index m, to obtain a differential complex

$$\dots \xrightarrow{\mathsf{d}^{k+n-m-2}} \Gamma^k(\mathcal{T}^{m+1}) \xrightarrow{\mathsf{d}^{k+n-m-1}} \Gamma^k_{-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m}} \Gamma^k_{-2}(\mathcal{T}^{m-1}) \xrightarrow{\mathsf{d}^{k+n-m+1}} \dots \tag{IX.64}$$

In the next section, we will determine the homology spaces of these differential complexes.

Remark IX.5.1.
The results in this section generalize ideas of [34], in particular the proofs of their Lemma 3, Theorem 5 and Theorem 7. But the distributional complexes in this section can also be related to the double complex of the preceding section. We identify the maximal complex (IX.55) / (IX.60) as the total complex of the double complex (IX.47), skipping the left-most column and the top-most-row of that diagram. The two families of broken complexes, (IX.53) and (IX.58), exemplify the two canonical filtrations of the total complex. We refer to [31] for more background on notion of homological algebra. Although the underlying ideas are similar, our presentation is specifically tailored towards finite element analysis and addresses the harmonic spaces of the broken complexes explicitly.

IX.6. Discrete Distributional Harmonic Forms

In this section we expand upon the homology theory of discrete distributional de Rham complexes. It will helpful to switch to the framework of Hilbert complexes for that purpose. This requires fixing a Hilbert space structure on the spaces of discrete distributional differential forms. For that reason, we henceforth assume for the remainder of this chapter that the spaces $\Lambda^k_{-1}(\mathcal{T}^m,\mathcal{U})$ are equipped with scalar products, which turns them into Hilbert spaces. Either by taking direct sums or by taking restrictions to subspaces, this yields a Hilbert space structure on the other spaces of discrete distributional differential forms discussed in this chapter. A specific choice of scalar product will be required only later when we address Poincaré-Friedrichs inequalities, but we remark that the scalar products (IX.33) are a possible choice.

Our goal is to construct isomorphisms between the harmonic spaces of Hilbert complexes of discrete distributional differential forms. To that end, we denote the harmonic spaces of these complexes by

$$\mathfrak{H}^k_{-b}(\mathcal{T}^m) := \left\{ \omega \in \Lambda^k_{-b}(\mathcal{T}^m) \;\middle|\; \begin{array}{l} \mathsf{d}^{k+n-m}\omega = 0, \\ \omega \perp \mathsf{d}^{k+n-m-1}\Lambda^{k-1}_{-b+1}(\mathcal{T}^m) \end{array} \right\}, \tag{IX.65}$$

$$\mathfrak{C}^k_{-b}(\mathcal{T}^m) := \left\{ \omega \in \Gamma^0_{-b}(\mathcal{T}^m) \;\middle|\; \begin{array}{l} \mathsf{d}^{k+n-m}\omega = 0, \\ \omega \perp \mathsf{d}^{k+n-m-1}\Gamma^k_{-b+1}(\mathcal{T}^{m+1}) \end{array} \right\}, \tag{IX.66}$$

The orthogonality is with respect to whatever scalar product is chosen on the differential complexes of discrete distributional differential forms.

We use the term *discrete distributional harmonic form* for the elements of the harmonic spaces $\mathfrak{H}^k_{-b}(\mathcal{T}^m)$ and $\mathfrak{C}^k_{-b}(\mathcal{T}^m)$. We sometimes write

$$\mathfrak{H}^k(\mathcal{T}^m) = \mathfrak{H}^k_0(\mathcal{T}^m), \quad \mathfrak{C}^k(\mathcal{T}^m) = \mathfrak{C}^k_0(\mathcal{T}^m).$$

The spaces of harmonic forms $\mathfrak{H}^k_{-1}(\mathcal{T}^m)$ and $\mathfrak{C}^k_{-1}(\mathcal{T}^m)$ are easily described:

Lemma IX.6.1.
Let $k, m \in \mathbb{Z}$. Then $\mathfrak{H}^k_0(\mathcal{T}^m) = \mathfrak{H}^k_{-1}(\mathcal{T}^m)$.

Proof. Unfolding definitions, we have $\mathfrak{H}^k_0(\mathcal{T}^m) \subseteq \mathfrak{H}^k_{-1}(\mathcal{T}^m)$. Conversely, we have $\omega \in \mathfrak{H}^k_{-1}(\mathcal{T}^m)$ if and only if $\mathsf{d}^k\omega = 0$ and ω is orthogonal to $\mathsf{d}^{k-1}\Lambda^{k-1}_0(\mathcal{T}^m)$. But in that case $\omega \in \Lambda^k_0(\mathcal{T}^m)$, and $\omega \in \mathfrak{H}^k_0(\mathcal{T}^m)$ follows by definitions. The proof is complete. □

Lemma IX.6.2.
Let $k, m \in \mathbb{Z}$. Then $\mathfrak{C}^k_0(\mathcal{T}^m) = \mathfrak{C}^k_{-1}(\mathcal{T}^m)$.

Proof. This is very similar to the proof of Lemma IX.6.1 above. First, we have $\mathfrak{C}^k_0(\mathcal{T}^m) \subseteq \mathfrak{C}^k_{-1}(\mathcal{T}^m)$. Conversely, we have $\omega \in \mathfrak{C}^k_{-1}(\mathcal{T}^m)$ if and only if $\mathsf{d}^{k+n-m}\omega = 0$ and ω is orthogonal to $\mathsf{d}^{k+n-m-1}\Gamma^{k-1}_0(\mathcal{T}^m)$. But in that case $\omega \in \Gamma^k_0(\mathcal{T}^m)$, and $\omega \in \mathfrak{C}^k_0(\mathcal{T}^m)$ follows by definitions. The proof is complete. □

Remark IX.6.3.
In the sequel, we prove statements about the spaces $\mathfrak{H}^k_{-b}(\mathcal{T}^m)$ and $\mathfrak{C}^k_{-b}(\mathcal{T}^m)$. The respective proofs are very similar in each case, but we provide full proofs for both for the sake of technical completeness.

To proceed with the agenda of this section, we require the additional assumptions already used in Section IX.4. This means that in the sequel, we assume that the local exactness condition, the geometric decomposition condition, and the local patch condition of $\mathcal{T}$ relative to $\mathcal{U}$ hold.

Moreover, we need to fix generalized inverses of the differential operators D^m_k and T^m_k. Specifically, we assume to have operators

$$\begin{aligned} \mathsf{E}^m_k &: \Lambda^k_{-1}(\mathcal{T}^{m-1}) \to \Lambda^k_{-1}(\mathcal{T}^m), \\ \mathsf{P}^m_k &: \Lambda^{k+1}_{-1}(\mathcal{T}^m) \to \Lambda^k_{-1}(\mathcal{T}^m) \end{aligned}$$

for $m, k \in \mathbb{Z}$ that satisfy

$$\begin{aligned} \mathsf{T}^m_k &= \mathsf{T}^m_k \mathsf{E}^m_k \mathsf{T}^m_k, \\ \mathsf{D}^m_k &= \mathsf{D}^m_k \mathsf{P}^m_k \mathsf{D}^m_k. \end{aligned}$$

A possible choice of such generalized inverses are the Moore-Penrose pseudoinverses [73] of D^m_k and T^m_k, but no specific choice needs to be made at this point. We construct such operators explicitly in Sections IX.7 and IX.8.

We now construct the following operators. When $m, k \in \mathbb{Z}$ and $b \in \mathbb{N}$, then we define

$$R^m_{k,b} : \Lambda^k_{-b}(\mathcal{T}^m) \to \Lambda^k_{-b}(\mathcal{T}^m), \quad \omega \mapsto \omega + (-1)^{n-m+b}\mathsf{d}^{n-m+k-1}\mathsf{E}^{m-b+2}_{k-b+1}\omega, \tag{IX.67}$$

$$S^k_{m,b} : \Gamma^k_{-b}(\mathcal{T}^m) \to \Gamma^k_{-b}(\mathcal{T}^m), \quad \omega \mapsto \omega + (-1)^{n-m+b}\mathsf{d}^{k+n-m-1}\mathsf{P}^{m+b-1}_{k+b-1}\omega, \tag{IX.68}$$

The main application of these operators is shown in the following two lemmas.

Lemma IX.6.4.
Let $m, k \in \mathbb{Z}$, let $b \in \mathbb{N}$ with $m \leq n$, and let $\omega \in \Lambda^k_{-b}(\mathcal{T}^m)$ with $\mathsf{d}^{k+n-m}\omega \in \Lambda^{k+1}_{-b}(\mathcal{T}^m)$.

- We have $\mathsf{d}^{k+n-m}\omega \in \mathsf{d}^{k+n-m}\Lambda^k_{-b+1}(\mathcal{T}^m)$.
- If $b = 1$, then $\omega \in \Lambda^k_0(\mathcal{T}^m)$.

- If $b \geq 2$, then $\mathsf{d}^{k+n-m} R^m_{k,b}\omega = \mathsf{d}^{k+n-m}\omega$ with $R^m_{k,b}\omega \in \Lambda^k_{-b+1}(\mathcal{T}^m)$.

Proof. First we consider the case $b = 1$. If $\omega \in \Lambda^k_{-1}(\mathcal{T}^m)$ with $\mathsf{d}^{k+n-m}\omega \in \Lambda^{k+1}_{-1}(\mathcal{T}^m)$, then $\mathsf{T}^m_k\omega = 0$. But then $\omega \in \Lambda^k_0(\mathcal{T}^m)$ by definition.

Next we consider the case $b \geq 2$. Let $\omega \in \Lambda^k_{-b}(\mathcal{T}^m)$ with $\mathsf{d}^{k+n-m}\omega \in \Lambda^{k+1}_{-b}(\mathcal{T}^m)$, and write

$$\omega = \omega^0 + \dots + \omega^{b-1}, \quad \omega^j \in \Lambda^{k-j}_{-1}(\mathcal{T}^{m-j}).$$

By $\mathsf{d}^{k+n-m}\omega \in \Lambda^{k+1}_{-b}(\mathcal{T}^m)$ we conclude $\mathsf{T}^{m-b+1}_{k-b+1}\omega^{b-1} = 0$. Using the properties of the operator $\mathsf{E}^{m-b+2}_{k-b+1}$ and the homology properties of the vertical complex, we have $\mathsf{T}^{m-b+2}_{k-b+1}\mathsf{E}^{m-b+2}_{k-b+1}\omega^{b-1} = \omega^{b-1}$. In particular, $\mathsf{E}^{m-b+2}_{k-b+1}\omega^{b-1} \in \Lambda^{k-b+1}_{-1}(\mathcal{T}^{m-b+2})$, and we have

$$\begin{aligned}(-1)^{n-m+b}\mathsf{d}^{k+n-m-1}\mathsf{E}^{m-b+2}_{k-b+1}\omega^{b-1} &= \mathsf{D}^{m-b+2}_{k-b+1}\mathsf{E}^{m-b+2}_{k-b+1}\omega^{b-1} - \mathsf{T}^{m-b+2}_{k-b+1}\mathsf{E}^{m-b+2}_{k-b+1}\omega^{b-1} \\ &= \mathsf{D}^{m-b+2}_{k-b+1}\mathsf{E}^{m-b+2}_{k-b+1}\omega^{b-1} - \omega^{b-1}.\end{aligned}$$

We conclude that

$$\begin{aligned}R^m_{k,b}\omega &= \omega^0 + \dots + \omega^{b-1} + (-1)^{n-m+b}\mathsf{d}^{k+n-m-1}\mathsf{E}^{m-b+2}_{k-b+1}\omega^{b-1} \\ &= \omega^0 + \dots + \omega^{b-2} + \mathsf{D}^{m-b+2}_{k-b+1}\mathsf{E}^{m-b+2}_{k-b+1}\omega^{b-1},\end{aligned}$$

so $R^m_{k,b}\omega \in \Lambda^k_{-b+1}(\mathcal{T}^m)$. Furthermore,

$$\mathsf{d}^{k+n-m}R^m_{k,b}\omega = \mathsf{d}^{k+n-m}\omega + (-1)^{n-m+b}\mathsf{d}^{k+n-m}\mathsf{d}^{k+n-m-1}\mathsf{E}^{m-b+2}_{k-b+1}\omega^{b-1} = \mathsf{d}^{k+n-m}\omega.$$

This completes the proof. $\square$

Lemma IX.6.5.
Let $m, k \in \mathbb{Z}$, let $b \in \mathbb{N}$ with $0 \leq k$ and let $\omega \in \Gamma^k_{-b}(\mathcal{T}^m)$ with $\mathsf{d}^{k+n-m}\omega \in \Gamma^k_{-b}(\mathcal{T}^{m-1})$.

- We have $\mathsf{d}^{k+n-m}\omega \in \mathsf{d}^{k+n-m}\Gamma^k_{-b+1}(\mathcal{T}^m)$.
- If $b = 1$, then $\omega \in \Gamma^k_0(\mathcal{T}^m)$.
- If $b \geq 2$, then $\mathsf{d}^{k+n-m}S^k_{m,b}\omega = \mathsf{d}^{k+n-m}\omega$ with $S^k_{m,b}\omega \in \Gamma^k_{-b+1}(\mathcal{T}^m)$.

Proof. First we consider the case $b = 1$. If $\omega \in \Gamma^k_{-1}(\mathcal{T}^m)$ with $\mathsf{d}^{k+n-m}\omega \in \Gamma^{k-1}_{-1}(\mathcal{T}^m)$, then $\mathsf{D}^m_k\omega = 0$. But then $\omega \in \Gamma^k_0(\mathcal{T}^m)$ by definition.

Next we consider the case $b \geq 2$. Let $\omega \in \Gamma^k_{-b}(\mathcal{T}^m)$ with $\mathsf{d}^{k+n-m}\omega \in \Gamma^{k-1}_{-b}(\mathcal{T}^m)$, and write

$$\omega = \omega^0 + \dots + \omega^{b-1}, \quad \omega^j \in \Gamma^{k+j}_{-1}(\mathcal{T}^{m+j}).$$

By $\mathsf{d}^{k+n-m}\omega \in \Gamma^k_{-b}(\mathcal{T}^{m-1})$ we conclude $\mathsf{D}^{m+b-1}_{k+b-1}\omega^{b-1} = 0$. Using the properties of the operator $\mathsf{P}^{m+b-1}_{k+b-2}$ and the homology of the horizontal complexes, we have $\mathsf{D}^{m+b-1}_{k+b-2}\mathsf{P}^{m+b-1}_{k+b-2}\omega^{b-1} = \omega^{b-1}$. In particular, $\mathsf{P}^{m+b-1}_{k+b-2}\omega^{b-1} \in \Lambda^{k+b-2}_{-1}(\mathcal{T}^{m+b-1})$, and we have

$$\begin{aligned}(-1)^{n-m-b+1}\mathsf{d}^{k+n-m-1}\mathsf{P}^{m+b-1}_{k+b-2}\omega^{b-1} &= \mathsf{D}^{m+b-1}_{k+b-2}\mathsf{P}^{m+b-1}_{k+b-2}\omega^{b-1} - \mathsf{T}^{m+b-1}_{k+b-2}\mathsf{P}^{m+b-1}_{k+b-2}\omega^{b-1} \\ &= \omega^{b-1} - \mathsf{T}^{m+b-1}_{k+b-2}\mathsf{P}^{m+b-1}_{k+b-2}\omega^{b-1}.\end{aligned}$$

We conclude that

$$\begin{aligned} S^k_{m,b}\omega &= \omega^0 + \cdots + \omega^{b-1} + (-1)^{n-m-b}\mathsf{d}^{k+n-m-1}\mathsf{P}^{m+b-1}_{k+b-2}\omega^{b-1} \\ &= \omega^0 + \cdots + \omega^{b-2} + \mathsf{T}^{m+b-1}_{k+b-2}\mathsf{P}^{m+b-1}_{k+b-2}\omega^{b-1}, \end{aligned}$$

so $S^k_{m,b}\omega \in \Gamma^k_{-b+1}(\mathcal{T}^m)$. Furthermore,

$$\mathsf{d}^{k+n-m}S^k_{m,b}\omega = \mathsf{d}^{k+n-m}\omega + (-1)^{n-m-b}\mathsf{d}^{k+n-m}\mathsf{d}^{k+n-m-1}\mathsf{P}^{m+b-1}_{k+b-2}\omega^{b-1} = \mathsf{d}^{k+n-m}\omega.$$

This completes the proof. □

Another auxiliary result restricts the class of discrete distributional differential forms that are candidates for being discrete distributional harmonic forms.

Lemma IX.6.6.
Let $m, k \in \mathbb{Z}$ and let $b \in \mathbb{N}$ with $1 \leq k$, with $m \leq n$, and with $2 \leq b < k+1$. If $\omega \in \Lambda^k_{-b+1}(\mathcal{T}^m)$ with $\omega \neq 0$ and $\mathsf{d}^{k+n-m}\omega = 0$, then ω is not orthogonal to $\mathsf{d}^{k+n-m-1}\Lambda^{k-1}_{-b+1}(\mathcal{T}^m)$.

Proof. Suppose that $\omega \in \Lambda^k_{-b}(\mathcal{T}^m)$ with

$$\omega = \omega^0 + \cdots + \omega^{b-2}, \quad \omega^j \in \Lambda^{k-j}_{-1}(\mathcal{T}^{m-j}),$$

and assume that $\mathsf{d}^{k+n-m}\omega = 0$. The idea of the proof is to construct $\xi \in \Lambda^{k-1}_{-b+1}(\mathcal{T}^m)$ such that $\mathsf{d}^{k-1}\xi$ is not orthogonal to ω. We then define

$$\xi^0 := (-1)^{n-m}\mathsf{P}^m_{k-1}\omega^0 \in \Lambda^{k-1}_{-1}(\mathcal{T}^m),$$

and recursively

$$\xi^j := (-1)^{j+n-m}\mathsf{P}^{m-j}_{k-j-1}\left(\omega^j + (-1)^{n-m+j}\mathsf{T}^{m-j+1}_{k-j}\xi^{j-1}\right) \in \Lambda^{k-j-1}_{-1}(\mathcal{T}^{m-j})$$

for $1 \leq j \leq b-2$. By the homology of the horizontal complexes and $\mathsf{D}^m_k\omega^0 = 0$ we have

$$\mathsf{d}^{k+n-m-1}\xi^0 = \omega^0 - \mathsf{T}^m_{k-1}\xi^0.$$

Now assume that we have already shown

$$\mathsf{d}^{k+n-m-1}(\xi^0 + \cdots + \xi^j) = \omega^0 + \cdots + \omega^j - (-1)^{n-m+j}\mathsf{T}^{m-j}_{k-j-1}\xi^j$$

for some index $0 \leq j < b-2$. This implies in particular

$$\begin{gathered} (-1)^{j+n-m}\mathsf{D}^{m-j}_{k-j-1}\xi^j = \omega^j \text{ if } j = 0, \\ (-1)^{j+n-m}\mathsf{D}^{m-j}_{k-j-1}\xi^j = \omega^j + (-1)^{n-m+j}\mathsf{T}^{m-j+1}_{k-j}\xi^{j-1} \text{ if } j > 0. \end{gathered}$$

So we find that

$$\mathsf{D}^{m-j-1}_{k-j-1}\mathsf{T}^{m-j}_{k-j-1}\xi^j = \mathsf{T}^{m-j}_{k-j}\mathsf{D}^{m-j}_{k-j-1}\xi^j = (-1)^{j+n-m}\mathsf{T}^{m-j}_{k-j}\omega^j,$$

and calculate, using $\mathsf{d}^{k+n-m}\omega = 0$, that

$$\begin{aligned} &(-1)^{n-m+j+1}\mathsf{D}^{m-j-1}_{k-j-1}\left(\omega^{j+1} - (-1)^{n-m+j}\mathsf{T}^{m-j}_{k-j-1}\xi^j\right) \\ &= (-1)^{n-m+j+1}\mathsf{D}^{m-j-1}_{k-j-1}\omega^{j+1} + \mathsf{D}^{m-j-1}_{k-j-1}\mathsf{T}^{m-j}_{k-j-1}\xi^j \\ &= (-1)^{n-m+j+1}\mathsf{D}^{m-j-1}_{k-j-1}\omega^{j+1} + (-1)^{j+n-m}\mathsf{T}^{m-j}_{k-j}\omega^j = 0. \end{aligned}$$

By the homology of the horizontal complex we find

$$(-1)^{j+n-m-1}\mathsf{D}^{m-j-1}_{k-j-2}\xi^{j+1} = \omega^{j+1} + (-1)^{n-m+j}\mathsf{T}^{m-j}_{k-j-1}\xi^j.$$

We combine these findings. We have

$$\begin{aligned} &\mathsf{d}^{k+n-m-1}\left(\xi^0 + \dots + \xi^j + \xi^{j+1}\right) \\ &= \omega^0 + \dots + \omega^j - (-1)^{n-m+j}\mathsf{T}^{m-j}_{k-j-1}\xi^j \\ &\qquad + (-1)^{n-m+j+1}\mathsf{D}^{m-j-1}_{k-j-2}\xi^{j+1} + (-1)^{n-m+j}\mathsf{T}^{m-j-1}_{k-j-2}\xi^{j+1} \\ &= \omega^0 + \dots + \omega^{j+1} - (-1)^{n-m+j+1}\mathsf{T}^{m-j-1}_{k-j-2}\xi^{j+1}. \end{aligned}$$

Iteration of this procedure provides us with

$$\mathsf{d}^{k-1}(\xi^0 + \dots + \xi^{b-2}) = \omega^0 + \dots + \omega^{b-2} - (-1)^{n-m+b}\mathsf{T}^{m-b+2}_{k-b+1}\xi^{b-2},$$

from which we deduce

$$\left\langle \mathsf{d}^{k-1}(\xi^0 + \dots + \xi^{b-2}), \omega \right\rangle_{-h} = \|\omega\|^2_{-h}.$$

Hence, if $\omega \neq 0$, then ω is not orthogonal to $\mathsf{d}^{k+n-m-1}\Lambda^{k-1}_{-b+1}(\mathcal{T}^m)$. The proof is complete. □

Lemma IX.6.7.
Let $m, k \in \mathbb{Z}$ and let $b \in \mathbb{N}$ with $m \leq n-1$, with $0 \leq k$, and with $2 \leq b < n-m+1$. If $\omega \in \Gamma^k_{-b+1}(\mathcal{T}^m)$ with $\omega \neq 0$ and $\mathsf{d}^{k+n-m}\omega = 0$, then ω is not orthogonal to $\mathsf{d}^{k+n-m-1}\Gamma^k_{-b+1}(\mathcal{T}^{m+1})$.

Proof. This is similar to the previous proof. Suppose that $\omega \in \Gamma^k_{-b}(\mathcal{T}^m)$ with

$$\omega = \omega^0 + \dots + \omega^{b-2}, \quad \omega^j \in \Gamma^{k+j}_{-1}(\mathcal{T}^{m+j}),$$

and assume that $\mathsf{d}^{k+n-m}\omega = 0$. We then define

$$\xi^0 := (-1)^{n-m+1}\mathsf{E}^{m+1}_k\omega^0 \in \Gamma^k_{-1}(\mathcal{T}^{m+1}),$$

and recursively

$$\xi^j := (-1)^{j+n-m}\mathsf{E}^{m-j}_{k-j-1}\left(\omega^j + (-1)^{n-m+j-1}\mathsf{D}^{m+j}_{k+j-1}\xi^{j-1}\right) \in \Gamma^{k+j}_{-1}(\mathcal{T}^{m+j+1})$$

for $1 \leq j \leq b-2$. By the homology of the vertical complexes and $\mathsf{T}^m_k\omega^0 = 0$ we have

$$\mathsf{d}^{k+n-m-1}\xi^0 = \omega^0 - \mathsf{D}^m_{k-1}\xi^0.$$

Now assume that we have already shown

$$\mathsf{d}^{k+n-m-1}(\xi^0 + \cdots + \xi^j) = \omega^0 + \cdots + \omega^j - (-1)^{n-m+j}\mathsf{D}^{m+j+1}_{k+j}\xi^j$$

for some index $0 \leq j < b-2$. This implies in particular

$$\begin{aligned}(-1)^{j+n-m}\mathsf{T}^{m+j+1}_{k+j}\xi^j &= \omega^j \text{ if } j = 0,\\ (-1)^{j+n-m}\mathsf{T}^{m+j+1}_{k+j}\xi^j &= \omega^j + (-1)^{n-m+j-1}\mathsf{D}^{m+j}_{k+j-1}\xi^{j-1} \text{ if } j > 0.\end{aligned}$$

So we find that

$$\mathsf{T}^{m+j+1}_{k+j+1}\mathsf{D}^{m+j+1}_{k+j}\xi^j = \mathsf{D}^{m+j}_{k+j}\mathsf{T}^{m+j+1}_{k+j}\xi^j = (-1)^{j+n-m}\mathsf{D}^{m+j}_{k+j}\omega^j,$$

and calculate, using $\mathsf{d}^{k+n-m}\omega = 0$, that

$$\begin{aligned}&(-1)^{n-m+j}\mathsf{T}^{m+j+1}_{k+j+1}\left(\omega^{j+1} + (-1)^{n-m+j}\mathsf{D}^{m+j+1}_{k+j}\xi^j\right)\\ &= (-1)^{n-m+j}\mathsf{T}^{m+j+1}_{k+j+1}\omega^{j+1} + \mathsf{T}^{m+j+1}_{k+j+1}\mathsf{D}^{m+j+1}_{k+j}\xi^j\\ &= (-1)^{n-m+j}\mathsf{T}^{m+j+1}_{k+j+1}\omega^{j+1} + (-1)^{j+n-m}\mathsf{D}^{m+j}_{k+j}\omega^j = 0.\end{aligned}$$

By the homology of the vertical complexes we find

$$(-1)^{j+n-m-1}\mathsf{T}^{m+j+2}_{k+j+1}\xi^{j+1} = \omega^{j+1} + (-1)^{n-m+j}\mathsf{D}^{m+j+1}_{k+j}\xi^j.$$

We combine these findings. We have

$$\begin{aligned}&\mathsf{d}^{k+n-m-1}\left(\xi^0 + \cdots + \xi^j + \xi^{j+1}\right)\\ &= \omega^0 + \cdots + \omega^j - (-1)^{n-m+j}\mathsf{D}^{m+j+1}_{k+j}\xi^j\\ &\qquad + (-1)^{n-m+j+1}\mathsf{T}^{m+j+2}_{k+j+1}\xi^{j+1} + (-1)^{n-m+j}\mathsf{D}^{m+j+2}_{k+j+1}\xi^{j+1}\\ &= \omega^0 + \cdots + \omega^{j+1} - (-1)^{n-m+j+1}\mathsf{D}^{m+j+2}_{k+j+1}\xi^{j+1}.\end{aligned}$$

Iteration of this procedure provides us with

$$\mathsf{d}^{k-1}(\xi^0 + \cdots + \xi^{b-2}) = \omega^0 + \cdots + \omega^{b-2} - (-1)^{n-m+b}\mathsf{D}^{m+b-1}_{k+b-2}\xi^{b-2},$$

from which we deduce

$$\left\langle \mathsf{d}^{k-1}(\xi^0 + \cdots + \xi^{b-2}), \omega\right\rangle_{-h} = \|\omega\|^2_{-h}, .$$

Hence, if $\omega \neq 0$, then ω is not orthogonal to $\mathsf{d}^{k+n-m-1}\Gamma^{k-1}_{-b+1}(\mathcal{T}^m)$. The proof is complete. □

The harmonic spaces $\mathfrak{H}^k_{-b}(\mathcal{T}^n)$ and $\mathfrak{C}^0_{-b}(\mathcal{T}^m)$ for $b \geq 2$ are constructed in a recursive manner.

Lemma IX.6.8.
Let $m, k \in \mathbb{Z}$ and $b \in \mathbb{N}$. with $1 \leq k$, with $0 < m \leq n$, and with $2 \leq b \leq k+1$. Let

$$Q^m_{k,b} : \Lambda^k_{-b}(\mathcal{T}^m) \to \ker\left(\mathsf{d}^k : \Lambda^k_{-b}(\mathcal{T}^m) \to \Lambda^{k+1}_{-b-1}(\mathcal{T}^m)\right)$$

be the orthogonal projection. Then the operator $Q^m_{k,b}R^{m*}_{k,b}$ acts as an isomorphism from $\mathfrak{H}^k_{-b+1}(\mathcal{T}^m)$ to $\mathfrak{H}^k_{-b}(\mathcal{T}^m)$.

Proof. Let $\omega \in \Lambda^{k}_{-b}(\mathcal{T}^m)$. We can uniquely write

$$\omega = \omega^0 + \cdots + \omega^{b-1}, \quad \omega^j \in \Lambda^{k-j}_{-1}(\mathcal{T}^{m-j}).$$

By construction of $R^m_{k,b}$, we have

$$\omega - R^m_{k,b}\omega \in \mathsf{d}^{k-1}\Lambda^{k-1}_{-b+1}(\mathcal{T}^m).$$

This implies in particular that

$$\begin{aligned} \mathsf{d}^k\omega = 0 \quad &\Longleftrightarrow \quad \mathsf{d}^k R^m_{k,b}\omega = 0, \\ \omega \in \mathsf{d}^{k-1}\Lambda^{k-1}_{-b+1}(\mathcal{T}^m) \quad &\Longleftrightarrow \quad R^m_{k,b}\omega \in \mathsf{d}^{k-1}\Lambda^{k-1}_{-b+1}(\mathcal{T}^m). \end{aligned}$$

From the last equivalence and the abstract Hodge decomposition, we conclude that

$$\mathsf{d}^k\omega = 0, \quad \omega \perp \mathfrak{H}^k_{-b}(\mathcal{T}^m) \quad \Longleftrightarrow \quad \mathsf{d}^k R^m_{k,b}\omega = 0, \quad R^m_{k,b}\omega \perp \mathfrak{H}^k_{-b}(\mathcal{T}^m).$$

We use Lemma IX.6.4 to first observe

$$\mathsf{d}^k R^m_{k,b}\omega = 0 \quad \Longrightarrow \quad \mathsf{d}^k\omega = 0 \quad \Longrightarrow \quad R^m_{k,b}\omega \in \Lambda^k_{-b+1}(\mathcal{T}^m)$$

and second to find

$$R^m_{k,b}\omega \in \mathsf{d}^{k-1}\Lambda^{k-1}_{-b+1}(\mathcal{T}^m) \quad \Longleftrightarrow \quad R^m_{k,b}\omega \in \mathsf{d}^{k-1}\Lambda^{k-1}_{-b+2}(\mathcal{T}^m).$$

In that way, we derive

$$\begin{aligned} & \mathsf{d}^k R^m_{k,b}\omega = 0, \quad R^m_{k,b}\omega \perp \mathfrak{H}^k_{-b}(\mathcal{T}^m) \\ \Longleftrightarrow \quad & R^m_{k,b}\omega \in \mathsf{d}^{k-1}\Lambda^{k-1}_{-b+1}(\mathcal{T}^m) \\ \Longleftrightarrow \quad & R^m_{k,b}\omega \in \mathsf{d}^{k-1}\Lambda^{k-1}_{-b+2}(\mathcal{T}^m) \\ \Longleftrightarrow \quad & \mathsf{d}^k R^m_{k,b}\omega = 0, \quad R^m_{k,b}\omega \perp \mathfrak{H}^{k-1}_{-b+1}(\mathcal{T}^m) \\ \Longleftrightarrow \quad & \mathsf{d}^k\omega = 0, \quad \omega \perp \left(R^m_{k,b}\right)^* \mathfrak{H}^{k-1}_{-b+1}(\mathcal{T}^m). \end{aligned}$$

Using the properties of the Hodge decomposition on Hilbert complexes, we conclude that the projection of $\left(R^m_{k,b}\right)^* \mathfrak{H}^k_{-b+1}(\mathcal{T}^m)$ onto $\ker \mathsf{d}^k \cap \Lambda^k_{-b}(\mathcal{T}^m)$ equals $\mathfrak{H}^k_{-b}(\mathcal{T}^m)$. Furthermore, Lemma IX.6.6 implies that for $p \in \mathfrak{H}^k_{-b+1}(\mathcal{T}^m)$ we have

$$\langle p, Q^k_b R^{m*}_{k,b} p\rangle = \langle p, R^{m*}_{k,b} p\rangle = \langle R^m_{k,b} p, p\rangle = \langle p, p\rangle.$$

This means that

$$Q^k_b R^{m*}_{k,b} : \mathfrak{H}^k_{-b+1}(\mathcal{T}^m) \to \mathfrak{H}^k_{-b}(\mathcal{T}^m)$$

is not only surjective but also injective. The proof is complete. □

Lemma IX.6.9.
Let $m, k \in \mathbb{Z}$ and $b \in \mathbb{N}$. with $0 \leq k$, with $0 \leq m < n$, and with $2 \leq b \leq n - m + 1$. Let

$$P^k_{m,b} : \Gamma^k_{-b}(\mathcal{T}^m) \to \ker\left(\mathsf{d}^{k+n-m} : \Gamma^k_{-b}(\mathcal{T}^m) \to \Gamma^k_{-b-1}(\mathcal{T}^{m-1})\right)$$

be the orthogonal projection. Then the operator $P^m_{k,b} S^{k*}_{m,b}$ acts as an isomorphism from $\mathfrak{C}^k_{-b-1}(\mathcal{T}^m)$ to $\mathfrak{C}^k_{-b}(\mathcal{T}^m)$.

Proof. Let $\omega \in \Gamma^k_{-b}(\mathcal{T}^m)$. We can write

$$\omega = \omega^0 + \cdots + \omega^{b-1}, \quad \omega^j \in \Gamma^{k+j}_{-1}(\mathcal{T}^{m+j}).$$

By construction of $S^m_{k,b}$ we have

$$\omega - S^m_{k,b}\omega \in \mathsf{d}^{k+n-m-1}\Gamma^k_{-b+1}(\mathcal{T}^{m+1}).$$

This implies in particular that

$$\begin{aligned} \mathsf{d}^{k+n-m}\omega = 0 \quad &\iff \quad \mathsf{d}^{k+n-m}S^k_{m,b}\omega = 0, \\ \omega \in \mathsf{d}^{k+n-m-1}\Gamma^k_{-b+1}(\mathcal{T}^{m+1}) \quad &\iff \quad S^k_{m,b}\omega \in \mathsf{d}^{k+n-m-1}\Gamma^k_{-b+1}(\mathcal{T}^{m+1}). \end{aligned}$$

From the last equivalence and the abstract Hodge decomposition, we conclude that

$$\mathsf{d}^{k+n-m}\omega = 0, \quad \omega \perp \mathfrak{C}^k_{-b}(\mathcal{T}^m) \quad \iff \quad \mathsf{d}^{k+n-m}S^k_{m,b}\omega = 0, \quad S^k_{m,b}\omega \perp \mathfrak{C}^k_{-b}(\mathcal{T}^m).$$

We use Lemma IX.6.5 to first observe

$$\mathsf{d}^{k+n-m}S^k_{m,b}\omega = 0 \quad \Longrightarrow \quad \mathsf{d}^{k+n-m}\omega = 0 \quad \Longrightarrow \quad S^k_{m,b}\omega \in \Gamma^k_{-b+1}(\mathcal{T}^m)$$

and second to find

$$S^k_{m,b}\omega \in \mathsf{d}^{k+n-m-1}\Gamma^k_{-b+1}(\mathcal{T}^{m+1}) \quad \iff \quad S^k_{m,b}\omega \in \mathsf{d}^{k+n-m-1}\Gamma^k_{-b+2}(\mathcal{T}^{m+1}).$$

In that way, we derive

$$\begin{aligned} & \mathsf{d}^{k+n-m}S^k_{m,b}\omega = 0, \quad S^k_{m,b}\omega \perp \mathfrak{C}^k_{-b}(\mathcal{T}^m) \\ \iff \quad & S^k_{m,b}\omega \in \mathsf{d}^{k+n-m-1}\Gamma^k_{-b+1}(\mathcal{T}^{m+1}). \\ \iff \quad & S^k_{m,b}\omega \in \mathsf{d}^{k+n-m-1}\Gamma^k_{-b+2}(\mathcal{T}^{m+1}) \\ \iff \quad & \mathsf{d}^{k+n-m}S^k_{m,b}\omega = 0, \quad S^k_{m,b}\omega \perp \mathfrak{C}^{k-1}_{-b+1}(\mathcal{T}^m) \\ \iff \quad & \mathsf{d}^{k+n-m}\omega = 0, \quad \omega \perp S^{k*}_{m,b}\mathfrak{C}^{k-1}_{-b+1}(\mathcal{T}^m). \end{aligned}$$

Using the properties of the Hodge decomposition on Hilbert complexes, we conclude that the projection of $S^{k*}_{m,b}\mathfrak{C}^{k-1}_{-b+1}(\mathcal{T}^m)$ onto $\ker \mathsf{d}^{k+n-m} \cap \Gamma^k_{-b}(\mathcal{T}^m)$ equals $\mathfrak{C}^k_{-b}(\mathcal{T}^m)$. Furthermore, Lemma IX.6.7 implies that for $p \in \mathfrak{C}^k_{-b+1}(\mathcal{T}^m)$ we have

$$\langle p, P^k_b S^{k*}_{m,b} p \rangle = \langle p, S^{k*}_{m,b} p \rangle = \langle S^k_{m,b} p, p \rangle = \langle p, p \rangle.$$

This means that

$$P^k_b S^{k*}_{m,b} : \mathfrak{C}^k_{-b+1}(\mathcal{T}^m) \to \mathfrak{C}^k_{-b}(\mathcal{T}^m)$$

is not only surjective but also injective. This completes the proof. □

Remark IX.6.10.
If $p \in \mathfrak{H}^k_{-b}(\mathcal{T}^n)$, then generally $R^*_{k,b+1}p \notin \mathfrak{H}^k_{-b}(\mathcal{T}^n)$. However, in the special case $k = n$ the orthogonal projection is redundant because $\mathsf{d}^n \Lambda^n_{-b}(\mathcal{T}^n) = 0$.

Remark IX.6.11.
The requirement of an orthogonal projection in the construction of the discrete distributional harmonic forms seems conceptually unsatisfying. However, one can see that, if we leave out this projection operators at every stage, the resulting construction produces at each stage a space of discrete distributional harmonic forms whose projection onto the discrete distributional harmonic forms is surjective. Analogous observations apply to the operators $S_{m,b}$.

The main result of this section is now evident. We generalize Theorem IX.4.1.

Theorem IX.6.12.
Let $k, m \in \mathbb{Z}$ and $b \in \mathbb{N}$ with $2 \leq b \leq n - m + 1$, with $0 \leq m \leq n$, and with $0 \leq k \leq n$. Under the assumptions of this section, we have isomorphisms

$$\begin{aligned}\mathfrak{C}^{k}(\mathcal{T}^{m}) &= \mathfrak{C}^{k}_{-1}(\mathcal{T}^{m}) \simeq \cdots \simeq \mathfrak{C}^{k}_{-b}(\mathcal{T}^{m}) \\ &= \mathfrak{H}^{k+b-1}_{-b}(\mathcal{T}^{m+b-1}) \simeq \cdots \simeq \mathfrak{H}^{k+b-1}_{-1}(\mathcal{T}^{m+b-1}) = \mathfrak{H}^{k+b-1}(\mathcal{T}^{m+b-1})\end{aligned}. \tag{IX.69}$$

Proof. This follows from iterated application of Lemmas IX.6.8 and IX.6.9, together with Lemma IX.6.15 and the fact $\mathfrak{H}^{k}_{-b}(\mathcal{T}^{m}) = \mathfrak{C}^{0}_{-b}(\mathcal{T}^{m-b-1})$ for $b \geq 2$. □

Remark IX.6.13.
With a different choice of indexing convention, Theorem IX.69 states that

$$\begin{aligned}\mathfrak{C}^{k-b+1}(\mathcal{T}^{m-b+1}) &= \mathfrak{C}^{k-b+1}_{-1}(\mathcal{T}^{m-b+1}) \simeq \cdots \simeq \mathfrak{C}^{k-b+1}_{-b}(\mathcal{T}^{m-b+1}) \\ &= \mathfrak{H}^{k}_{-b}(\mathcal{T}^{m}) \simeq \cdots \simeq \mathfrak{H}^{k}_{-1}(\mathcal{T}^{m}) = \mathfrak{H}^{k}(\mathcal{T}^{m})\end{aligned}$$

for $k, m \in \mathbb{Z}$ and $b \in \mathbb{N}$ with $0 \leq m \leq m$, with $0 \leq k \leq n$, and with $2 \leq b \leq k + 1$.

Corollary IX.6.14.
Under the assumptions of this section, we have isomorphisms between harmonic spaces:

$$\begin{aligned}\mathcal{H}_{n-k}(\mathcal{T},\mathcal{U}) \simeq \mathfrak{C}^{0}(\mathcal{T}^{n-k}) &= \mathfrak{C}^{0}_{-1}(\mathcal{T}^{n-k}) \simeq \cdots \simeq \mathfrak{C}^{0}_{-k-1}(\mathcal{T}^{n-k}) \\ &= \mathfrak{H}^{k}_{-k-1}(\mathcal{T}^{n}) \simeq \cdots \simeq \mathfrak{H}^{k}_{-1}(\mathcal{T}^{n}) = \mathfrak{H}^{k}(\mathcal{T}^{n})\end{aligned}$$

for $0 \leq k \leq n$.

Proof. This follows the previous theorem together with the local exactness condition and the observation $\mathfrak{H}^{0}(\mathcal{T}^{n}) = \mathfrak{C}^{0}(\mathcal{T}^{n})$. □

The identity $\mathfrak{H}^{0}(\mathcal{T}^{n}) = \mathfrak{C}^{0}(\mathcal{T}^{n})$ is evident since this is precisely the subspace of $\Lambda^{0}_{-1}(\mathcal{T}^{n}) = \Gamma^{0}_{-1}(\mathcal{T}^{n})$ whose members have vanishing horizontal and vertical derivative. More generally, the following result is true.

Lemma IX.6.15.
Let $k, m \in \mathbb{Z}$ with $k > 1$ and $m < n$. Then $\mathfrak{C}^{k}_{-1}(\mathcal{T}^{m}) \simeq \mathfrak{H}^{k}_{-1}(\mathcal{T}^{m})$.

Proof. If $\xi \in \Gamma^{k}_{0}(\mathcal{T}^{m+1})$, then

$$\mathsf{D}^{m}_{k-1}\mathsf{T}^{m+1}_{k-1}\mathsf{P}^{m+1}_{k}\xi = \mathsf{T}^{m+1}_{k}\xi$$

by the homology of the vertical complexes. Conversely, if $\xi' \in \Lambda^{k-1}_{0}(\mathcal{T}^{m})$, then

$$\mathsf{T}^{m+1}_{k}\mathsf{D}^{m+1}_{k-1}\mathsf{E}^{m+1}_{k-1}\xi' = \mathsf{D}^{m}_{k-1}\xi'$$

by the homology of the horizontal complexes. This shows that

$$\mathsf{d}^{k+n-m-1}\Gamma^{k}_{0}(\mathcal{T}^{m+1}) = \mathsf{d}^{k+n-m-1}\Lambda^{k-1}_{0}(\mathcal{T}^{m}).$$

The desired statement follows as a consequence. □

At the extreme indices, the following result holds.

Lemma IX.6.16.
Let $0 \leq m \leq n$. Then

$$\mathfrak{H}^{0}(\mathcal{T}^{m}) = \mathfrak{C}^{0}(\mathcal{T}^{m}) \oplus \mathsf{T}^{m+1}_{0}\Gamma^{0}_{0}(\mathcal{T}^{m+1}). \tag{IX.70}$$

Let $0 \leq k \leq n$. Then

$$\mathfrak{C}^{k}(\mathcal{T}^{n}) = \mathfrak{H}^{k}(\mathcal{T}^{n}) \oplus \mathsf{D}^{n}_{k-1}\Lambda^{k-1}_{0}(\mathcal{T}^{n}). \tag{IX.71}$$

Proof. Let $0 \leq m \leq n$ and $\omega \in \Lambda^{0}_{-1}(\mathcal{T}^{m})$. Then $\omega \in \mathfrak{H}^{0}(\mathcal{T}^{m})$ if and only if $\mathsf{d}^{k+n-m}\omega = 0$, which is the case if and only if $\omega \in \Gamma^{0}(\mathcal{T}^{m})$ with $\mathsf{T}^{m}_{0}\omega = 0$. Now (IX.70) follows by the Hodge decomposition.

Analogously, Let $0 \leq k \leq n$ and $\omega \in \Gamma^{k}_{-1}(\mathcal{T}^{n})$. Then $\omega \in \mathfrak{C}^{k}(\mathcal{T}^{n})$ if and only if $\mathsf{d}^{k}\omega = 0$, which is the case if and only if $\omega \in \Lambda^{k}(\mathcal{T}^{n})$ with $\mathsf{T}^{n}_{k}\omega = 0$. Similar as above, (IX.71) follows by the Hodge decomposition. □

This completes our description of the harmonic spaces of Hilbert complexes of discrete distributional differential forms.

IX.7. Inequalities on Horizontal Complexes

So far we have addressed the homology theory of complexes of discrete distributional differential forms. We have constructed explicit isomorphisms that help us to determine the discrete distributional harmonic forms. But apart from harmonic spaces, Poincaré-Friedrichs inequalities are another fundamental topic in the theory of Hilbert complexes. The remainder of this chapter will be devoted to estimating the constants in Poincaré-Friedrichs inequalities. We initiate these efforts with analyzing inequalities of horizontal complexes.

As a base for this discussion, we make additional specifications on the class of discrete de Rham complexes (IX.15) associated to simplices: we assume additionally that we use polynomial de Rham complexes. Recall that for each $C \in \mathcal{T}^{m}$ we have a fixed reference transformation $\varphi_{C} : \Delta_{m} \to C$, as described in Chapter II. We say that the *polynomial order R condition* holds if for each $m, k \in \mathbb{Z}$ we have

$$\varphi^{*}_{C}\Lambda^{k}(C) \subseteq \mathcal{P}_{R}\Lambda^{k}(\Delta_{m}), \quad C \in \mathcal{T}^{m}.$$

This means that on each simplex the differential complexes consist of spaces of polynomial differential forms of order at most R. For the remainder of this chapter, we assume that the polynomial order R condition holds for some $R \in \mathbb{N}_{0}$.

Example IX.7.1.
Suppose that $\mathcal{T}$ is a finite triangulation and that $\mathcal{U} \subseteq \mathcal{T}$ is a subcomplex. Suppose furthermore that we have fixed admissible sequence types $\mathcal{P}_F \in \mathscr{A}$ for each $F \in \mathcal{T}$ such that the hierarchy condition holds (see Chapter IV). In that case we obtain differential complexes (IX.15) of the form

$$\dots \xrightarrow{\mathsf{d}_C^{k-1}} \mathcal{P}\Lambda^k(C) \xrightarrow{\mathsf{d}_C^k} \mathcal{P}\Lambda^{k+1}(C) \xrightarrow{\mathsf{d}_C^{k+1}} \dots$$

as described in Example IX.1.2. Trivially, the polynomial order R assumption holds for some $R \in \mathbb{N}$ sufficiently large.

A first application is an inverse inequality.

Lemma IX.7.2.
Let $m, k \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. There exists a constant $\check{\mu} \geq 0$, depending only on R, m, k, and $\mu(\mathcal{T})$, such that for all $\omega \in \Lambda^k(C)$ and $C \in \mathcal{T}^m \setminus \mathcal{U}^m$ we have

$$h_C^\alpha \left\| \mathsf{d}_C^k \omega_C \right\|_{L^2\Lambda^{k+1}(C)}^2 \leq \check{\mu} h_C^{\alpha-2} \left\| \omega_C \right\|_{L^2\Lambda^k(C)}^2 . \tag{IX.72}$$

Proof. Let $C \in \mathcal{T}^m \setminus \mathcal{U}^m$ and $\omega_C \in \Lambda^k(C)$. We have

$$\begin{aligned} \left\| \mathsf{d}_C^k \omega_C \right\|_{L^2\Lambda^{k+1}(C)} &= \left\| \varphi_C^{-*} \mathsf{d}_{\Delta_m}^k \varphi_C^* \omega_C \right\|_{L^2\Lambda^{k+1}(C)} \\ &\leq \sigma_{\max}\left(\mathrm{D}\, \varphi_C^{-1} \right)^{k+1} \det\left(\mathrm{D}\, \varphi_C \right)^{\frac{1}{2}} \left\| \mathsf{d}_{\Delta_m}^k \varphi_C^* \omega_C \right\|_{L^2\Lambda^{k+1}(\Delta_m)} \\ &\leq \check{\mu}_{R,m} \sigma_{\max}\left(\mathrm{D}\, \varphi_C^{-1} \right)^{k+1} \det\left(\mathrm{D}\, \varphi_C \right)^{\frac{1}{2}} \left\| \varphi_C^* \omega_C \right\|_{L^2\Lambda^k(\Delta_m)} , \end{aligned}$$

where $\check{\mu}_{R,m}$ depends only on R and m. Then we use

$$\left\| \varphi_C^* \omega_C \right\|_{L^2\Lambda^k(\Delta_m)} \leq \sigma_{\max}\left(\mathrm{D}\, \varphi_C \right)^k \det\left(\mathrm{D}\, \varphi_C^{-1} \right)^{\frac{1}{2}} \left\| \omega_C \right\|_{L^2\Lambda^k(C)} .$$

In combination,

$$\left\| \mathsf{d}_C^k \omega_C \right\|_{L^2\Lambda^{k+1}(C)} \leq \check{\mu}_{R,m} \cdot \sigma_{\max}\left(\mathrm{D}\, \varphi_C^{-1} \right) \cdot \kappa\left(\mathrm{D}\, \varphi_C \right)^k \cdot \left\| \omega_C \right\|_{L^2\Lambda^k(\Delta_m)} .$$

This completes the proof. □

Lemma IX.7.3.
Let $m, k \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. There exists $\check{\mu} \geq 0$, depending only on R, m, k, and $\mu(\mathcal{T})$, such that for all $\omega \in \Lambda_{-1}^k(\mathcal{T}^m)$ we have

$$\sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^\alpha \left\| (\mathsf{D}_k^m \omega)_C \right\|_{L^2\Lambda^{k+1}(C)}^2 \leq \check{\mu} \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{\alpha-2} \left\| \omega_C \right\|_{L^2\Lambda^k(C)}^2 . \tag{IX.73}$$

Proof. This follows because (IX.34) is the direct sum of differential complexes (IX.35) associated to the simplices in $C \in \mathcal{T}^m \setminus \mathcal{U}^m$. □

As a converse to the inverse inequality, we prove a Poincaré-Friedrichs inequality. We use the existence of an L^2 bounded generalized inverse on the reference simplex that preserves polynomial differential forms. Specifically, Proposition 4.2 of [64] and Lemma 3.8 of [9] imply the existence of a bounded operator

$$\mathsf{P}^k_{\Delta_m} : L^2\Lambda^{k+1}(\Delta_m) \to H\Lambda^k(\Delta_m)$$

such that

$$\mathsf{d}^k_{\Delta_m}\mathsf{P}^k_{\Delta_m}\mathsf{d}^k_{\Delta_m}\xi = \mathsf{d}^k_{\Delta_m}\xi, \quad \xi \in H\Lambda^k(\Delta_m),$$

and

$$\mathsf{P}^k_{\Delta_m}\big(\mathcal{P}_r\Lambda^{k+1}(\Delta_m)\big) \subseteq \mathcal{P}^-_r\Lambda^k(\Delta_m).$$

The operator $\mathsf{P}^k_{\Delta_m}$ is an averaged Poincaré operator. We then define for each simplex

$$\mathsf{P}^k_C := \varphi_C^{-*}\mathsf{P}^k_{\Delta_m}\varphi_C^*, \quad C \in \mathcal{T}^m \setminus \mathcal{U}^m, \quad k,m \in \mathbb{Z},$$

and combine these simplexwise operators to

$$\mathsf{P}^m_k : \Lambda^{k+1}(\mathcal{T}^m) \to \Lambda^k(\mathcal{T}^m), \quad \omega \mapsto \sum_{C\in\mathcal{T}^m\setminus\mathcal{U}^m} \mathsf{P}^k_C\omega_C.$$

We carry out the following estimates.

Lemma IX.7.4.
Let $m,k \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. There exists a constant $\hat\mu \geq 0$, depending only on R, m, k, and $\mu(\mathcal{T})$, such that for all $C \in \mathcal{T}^m \setminus \mathcal{U}^m$ and $\omega \in \Lambda^{k+1}(C)$ we have

$$h_C^\alpha \left\|\mathsf{P}^k_C\omega_C\right\|^2_{L^2\Lambda^k(C)} \leq \hat\mu h_C^{\alpha-2}\left\|\omega_C\right\|^2_{L^2\Lambda^{k+1}(C)}. \tag{IX.74}$$

Proof. Let $C \in \mathcal{T}^m \setminus \mathcal{U}^m$ and $\omega_C \in \Lambda^k(C)$. We have

$$\begin{aligned}\left\|\mathsf{P}^k_C\omega_C\right\|_{L^2\Lambda^k(C)} &= \left\|\varphi_C^{-*}\mathsf{P}^k_{\Delta_m}\varphi_C^*\omega_C\right\|_{L^2\Lambda^k(C)}\\ &\leq \sigma_{\max}\left(\mathrm{D}\,\varphi_C^{-1}\right)^k \det\left(\mathrm{D}\,\varphi_C\right)^{\frac12}\left\|\mathsf{P}^k_{\Delta_m}\varphi_C^*\omega_C\right\|_{L^2\Lambda^k(\Delta_m)}\\ &\leq \hat\mu_m\sigma_{\max}\left(\mathrm{D}\,\varphi_C^{-1}\right)^k \det\left(\mathrm{D}\,\varphi_C\right)^{\frac12}\left\|\varphi_C^*\omega_C\right\|_{L^2\Lambda^{k+1}(\Delta_m)},\end{aligned}$$

where $\hat\mu_m$ depends only on m. Then we use that

$$\left\|\varphi_C^*\omega_C\right\|_{L^2\Lambda^{k+1}(\Delta_m)} \leq \sigma_{\max}\left(\mathrm{D}\,\varphi_C\right)^{k+1}\det\left(\mathrm{D}\,\varphi_C^{-1}\right)^{\frac12}\left\|\omega_C\right\|_{L^2\Lambda^k(C)}$$

In combination,

$$\left\|\mathsf{P}^k_C\omega_C\right\|_{L^2\Lambda^{k+1}(C)} \leq \hat\mu_m \cdot \sigma_{\max}\left(\mathrm{D}\,\varphi_C\right)\cdot\kappa\left(\mathrm{D}\,\varphi_C\right)^k\cdot\left\|\omega_C\right\|_{L^2\Lambda^{k+1}(\Delta_m)}.$$

This completes the proof. □

Lemma IX.7.5.
Let $m,k \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. There exists a constant $\hat\mu \geq 0$, depending only on R, m, k and $\mu(\mathcal{T})$, such that for $\omega \in \Lambda^k_{-1}(\mathcal{T}^m)$ we have

$$\sum_{C\in\mathcal{T}^m\setminus\mathcal{U}^m} h_C^\alpha \left\|(\mathsf{P}^m_k\omega)_C\right\|^2_{L^2\Lambda^k(C)} \leq \hat\mu \sum_{C\in\mathcal{T}^m\setminus\mathcal{U}^m} h_C^{\alpha-2}\left\|\omega_C\right\|^2_{L^2\Lambda^{k+1}(C)}. \tag{IX.75}$$

Proof. This follows because (IX.34) is the direct sum of (IX.35). □

IX.8. Inequalities on Vertical Complexes

We have investigated inequalities in horizontal differential complexes with respect to a mesh-dependent norm. We now conduct a similar investigation for vertical differential complexes. Some additional technical definitions are required, though.

First, the geometric decomposition assumption implies that for each $\omega \in \Gamma^k(C)$ with $C \in \mathcal{T} \setminus \mathcal{U}$ and $k \in \mathbb{Z}$ we have a unique decomposition

$$\omega = \sum_{F \in \Delta(C)} \operatorname{ext}^k_{F,C} \zeta^F_C, \quad \zeta^F_C \in \mathring{\Gamma}^k(F). \tag{IX.76}$$

We use this notation for a decomposition of the form (IX.76) without further notice in this section. More generally, for each $\omega \in \Gamma^k(\mathcal{T}^m)$ with $k, m \in \mathbb{Z}$ we have unique decompositions

$$\omega = \sum_{F \in \mathcal{T}^{[m]}} \zeta^F, \quad \zeta^F = \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} \operatorname{ext}^k_{F,C} \zeta^F_C.$$

The vertical differential operator T^m_k preserves the decomposition of ω into terms associated to simplices $F \in \mathcal{T}^{[m]}$. In particular, the vertical complex

$$\dots \xrightarrow{\mathsf{T}^{m+1}_k} \Gamma^k(\mathcal{T}^m) \xrightarrow{\mathsf{T}^m_k} \Gamma^k(\mathcal{T}^{m-1}) \xrightarrow{\mathsf{T}^{m-1}_k} \dots$$

is the direct sum of differential complexes

$$\dots \xrightarrow{\mathsf{T}^{m+1}_k} \Gamma^m_k(F) \xrightarrow{\mathsf{T}^m_k} \Gamma^{m-1}_k(F) \xrightarrow{\mathsf{T}^{m-1}_k} \dots \tag{IX.77}$$

over $F \in \mathcal{T}$. We will transform this differential complex to a reference setting, for which we use transformations to reference micropatches. For each $F \in \mathcal{T}$, we write $\Psi^*_F \Gamma^m_k(F)$ for the piecewise pullback of $\Gamma^m_k(F)$ onto the m-simplices of the set $\widehat{\mathcal{M}}(\mathcal{T}, F) \setminus \widehat{\mathcal{N}}(\mathcal{T}, \mathcal{U}, F)$ along the reference transformation Ψ_F. For each $F \in \mathcal{T}$ we obtain the reference differential complexes

$$\dots \xrightarrow{\Psi^*_F \mathsf{T}^{m+1}_k \Psi^{-*}_F} \Psi^*_F \Gamma^m_k(F) \xrightarrow{\Psi^*_F \mathsf{T}^m_k \Psi^{-*}_F} \Psi^*_F \Gamma^{m-1}_k(F) \xrightarrow{\Psi^*_F \mathsf{T}^{m-1}_k \Psi^{-*}_F} \dots \tag{IX.78}$$

By the results of Chapter II, we may assume without loss of generality that there are only finitely many differential complexes of the form (IX.78). This allows for an inverse inequality with uniformly bounded constants on the differential complexes (IX.77). For each $F \in \mathcal{T}$, we fix an operator

$$\widehat{\mathsf{E}}^m_{k,F} : \Psi^*_F \Gamma^{m-1}_k(F) \to \Psi^*_F \Gamma^m_k(F)$$

that satisfies

$$\mathsf{T}^m_k \Psi^{-*}_F \widehat{\mathsf{E}}^m_{k,F} \Psi^*_F \mathsf{T}^m_k \zeta^F = \mathsf{T}^m_k \zeta^F, \quad \zeta^F \in \Gamma^m_k(F).$$

We eventually define

$$\mathsf{E}^m_k : \Gamma^k(\mathcal{T}^{m-1}) \to \Gamma^k(\mathcal{T}^{m-1}), \qquad \sum_{F \in \mathcal{T}^{[m-1]}} \zeta^F \to \sum_{F \in \mathcal{T}^{[m-1]}} \Psi^{-*}_F \widehat{\mathsf{E}}^m_{k,F} \Psi^*_F \zeta^F.$$

By construction we have the identity

$$\mathsf{T}_k^m \mathsf{E}_k^m \mathsf{T}_k^m \omega = \mathsf{T}_k^m \omega, \quad \omega \in \Gamma^k(\mathcal{T}^m).$$

Again, by the results of Chapter II, we may assume without loss of generality that the collection $(\widehat{\mathsf{E}}_{k,F}^m)_{F \in \mathcal{T}}$ has a cardinality that can be bounded in terms of k, m, and $\mu(\mathcal{T})$. This observation will be crucial for proving the next lemma.

Theorem IX.8.1.
Let $\alpha \in \mathbb{R}$. There exist constants $\grave{\mu}, \acute{\mu} \geq 0$ such that

$$\sum_{Q \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1}} h_Q^\alpha \|(\mathsf{T}_k^m \omega)_Q\|_{L^2\Lambda^k(Q)}^2 \leq \grave{\mu} \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{\alpha-1} \|\omega_C\|_{L^2\Lambda^k(C)}^2,$$
$$\sum_{T \in \mathcal{T}^{m+1} \setminus \mathcal{U}^{m+1}} h_T^\alpha \|(\mathsf{E}_k^m \omega)_T\|_{L^2\Lambda^k(T)}^2 \leq \acute{\mu} \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{\alpha+1} \|\omega_C\|_{L^2\Lambda^k(C)}^2.$$

The constants $\grave{\mu}$ and $\acute{\mu}$ depend only on R, α, k, m, $\mu(\mathcal{T})$, and $\mu_{\mathrm{lqu}}(\mathcal{T})$.

Proof. We use the geometric decomposition. We write

$$\omega = \sum_{F \in \mathcal{T}^{[m]} \setminus \mathcal{U}^m} \zeta^F,$$
$$\mathsf{T}_k^m \omega = \sum_{F \in \mathcal{T}^{[m-1]} \setminus \mathcal{U}^{m-1}} \grave{\zeta}^F,$$
$$\mathsf{E}_k^m \omega = \sum_{F \in \mathcal{T}^{[m+1]} \setminus \mathcal{U}^{m+1}} \acute{\zeta}^F$$

for the respective decompositions of ω, $\mathsf{T}_k^m \omega$ and $\mathsf{E}_k^m \omega$ into terms associated to local patches as in Lemma IX.3.2. By construction, for $F \in \mathcal{T}$ we have

$$\mathsf{T}_k^m \zeta^F = \grave{\zeta}^F, \quad \mathsf{E}_k^m \zeta^F = \acute{\zeta}^F. \tag{IX.79}$$

Moreover, as in (IX.76) we write

$$\zeta_F = \sum_{\substack{C \in \mathcal{M}(\mathcal{T},F)^m \\ C \notin \mathcal{N}(\mathcal{T},\mathcal{U},F)^m}} \mathrm{ext}_{F,C}^k \, \zeta_C^F, \quad F \in \mathcal{T}^{[m]} \setminus \mathcal{U}^{[m]},$$
$$\grave{\zeta}_F = \sum_{\substack{Q \in \mathcal{M}(\mathcal{T},F)^{m-1} \\ Q \notin \mathcal{N}(\mathcal{T},\mathcal{U},F)^{m-1}}} \mathrm{ext}_{F,Q}^k \, \grave{\zeta}_Q^F, \quad F \in \mathcal{T}^{[m-1]} \setminus \mathcal{U}^{[m-1]},$$
$$\acute{\zeta}_F = \sum_{\substack{T \in \mathcal{M}(\mathcal{T},F)^{m+1} \\ T \notin \mathcal{N}(\mathcal{T},\mathcal{U},F)^{m+1}}} \mathrm{ext}_{F,T}^k \, \acute{\zeta}_T^F \quad F \in \mathcal{T}^{[m+1]} \setminus \mathcal{U}^{[m+1]}.$$

Using the triangle inequality and definitions, we observe that

$$\sum_{\substack{Q \in \mathcal{T}^{m-1} \\ Q \notin \mathcal{U}^{m-1}}} h_Q^\alpha \|(\mathsf{T}_k^m \omega)_Q\|_{L^2\Lambda^k(Q)}^2 \leq \sum_{\substack{F \in \mathcal{T}^{[m]} \\ Q \in \mathcal{M}(\mathcal{T},F)^{m-1} \\ Q \notin \mathcal{N}(\mathcal{T},\mathcal{U},F)^{m-1}}} \mu_{\mathrm{lqu}}(\mathcal{T})^\alpha h_F^\alpha \left\| \mathrm{ext}_{F,Q}^k \, \grave{\zeta}_Q^F \right\|_{L^2\Lambda^k(Q)}^2,$$

and such that

$$\sum_{\substack{T \in \mathcal{T}^{m+1} \\ T \notin \mathcal{U}^{m+1}}} h_T^\alpha \|(\mathsf{E}_k^m \omega)_T\|^2_{L^2\Lambda^k(T)} \leq \sum_{\substack{F \in \mathcal{T}^{[m]} \\ T \in \mathcal{N}(\mathcal{T},F)^{m+1} \\ T \notin \mathcal{N}(\mathcal{T},\mathcal{U},F)^{m+1}}} \mu_{\mathrm{lqu}}(\mathcal{T})^\alpha h_F^\alpha \left\| \operatorname{ext}^k_{F,T} \acute{\zeta}_T^F \right\|^2_{L^2\Lambda^k(T)}.$$

Next, we use our observation on the maximum number of reference differential complexes (IX.77). Let $F \in \mathcal{T}^{[m-1]}$. For $Q \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1}$ with $F \in \Delta(Q)$ we find that

$$h_F^\alpha \left\| \operatorname{ext}^k_{F,Q} \grave{\zeta}_Q^F \right\|^2_{L^2\Lambda^k(Q)} \leq \grave{\mu}_F h_F^{\alpha-1} \left\| \operatorname{ext}^k_{F,Q} \zeta_C^F \right\|^2_{L^2\Lambda^k(C)}$$

with a constant $\grave{\mu}_F$ that depends only on R, m and $\mu(\mathcal{T})$. On the other hand, we find that

$$\sum_{\substack{T \in \mathcal{M}(\mathcal{T},F)^{m+1} \\ T \notin \mathcal{N}(\mathcal{T},\mathcal{U},F)^{m+1}}} h_F^\alpha \left\| \operatorname{ext}^k_{F,T} \acute{\zeta}_T^F \right\|^2_{L^2\Lambda^k(T)} \leq \acute{\mu}_F \sum_{\substack{C \in \mathcal{M}(\mathcal{T},F)^{m} \\ C \notin \mathcal{N}(\mathcal{T},\mathcal{U},F)^{m}}} h_F^{\alpha+1} \left\| \operatorname{ext}^k_{F,T} \zeta_C^F \right\|^2_{L^2\Lambda^k(C)}$$

with a constant $\acute{\mu}_F$ that depends only on R, m and $\mu(\mathcal{T})$. In particular, the constants $\grave{\mu}_F$ and $\acute{\mu}_F$ can be bounded independently of F.

Finally, another scaling argument implies the existence of μ'', depending only on R, m, k, $\mu(\mathcal{T})$, and $\mu_{\mathrm{lqu}}(\mathcal{T})$, such that

$$\begin{aligned}
&\sum_{F \in \mathcal{T}^{[m]}} \sum_{\substack{C \in \mathcal{M}(\mathcal{T},F)^{m} \\ C \notin \mathcal{N}(\mathcal{T},\mathcal{U},F)^{m}}} h_F^{\alpha\pm1} \left\| \operatorname{ext}^k_{F,C} \zeta_C^F \right\|^2_{L^2\Lambda^k(C)} \\
&\quad \leq \sum_{F \in \mathcal{T}^{[m]}} \sum_{\substack{C \in \mathcal{M}(\mathcal{T},F)^{m} \\ C \notin \mathcal{N}(\mathcal{T},\mathcal{U},F)^{m}}} h_F^{\alpha\pm1} \left\| \operatorname{ext}^k_{F,C} \zeta_C^F \right\|^2_{L^2\Lambda^k(C)} \\
&\quad \leq \mu'' \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{\alpha\pm1} \|\omega_C\|^2_{L^2\Lambda^k(C)}.
\end{aligned}$$

This completes the proof. □

IX.9. Hilbert Chain Complexes

Earlier in this thesis we have learned about the distinguished role of the complex of Whitney forms in the theory of finite element differential forms, which is primarily due to its duality relation with the simplicial chain complex. This has related the cohomology spaces of the complex of Whitney forms to the homology spaces of the simplicial chain complex. In this section we use this relation to analyze Poincaré-Friedrichs constants.

The general idea is as follows: the L^2 scalar product on n-simplices gives rise to a Hilbert space structure on the Whitney forms. For example, if $\mathcal{T}$ triangulates a domain, then this is just the Hilbert space structure described by the L^2 norm of differential forms. This Hilbert space structure induces a Hilbert space structure on the spaces of simplicial chains by the duality pairing. Thus we obtain Hilbert complexes whose Poincaré-Friedrichs constants we put into relation.

We assume that $\mathcal{T}$ is an n-dimensional simplicial complex and that $\mathcal{U}$ is a subcomplex. We recall the differential complex of Whitney forms of Chapter IV, which we visualize, in this case, as a differential complex from the right to the left:

$$0 \leftarrow \mathcal{W}\Lambda^n(\mathcal{T},\mathcal{U}) \xleftarrow{\mathsf{d}^{n-1}} \dots \xleftarrow{\mathsf{d}^0} \mathcal{W}\Lambda^0(\mathcal{T},\mathcal{U}) \leftarrow 0. \tag{IX.80}$$

A Hilbert space structure over $\mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U})$ is induced by the scalar product

$$\langle \phi, \psi \rangle_{L^2\Lambda^m} := \sum_{C \in \mathcal{T}^n \setminus \mathcal{U}^n} \langle \phi_C, \psi_C \rangle_{L^2\Lambda^m(C)}, \quad \phi, \psi \in \mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U}). \tag{IX.81}$$

We write $\mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U})_{L^2\Lambda^m}$ for the Hilbert space that results from equipping $\mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U})$ with that scalar product. Thus the differential complex (IX.80) gives rise to a Hilbert complex

$$0 \leftarrow \mathcal{W}\Lambda^n(\mathcal{T},\mathcal{U})_{L^2\Lambda^n} \xleftarrow{\mathsf{d}^{n-1}} \dots \xleftarrow{\mathsf{d}^0} \mathcal{W}\Lambda^0(\mathcal{T},\mathcal{U})_{L^2\Lambda^0} \leftarrow 0. \tag{IX.82}$$

We let $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}} \geq 0$ denote the Poincaré-Friedrichs constant of this Hilbert complex. Thus for every $\omega \in \mathsf{d}^m\mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U})$ being the exterior derivative of a Whitney m-form there exists $\xi \in \mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U})$ such that

$$\|\xi\|_{L^2\Lambda^m} \leq \mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}} \|\omega\|_{L^2\Lambda^{m+1}}.$$

Remark IX.9.1.
In the sequel, we derive generalized Poincaré-Friedrichs inequalities whose constants can be expressed in terms of $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$, but we do not enlarge upon characterizing $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$ any further in this chapter. In typical applications, however, $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$ can be estimated in terms of the mesh quality and the Poincaré-Friedrichs constant of the L^2 de Rham complex. Previous findings in the literature have accomplished this in the cases $\Gamma_N = \emptyset$ and $\Gamma_N = \partial\Omega$; see [9, 58] and Theorem 3.6 of [11]. If $\mathcal{T}$ triangulates a weakly Lipschitz domain and $\mathcal{U}$ triangulates an admissible boundary patch, then a bound for the Poincaré-Friedrichs constant $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$ in terms of the mesh regularity and geometric properties of the triangulated domain can be proven, as has been outlined in Chapter VIII.

Next we recall the differential complex

$$0 \to \Gamma^0(\mathcal{T}^n) \xrightarrow{\mathsf{T}^n_0} \dots \xrightarrow{\mathsf{T}^1_0} \Gamma^0(\mathcal{T}^0) \to 0. \tag{IX.83}$$

Due to the local exactness condition, the spaces in this complex are spanned by the indicator functions 1_C of the simplices $C \in \mathcal{T} \setminus \mathcal{U}$. We equip each space $\Gamma^0(\mathcal{T}^m)$ with the scalar product

$$\langle \omega, \eta \rangle_{-h} := \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{m-n} \omega_C \eta_C, \quad \omega, \eta \in \Gamma^0(\mathcal{T}^m), \quad m \in \mathbb{Z}. \tag{IX.84}$$

This makes (IX.83) into a Hilbert complex. We let $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$ denote the Poincaré-Friedrichs constant of this Hilbert complex. The next result relates $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$ to the Poincaré-Friedrichs constant $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$ of (IX.82).

Theorem IX.9.2.
The Poincaré-Friedrichs constant $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$ of (IX.83) can be bounded in terms of $\mu(\mathcal{T})$, $\mu_{\mathrm{lqu}}(\mathcal{T})$, n, and $\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$.

Proof. We recall that we have a non-degenerate pairing of $\mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U})$ with the spaces of simplicial chains $\mathcal{C}_m(\mathcal{T},\mathcal{U})$ via integration. The simplices $\mathcal{T}^m \setminus \mathcal{U}^m$ are a basis of $\mathcal{C}_m(\mathcal{T},\mathcal{U})$, and the Whitney forms ϕ^C for $C \in \mathcal{T}^m \setminus \mathcal{U}^m$ are the dual basis with respect to this bilinear pairing, up to a constant factor $1/m!$, as described in Equation (III.8).

There exists a scalar product $\langle \cdot,\cdot \rangle_{\aleph}$ over $\mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U})$ with respect to which $(\phi^C)_{C\in\mathcal{T}^m\setminus\mathcal{U}^m}$ constitutes an orthogonal basis of $\mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U})$ such that

$$\|\phi^C\|_{\aleph} = h_C^{\frac{n}{2}-m}, \quad C \in \mathcal{T}^m \setminus \mathcal{U}^m. \tag{IX.85}$$

Using Lemma V.2.3 we now see that there exists $\mu_{\aleph} > 0$ such that

$$\mu_{\aleph}^{-1}\|\phi\|_{L^2\Lambda^m} \leq \|\phi\|_{\aleph} \leq \mu_{\aleph}\|\phi\|_{L^2\Lambda^m}, \quad \phi \in \mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U}), \tag{IX.86}$$

where $\mu_{\aleph}$ depends only on k, n, $\mu(\mathcal{T})$, and $\mu_{\mathrm{lqu}}(\mathcal{T})$. We write $\mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U})_{\aleph}$ for the vector space $\mathcal{W}\Lambda^m(\mathcal{T},\mathcal{U})$ equipped with the scalar product $\langle \cdot,\cdot \rangle_{\aleph}$. We have got an isomorphism of Hilbert complexes:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathcal{W}^n(\mathcal{T},\mathcal{U})_{L^2\Lambda^m} & \xleftarrow{\mathsf{d}^{n-1}} & \dots & \xleftarrow{\mathsf{d}^0} & \mathcal{W}^0(\mathcal{T},\mathcal{U})_{L^2\Lambda^m} & \longleftarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longleftarrow & \mathcal{W}^n(\mathcal{T},\mathcal{U})_{\aleph} & \xleftarrow{\mathsf{d}^{n-1}} & \dots & \xleftarrow{\mathsf{d}^0} & \mathcal{W}^0(\mathcal{T},\mathcal{U})_{\aleph} & \longleftarrow & 0.
\end{array}$$

We conclude that the Poincaré-Friedrichs constant of the bottom row complex is bounded by $\mu_{\aleph}^2\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$.

The scalar product $\langle \cdot,\cdot \rangle_{\aleph}$ over $\mathcal{W}^m(\mathcal{T},\mathcal{U})$ induces a scalar product $\langle \cdot,\cdot \rangle_{\aleph}$ over $\mathcal{C}_m(\mathcal{T},\mathcal{U})$ via duality. We denote by $\mathcal{C}_m(\mathcal{T},\mathcal{U})_{\aleph}$ the vector space $\mathcal{C}_m(\mathcal{T},\mathcal{U})$ equipped with the scalar product $\langle \cdot,\cdot \rangle_{\aleph}$. Note that $\mathcal{T}^m \setminus \mathcal{U}^m$ constitutes an orthogonal basis of $\mathcal{C}_m(\mathcal{T},\mathcal{U})_{\aleph}$, and that

$$\|C\|_{\aleph} = h_C^{m-\frac{n}{2}}, \quad C \in \mathcal{T}^m \setminus \mathcal{U}^m. \tag{IX.87}$$

Thus, the Poincaré-Friedrichs constant of the Hilbert complex

$$0 \longrightarrow \mathcal{C}_n(\mathcal{T},\mathcal{U})_{\aleph} \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \mathcal{C}_0(\mathcal{T},\mathcal{U})_{\aleph} \longrightarrow 0$$

is bounded by $\mu_{\aleph}^2\mu^{\mathcal{W}}_{\mathcal{T},\mathcal{U}}$ too. Next, via the identification of simplices with their indicator functions, we have an isomorphism of Hilbert complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}_n(\mathcal{T},\mathcal{U})_{\aleph} & \xrightarrow{\partial_n} & \dots & \xrightarrow{\partial_1} & \mathcal{C}_0(\mathcal{T},\mathcal{U})_{\aleph} & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & \Gamma^0(\mathcal{T}^n) & \xrightarrow{\mathsf{T}_0^n} & \dots & \xrightarrow{\mathsf{T}_0^1} & \Gamma^0(\mathcal{T}^0) & \longrightarrow & 0.
\end{array}$$

Elementary computations show that

$$\mu(\mathcal{T})^{-1}\|C\|_{\aleph} \leq h_C^{\frac{m-n}{2}} \|1_C\|_{L^2\Lambda^0(C)} \leq \|C\|_{\aleph}.$$

We conclude that the Poincaré-Friedrichs constant $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$ of the Hilbert complex (IX.83) satisfies the desired bound. The proof is complete. □

As explained in Section VIII.1, the Poincaré-Friedrichs constant of a Hilbert complex bounds the operator norm of a generalized inverse of the differential. In our specific setting, this has the following application. There exists a linear operator $\mathscr{E}^m : \Gamma^0(\mathcal{T}^{m-1}) \to \Gamma^0(\mathcal{T}^m)$ such that

$$\mathsf{T}^m_0 \mathscr{E}^m \mathsf{T}^m_0 \xi = \mathsf{T}^m_0 \xi, \quad \xi \in \Gamma^0(\mathcal{T}^m),$$

and such that for all $\xi \in \Gamma^0(\mathcal{T}^m)$ we have

$$\sqrt{\sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{m-n} \|(\mathscr{E}^m \xi)_C\|^2_{L^2\Lambda^0(C)}} \leq \mu^{\Gamma}_{\mathcal{T},\mathcal{U}} \sqrt{\sum_{F \in \mathcal{T}^{m-1} \setminus \mathcal{U}^{m-1}} h_F^{m-n-1} \|\xi_F\|^2_{L^2\Lambda^0(F)}}.$$

In particular, for $m \in \mathbb{Z}$ and $\xi \in \mathsf{T}^m_0 \Gamma^0(\mathcal{T}^m)$ we have

$$\mathsf{T}^m_0 \mathscr{E}^m \xi = \xi.$$

IX.10. Poincaré-Friedrichs Inequalities

We finish this chapter with the derivation of Poincaré-Friedrichs constants of Hilbert complexes of discrete distributional differential forms whose Hilbert space structure is induced by a mesh-dependent scalar product. The agenda of this chapter is to express the Poincaré-Friedrichs constant of these Hilbert complexes in terms of the Poincaré-Friedrichs constant $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$ introduced previously.

The Poincaré-Friedrichs inequalities are proven with respect to a mesh-dependent scalar product. Earlier in this chapter, we have developed isomorphisms between harmonic spaces of complexes of discrete distributional differential forms with respect to a general class of scalar products (see Section IX.6). For this section, we recall the definition

$$\langle \omega, \eta \rangle_{-h} = \sum_{C \in \mathcal{T}^m \setminus \mathcal{U}^m} h_C^{m-n} \langle \omega_C, \eta_C \rangle_{L^2\Lambda^k(C)}, \quad \omega, \eta \in \Lambda^k_{-1}(\mathcal{T}^m). \tag{IX.88}$$

Since $\Lambda^k_{-b}(\mathcal{T}^m)$ and $\Gamma^k_{-b}(\mathcal{T}^m)$ are defined as direct sums of spaces of the form $\Lambda^k(\mathcal{T}^m)$, this yields scalar products on these spaces too. Note that generalizes the scalar product (IX.84) considered previously.

Example IX.10.1.
At this point we consider a motivational example. Suppose that $\mathcal{T}$ triangulates a contractible domain and that $\mathcal{U} = \emptyset$. Let $\omega \in \mathsf{d}^0\Lambda^0(\mathcal{T}^n)$ be the gradient of a function in the conforming finite element space $\Lambda^0(\mathcal{T}^n)$. We show how to construct

a preimage. We let $\xi^0 = \mathsf{P}_0^n \omega$ so that $\mathsf{D}_0^n \xi^0 = \omega$. But $\xi^0 \in \Lambda^0_{-1}(\mathcal{T}^n)$ is discontinuous in general. We write $\eta = \omega - \mathsf{d}^0 \xi^0$. Then

$$\eta = \omega - \mathsf{D}_0^n \xi^0 + \mathsf{T}_0^n \xi^0 = \mathsf{T}_0^n \xi^0.$$

Note that η represents the inter-element jumps of ξ^0. Due to the identities

$$\mathsf{T}_0^{n-1} \eta = \mathsf{T}_0^{n-1} \mathsf{T}_0^n \xi^0 = 0,$$
$$\mathsf{D}_0^{n-1} \eta = \mathsf{D}_0^{n-1} \mathsf{T}_0^n \xi^0 = \mathsf{T}_1^n \mathsf{D}_0^n \xi^0 = \mathsf{T}_1^n \omega = 0$$

we conclude that $\eta \in \Gamma^0(\mathcal{T}^{n-1})$ with $\mathsf{T}_0^{n-1}\eta = 0$. Since the domain is contractible, there exists $\widetilde{\xi} \in \Gamma^0(\mathcal{T}^n)$ such that $\mathsf{T}_0^n \widetilde{\xi} = \eta$. But then

$$\mathsf{d}^0 \left(\xi^0 + \widetilde{\xi} \right) = \omega - \eta + \mathsf{d}^0 \widetilde{\xi} = \omega - \eta + \mathsf{T}_0^n \widetilde{\xi} = \omega$$

We set $\xi := \xi^0 + \widetilde{\xi}$, so that $\mathsf{d}^0 \xi = \omega$. Now $\xi \in \Lambda^0(\mathcal{T}^n)$ is the desired preimage in the conforming finite element space. The only non-local operation in the construction of ξ has been finding $\widetilde{\xi}$, which is independent of any polynomial order.

The driving motivation in this section is to generalize the previous example. We derive Poincaré-Friedrichs constants for the Hilbert complex

$$0 \to \Lambda^0_{-1}(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda^n_{-n-1}(\mathcal{T}^n) \longrightarrow 0. \tag{IX.89}$$

Given $\omega \in \mathsf{d}^k \Lambda^k_{-k-1}(\mathcal{T}^n)$ being the distributional exterior derivative of a discrete distributional differential form in $\Lambda^{k-1}_{-k}(\mathcal{T}^n)$, we explicitly construct $\xi \in \Lambda^{k-1}_{-k}(\mathcal{T}^n)$ satisfying $\mathsf{d}^{k-1}\xi = \omega$. Together with the results of Sections IX.7, IX.8, and IX.9, the construction of ξ reveals an estimate for the Poincaré-Friedrichs constant. In the second part of this section we modify the construction to accommodate special structure in the preimage, which yields Poincaré-Friedrichs inequalities for the other discrete distributional de Rham complexes.

Throughout this section we assume that

$$\omega \in \mathsf{d}^k \Lambda^k_{-k-1}(\mathcal{T}^n)$$

is a fixed but arbitrary discrete distributional differential form that lies within the range of

$$\mathsf{d}^k : \Lambda^k_{-k-1}(\mathcal{T}^n) \to \Lambda^{k+1}_{-k-2}(\mathcal{T}^n)$$

There is a unique way to write ω as

$$\omega = \omega_0 + \dots + \omega_{k+1}, \quad \omega_i \in \Lambda^{k+1-i}_{-1}(\mathcal{T}^{n-i}), \quad 0 \leq i \leq k+1.$$

We explicitly construct $\xi \in \Lambda^k_{-b}(\mathcal{T}^n)$ such that $\mathsf{d}^k \xi = \omega$ in the following manner. We define $\xi \in \Lambda^k_{-k-1}(\mathcal{T}^n)$ as

$$\xi := \xi_0 + \dots + \xi_k + \widetilde{\xi}, \tag{IX.90}$$

where we first define $\xi^0 \in \Lambda^k_{-1}(\mathcal{T}^n)$ by

$$\xi^0 := \mathsf{P}^n_k \omega^0, \tag{IX.91}$$

then recursively define $\xi^i \in \Lambda^{k-i}_{-1}(\mathcal{T}^{n-i})$ for $1 \leq i \leq k$ by

$$\xi^i := (-1)^i \mathsf{P}^{n-i}_{k-i} \omega^i + \mathsf{P}^{n-i}_{k-i} \mathsf{T}^{n-i+1}_{k-i+1} \xi^{i-1}, \tag{IX.92}$$

and eventually define $\widetilde{\xi} \in \Gamma^0(\mathcal{T}^{n-k})$ by

$$\widetilde{\xi} := (-1)^{k+1} \mathscr{E}^m \omega^{k+1} - \mathscr{E}^m \mathsf{T}^{n-k}_0 \xi^k. \tag{IX.93}$$

Here we have used the horizontal antiderivative P^m_k and the operator $\mathscr{E}^m$ from Section IX.9. The basic idea of constructing a preimage in this manner has been known in differential topology [31, II.9] for a long time. We also observe that a similar construction was already used in the proof of Lemma IX.6.6. First we verify the following identity.

Lemma IX.10.2.
Let ω and ξ be defined as above. Then $\mathsf{d}^k \xi = \omega$.

Proof. First, we have $\mathsf{D}^n_{k+1} \omega_0 = 0$ by assumption. The local exactness condition thus implies $\mathsf{D}^n_k \xi^0 = \omega^0$. Moreover, $\mathsf{d}^k \omega = 0$ implies that

$$\begin{aligned} -\mathsf{D}^{n-1}_k \left(\omega^1 + \mathsf{T}^n_k \xi^0 \right) &= -\mathsf{D}^{n-1}_k \omega^1 - \mathsf{D}^{n-1}_k \mathsf{T}^n_k \xi^0 \\ &= -\mathsf{D}^{n-1}_k \omega^1 - \mathsf{T}^n_{k+1} \mathsf{D}^n_k \xi^0 = -\mathsf{D}^{n-1}_k \omega^1 - \mathsf{T}^n_{k+1} \omega^0 = 0. \end{aligned}$$

Next we use an induction argument. Let us assume that we already have

$$\begin{aligned} \omega - \mathsf{d}^k \left(\xi^0 + \dots + \xi^i \right) &= (-1)^i \mathsf{T}^{n-i}_{k-i} \xi^i + \omega^{i+1} + \dots + \omega^b, \\ \mathsf{D}^{n-i-1}_{k-i-1} \omega^{i+1} &= (-1)^{i+1} \mathsf{D}^{n-i-1}_{k-i-1} \mathsf{T}^{n-i}_{k-i} \xi^i. \end{aligned}$$

for some $0 \leq i < k$. But then

$$\mathsf{D}^{n-i-1}_{k-i-1} \xi^{i+1} = (-1)^{i+1} \omega^{i+1} - \mathsf{T}^{n-i}_{k-i} \xi^i.$$

Thus we find

$$\omega - \mathsf{d}^k \left(\xi^0 + \dots + \xi^i + \xi^{i+1} \right) = (-1)^{i+1} \mathsf{T}^{n-i-1}_{k-i-1} \xi^{i+1} + \omega^{i+2} + \dots + \omega^b \in \mathsf{d}\Lambda^k_{-b}(\mathcal{T}^n),$$

and

$$\begin{aligned} \mathsf{D}^{n-i+2}_{k-i+2} \left((-1)^{i+1} \mathsf{T}^{n-i-1}_{k-i-1} \xi^{i+1} + \omega^{i+2} \right) &= (-1)^{i+1} \mathsf{D}^{n-i-2}_{k-i-1} \mathsf{T}^{n-i-1}_{k-i-1} \xi^{i+1} + \mathsf{D}^{n-i-2}_{k-i-1} \omega^{i+2} \\ &= \mathsf{T}^{n-i+1}_{k-i} \omega^{i+1} + \mathsf{D}^{n-i-2}_{k-i-1} \omega^{i+2} = 0. \end{aligned}$$

Hence the assumptions for index i hold again for the index $i + 1 \leq k$.

Iteration of this argument eventually provides

$$\omega - \mathsf{d}^k \left(\xi^0 + \dots + \xi^k \right) = \omega^{k+1} + (-1)^{k+1} \mathsf{T}^{n-k}_0 \xi^k.$$

Let $\eta := \omega - \mathsf{d}^k \left(\xi^0 + \cdots + \xi^k \right)$. Then $\eta \in \Lambda^0_{-1}(\mathcal{T}^{n-k-1})$, and $\eta \in \mathsf{d}\Lambda^k_{-k-1}(\mathcal{T}^n)$. By construction we have

$$\mathsf{D}^{n-k-1}_0 \eta = 0, \quad \mathsf{T}^{n-k-1}_0 \eta = 0.$$

In particular $\eta \in \Gamma^0(\mathcal{T}^{n-k-1})$. Since $\eta \in \mathsf{d}^k \Lambda^k_{-k-1}(\mathcal{T}^n)$, we know that η is orthogonal to $\mathfrak{H}^{k+1}_{-k-2}(\mathcal{T}^n) = \mathfrak{C}^{k+1}_{-k-2}(\mathcal{T}^{n-k-1})$. Our next goal is showing that η is orthogonal to $\mathfrak{C}^0(\mathcal{T}^{n-k-1})$. To see this, let $p \in \mathfrak{C}^{k+1}_{-k-2}(\mathcal{T}^{n-k-1})$ be arbitrary. Then there is a unique way to write

$$p = p^0 + \cdots + p^k + p^{k+1}, \quad p^i \in \Lambda^{k+i}(\mathcal{T}^{n+i}), \quad 0 \leq i \leq k+1.$$

From Lemma IX.6.9 we conclude that $p^0 \in \mathfrak{C}^0(\mathcal{T}^{n-k})$. Hence

$$\begin{aligned}\left\langle p^0, (-1)^{k+1} \mathsf{T}^{n-k}_0 \xi^k + \omega^{k+1} \right\rangle_{-h} &= \left\langle p, (-1)^{k+1} \mathsf{T}^{n-k}_0 \xi^k + \omega^{k+1} \right\rangle_{-h} \\ &= \left\langle p, \omega - \mathsf{d}^k \left(\xi^0 + \cdots + \xi^k \right) \right\rangle_{-h} = \langle p, \omega \rangle_{-h} = 0\end{aligned}$$

by assumption on ω. Thus η is orthogonal to $\mathfrak{C}^0(\mathcal{T}^{n-k-1})$, and we conclude that

$$(-1)^{k+1} \mathsf{T}^{n-k}_0 \left(\widetilde{\xi} + \xi_k \right) = \omega^{k+1}.$$

This completes the proof. □

Next we bound the $\|\cdot\|_{-h}$ norm of ξ in terms of the $\|\cdot\|_{-h}$ norm of ω. In fact, we more generally prove a family of inequalities parametrized over $\alpha \in \mathbb{R}$. The special case $\alpha = 0$ gives the desired Poincaré-Friedrichs inequality. The special case $\alpha = 1$ is of technical use in another proof further below.

Lemma IX.10.3.
Let ω and ξ be as above. For any $\alpha \in \mathbb{R}$ there exists $\mu_{(\alpha)}$, depending only on $\hat{\mu}$, $\grave{\mu}$, and α, such that

$$\sum_{\substack{C \in \mathcal{T}^{n-i} \\ C \notin \mathcal{U}^{n-i}}} h_C^{\alpha-i} \left\| \xi^i_C \right\|^2_{L^2\Lambda^{k-i}(C)} \leq \mu_{(\alpha)} \sum_{j=0}^{i} \sum_{\substack{C \in \mathcal{T}^{n-j} \\ C \notin \mathcal{U}^{n-j}}} h_C^{\alpha-j+2} \left\| \omega^j_C \right\|^2_{L^2\Lambda^{k-j+1}(C)}. \tag{IX.94}$$

Moreover, we have

$$\sum_{\substack{C \in \mathcal{T}^{n-k} \\ C \notin \mathcal{U}^{n-k}}} h_C^{-k} \left\| \widetilde{\xi}_C \right\|^2_{L^2\Lambda^0(C)} \leq \widetilde{\mu} \sum_{i=0}^{k+1} \sum_{\substack{C \in \mathcal{T}^{n-k-1} \\ C \notin \mathcal{U}^{n-k-1}}} h_C^{-i} \left\| \omega^i_C \right\|^2_{L^2\Lambda^{k-i+1}(C)} \tag{IX.95}$$

with a constant $\widetilde{\mu}$ that depends only on $\hat{\mu}$, $\grave{\mu}$, and $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$.

Proof. From Theorem IX.7.5 we find

$$\sum_{C \in \mathcal{T}^n \setminus \mathcal{U}^n} h_C^{\alpha} \left\| \xi^0_C \right\|^2_{L^2\Lambda^k(C)} \leq \hat{\mu} \sum_{C \in \mathcal{T}^n \setminus \mathcal{U}^n} h_C^{\alpha+2} \left\| \omega^0_C \right\|^2_{L^2\Lambda^{k+1}(C)}. \tag{IX.96}$$

This shows (IX.94) in the case $i = 0$.

Suppose that (IX.94) holds for some index $0 \leq i \leq k-1$. Using the construction of ξ^{i+1} and Theorem IX.7.5 and Theorem IX.8.1, we then have

$$\begin{aligned}
&\sum_{\substack{C \in \mathcal{T}^{n-i-1} \\ C \notin \mathcal{U}^{n-i-1}}} h_C^{\alpha-i-1} \left\| \xi_C^{i+1} \right\|_{L^2\Lambda^{k-i-1}(C)}^2 \\
&= \sum_{\substack{C \in \mathcal{T}^{n-i-1} \\ C \notin \mathcal{U}^{n-i-1}}} h_C^{\alpha-i-1} \left\| \mathsf{P}_{k-i-1}^{n-i-1} \omega_C^{i+1} - \mathsf{P}_{k-i-1}^{n-i-1} (\mathsf{T}_{n-i}^{n-i} \xi^i)_C \right\|_{L^2\Lambda^{k-i-1}(C)}^2 \\
&\leq \hat{\mu} \sum_{\substack{C \in \mathcal{T}^{n-i-1} \\ C \notin \mathcal{U}^{n-i-1}}} h_C^{\alpha-i+1} \left\| \omega_C^{i+1} - (\mathsf{T}_{n-i}^{n-i} \xi^i)_C \right\|_{L^2\Lambda^{k-i}(C)}^2 \\
&\leq 2\hat{\mu} \sum_{\substack{C \in \mathcal{T}^{n-i-1} \\ C \notin \mathcal{U}^{n-i-1}}} h_C^{\alpha-i+1} \left\| \omega_C^{i+1} \right\|_{L^2\Lambda^{k-i}(C)}^2 + 2\hat{\mu}\grave{\mu} \sum_{\substack{C \in \mathcal{T}^{n-i} \\ C \notin \mathcal{U}^{n-i}}} h_C^{\alpha-i} \left\| \xi_C^i \right\|_{L^2\Lambda^{k-i}(C)}^2 \\
&\leq \mu_{(\alpha)} \sum_{j=0}^{i+1} \sum_{\substack{C \in \mathcal{T}^{n-j} \\ C \notin \mathcal{U}^{n-j}}} h_C^{\alpha-j+2} \left\| \omega_C^j \right\|_{L^2\Lambda^{k-j+1}(C)}^2,
\end{aligned}$$

where $\mu_{(\alpha)}$ is a constant that depends only on $\hat{\mu}$, $\grave{\mu}$, and α. We conclude that (IX.94) holds for $i+1$ too. An induction argument shows (IX.94) for all $0 \leq i \leq k$.

In order to show (IX.95), let $\eta \in \Gamma^0(\mathcal{T}^{n-k-1})$ be defined by

$$\eta := \omega - \mathsf{d}^k \left(\xi^0 + \cdots + \xi^k \right),$$

as in the proof of the Theorem IX.10.2. We find by Theorem IX.9.2 that

$$\begin{aligned}
\sum_{\substack{C \in \mathcal{T}^{n-k} \\ C \notin \mathcal{U}^{n-k}}} h_C^{-k} \left\| \widetilde{\xi}_C \right\|_{L^2\Lambda^0(C)}^2 &\leq \left(\mu_{\mathcal{T},\mathcal{U}}^{\Gamma} \right)^2 \sum_{\substack{C \in \mathcal{T}^{n-k-1} \\ C \notin \mathcal{U}^{n-k-1}}} h_C^{-k-1} \left\| \eta_C \right\|_{L^2\Lambda^0(C)}^2 \\
&= \left(\mu_{\mathcal{T},\mathcal{U}}^{\Gamma} \right)^2 \sum_{\substack{C \in \mathcal{T}^{n-k-1} \\ C \notin \mathcal{U}^{n-k-1}}} h_C^{-k-1} \left\| \omega_C^{k+1} - (\mathsf{T}_0^{n-k} \xi^k)_C \right\|_{L^2\Lambda^0(C)}^2.
\end{aligned}$$

To estimate the last term, we use

$$\sum_{C \in \mathcal{T}^{n-k-1} \setminus \mathcal{U}^{n-k-1}} h_C^{-k-1} \left\| (\mathsf{T}_0^{n-k} \xi^k)_C \right\|_{L^2\Lambda^0(C)}^2 \leq \grave{\mu} \sum_{C \in \mathcal{T}^{n-k} \setminus \mathcal{U}^{n-k}} h_C^{-k-2} \left\| \xi_C^k \right\|_{L^2\Lambda^0(C)}^2.$$

Finally, we apply (IX.94) with $i = k$ and $\alpha = -2$ to obtain (IX.95). This completes the proof. □

Corollary IX.10.4.
The Hilbert complex

$$0 \to \Lambda_{-1}^0(\mathcal{T}^n) \xrightarrow{\mathsf{d}^0} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda_{-n-1}^n(\mathcal{T}^n) \to 0$$

satisfies Poincaré-Friedrichs inequalities with respect to the scalar product $\langle \cdot, \cdot \rangle_{-h}$. The Poincaré-Friedrichs constant depends only on $\mu_{\mathcal{T},\mathcal{U}}^{\Gamma}$, $\mu(\mathcal{T})$, $\mu_{\mathrm{lqu}}(\mathcal{T})$, n, and R.

We have proven Poincaré-Friedrichs-type inequalities for the "maximal" complexes of discrete distributional differential forms (IX.55) / (IX.60). We would like to obtain analogous inequalities for the subcomplexes (IX.52) – (IX.58), and the key idea is again to explicitly construct a preimage under the distributional exterior derivative. In other words, if ω features additional properties, then we construct ξ with additional properties.

For example, let $0 \leq m \leq n$ and suppose that $\omega_i = 0$ for $0 \leq i < n-m$. By definition we then have $\omega \in \Lambda^{k+1}_{-k+n-m-1}(\mathcal{T}^m)$. We would like to have $\xi \in \Lambda^{k}_{-k+n-m}(\mathcal{T}^m)$ in that case, since that would immediately imply a Poincaré-Friedrichs inequality for the Hilbert complex

$$\dots \xrightarrow{\mathsf{d}^{k+n-m-1}} \Lambda^{k}_{-k+n-m}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m}} \Lambda^{k+1}_{-k+n-m-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m+1}} \dots$$

with respect to the norm $\|\cdot\|_{-h}$. But it is trivially seen that our construction of ξ satisfies $\xi_i = 0$ for $0 \leq i \leq n-m$ in that case. Hence we can formulate the following corollary.

Corollary IX.10.5.
Let $0 \leq m \leq n$. The Hilbert complex

$$0 \to \Lambda^{0}_{-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{n-m}} \dots \xrightarrow{\mathsf{d}^{n-1}} \Lambda^{m}_{-m-1}(\mathcal{T}^m) \to 0$$

satisfies Poincaré-Friedrichs inequalities with respect to the scalar product $\langle\cdot,\cdot\rangle_{-h}$. The Poincaré-Friedrichs constant depends only on $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$, $\mu(\mathcal{T})$, $\mu_{\mathrm{lqu}}(\mathcal{T})$, n, and R.

Another interesting special case, whose analysis is more complicated, is ω satisfying $\omega_i = 0$ for $b \leq i \leq k+1$ where $2 \leq b \leq k+2$. In that case $\omega \in \Lambda^{k+1}_{-b}(\mathcal{T}^n)$ and we would like to have $\xi \in \Lambda^{k}_{-b+1}(\mathcal{T}^n)$. Note that the special case $b = k+2$ is covered by the preceding construction, but $2 \leq b < k+2$ is not. Since ξ as constructed above only satisfies $\xi \in \Lambda^{k}_{-k+1}(\mathcal{T}^n)$ in general, some modifications are due.

If $\omega \in \Lambda^{k+1}_{-b}(\mathcal{T}^n)$, then $\omega = \omega^0 + \dots + \omega^{b-1}$. We have

$$\xi^i = \mathsf{P}^{n-i}_{k-i}\mathsf{T}^{n-i+1}_{k-i+1}\xi^{i-1}$$

for $b \leq i \leq k$, and we have

$$\widetilde{\xi} = -\mathscr{E}^{n-k}\mathsf{T}^{n-k}_{0}\xi^{k}.$$

Unfolding this recursive relation, we find that

$$\xi^i = \mathsf{P}^{n-i}_{k-i}\mathsf{T}^{n-i+1}_{k-i+1}\cdots\mathsf{P}^{n-b}_{k-b}\mathsf{T}^{n-b+1}_{k-b+1}\xi^{b-1}$$

for $b \leq i \leq k$, and that

$$\widetilde{\xi} = -\mathscr{E}^{n-k}\mathsf{T}^{n-k}_{0}\mathsf{P}^{n-k}_{0}\mathsf{T}^{n-k+1}_{1}\cdots\mathsf{P}^{n-b}_{k-b}\mathsf{T}^{n-b+1}_{k-b+1}\xi^{b-1}.$$

By iterative application of Lemma IX.6.4 we find

$$R_{k,i}\dots R_{k,k+1}\xi \in \Lambda^{k}_{-i}(\mathcal{T}^n), \quad \mathsf{d}^k R_{k,i}\cdots R_{k,k+1}\xi = \omega$$

for $b \leq i \leq k+1$. In particular, the case $i = b$ provides us with a member of $\Lambda^k_{-b+1}(\mathcal{T}^n)$ whose distributional exterior derivative equals ω. This is the desired preimage.

It remains to provide the relevant norm estimates. We have

$$R_{k,b} \dots R_{k,k+1}\xi = \xi^0 + \dots + \xi^{b-2} + \theta^{b-1} + \dots + \theta^k + \widetilde{\theta},$$

where $\theta^i \in \Lambda^{k-b+2}_{-1}(\mathcal{T}^{n-b+2})$ is given by

$$\theta^i = (-1)^{i-b} \mathsf{D}^{n-b+2}_{k-b+1} \mathsf{E}^{n-b+2}_{k-b+1} \cdots \mathsf{D}^{n-i+1}_{k-i} \mathsf{E}^{n-i+1}_{k-i} \xi^i$$

for $b - 1 \leq i \leq k$, and where $\widetilde{\theta} \in \Lambda^{k-b+2}_{-1}(\mathcal{T}^{n-b+2})$ is given by

$$\widetilde{\theta} = (-1)^{k-b+1} \mathsf{D}^{n-b+2}_{k-b+1} \mathsf{E}^{n-b+2}_{k-b+1} \cdots \mathsf{D}^{n-k+1}_{0} \mathsf{E}^{n-k+1}_{0} \widetilde{\xi}.$$

The terms $\xi^0, \dots, \xi^{b-2}$ have been estimated earlier. It remains to treat the terms $\theta^{b-1}, \dots, \theta^k$, and $\widetilde{\theta}$.

We derive estimates via repeated application of Lemma IX.7.3, Lemma IX.7.5, and Lemma IX.8.1. Let $\alpha \in \mathbb{R}$ and let μ denote a generic constant that depends only on $\hat{\mu}$, $\check{\mu}$, $\acute{\mu}$, $\grave{\mu}$, $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$, and α, and whose value may change from line to line. Skipping over a series of repeated estimates, we obtain

$$\begin{aligned} \sum_{\substack{C \in \mathcal{T}^{n-b+2} \\ C \notin \mathcal{U}^{n-b+2}}} h_C^{\alpha-b+2} \left\| \theta^i_C \right\|^2_{L^2\Lambda^{k-b+2}(C)} &\leq \mu \sum_{\substack{C \in \mathcal{T}^{n-i} \\ C \notin \mathcal{U}^{n-i}}} h_C^{\alpha-i} \left\| \xi^i_C \right\|^2_{L^2\Lambda^{k-i}(C)} \\ \leq \mu \sum_{\substack{C \in \mathcal{T}^{n-b+1} \\ C \notin \mathcal{U}^{n-b+1}}} h_C^{\alpha-b+1} \left\| \xi^{b-1}_C \right\|^2_{L^2\Lambda^{k-b-1}(C)} &\leq \mu \sum_{j=0}^{b-1} \sum_{\substack{C \in \mathcal{T}^{n-j} \\ C \notin \mathcal{U}^{n-j}}} h_C^{\alpha-b+3} \left\| \omega^j_C \right\|^2_{L^2\Lambda^{k-j}(C)} \end{aligned}$$

Next, with application of Lemma IX.10.3, and Theorem IX.9.2, we find

$$\begin{aligned} \sum_{\substack{C \in \mathcal{T}^{n-b+2} \\ C \notin \mathcal{U}^{n-b+2}}} h_C^{-b+2} \left\| \widetilde{\theta}_C \right\|^2_{L^2\Lambda^{k-b+2}(C)} &\leq \mu \sum_{\substack{C \in \mathcal{T}^{n-k} \\ C \notin \mathcal{U}^{n-k}}} h_C^{-k} \left\| \widetilde{\xi}_C \right\|^2_{L^2\Lambda^{0}(C)} \\ &\leq \mu \sum_{\substack{C \in \mathcal{T}^{n-k} \\ C \notin \mathcal{U}^{n-k}}} h_C^{-k-2} \left\| \xi^k_C \right\|^2_{L^2\Lambda^{0}(C)} \leq \mu \sum_{\substack{C \in \mathcal{T}^{n-b-1} \\ C \notin \mathcal{U}^{n-b-1}}} h_C^{-b-1} \left\| \xi^{b-1}_C \right\|^2_{L^2\Lambda^{k-b-1}(C)} \\ &\leq \mu \sum_{j=0}^{b-1} \sum_{\substack{C \in \mathcal{T}^{n-j} \\ C \notin \mathcal{U}^{n-j}}} h_C^{-b+1} \left\| \omega^j_C \right\|^2_{L^2\Lambda^{k-j}(C)}. \end{aligned}$$

This proves the desired Poincaré-Friedrichs inequality.

Finally, we treat the case $b = 1$. Suppose that $\omega = \omega^0$, so that $\omega \in \Lambda^{k+1}_{-1}(\mathcal{T}^n)$. Since $\omega \in \mathsf{d}^k \Lambda^k_{-1}(\mathcal{T}^n)$, we find that $\omega \in \Lambda^{k+1}(\mathcal{T}^n)$ and $\omega \in \mathsf{d}^k \Lambda^k(\mathcal{T}^n)$. The construction of ξ then assures that $\xi \in \Lambda^k(\mathcal{T}^n)$. This gives the desired Poincaré-Friedrichs inequality.

Conclusively, we have bounded the Poincaré-Friedrichs constants of discrete distributional de Rham complexes. We summarize those findings in the following main result.

Theorem IX.10.6.
The Hilbert complexes

$$\dots \xrightarrow{\mathsf{d}^{k+n-m-2}} \Lambda^{k-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m-1}} \Lambda^{k}_{-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m}} \Lambda^{k+1}_{-2}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m+1}} \dots$$

and

$$\dots \xrightarrow{\mathsf{d}^{k+n-m-2}} \Gamma^{k}(\mathcal{T}^{m+1}) \xrightarrow{\mathsf{d}^{k+n-m-1}} \Gamma^{k}_{-1}(\mathcal{T}^m) \xrightarrow{\mathsf{d}^{k+n-m}} \Gamma^{k}_{-2}(\mathcal{T}^{m-1}) \xrightarrow{\mathsf{d}^{k+n-m+1}} \dots$$

satisfy Poincaré-Friedrichs inequalities with respect to the scalar product $\langle \cdot, \cdot \rangle_{-h}$. The Poincaré-Friedrichs constants depend only on $\mu^{\Gamma}_{\mathcal{T},\mathcal{U}}$, $\mu(\mathcal{T})$, $\mu_{\mathrm{lqu}}(\mathcal{T})$, n, and R.

Remark IX.10.7.
In [34, Section 3.4], Braess and Schöberl employed the scalar product $\langle \cdot, \cdot \rangle_{h}$, defined as in (IX.32). They proved Poincaré-Friedrichs inequalities with respect to that scalar product when the underlying triangulation is a local patch, essentially relying on a scaling argument. It is easily seen that their Lemma 9 holds for our scalar product $\langle \cdot, \cdot \rangle_{-h}$ in a similar manner if the distributional finite element complex is considered on a local patch. In the light of the result of this chapter, we are inclined to consider $\langle \cdot, \cdot \rangle_{-h}$ as the "natural" scalar product for distributional finite element spaces.

Example IX.10.8.
Let $\mathcal{T}$ again be a triangulation of a connected domain and let $\mathcal{U} = \emptyset$. The previous results imply that for any $f \in \Lambda^n(\mathcal{T}^n)$ we can construct $\sigma \in \Lambda^{n-1}(\mathcal{T}^n)$ such that $\mathsf{d}^{n-1}\sigma = f$. The construction consists of local operations and one single global computation: given a 0-chain $s \in \mathcal{C}_0(\mathcal{T})$ (i.e. a linear combination of oriented points), we need to find a 1-chain $s \in \mathcal{C}_1(\mathcal{T})$ (i.e. a linear combination of oriented edges) such that $\partial S = s$. The condition number of the latter global problem with respect to the norm $\| \cdot \|_{-h}$ depends only on the Poincaré-Friedrichs constant of the complex of Whitney forms with respect to the L^2 product and the mesh regularity. It does not depend on any polynomial order of the finite element spaces.

X. Flux Reconstruction and Applications

In this chapter we approach the work of Braess and Schöberl from a different angle and elaborate upon equilibrated a posteriori error estimation in finite element exterior calculus. We obtain a practically relevant result: we generalize the publication of Braess and Schöberl [34] to the case of higher order edge elements.

A priori error estimates for finite element methods bound the approximation error of the Galerkin solution using only data available *prior* to the computation of the Galerkin solution. But it is reasonable to assume that *posterior* to computing an approximate solution we can derive sharper error estimates: after all, the approximate solution is additional information explicitly known. The continuity of research on a posteriori error estimation (see, e.g., the monographs [4, 156, 172]) may be partially explained by their critical role for many adaptive finite element methods [48, 51].

One of the most important residual error estimators, found in many introductory textbooks on finite element methods [32, 83], is the classical residual error estimator. We demonstrate the basic idea by the means of the Poisson equation. Let Ω be a Lipschitz domain. It is standard that for every $f \in L^2(\Omega)$ there exists $u \in H^1_0(\Omega)$ such that $-\operatorname{div} \nabla u = f$. For $u_h \in H^1_0(\Omega)$, conceived as an approximation of u, we define the residual $r_h \in H^{-1}(\Omega)$ as the functional $f + \operatorname{div} \nabla u$ in the dual space of $H^1_0(\Omega)$. It follows from definitions and basic facts that the H^{-1} norm of r_h is comparable to the H^1 norm of the approximation error $u - u_h$.

When we choose u_h to be the (piecewise polynomial) Galerkin solution of a finite element method, then the distribution r_h can be represented as the sum of integrals over full-dimensional simplices and trace integrals over faces (each against a polynomial weight). The latter integrals over faces are also known as *jump terms* in this context. Since r_h features this special structure, the whole trick is now to estimate the H^{-1} norm of r_h in terms of a mesh-dependent norm of r_h. We refer to the monograph of Braess [32] for further details.

The classical residual error estimator leaves room for improvement. The estimate typically involves unknown constants that are not easy to estimate in practice, and the resulting error bounds are generally not regarded as sharp. The research in adaptive finite element methods has driven much of the development of alternative methods for a posteriori error estimation, of which *equilibrated a posteriori error estimators* [3, 32, 33, 118, 155, 171] are one important example.

We let the aforementioned Poisson problem again serve as a basic example. Suppose that $\boldsymbol{\sigma} \in \mathbf{H}(\operatorname{div}, \Omega)$ is a square-integrable vector field with divergence in $L^2(\Omega)$

that solves the flux equation $-\operatorname{div}\boldsymbol{\sigma} = f$. The Prager-Synge theorem (see [34]) states that

$$\|\nabla u_h - \boldsymbol{\sigma}\|^2_{\mathbf{L}^2(\Omega)} = \|\nabla u_h - \nabla u\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla u - \boldsymbol{\sigma}\|^2_{\mathbf{L}^2(\Omega)}.$$

An estimate for the error norm $\|\nabla u - \nabla u_h\|_{\mathbf{L}^2(\Omega)}$ in terms of $\|\nabla u_h - \boldsymbol{\sigma}\|_{\mathbf{L}^2(\Omega)}$ follows immediately; the only condition is that $\boldsymbol{\sigma}$ solves the flux equation. Of course, in order to make this error estimate computationally feasible, we need to obtain such a flux $\boldsymbol{\sigma}$ in the first place. Under the assumption that $f \in \mathcal{P}^r_{\mathrm{DC}}(\mathcal{T})$ is piecewise polynomial of order r with respect to a triangulation $\mathcal{T}$ of the domain, this is computationally feasible: there exists $\boldsymbol{\sigma}_h \in \mathbf{RT}^r(\mathcal{T})$ in the Raviart-Thomas space solving the flux equation $-\operatorname{div}\boldsymbol{\sigma}_h = f$. The mixed finite element method for the Poisson equation determines such a solution.

The mixed finite element method, however, comes at the cost of solving a global finite element problem. These computational costs can be circumvented by a procedure called *local equilibration* in the literature, which is also the namesake for the whole method, and which we outline as follows. Let $\mathcal{U}$ be a subcomplex of $\mathcal{T}$ that triangulates the boundary and let $\mathcal{P}^r(\mathcal{T},\mathcal{U})$ denote the Lagrange space of order r with Dirichlet boundary conditions. We assume that $u_h \in \mathcal{P}^r(\mathcal{T},\mathcal{U})$ satisfies the Galerkin property

$$\int_\Omega \nabla u_h \cdot \nabla v_h \,\mathrm{d}x = \int_\Omega f v_h \,\mathrm{d}x, \quad v_h \in \mathcal{P}^r(\mathcal{T},\mathcal{U}).$$

For simplicity we assume that f is piecewise of polynomial order $r-1$. We let $\mathbf{RT}^r_{-1}(\mathcal{T})$ denote the subspace of $\mathbf{L}^2(\Omega)$ whose members are piecewise in the Raviart-Thomas space of order r. Writing ϕ_V for the hat function associated with any vertex $V \in \mathcal{T}^0$, one can show that the distribution $r_{h,V} := \phi_V \cdot r_h$ is supported in the local patch around V and that there exists $\boldsymbol{\varrho}_{h,V} \in \mathbf{RT}^r_{-1}(\mathcal{T})$, supported in the same local patch, such that $-\operatorname{div}\boldsymbol{\varrho}_{h,V} = r_{h,V}$. Now let $\boldsymbol{\varrho}_h$ be the sum of all $\boldsymbol{\varrho}_{h,V}$ over all vertices $V \in \mathcal{T}^0$. We observe $-\operatorname{div}\boldsymbol{\varrho}_h = r_h$. Letting $\boldsymbol{\sigma}_h = \boldsymbol{\varrho}_h + \nabla u_h$, we discover that

$$\begin{aligned}
-\operatorname{div}\boldsymbol{\sigma}_h &= -\operatorname{div}\boldsymbol{\varrho}_h - \operatorname{div}\nabla u_h \\
&= r_h - \operatorname{div}\nabla u_h \\
&= f + \operatorname{div}\nabla u_h - \operatorname{div}\nabla u_h = f.
\end{aligned}$$

In particular, $\boldsymbol{\sigma}_h \in \mathbf{RT}^r(\mathcal{T})$. The key observation is that $\boldsymbol{\sigma}_h$ is constructed using only local operations, which are independent from each other and hence parallelizable. Computational experiments indicate that this estimator is competitive [33, 47]. We remark the locally reconstructed flux is generally different from the flux determined by a mixed finite element method.

Whereas numerous publications treat a posteriori error estimation for the Poisson problem, much less is known for the curl curl problem. Several publications have adapted the classical residual error estimator to Maxwell's equations [16, 144, 160] but little has been published on different error estimators, even though the experience with the Poisson problem suggests that this is of practical relevance.

Important progress has been accomplished by Braess and Schöberl [34] who approached a generalization of the equilibrated residual error estimator to the finite element method for the curl curl equation based on Nédélec elements. For a brief outline of the idea, which has many parallels to the equilibrated error estimator for the Poisson problem, let us assume that Ω is a contractible Lipschitz domain in $\mathbb{R}^3$ and that $\mathbf{f} \in \mathbf{H}_0(\operatorname{div}, \Omega)$ with $\operatorname{div} \mathbf{f} = 0$. Then there exists $\boldsymbol{\sigma} \in \mathbf{H}_0(\operatorname{curl}, \Omega)$ solving the flux equation $\operatorname{curl} \boldsymbol{\sigma} = \mathbf{f}$. Furthermore, there exists a solution $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega)$ to the curl curl problem

$$\operatorname{curl} \mathbf{u} \in \mathbf{H}_0(\operatorname{curl}, \Omega), \quad \operatorname{curl} \operatorname{curl} \mathbf{u} = \mathbf{f}.$$

To select a unique solution, we may require $\mathbf{u} \in \mathbf{H}_0(\operatorname{div}, \Omega)$ and, say, the divergence free constraint $\operatorname{div} \mathbf{u} = 0$.

Now suppose that $\mathbf{u}_h \in \mathbf{H}(\operatorname{curl}, \Omega)$ and that $\boldsymbol{\xi} \in \mathbf{H}_0(\operatorname{curl}, \Omega)$ with $\operatorname{curl} \boldsymbol{\xi} = \mathbf{f}$. Then one can show via a generalized Prager-Synge theorem [34, Theorem 11] that

$$\| \operatorname{curl} \mathbf{u}_h - \boldsymbol{\xi} \|^2_{\mathbf{L}^2(\Omega)} = \| \operatorname{curl} \mathbf{u}_h - \operatorname{curl} \mathbf{u} \|^2_{\mathbf{L}^2(\Omega)} + \| \operatorname{curl} \mathbf{u} - \boldsymbol{\xi} \|^2_{\mathbf{L}^2(\Omega)}.$$

Similar as before, we obtain an estimate of the error $\mathbf{u}_h - \mathbf{u}$ in the $\mathbf{H}(\operatorname{curl}, \Omega)$ seminorm in terms of the $\mathbf{L}^2$ norm of $\operatorname{curl} \mathbf{u}_h - \boldsymbol{\xi}$. For a computational application we need an algorithm to compute $\boldsymbol{\xi} \in \mathbf{H}_0(\operatorname{curl}, \Omega)$ solving the flux equation. Let $\mathcal{T}$ be a triangulation of the domain Ω and let $\mathcal{U}$ be the induced triangulation of the boundary $\partial\Omega$. We additionally assume that $\mathbf{f} \in \mathbf{RT}^r(\mathcal{T}, \mathcal{U})$ for some $r \in \mathbb{N}_0$. Now $\operatorname{div} \mathbf{f} = 0$ implies the existence of $\boldsymbol{\xi}_h \in \mathbf{Nd}^r(\mathcal{T}, \mathcal{U})$ solving the flux equation $\operatorname{curl} \boldsymbol{\xi}_h = \mathbf{f}$. The flux $\boldsymbol{\xi}_h$ can obtained via a mixed finite element method.

The computational costs of a global problem can again be avoided with a localized flux reconstruction. A prerequisite is that $\mathbf{u}_h \in \mathbf{Nd}^r(\mathcal{T})$ satisfies the Galerkin property

$$\int_\Omega \operatorname{curl} \mathbf{u}_h \cdot \operatorname{curl} \mathbf{v}_h \, \mathrm{d}x = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, \mathrm{d}x, \quad \mathbf{v}_h \in \mathbf{Nd}^r(\mathcal{T}).$$

Following the same philosophy as for Poisson problem, we may define the residual, decompose it into divergence-free distributions localized over patches, and solve the (distributional) flux equation on each patch locally. The crux of the construction, however, is finding the residual decomposition, which has been accomplished for $r = 0$ in the aforementioned publication by Braess and Schöberl. Generalizations to higher order edge elements have remained elusive as of now.

In this thesis we reassess equilibrated error estimators in the framework of finite element exterior calculus and improve upon the situation. We emphasize that we do not elaborate upon the details of the classical residual error estimator, for which a comprehensive study from the perspective of finite element exterior calculus has already been accomplished by Demlow and Hirani [72]. The major contribution of this chapter are algorithms for *partially localized flux reconstruction*, which builds immediately upon the concepts of Chapter IV.

Partially localized flux reconstruction seems to be a new tool in the theory of finite element methods, and our construction of finite element spaces in Chapter IV provides the underlying formalism. In this context, *flux reconstruction* refers to computing a generalized inverse of the exterior derivative between finite element spaces of differential forms, i.e., solving the flux equation. Specifically, we want to compute a generalized inverse for the mapping $\operatorname{curl} : \mathbf{Nd}^r(\mathcal{T},\mathcal{U}) \to \mathbf{RT}^r(\mathcal{T},\mathcal{U})$ from order r Nédélec elements to order r Raviart-Thomas elements.

Algorithmically we can tackle the problem either with a mixed finite element method or by solving normal equations, both of which involve global problems over higher order finite element spaces. Our framework, however, reduces the global problem to the lowest-order case. For example, assume that $\boldsymbol{\omega} \in \mathbf{RT}^r(\mathcal{T},\mathcal{U})$ is the curl of a member of $\mathbf{Nd}^r(\mathcal{T},\mathcal{U})$. We decompose $\boldsymbol{\omega} = \boldsymbol{\omega}^{\mathrm{lo}} + \operatorname{curl} \boldsymbol{\xi}^{\mathrm{hi}}$, where $\boldsymbol{\omega}^{\mathrm{lo}} \in \mathbf{RT}^0(\mathcal{T},\mathcal{U})$ is the canonical interpolation of $\boldsymbol{\omega}$ onto the lowest-order Raviart-Thomas space, and where $\boldsymbol{\xi}^{\mathrm{hi}} \in \mathbf{Nd}^r(\mathcal{T},\mathcal{U})$ is constructed by solving independent local problems. These local problems are associated to single tetrahedra, and their stability and complexity depends only on the local polynomial order and mesh quality; they are independent of each other and hence amenable for parallelization. One can show the existence $\boldsymbol{\xi}^{\mathrm{lo}} \in \mathbf{Nd}^0(\mathcal{T},\mathcal{U})$ with $\operatorname{curl} \boldsymbol{\xi}^{\mathrm{lo}} = \boldsymbol{\omega}^{\mathrm{lo}}$, computed by solving a global problem only on a smaller lowest-order space. Eventually $\boldsymbol{\xi} := \boldsymbol{\xi}^{\mathrm{lo}} + \boldsymbol{\xi}^{\mathrm{hi}} \in \mathbf{Nd}^r(\mathcal{T},\mathcal{U})$ satisfies $\operatorname{curl} \boldsymbol{\xi} = \boldsymbol{\omega}$. The flux reconstruction is *partially localized* in the sense that only lowest-order terms require a global computation.

A minor application of theoretical interest is determining the cohomology spaces of finite element de Rham complex of higher polynomial order. Specifically, the finite element interpolant from higher order finite element de Rham complexes onto the Whitney forms induces isomorphisms on cohomology.

A major application, however, solves the open problem in the theory of a equilibrated posteriori error estimators. We devise a fully localized flux reconstruction for the operator $\operatorname{curl} : \mathbf{Nd}^r(\mathcal{T},\mathcal{U}) \to \mathbf{RT}^r(\mathcal{T},\mathcal{U})$ provided that the Galerkin solution is given as additional information. Efficient algorithms for finite element flux reconstruction are critical to make the estimator feasible in computations [17, 18, 32, 84]. The partially localized flux reconstruction of this chapter finally enables a *fully localized flux reconstruction* and thus the equilibrated a posteriori error estimator for edge elements of arbitrary and possibly non-uniform polynomial order.

X.1. Partially Localized Flux Reconstruction

We introduce *partially localized flux reconstructions* in the calculus of differential forms. This section can be read as a direct continuation of Chapter IV. Let $\mathcal{T}$ be a simplicial complex and let $\mathcal{U} \subseteq \mathcal{T}$ be a simplicial subcomplex. We recall the complex of Whitney forms:

$$\dots \xrightarrow{\mathsf{d}^{k-1}} \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k}} \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k+1}} \dots$$

Additionally, we let $\mathcal{P} : \mathcal{T} \to \mathscr{A}$ be a hierarchical association of admissible sequence types to the simplices of $\mathcal{T}$. Following the construction principles of Chapter IV, we have a finite element de Rham complex with boundary conditions:

$$\dots \xrightarrow{\mathsf{d}^{k-1}} \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k}} \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) \xrightarrow{\mathsf{d}^{k+1}} \dots$$

The canonical interpolation onto the Whitney forms

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\mathsf{d}^{k-1}} & \mathcal{P}\Lambda^{k}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k}} & \mathcal{P}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots \\
 & & \Big\downarrow{\scriptstyle I^{k}_{\mathcal{W}}} & & \Big\downarrow{\scriptstyle I^{k+1}_{\mathcal{W}}} & & \\
\dots & \xrightarrow{\mathsf{d}^{k-1}} & \mathcal{W}\Lambda^{k}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k}} & \mathcal{W}\Lambda^{k+1}(\mathcal{T},\mathcal{U}) & \xrightarrow{\mathsf{d}^{k+1}} & \dots
\end{array}$$

gives a morphism of differential complexes. For the purpose of this section we use in particular the differential complexes

$$\dots \xrightarrow{\mathsf{d}^{k-1}_{F}} \mathring{\underline{\mathcal{P}}}\Lambda^{k}(F) \xrightarrow{\mathsf{d}^{k}_{F}} \mathring{\underline{\mathcal{P}}}\Lambda^{k+1}(F) \xrightarrow{\mathsf{d}^{k+1}_{F}} \dots$$

associated to $F \in \mathcal{T}$. We focus on constructively solving the *flux equation*

$$\mathsf{d}^{k-1}\xi = \omega, \tag{X.1}$$

where $\omega \in \mathcal{P}\Lambda^{k}(\mathcal{T},\mathcal{U})$ is the given data and $\xi \in \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$ is sought in the finite element space. Even if a solution exists, it might not be unique. Under the assumption that a solution exists, the problem of *flux reconstruction* is to find any solution to (X.1) in the finite element space. Moreover, we want to compute that solution in an efficient manner.

Flux reconstruction amounts to determining a generalized inverse of the operator

$$\mathsf{d}^{k-1} : \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^{k}(\mathcal{T},\mathcal{U}) \tag{X.2}$$

between finite element spaces. In this chapter we develop a method to reduce this problem to the lowest-order case. It then only remains to find a generalized inverse of the operator

$$\mathsf{d}^{k-1} : \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^{k}(\mathcal{T},\mathcal{U}) \tag{X.3}$$

between the spaces of Whitney forms. The higher order aspects of the problem are treated in local problems associated to simplices which are solved independently from each other. This is a fundamental result on the structure of higher order finite element spaces that is not only of theoretical interest but also relevant in numerical algorithms.

Before formulating the main result, we make our assumptions more precise. First we fix a generalized inverse of the exterior derivative (X.3) between Whitney forms. Specifically, we assume that we have a linear mapping

$$\mathsf{P}^{k}_{\mathcal{W}} : \mathcal{W}\Lambda^{k}(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \tag{X.4}$$

such that

$$\mathsf{d}^{k-1}\mathsf{P}^{k}_{\mathcal{W}}\mathsf{d}^{k-1}\xi = \mathsf{d}^{k-1}\xi, \quad \xi \in \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}). \tag{X.5}$$

In particular, $\omega = \mathsf{d}^{k-1}\mathsf{P}^{k}_{\mathcal{W}}\omega$ whenever $\omega \in \mathcal{W}\Lambda^{k}(\mathcal{T},\mathcal{U})$ is the exterior derivative of a Whitney form in $\mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$.

Similarly, for each simplex $F \in \mathcal{T}$ we fix a generalized derivative

$$\mathsf{P}_F^k : \mathring{\underline{\mathcal{P}}}\Lambda^k(F) \to \mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F) \tag{X.6}$$

such that

$$\mathsf{d}_F^{k-1}\mathsf{P}_F^k\mathsf{d}_F^{k-1}\xi = \mathsf{d}_F^{k-1}\xi, \quad \xi \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F). \tag{X.7}$$

We have $\omega = \mathsf{d}_F^{k-1}\mathsf{P}_F^k\omega$ whenever $\omega \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ is the exterior derivative of a member of $\mathring{\underline{\mathcal{P}}}\Lambda^{k-1}(F)$. The existence of a mapping $\mathsf{P}_{\mathcal{W}}^k$ and mappings P_F^k with such properties is an elementary fact of linear algebra.

Remark X.1.1.
There is no canonical choice in fixing the generalized inverses. Upon fixing a Hilbert space structure on the Whitney forms, however, the Moore-Penrose pseudoinverse of $\mathsf{d}^{k-1} : \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$ is a natural choice. This Moore-Penrose pseudoinverse provides the least-squares solution of the problem. Entirely analogous statements hold for choosing the generalized inverses P_F^k.

Assuming to have fixed generalized inverses as above, we introduce the *partially localized flux reconstruction* without further ado.

Theorem X.1.2.
Suppose that $\omega \in \mathcal{P}\Lambda^k(\mathcal{T})$ with $\mathsf{d}^k\omega = 0$. For $m \in \{k, \dots, n\}$ we let

$$\xi^m := \sum_{F \in \mathcal{T}^m} \operatorname{Ext}_F^{k-1} \mathsf{P}_F^k \operatorname{tr}_F^k \left(\omega - I_{\mathcal{W}}^k\omega - \sum_{l=k}^{m-1} \mathsf{d}^{k-1}\xi^l \right). \tag{X.8}$$

Then

$$I_{\mathcal{W}}^k\omega + \mathsf{d}^{k-1}\left(\sum_{m=k}^{n} \xi^m \right) = \omega. \tag{X.9}$$

In particular, if there exists $\xi \in \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$ with $\mathsf{d}^{k-1}\xi = \omega$, then

$$\mathsf{d}^{k-1}\left(\mathsf{P}_{\mathcal{W}}^k I_{\mathcal{W}}^k\omega + \sum_{m=k}^{n} \xi^m \right) = \omega. \tag{X.10}$$

Proof. We use the modified geometric decomposition (Lemma IV.3.9) to write

$$\omega = I_{\mathcal{W}}^k\omega + \sum_{m=k}^{n} \sum_{F \in \mathcal{T}^m} \operatorname{Ext}_F^k \mathring{\omega}_F,$$

where $\mathring{\omega}_F \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ for each $F \in \mathcal{T}$. We thus find for $F \in \mathcal{T}^k$ that

$$\operatorname{tr}_F^k \left(\omega - I_{\mathcal{W}}^k\omega \right) \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F).$$

The proof is completed by an induction argument. For each $F \in \mathcal{T}$ we set

$$\theta_F := \operatorname{tr}_F^k \left(\omega - I_{\mathcal{W}}^k\omega - \sum_{l=k}^{\dim F-1} \mathsf{d}^{k-1}\xi^l \right).$$

Let $m \in \{k, \dots, n-1\}$. Suppose that $\theta_f \in \mathring{\underline{\mathcal{P}}}\Lambda^k(f)$ for each $f \in \mathcal{T}^m$, which is certainly true if $m = k$. Then ξ^m as in (X.8) is well-defined. We find

$$\begin{aligned}
\mathsf{d}_f^k \theta_f &= \mathsf{d}_f^k \operatorname{tr}_f^k \left(\omega - I_{\mathcal{W}}^k \omega - \sum_{l=k}^{m-1} \mathsf{d}^{k-1} \xi^l \right) \\
&= \operatorname{tr}_f^k \left(\mathsf{d}^k \omega - \mathsf{d}^k I_{\mathcal{W}}^k \omega - \mathsf{d}^k \sum_{l=k}^{m-1} \mathsf{d}^{k-1} \xi^l \right) \\
&= \operatorname{tr}_f^k \left(\mathsf{d}^k \omega - I_{\mathcal{W}}^{k+1} \mathsf{d}^k \omega \right) = 0,
\end{aligned}$$

since $\mathsf{d}^k \omega = 0$, and conclude that $\mathsf{d}_f^{k-1} P_f^k \theta_f = \theta_f$. In particular,

$$\operatorname{tr}_f^k \mathsf{d}^{k-1} \xi^m = \mathsf{d}_f^{k-1} P_f \theta_f = \operatorname{tr}_f^k \left(\omega - I_{\mathcal{W}}^k \omega - \sum_{l=k}^{m-1} \mathsf{d}^{k-1} \xi^l \right). \tag{X.11}$$

If $m < n$, then $\theta_F \in \mathring{\underline{\mathcal{P}}}\Lambda^k(F)$ for each $F \in \mathcal{T}^{m+1}$. The argument may be iterated until $m = n$. In the latter case (X.11) provides (X.9).

Finally, if there exists $\xi \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$ with $\mathsf{d}^{k-1} \xi = \omega$, then

$$I_{\mathcal{W}}^k \omega = I_{\mathcal{W}}^k \mathsf{d}^{k-1} \xi = \mathsf{d}^{k-1} I_{\mathcal{W}}^{k-1} \xi.$$

and hence $\mathsf{d}^{k-1} \mathsf{P}_{\mathcal{W}}^k I_{\mathcal{W}}^k \xi = I_{\mathcal{W}}^{k-1} \xi$, which shows (X.10). This completes the proof. □

One implication of the theorem is that for every $\omega \in \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U})$ with $\mathsf{d}^k \omega = 0$ there exists $\xi^{\mathrm{hi}} \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$ such that

$$\omega = I_{\mathcal{W}}^k \omega + \mathsf{d}^k \xi^{\mathrm{hi}}.$$

If additionally ω is the exterior derivative of a member of $\mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$, then there exists $\xi^{\mathrm{lo}} \in \mathcal{W}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$ with $\mathsf{d}^{k-1} \xi^{\mathrm{lo}} = I_{\mathcal{W}}^k \omega$. Thus

$$\xi := \xi^{\mathrm{lo}} + \xi^{\mathrm{hi}}$$

is a solution of $\mathsf{d}^{k-1} \xi = \omega$ in the finite element space $\mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$.

As a simple first application we address the dimension of the cohomology classes of the finite element de Rham complex. This is a new proof of a result which has been shown before [11, 56, 132] with different techniques. Conceptually, this shows that the cohomological information are encoded completely in the lowest order component of the finite element de Rham complex.

Lemma X.1.3.
The commuting interpolant $I_{\mathcal{W}}^k : \mathcal{P}\Lambda^k(\mathcal{T}, \mathcal{U}) \to \mathcal{W}\Lambda^k(\mathcal{T}, \mathcal{U})$ induces isomorphisms on cohomology.

Proof. Let $\omega \in \mathcal{W}\Lambda^k(\mathcal{T}, \mathcal{U})$ with $\mathsf{d}^k \omega = 0$. If we have $\omega \notin \mathsf{d}^{k-1} \mathcal{W}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$, then in particular $\omega \notin \mathsf{d}^{k-1} \mathcal{P}\Lambda^{k-1}(\mathcal{T}, \mathcal{U})$, since the finite element interpolant commutes with the exterior derivative. Hence $I_{\mathcal{W}}^k$ is surjective on cohomology.

Conversely, suppose that $\omega \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$ with $\mathsf{d}^k\omega = 0$. If $\omega \notin \mathsf{d}^{k-1}\mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$, then there exists $\xi \in \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$ such that $\omega = \mathsf{d}^{k-1}\xi + I^k_{\mathcal{W}}\omega$. Since $I^k_{\mathcal{W}}$ commutes with the exterior derivative, we conclude $I^k_{\mathcal{W}}\omega \notin \mathsf{d}^{k-1}\mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$ from $\omega \notin \mathsf{d}^{k-1}\mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$. Hence $I^k_{\mathcal{W}}$ is injective on cohomology. □

The partially localized flux reconstruction is also relevant from a computational point of view. Consider again the flux equation $\mathsf{d}^{k-1}\xi = \omega$ for given $\omega \in \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U})$. Under the condition that ω is contained in the image of

$$\mathsf{d}^{k-1} : \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{P}\Lambda^k(\mathcal{T},\mathcal{U}) \tag{X.12}$$

there exists a solution $\xi \in \mathcal{P}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$ to the flux equation $\mathsf{d}^{k-1}\xi = \omega$. One possibility to computationally solve the flux equation is treating it as a least-squares problem: we fix a Hilbert space structure on the finite element spaces and compute the action of the Moore-Penrose pseudoinverse of the operator (X.12). This is a standard topic of numerical linear algebra. A drawback of this method is that the spectral properties of the operator (X.12) for higher polynomial order can be disadvantageous with regards to the L^2 norm. The condition number of the least-squares problem generally grows with the polynomial order, as does the size of the linear system of equations, which negatively affects the performance of the numerical methods. In particular, the stability and size of the problem on higher order spaces is comparable to computing the flux variable in a mixed finite element method.

How to avoid solving a global problem on a high order finite element space is now apparent by Theorem X.1.2. We invoke the following steps.

1. As outlined above, with a block of mutually independent local computations we split the main problem into two independent subproblems: one subproblem is to solve a flux equation $\mathsf{d}^{k-1}\xi^{lo} = I^k_{\mathcal{W}}\omega$ over the space of Whitney forms, whereas the second subproblem involves the higher order contributions.

2. In the first subproblem we seek a flux reconstruction $\xi^{\mathrm{lo}} \in \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U})$ for $I^k_{\mathcal{W}}\omega \in \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$. Hence we still need to solve a global least-squares problem but this time only for the operator $\mathsf{d}^{k-1} : \mathcal{W}\Lambda^{k-1}(\mathcal{T},\mathcal{U}) \to \mathcal{W}\Lambda^k(\mathcal{T},\mathcal{U})$ over finite element spaces of lowest order.

3. In the second subproblem we calculate ξ^{hi} by iterating over the dimension of the simplices in $\mathcal{T}$ from lowest to highest; at each step we solve a block of mutually independent local subproblems. In particular, at each step the computation is amenable to parallelization.

In this sense the flux reconstruction is partially localized: the only remaining global operation involves a finite element space of merely lowest order instead of the full finite element space. A fully localized flux reconstruction is feasible when additional structure is provided; this will be crucial to our application in the next section.

Remark X.1.4.
We can rearrange the construction of ξ^{hi} such that, instead of solving a sequence of parallelizable blocks of mutually independent local computations, we process only one parallelizable block of mutually independent local problems associated to full-dimensional simplices. This comes at the cost of redundant computations.

Remark X.1.5.
The L^2 stability of the global lowest-order problem depends only on the mesh quality and the domain, and the L^2 stability of the local problems depends only on the mesh quality and the polynomial order. Whether the dependency on the polynomial order can be dropped remains for future research (but see [33]).

Remark X.1.6.
We briefly compare the flux reconstruction of this chapter with the flux reconstruction encountered in Chapter IX. In the latter case, the purpose of the flux reconstruction was to derive Poincaré-Friedrichs inequalities. In this chapter, the problem of flux reconstruction is approached from a different point of view. In particular, we pose the problem over conforming finite element spaces. We emphasize at this point that the construction is similar but strictly different. The flux reconstruction in Chapter IX can be used on conforming finite element spaces too, but the structure of the construction is very different. A global flux reconstruction is performed as a subproblem in both cases. But the flux reconstruction in this chapter can be conducted independently from flux reconstruction on local patches (only the interpolant onto the Whitney forms needs to be computed), whereas the global flux reconstruction on simplicial chains in Chapter IX is both preceded and succeeded by local computations. Furthermore, the flux reconstruction of this chapter can be applied in a posteriori error estimation as we demonstrate in the next section.

X.2. Applications in A Posteriori Error Estimation

In this section we use the *partially localized* flux reconstruction to obtain a *fully localized* flux reconstruction for the equilibrated a posteriori estimation of the curl curl equation. The original construction of Braess and Schöberl works only for finite element spaces of lowest polynomial order. With the partially localized flux reconstruction at our disposal, it is no difficulty to generalize this to the case of higher order finite element spaces. We demonstrate the theory by the means of an example in three dimensions, which has already been alluded to in the introduction of this chapter. The construction works fully analogously in two dimensions.

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded weakly Lipschitz domain. We let $C^\infty(\overline{\Omega})$ denote the space of restrictions of smooth functions over $\mathbb{R}^n$ to Ω and define $\mathbf{C}^\infty(\overline{\Omega}) := C^\infty(\overline{\Omega})^3$. We let $C^\infty_c(\Omega)$ denote the space of smooth functions over Ω with support compactly contained in Ω. We let $\mathbf{C}^\infty_c(\Omega) := C^\infty_c(\Omega)^3$. Moreover, we let $L^2(\Omega)$ and $\mathbf{L}^2(\Omega) := L^2(\Omega)^3$ denote the Hilbert spaces of square-integrable functions and vector fields, respectively, over Ω. The corresponding scalar products and norms are written $\langle \cdot, \cdot \rangle_{L^2}$, $\| \cdot \|_{L^2}$, $\langle \cdot, \cdot \rangle_{\mathbf{L}^2}$, and $\| \cdot \|_{\mathbf{L}^2}$. The partial derivatives of such tensor fields are well-defined in the sense of distributions, and hence we may set

$$
\begin{aligned}
H^1(\Omega) &:= \left\{ v \in L^2(\Omega) \mid \operatorname{grad} v \in \mathbf{L}^2(\Omega) \right\}, \\
\mathbf{H}(\operatorname{curl}, \Omega) &:= \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \operatorname{curl} \mathbf{v} \in \mathbf{L}^2(\Omega) \right\}, \\
\mathbf{H}(\operatorname{div}, \Omega) &:= \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \operatorname{div} \mathbf{v} \in L^2(\Omega) \right\}.
\end{aligned}
$$

These are Hilbert spaces when equipped with the canonical norms. We also consider subspaces of these with boundary conditions imposed. Specifically, we define

$$\begin{aligned} H^1_0(\Omega) &:= \overline{C^\infty_c(\Omega)}^{H^1(\Omega)}, \\ \mathbf{H}_0(\mathrm{curl},\Omega) &:= \overline{\mathbf{C}^\infty_c(\Omega)}^{\mathbf{H}(\mathrm{curl},\Omega)}, \\ \mathbf{H}_0(\mathrm{div},\Omega) &:= \overline{\mathbf{C}^\infty_c(\Omega)}^{\mathbf{H}(\mathrm{div},\Omega)}. \end{aligned}$$

We can assemble the differential complexes

$$H^1(\Omega) \xrightarrow{\mathrm{grad}} \mathbf{H}(\mathrm{curl},\Omega) \xrightarrow{\mathrm{curl}} \mathbf{H}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \tag{X.13}$$

and

$$L^2(\Omega) \xleftarrow{\mathrm{div}} \mathbf{H}_0(\mathrm{div},\Omega) \xleftarrow{\mathrm{curl}} \mathbf{H}_0(\mathrm{curl},\Omega) \xleftarrow{\mathrm{grad}} H^1_0(\Omega). \tag{X.14}$$

Here, the differential operators have closed range. If moreover the domain is contractible, then the differential complexes (X.13) and (X.14) are exact. We also recall the integration by parts formulas

$$\langle \mathrm{grad}\, u, \mathbf{v}\rangle_{\mathbf{L}^2} = -\langle u, \mathrm{div}\,\mathbf{v}\rangle_{\mathbf{L}^2}, \quad u \in H^1(\Omega), \quad \mathbf{v} \in \mathbf{H}_0(\mathrm{div},\Omega), \tag{X.15}$$

$$\langle \mathrm{curl}\,\mathbf{u}, \mathbf{v}\rangle_{\mathbf{L}^2} = \langle \mathbf{u}, \mathrm{curl}\,\mathbf{v}\rangle_{\mathbf{L}^2}, \quad \mathbf{u} \in \mathbf{H}(\mathrm{curl},\Omega), \quad \mathbf{v} \in \mathbf{H}_0(\mathrm{curl},\Omega), \tag{X.16}$$

$$\langle \mathrm{div}\,\mathbf{u}, v\rangle_{L^2} = -\langle \mathbf{u}, \mathrm{grad}\, v\rangle_{L^2}, \quad \mathbf{u} \in \mathbf{H}(\mathrm{div},\Omega), \quad v \in H^1_0(\Omega). \tag{X.17}$$

Conceptually, the curl-curl *problem* for a given vector field $\mathbf{f}$ asks for a vector field $\mathbf{u}$ that satisfies $\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} = \mathbf{f}$. Specifically, we consider the following weak formulation of the problem in terms of Sobolev spaces. We assume that $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and search for $\mathbf{u} \in \mathbf{H}(\mathrm{curl},\Omega)$ with

$$\langle \mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v}\rangle_{\mathbf{L}^2} = \langle \mathbf{f}, \mathbf{v}\rangle_{\mathbf{L}^2}, \quad \mathbf{v} \in \mathbf{H}(\mathrm{curl},\Omega). \tag{X.18}$$

One can show that (X.18) has a solution. Without further conditions, there is no unique solution because the curl operator has a non-trivial kernel. To ensure uniqueness one may require the solution $\mathbf{u}$ to be orthogonal to the gradients of functions in $H^1(\Omega)$; one can show that this enforces $\mathbf{u} \in \mathbf{H}_0(\mathrm{div},\Omega)$ with $\mathrm{div}\,\mathbf{u} = 0$. Conditions to ensure uniqueness of $\mathbf{u}$, however, are not central to our exposition in this section.

Let us assume additionally that Ω is contractible and that $\mathbf{f} \in \mathbf{H}_0(\mathrm{div},\Omega)$ with $\mathrm{div}\,\mathbf{f} = 0$. Then the differential complex (X.14) is exact at $\mathbf{H}_0(\mathrm{div},\Omega)$ and $\mathbf{f}$ is the curl of a vector field in $\mathbf{H}_0(\mathrm{curl},\Omega)$. Under these conditions, any weak solution $\mathbf{u}$ of (X.18) satisfies $\mathrm{curl}\,\mathbf{u} \in H^1_0(\Omega)$ with $\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} = \mathbf{f}$, and thus is a strong solution.

In order to address a posteriori error estimation we fix a solution $\mathbf{u} \in \mathbf{H}(\mathrm{curl},\Omega)$ and let $\mathbf{u}_h \in \mathbf{H}(\mathrm{curl},\Omega)$ be arbitrary. Furthermore, we let $\boldsymbol{\sigma} \in H_0(\mathrm{curl},\Omega)$ with $\mathrm{curl}\,\boldsymbol{\sigma} = \mathbf{f}$. By the binomial theorem we see

$$\begin{aligned} \|\boldsymbol{\sigma} - \mathrm{curl}\,\mathbf{u}_h\|^2_{\mathbf{L}^2} &= \|\boldsymbol{\sigma} - \mathrm{curl}\,\mathbf{u} + \mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{u}_h\|^2_{\mathbf{L}^2} \\ &= \|\boldsymbol{\sigma} - \mathrm{curl}\,\mathbf{u}\|^2_{\mathbf{L}^2} + \|\,\mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{u}_h\|^2_{\mathbf{L}^2} \\ &\qquad - 2\,\langle \boldsymbol{\sigma} - \mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{u} - \mathrm{curl}\,\mathbf{u}_h\rangle_{\mathbf{L}^2}. \end{aligned}$$

Using (X.16) and $\operatorname{curl}\boldsymbol{\sigma} = \mathbf{f} = \operatorname{curl}\mathbf{u}$ we note

$$\langle \boldsymbol{\sigma} - \operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{u} - \operatorname{curl}\mathbf{u}_h \rangle_{\mathbf{L}^2} = \langle \operatorname{curl}(\boldsymbol{\sigma} - \operatorname{curl}\mathbf{u}), \mathbf{u} - \mathbf{u}_h \rangle_{\mathbf{L}^2} = 0.$$

Thus we conclude

$$\|\boldsymbol{\sigma} - \operatorname{curl}\mathbf{u}_h\|_{\mathbf{L}^2}^2 = \|\boldsymbol{\sigma} - \operatorname{curl}\mathbf{u}\|_{\mathbf{L}^2}^2 + \|\operatorname{curl}\mathbf{u} - \operatorname{curl}\mathbf{u}_h\|_{\mathbf{L}^2}^2. \tag{X.19}$$

Equation (X.19) generalizes what is known as *Prager-Synge identity* or *hypercircle identity* in the literature (see [34]).

This identity has the following practical significance. Let $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega)$ with $\operatorname{curl}\mathbf{u} \in H_0(\operatorname{curl}, \Omega)$ be a strong solution of (X.18). Given any exact solution $\boldsymbol{\sigma} \in H_0(\operatorname{curl}, \Omega)$ of $\operatorname{curl}\boldsymbol{\sigma} = \mathbf{f}$ and any $\mathbf{u}_h \in \mathbf{H}(\operatorname{curl}, \Omega)$, we obtain via (X.19) that

$$\|\boldsymbol{\sigma} - \operatorname{curl}\mathbf{u}_h\|_{\mathbf{L}^2} \geq \|\operatorname{curl}\mathbf{u} - \operatorname{curl}\mathbf{u}_h\|_{\mathbf{L}^2}. \tag{X.20}$$

The left-hand side of (X.20) is given in terms of known objects and dominates the right-hand side of (X.20), which depends on the generally unknown true solution $\mathbf{u}$. Seeing $\mathbf{u}_h$ as an approximation of $\mathbf{u}$, we may regard (X.20) as an error estimate in the $\mathbf{H}(\operatorname{curl}, \Omega)$ seminorm.

In a typical application, $\mathbf{u}_h$ is the Galerkin solution of a finite element method. We can apply (X.20) to obtain an upper bound on one component of the error in the $\mathbf{H}(\operatorname{curl}, \Omega)$ norm provided that an exact solution $\boldsymbol{\sigma} \in H_0(\operatorname{curl}, \Omega)$ of $\operatorname{curl}\boldsymbol{\sigma} = \mathbf{f}$ is available. Note that $\operatorname{curl}\mathbf{u}$ is generally unknown and hence not a candidate for $\boldsymbol{\sigma}$. But numerical algorithms for flux reconstruction make (X.20) productive for applications.

As a technical preparation, we consider finite element de Rham complexes over the domain Ω. Let $\mathcal{T}$ be a simplicial complex triangulating Ω and let $\mathcal{U}$ denote the subcomplex of $\mathcal{T}$ triangulating $\partial\Omega$. We focus on higher order finite element spaces of uniform order; the generalization to spaces of non-uniform polynomial order is straight forward.

Let $r \in \mathbb{N}_0$. With respect to $\mathcal{T}$ we let $\mathcal{P}^r(\mathcal{T})$ denote the *Lagrange space*, let $\mathbf{Nd}^r(\mathcal{T})$ denote the *Nédélec space*, let $\mathbf{RT}^r(\mathcal{T})$ denote the *Raviart-Thomas space*, and let $\mathcal{P}^r_{\mathrm{DC}}(\mathcal{T})$ denote the space of piecewise polynomial functions, each with polynomial order r. From spaces of this form we may assemble the finite element de Rham complexes

$$\mathcal{P}^{r+1}(\mathcal{T}) \xrightarrow{\operatorname{grad}} \mathbf{Nd}^r(\mathcal{T}) \xrightarrow{\operatorname{curl}} \mathbf{RT}^r(\mathcal{T}) \xrightarrow{\operatorname{div}} \mathcal{P}^r_{\mathrm{DC}}(\mathcal{T}). \tag{X.21}$$

Next we recall finite element spaces with boundary conditions. We let $\mathcal{P}^r(\mathcal{T},\mathcal{U})$, $\mathbf{Nd}^r(\mathcal{T},\mathcal{U})$, $\mathbf{RT}^r(\mathcal{T},\mathcal{U})$ denote the subspaces of $\mathcal{P}^r(\mathcal{T})$, $\mathbf{Nd}^r(\mathcal{T})$, $\mathbf{RT}^r(\mathcal{T})$ with Dirichlet, tangential, and normal boundary conditions along $\partial\Omega$, respectively. Again we may assemble a finite element de Rham complex

$$\mathcal{P}^r_{\mathrm{DC}}(\mathcal{T}) \xleftarrow{\operatorname{div}} \mathbf{Nd}^r(\mathcal{T},\mathcal{U}) \xleftarrow{\operatorname{curl}} \mathbf{RT}^r(\mathcal{T},\mathcal{U}) \xleftarrow{\operatorname{grad}} \mathcal{P}^{r+1}(\mathcal{T},\mathcal{U}). \tag{X.22}$$

The differential complex (X.21) is a finite-dimensional subcomplex of (X.13) and the differential complex (X.22) is a finite-dimensional subcomplex of (X.14).

Let $\mathbf{f} \in \mathbf{H}_0(\operatorname{div},\Omega)$ be as before except for the additional assumption that $\mathbf{f} \in \mathbf{RT}^r(\mathcal{T},\mathcal{U})$. Then there exists a member of $\mathbf{Nd}^r(\mathcal{T},\mathcal{U})$ whose curl equals $\mathbf{f}$. In order to utilize the error estimate (X.20) in practical computations, it remains to algorithmically construct a generalized inverse for the operator

$$\operatorname{curl} : \mathbf{Nd}^r(\mathcal{T},\mathcal{U}) \to \mathbf{RT}^r(\mathcal{T},\mathcal{U}). \tag{X.23}$$

One possibility is solving a least-squares problem over the whole finite element space. We have seen, however, that a global computation over only lowest-order finite element spaces is sufficient. In the light of Theorem X.1.2 and the subsequent discussions in the previous section, we decompose

$$\mathbf{f} = \mathbf{f}_0 + \operatorname{curl} \boldsymbol{\xi}_r,$$

where $\mathbf{f}_0 \in \mathbf{RT}^0(\mathcal{T},\mathcal{U})$ is the canonical interpolation of $\mathbf{f}$ onto the lowest-order Raviart-Thomas space with homogeneous normal boundary conditions and where $\boldsymbol{\xi}_r \in \mathbf{Nd}^r(\mathcal{T},\mathcal{U})$ is computed through a number of local problems over simplices whose computation is parallelizable. This reduces the flux reconstruction problem to the special case $r = 0$.

The partially localized flux reconstruction can be extended to a *fully localized* flux reconstruction if additional information is given. Specifically, assume that $\mathbf{u}_h \in \mathbf{Nd}^r(\mathcal{T})$ satisfies the Galerkin condition

$$\langle \operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v}_h \rangle_{L^2} = \langle \mathbf{f}, \mathbf{v}_h \rangle_{L^2}, \quad \mathbf{v}_h \in \mathbf{Nd}^r(\mathcal{T}). \tag{X.24}$$

As a first step towards the fully localized flux reconstruction, we compute the decomposition $\mathbf{f} = \mathbf{f}_0 + \operatorname{curl} \boldsymbol{\xi}_r$ with $\mathbf{f}_0 \in \mathbf{RT}^0(\mathcal{T},\mathcal{U})$ and $\boldsymbol{\xi}_r \in \mathbf{Nd}^r(\mathcal{T},\mathcal{U})$. This can achieved by independent local computations.

For the next step we observe that both $\operatorname{curl} \mathbf{u}_h$ and $\boldsymbol{\xi}_r$ are members of $\mathcal{P}^r_{\mathrm{DC}}(\mathcal{T})^3$, i.e., they are vector fields piecewise polynomial of order r. We let $\boldsymbol{\gamma}_h \in \mathcal{P}^0_{\mathrm{DC}}(\mathcal{T})^3$ denote the L^2 orthogonal projection of $\boldsymbol{\xi}_r - \operatorname{curl} \mathbf{u}_h$ onto the space $\mathcal{P}^0_{\mathrm{DC}}(\mathcal{T})^3$ of piecewise constant vector fields. Note $\boldsymbol{\gamma}_h$ can be computed for each simplex independently. By construction we have

$$\langle \boldsymbol{\gamma}_h, \boldsymbol{\tau}_h \rangle_{\mathbf{L}^2} = \langle \boldsymbol{\xi}_r - \operatorname{curl} \mathbf{u}_h, \boldsymbol{\tau}_h \rangle_{\mathbf{L}^2}, \quad \boldsymbol{\tau}_h \in \mathcal{P}^0_{\mathrm{DC}}(\mathcal{T})^3.$$

Using the Galerkin orthogonality (X.24) we verify

$$\begin{aligned}
0 &= \langle \mathbf{f}, \mathbf{v}_h \rangle_{\mathbf{L}^2} - \langle \operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v}_h \rangle_{\mathbf{L}^2} \\
&= \langle \mathbf{f}_0 + \operatorname{curl} \boldsymbol{\xi}_h, \mathbf{v}_h \rangle_{\mathbf{L}^2} - \langle \operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v}_h \rangle_{\mathbf{L}^2} \\
&= \langle \mathbf{f}_0, \mathbf{v}_h \rangle_{\mathbf{L}^2} + \langle \boldsymbol{\xi}_r - \operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v}_h \rangle_{\mathbf{L}^2} \\
&= \langle \mathbf{f}_0, \mathbf{v}_h \rangle_{\mathbf{L}^2} + \langle \boldsymbol{\gamma}_h, \operatorname{curl} \mathbf{v}_h \rangle_{\mathbf{L}^2}
\end{aligned}$$

for every $\mathbf{v}_h \in \mathbf{Nd}^0(\mathcal{T})$; here we have used that $\operatorname{curl} \mathbf{v}_h \in \mathcal{P}^0_{\mathrm{DC}}(\mathcal{T})^3$. Moreover, $\operatorname{div} \mathbf{f}_0 = 0$ since the finite element interpolant commutes with the differential operators. The next crucial step is using the fully localized flux reconstruction for lowest-order finite element spaces in [34]. The original construction of Braess and Schöberl gives $\boldsymbol{\varrho}_h \in \mathbf{RT}^0(\mathcal{T},\mathcal{U})$ with

$$\langle \boldsymbol{\varrho}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2} = \langle \mathbf{f}_0, \mathbf{v} \rangle_{\mathbf{L}^2} + \langle \boldsymbol{\gamma}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2}, \quad \mathbf{v} \in \mathbf{H}(\operatorname{curl}, \Omega), \tag{X.25}$$

where $\boldsymbol{\varrho}_h$ is computed by solving local independent problems over element patches around vertices. We refer specifically to Section 4.2 and Section 4.4 of [34] for the details in the literature, but we outline the construction in Remark X.2.4 below too.

This leads us to

$$\begin{aligned}
&\langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{L}^2} - \langle \operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2} \\
&= \langle \mathbf{f}_0, \mathbf{v} \rangle_{\mathbf{L}^2} + \langle \boldsymbol{\xi}_r - \operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2} \\
&= \langle \mathbf{f}_0, \mathbf{v} \rangle_{\mathbf{L}^2} + \langle \boldsymbol{\gamma}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2} + \langle \boldsymbol{\xi}_r - \operatorname{curl} \mathbf{u}_h - \boldsymbol{\gamma}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2} \\
&= \langle \boldsymbol{\varrho}_h + \boldsymbol{\xi}_r - \operatorname{curl} \mathbf{u}_h - \boldsymbol{\gamma}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2}
\end{aligned}$$

for all $\mathbf{v} \in \mathbf{H}(\operatorname{curl})$. Upon setting

$$\boldsymbol{\sigma}_h := \boldsymbol{\varrho}_h + \boldsymbol{\xi}_r - \boldsymbol{\gamma}_h,$$

this can be rewritten as

$$\langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{L}^2} - \langle \operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2} = \langle \boldsymbol{\sigma}_h - \operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2}, \quad \mathbf{v} \in \mathbf{H}(\operatorname{curl}, \Omega),$$

which immediately implies

$$\langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{L}^2} = \langle \boldsymbol{\sigma}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2}, \quad \mathbf{v} \in \mathbf{H}(\operatorname{curl}, \Omega).$$

We conclude that $\boldsymbol{\sigma}_h \in \mathbf{H}_0(\operatorname{curl}, \Omega)$. By construction we have $\boldsymbol{\sigma}_h \in \mathbf{Nd}^r(\mathcal{T},\mathcal{U})$ with

$$\mathbf{f} = \operatorname{curl} \boldsymbol{\sigma}_h.$$

Constructing $\boldsymbol{\sigma}_h$ has involved only local computations. This completes the fully localized flux reconstruction and enables the a posteriori error estimate (X.20).

Remark X.2.1.
Our techniques apply similarly to higher order flux reconstruction for edge elements in dimension two. Again, the lowest-order case is treated in [34]. Moreover, we may treat the curl curl problem with mixed boundary conditions in an entirely analogous manner as long as the differential complexes are exact.

Remark X.2.2.
Our construction has assumed a contractible domain and that $\mathbf{f} \in \mathbf{RT}^r(\mathcal{T},\mathcal{U})$ with $\operatorname{div} \mathbf{f} = 0$. We remark that the condition $\mathbf{f} \in H_0(\Omega, \operatorname{div})$ has appeared in the discussions of Demlow and Hirani, who have discussed it under the label *Hodge imbalance* [72]. In general, the flux equation $\operatorname{curl} \xi = \mathbf{f}$ is not solvable exactly. We expect this to hamper the estimator in practical computations.

Remark X.2.3.
With Remark X.1.4 in mind, we see that $\boldsymbol{\xi}_r$ and $\boldsymbol{\gamma}_h$ are computable on each simplex

using only the information given on that simplex. At the cost of redundant computations, we may rearrange the calculations so that $\boldsymbol{\sigma}_h$ is constructed with a single parallelizable block of problems associated to patches.

Via Remark X.1.5 we furthermore see that the stability of the construction of $\boldsymbol{\sigma}_h$ depends only on the mesh quality, the domain, and the polynomial order of the finite element spaces. We conjecture that the last dependence can be dropped, i.e. that equilibrated a posteriori error estimators for edge elements are robust with respect to the polynomial order (see [33]).

Remark X.2.4.
We have referred to the original publication of Braess and Schöberl for the details of how to construct $\boldsymbol{\varrho}_h$. In order to give a self-contained exposition, we provide more details in this lengthy remark. We define the distributional vector field $\boldsymbol{\beta}$ by

$$\langle \boldsymbol{\beta}, \mathbf{v} \rangle := \langle \mathbf{f}_0, \mathbf{v} \rangle_{\mathbf{L}^2} + \langle \boldsymbol{\gamma}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2}, \quad \mathbf{v} \in \mathbf{H}(\operatorname{curl}, \Omega). \tag{X.26}$$

Since $\boldsymbol{\gamma}_h$ is piecewise constant, integration by parts shows

$$\langle \boldsymbol{\gamma}_h, \operatorname{curl} \mathbf{v} \rangle_{\mathbf{L}^2} = \sum_{T \in \mathcal{T}^3} \int_T \boldsymbol{\gamma}_h \cdot \operatorname{curl} \mathbf{v} \, \mathrm{d}s = \sum_{\substack{T \in \mathcal{T}^3 \\ F \in \Delta(T)^2}} \int_F \mathbf{v} \cdot (\boldsymbol{\gamma}_h \times \vec{n}_{T,F}) \, \mathrm{d}s$$

for all $\mathbf{v} \in \mathbf{H}(\operatorname{curl}, \Omega)$, where $\vec{n}_{T,F}$ is the outward unit formal of T along the boundary face F. We conclude that there exist unique constant vector fields $\boldsymbol{\beta}_T \in \mathbf{RT}^0(T)$ for $T \in \mathcal{T}^3$ and unique constant tangential vector fields $\boldsymbol{\beta}_F \in \mathbf{RT}^0(F)$ over $F \in \mathcal{T}^2$ such that we can represent $\boldsymbol{\beta}$ by

$$\langle \boldsymbol{\beta}, \mathbf{v} \rangle = \sum_{T \in \mathcal{T}^3} \int_T \boldsymbol{\beta}_T \cdot \mathbf{v} \, \mathrm{d}x + \sum_{F \in \Delta(T)^2} \int_F \boldsymbol{\beta}_F \cdot \tau_F(\mathbf{v}) \, \mathrm{d}s \tag{X.27}$$

for all $\mathbf{v} \in \mathbf{H}(\operatorname{curl}, \Omega)$. Here $\tau_F(\mathbf{v})$ denotes the tangential component of $\mathbf{v}$ along F. We have $\mathbf{f}_{0|T} = \boldsymbol{\beta}_T$ for each $T \in \mathcal{T}^3$, whereas $\boldsymbol{\beta}_F$ for each $F \in \mathcal{T}^2$ is the jump term along F induced by the integration by parts over adjacent tetrahedra.

The idea is to decompose $\boldsymbol{\beta}$ into the sum of locally supported divergence-free distributional vector fields $\boldsymbol{\beta}^V$ associated to vertices $V \in \mathcal{T}^0$. The vanishing divergence of each $\boldsymbol{\beta}^V$ then proves solvability of the local flux equation $\operatorname{curl} \boldsymbol{\varrho}_h^V = \boldsymbol{\beta}^V$. Summing the local solutions yields a solution $\boldsymbol{\varrho}_h$ to the global flux equation $\operatorname{curl} \boldsymbol{\varrho}_h = \boldsymbol{\beta}$.

We gather more information about the summands in the representation (X.27). Since $\mathbf{f}_0$ is divergence free, we find for each $T \in \mathcal{T}^3$ that

$$0 = \int_T \operatorname{div} \mathbf{f}_{0|T} \, \mathrm{d}x = \int_T \operatorname{div} \boldsymbol{\beta}_T \, \mathrm{d}x = \sum_{\substack{F \in \Delta(T)^2 \\ F \notin \mathcal{U}}} \int_F \boldsymbol{\beta}_T \cdot \vec{n}_{T,F} \, \mathrm{d}s. \tag{X.28}$$

For $v \in C^\infty(\overline{\Omega})$ and $F \in \mathcal{T}^2$ we observe

$$\begin{aligned} 0 &= \int_F \boldsymbol{\beta}_F \cdot \tau_F (\operatorname{grad} v) \, \mathrm{d}s = \int_F \boldsymbol{\beta}_F \cdot \operatorname{grad} \tau_F (v) \, \mathrm{d}s \\ &= - \int_F \operatorname{div} \boldsymbol{\beta}_F \cdot \tau_F (v) \, \mathrm{d}s - \sum_{\substack{E \in \Delta(F)^1 \\ E \notin \mathcal{U}}} \int_E \tau_F (v) \, \vec{n}_{F,E} \cdot \boldsymbol{\beta}_F \, \mathrm{d}e. \end{aligned}$$

Since $v \in C^\infty(\overline{\Omega})$ was arbitrary, we find for all $F \in \mathcal{T}^2$ that

$$0 = \int_F \operatorname{div} \boldsymbol{\beta}_F \, \mathrm{d}s = \sum_{E \in \Delta(F)^1} \int_E \boldsymbol{\beta}_F \cdot \vec{n}_{F,E} \, \mathrm{d}e, \tag{X.29}$$

and we find for all $E \in \mathcal{T}^1$ that

$$0 = \sum_{\substack{F \in \mathcal{T}^2 \\ E \in \Delta(F)^1}} \int_E \vec{n}_{F,E} \cdot \boldsymbol{\beta}_F \, \mathrm{d}e. \tag{X.30}$$

We equip every $E \in \mathcal{T}^1$ with an arbitrary but fixed orientation, so that one vertex of E is the back vertex and the other is the front vertex. For every tetrahedron T containing the edge E we let $F^1(T,E)$ and $F^2(T,E)$ denote the two faces of T that are opposite to the back and the front vertex of E, respectively. Similarly, for every face F containing the edge E we let $E^1(F,E)$ and $E^2(F,E)$ denote the two edges of F that are opposite to the back and the front vertex of E, respectively. It is then possible to show that

$$\begin{aligned} 0 = \frac{1}{6} \sum_{\substack{F \in \mathcal{T}^2 \\ E \in \Delta(F)}} \int_{E^1(F,E)} \boldsymbol{\beta}_F \cdot \vec{n}_{F,E^1(T,E)} \, \mathrm{d}e - \int_{E^2(F,E)} \boldsymbol{\beta}_F \cdot \vec{n}_{F,E^2(T,E)} \, \mathrm{d}e \\ + \frac{1}{12} \sum_{\substack{T \in \mathcal{T}^3 \\ E \in \Delta(T)}} \int_{F^1(T,E)} \boldsymbol{\beta}_T \cdot \vec{n}_{T,F^1(T,E)} \, \mathrm{d}s - \int_{F^2(T,E)} \boldsymbol{\beta}_T \cdot \vec{n}_{T,F^2(T,E)} \, \mathrm{d}s. \end{aligned} \tag{X.31}$$

This is precisely Equation (4.9) of [34] except for a different sign convention, and can be seen by evaluating $\boldsymbol{\beta}$ on the basis vector field in $\mathbf{Nd}^0(\mathcal{T})$ associated with E, and then recalling the Galerkin orthogonality

$$\langle \boldsymbol{\beta}, \mathbf{v}_h \rangle = 0, \quad \mathbf{v}_h \in \mathbf{Nd}^0(\mathcal{T}).$$

Having gathered these properties of $\boldsymbol{\beta}$, we develop the localized decomposition. Let $V \in \mathcal{T}^0$ be any vertex of the triangulation. First we define vector fields over tetrahedra. Whenever $T \in \mathcal{T}^3$ with $V \in \Delta(T)^0$ is a tetrahedron containing V as a vertex, and $F \in \Delta(T)^2$ is a face of T containing V as a vertex, then we let $F^1(T,V)$ and $F^2(T,V)$ denote the other two faces of T containing V in arbitrary order, and furthermore we let $F^o(T,V)$ denote the face of T opposite to V. We define the vector field $\boldsymbol{\beta}_T^V \in \mathbf{RT}^0(T)$ by requiring

$$\begin{aligned} \int_F \boldsymbol{\beta}_T^V \cdot \vec{n}_{T,F} \, \mathrm{d}s = \frac{1}{3} \int_F \boldsymbol{\beta}_T \cdot \vec{n}_{T,F} \, \mathrm{d}s + \frac{1}{12} \int_{F^o(T,V)} \boldsymbol{\beta}_T \cdot \vec{n}_{T,F^o(T,V)} \, \mathrm{d}s \\ - \frac{1}{24} \int_{F^1(T,V)} \boldsymbol{\beta}_T \cdot \vec{n}_{T,F^1(T,V)} \, \mathrm{d}s - \frac{1}{24} \int_{F^2(T,V)} \boldsymbol{\beta}_T \cdot \vec{n}_{T,F^2(T,V)} \, \mathrm{d}s \end{aligned}$$

for each $F \in \Delta(T)^2$ that contains V as a vertex, and by requiring

$$\int_{F^o(T,V)} \boldsymbol{\beta}_T^V \cdot \vec{n}_{T,F^o(T,V)} \, \mathrm{d}s = 0$$

on the face opposite to V. Since these integrals are the degrees of freedom of $\mathbf{RT}^0(T)$, this uniquely defines $\boldsymbol{\beta}_T^V$.

Additionally we define vector fields over faces. Whenever $F \in \mathcal{T}^2$ with $V \in \Delta(T)^0$ is a face containing V as a vertex, and $E \in \Delta(F)^1$ is an edge of F containing V as a vertex again, then we let $E^1(F,V)$ denote the edge of F containing V and opposite to E, and we let $E^o(F,V)$ denote the edge of F opposite to V. Moreover, whenever $T \in \mathcal{T}^3$ is a tetrahedron containing F, then we let $F'(T,E,V)$ be the face of T containing V but not containing E. We define the vector field $\boldsymbol{\beta}_F^V \in \mathbf{RT}^0(F)$ by requiring[i]

$$\begin{aligned}\int_E \boldsymbol{\beta}_F^V \cdot \vec{n}_{F,E} \,\mathrm{d}e = \frac{1}{2}\int_E &\boldsymbol{\beta}_F \cdot \vec{n}_{F,E} \,\mathrm{d}e \\ &+ \frac{1}{6}\int_{E^o(F,V)} \boldsymbol{\beta}_F \cdot \vec{n}_{F,E^o(F,V)} \,\mathrm{d}e - \frac{1}{6}\int_{E^1(F,V)} \boldsymbol{\beta}_F \cdot \vec{n}_{F,E^1(F,V)} \,\mathrm{d}e \\ &+ \frac{1}{24} \sum_{\substack{T\in\mathcal{T}^3 \\ F\in\Delta(T)^2}} \int_{F^o(T,V)} \boldsymbol{\beta}_T \cdot \vec{n}_{F^o(T,V)} \,\mathrm{d}s - \int_{F'(T,E,V)} \boldsymbol{\beta}_T \cdot \vec{n}_{F'(T,E,V)} \,\mathrm{d}s.\end{aligned}$$

Similarly as for the terms on the tetrahedra, we set

$$\int_{E^o(F,V)} \boldsymbol{\beta}_F^V \cdot \vec{n}_{F,E^o(F,V)} \,\mathrm{d}e = 0$$

over the edge opposite to V. Since these integrals are the degrees of freedom of $\mathbf{RT}^0(F)$, this uniquely defines $\boldsymbol{\beta}_F^V$.

For each $V \in \mathcal{T}^0$ we define the distributional vector field $\boldsymbol{\beta}^V$ by

$$\left\langle \boldsymbol{\beta}^V, \mathbf{v} \right\rangle := \sum_{\substack{T\in\mathcal{T}^3 \\ V\in\Delta(T)^0}} \left\langle \boldsymbol{\beta}_T^V, \mathbf{v} \right\rangle + \sum_{\substack{F\in\mathcal{T}^2 \\ V\in\Delta(T)^0}} \left\langle \boldsymbol{\beta}_F^V, \mathbf{v} \right\rangle, \quad \mathbf{v} \in \mathbf{C}^\infty(\overline{\Omega}).$$

It is easily checked that $\boldsymbol{\beta}$ is again the sum of all $\boldsymbol{\beta}^V$ over the vertices of $\mathcal{T}$.

$$\boldsymbol{\beta} = \sum_{V\in\mathcal{T}^0} \boldsymbol{\beta}^V.$$

To see this, we use symmetries in the definition to find

$$\sum_{V\in\mathcal{T}^0} \int_F \boldsymbol{\beta}_T^V \cdot \vec{n}_{T,F} \,\mathrm{d}s = \sum_{V\in\Delta(F)^0} \int_F \boldsymbol{\beta}_T^V \cdot \vec{n}_{T,F} \,\mathrm{d}s = \int_F \boldsymbol{\beta}_T \cdot \vec{n}_{T,F} \,\mathrm{d}s$$

for every $T \in \mathcal{T}^3$ and $F \in \Delta(T)^2$, and to find

$$\sum_{V\in\mathcal{T}^0} \int_E \boldsymbol{\beta}_F^V \cdot \vec{n}_{F,E} \,\mathrm{d}e = \sum_{V\in\Delta(E)^0} \int_E \boldsymbol{\beta}_F^V \cdot \vec{n}_{F,E} \,\mathrm{d}e = \int_E \boldsymbol{\beta}_F \cdot \vec{n}_{F,E} \,\mathrm{d}e,$$

[i] To the author's best understanding, the definition of the local face terms in Equation (4.18) of [34] are subject to a sign error.

for every $F \in \mathcal{T}^2$ and $E \in \Delta(T)^1$.

We want to show that each $\boldsymbol{\beta}^V$ has vanishing distributional divergence. Let $V \in \mathcal{T}^0$ be an arbitrary but fixed vertex. For every $v \in C^\infty(\overline{\Omega})$ we observe

$$\begin{aligned}\left\langle \boldsymbol{\beta}^V, \operatorname{grad} v \right\rangle = &\sum_{\substack{T \in \mathcal{T}^3 \\ V \in \Delta(T)^0}} \left(\sum_{\substack{F \in \Delta(T)^2 \\ V \in \Delta(F)^0}} \int_F v \cdot \left(\vec{n}_{T,F} \cdot \boldsymbol{\beta}_T^V \right) \mathrm{d}s - \int_T v \cdot \operatorname{div} \boldsymbol{\beta}_T^V \,\mathrm{d}x \right) \\ &+ \sum_{\substack{F \in \mathcal{T}^2 \\ V \in \Delta(F)^0}} \left(\sum_{\substack{E \in \Delta(F)^1 \\ V \in \Delta(E)^0}} \int_E v \cdot \left(\vec{n}_{F,E} \cdot \boldsymbol{\beta}_F^V \right) \mathrm{d}s - \int_F v \cdot \operatorname{div} \boldsymbol{\beta}_F^V \,\mathrm{d}s \right).\end{aligned}$$

For every tetrahedron $T \in \mathcal{T}^3$ containing V we see

$$\int_T \operatorname{div} \boldsymbol{\beta}_T^V \,\mathrm{d}x = \sum_{F \in \Delta(T)^2} \int_F \boldsymbol{\beta}_T^V \cdot \vec{n}_{T,F} \,\mathrm{d}s = \frac{1}{3} \sum_{F \in \Delta(T)^2} \int_F \boldsymbol{\beta}_T \cdot \vec{n}_{T,F} \,\mathrm{d}s = 0$$

as a consequence of (X.28). When $F \in \mathcal{T}^2$ is a face containing V, then

$$\int_F \operatorname{div} \boldsymbol{\beta}_F^V \,\mathrm{d}s - \sum_{\substack{T \in \mathcal{T}^3 \\ F \in \Delta(T)^2}} \int_F \boldsymbol{\beta}_T^V \cdot \vec{n}_{T,F} \,\mathrm{d}s = 0$$

follows by a direct combination of (X.28) and (X.29). Lastly, when $E \in \mathcal{T}^1$ contains the vertex V, then (X.30) and (X.31) imply that

$$0 = \sum_{\substack{F \in \mathcal{T}^2 \\ E \in \Delta(F)^1}} \int_E \vec{n}_{F,E} \cdot \boldsymbol{\beta}_F^V \,\mathrm{d}e.$$

In summary, the distributional divergence of $\boldsymbol{\beta}^V$ vanishes. Since $\boldsymbol{\beta}^V$ is supported only over the macropatch around V, it is possible to construct $\boldsymbol{\varrho}_h^V \in \mathbf{L}^2(\Omega)$ with support in the local macropatch around V that is piecewise in the Raviart-Thomas space of lowest order and that satisfies $\operatorname{curl} \boldsymbol{\varrho}_h^V = \boldsymbol{\beta}^V$. Summing over V, we obtain $\boldsymbol{\varrho}_h$ as desired. This finishes our remark on the localized flux reconstruction.

XI. Conclusions and Perspectives

This thesis has addressed the theoretical foundations of finite element exterior calculus. We have contributed to the understanding of basis constructions, we have clarified and extended the applications of smoothed projections, and we have made important progress in the theory of a posteriori error estimation. These mathematical investigations open perspectives for future research.

We have commenced our mathematical investigations with one of the most basic concepts of mathematics, namely simplices and triangulations, in Chapter II. We have approached how to quantify the regularity of simplices and simplicial triangulations. The purpose of this exposition was providing rigorous and explicit proofs for several mathematical results which are usually treated as mathematical folklore. An interesting qualitative observation is that global properties of the triangulated domain enter local estimates (see, e.g., Remark II.4.7). The quantitative bounds in Chapter II, however, are generally far from sharp, and more technical effort may produce more precise results. In addition, similar tracks of research emerged in different areas of geometry [50, 78, 133, 180], and connecting these developments may lead to results interesting to a broader mathematical audience.

In Chapter III we have outlined finite element spaces of differential forms with particular attention to the construction of geometrically decomposed bases for the $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ families of finite element spaces. We have elaborated several contributions in the literature that have not found as much attention yet, and we have also provided some new results. This includes a new presentation of geometrically decomposed bases, and we have given a detailed analysis of the two isomorphic pairs

$$\mathcal{P}_r\Lambda^k(T) \simeq \mathring{\mathcal{P}}_{r+n-k+1}^-\Lambda^{n-k}(T), \quad \mathring{\mathcal{P}}_{r+n-k+1}\Lambda^k(T) \simeq \mathcal{P}_r^-\Lambda^{n-k}(T)$$

in finite element exterior calculus. This has enabled the identification of linear dependencies in the canonical spanning sets and has produced explicit formulas for the canonical duality pairings. Here, our major point of reference has been a recent publication by Christiansen and Rapetti [57].

Future work could integrate ideas of Chapter III into a fully self-contained exposition that can be used in introductions to finite element exterior calculus. Moreover, the techniques of finite element exterior calculus can potentially contribute to the already considerable efforts of research in higher order finite element methods, which have addressed properties of finite element bases, such as sparsity, hierarchical structure, condition numbers of finite element matrices, or fast evaluation of finite element matrices (see, e.g., [20, 21, 22, 23, 24, 116, 119, 120, 136, 161]).

We have then proceeded from finite element spaces on single simplices to finite element de Rham complexes over entire triangulations. Combining ideas from different sources ([9, 11, 56, 69, 183]), we have set up the theory in a new manner that emphasized the locality of the higher order parts of finite element spaces. The resulting framework formalizes finite element de Rham complexes of non-uniform polynomial order in finite element exterior calculus.

A short-coming of our exposition is that our representation of the degrees of freedom involves generally non-canonical Riemannian metrics, whereas the degrees of freedom of finite element spaces of uniform polynomial order have well-known metric independent descriptions [9]. This open problem might be subject to future research but does not affect our applications in this work.

Finite element de Rham complexes with non-uniform polynomial order have been only of secondary interest in this thesis: their theory has emerged naturally from our way of constructing the finite element de Rham complexes, motivated by preparing the partially localized flux reconstruction in Chapter X. Of course, finite element spaces of non-uniform polynomial order are an active topic of research in their own right and constitutive for *hp*-adaptive finite element methods. This thesis prepares an access for research on mixed *hp*-adaptive finite element methods in finite element exterior calculus.

We have invested considerable effort in a detailed exposition of the smoothed projection in finite element exterior calculus and its applications to a priori error estimates in Chapters V, VI, VII, and VIII. On the one hand, this has extended finite element exterior calculus to a broader class of domains and boundary conditions. On the other hand, our detailed calculations have revealed some interesting qualitative relations (see, e.g., Remark VII.8.3 or Remark VII.8.13), and have pointed out non-trivial gaps (and small mistakes) in the existing literature (see Remarks VII.8.12 and VII.8.9). As a remedy for the latter, we have introduced new mathematical techniques into numerical analysis based on Lipschitz topology and geometric measure theory, which may be helpful in future research.

The construction of our smoothed projection has followed the line of thought of previous publications (which is why we call it a *smoothed projection*) but there have been considerable modifications. We have implemented the first step of extending a differential form by reflection across the boundary with a result from Lipschitz topology that seems to be a new tool for numerical analysis. We have accommodated the possibility of partial boundary conditions with a bi-Lipschitz deformation, again with reference to Lipschitz topology. The extended differential form is then mollified with a generalization of the classical convolution by a smooth mollifier that allows us to locally control the mollification radius.

The next step towards the smoothed projection has instantiated this smoothing operator with a function that indicates the local mesh size. Here we have identified a small mistake in the literature, which, however, has also pointed out a hitherto overlooked qualitative property of smoothing operators (see Remark VII.8.12). The final stage in the construction of the smoothing operator has employed the Schöberl trick to bound the interpolation error over the finite element spaces. During the course of this research, a gap in the proof of Lemma 5.5 of [9] was identified (which applies similarly to Lemma 4.2 in [58]). This has been our driving motivation for using geometric measure theory in numerical analysis.

Since the construction and analysis of the smoothed projection involves coordinate transformations of low regularity, we have related our entire discussion of the smoothed projection with research on analysis on "rough" spaces. This includes Lipschitz topology and geometric measure theory, and we have pointed out the class of weakly Lipschitz domains as a natural choice of geometric ambient in the theory of finite element methods. Furthermore, we have related the theory of finite element exterior calculus with the analysis of $W^{p,q}$ differential forms on Lipschitz manifolds. Our analysis of the Hodge Laplace equation with mixed boundary conditions is based on recent contributions in global analysis [99], and we have emphasized the role of de Rham complexes with partial boundary conditions.

There are several interesting perspectives for future research on smoothed projections. One direction is to further generalize the admissible geometric background. In this thesis we have only considered domains in $\mathbb{R}^n$, but the numerical analysis of partial differential equations on manifolds is an active topic of research [60, 71, 80]. Finite element exterior calculus uses the language of differential geometry and thus admits a natural background for such research. In the opposite direction of research, local additional regularity is used, for example, in the design of a priori hp-adaptive methods [49, 162, 163, 164], and even in the case of scalar-valued problems, this is still an area of active research with numerous open problems.

The smoothed projection can be seen as a generalization of the classical smoothing operator on $\mathbb{R}^n$, which is defined by convolution with the classical mollifier. Apart from numerical analysis, this idea has emerged in the global analysis on manifolds [66, 97, 100, 101]. Relating the developments in both areas is an intriguing research perspective, and one result of this thesis can be regarded as a step towards that direction: we construct a commuting mollification operator over weakly Lipschitz domains from $W^{p,q}$ de Rham complexes into the complex of smooth differential forms (see Theorem VII.4.1). In particular, this smoothing operator preserves partial boundary conditions. As an application, we have proven a result of purely analytical interest: we have shown the density of smooth differential forms in the $W^{p,q}$ classes of differential forms over weakly Lipschitz domains with partial boundary conditions.

We do not leave unmentioned that a considerably more detailed study of smoothing operators is feasible, as demonstrated by Karkulik and Melenk [117]. Moreover, whereas explicit calculations have been a guideline in the construction of the smoothed projection, we have used abstract existence results at several points when fixing Lipschitz collars. It seems plausible that explicit constructions of Lipschitz collars are possible for polyhedral domains.

We have concluded our research on smoothed projections with a priori error estimates for the Hodge Laplace equation with mixed boundary conditions. We have omitted eigenvalue problems, which are another straightforward application of the Galerkin theory of Hilbert complexes. With regards to partial boundary conditions, two particularly noteworthy topics justify further research. First, not many Gaffney-type inequalities are available in the case of partial boundary conditions, besides a general $H^{\frac{1}{2}}$ estimate ([113]). Second, the approximation of harmonic forms has attracted special attention in numerical analysis [70] and we have seen that non-trivial harmonic forms appear in the presence of partial boundary conditions, even if the domain itself has a simple topology.

After a priori error estimates we have addressed a posteriori error estimation in finite element exterior calculus. The classical residual error estimator was treated by Demlow and Hirani [72]. We have directed our interest instead towards equilibrated a posteriori error estimation [4, 156, 172] and focused on Braess and Schöberl's equilibrated a posteriori error estimator for edge elements [34].

In Chapter IX, we have investigated differential complexes of discrete distributional differential forms. These generalize the distributional finite element sequences of Braess and Schöberl [34]. During the development of this PhD thesis the homology theory of discrete distributional differential forms was completed and the Poincaré-Friedrichs inequalities were successfully analyzed. An aspect that deserves further attention are duality relations between discrete distributional differential forms and conforming finite element spaces.

The final chapter has approached the seminal contribution of Braess and Schöberl from a different perspective. Here we have introduced partially localized flux reconstructions, which build upon the principle that has already been central to Chapter IV: the global properties of the finite element space are encoded in the lowest-order part, whereas the higher-order part is localized. In Chapter X we have reduced the problem of flux reconstruction between higher-order finite element spaces to flux reconstruction between lowest-order finite element spaces, using only local computations. This has extended Braess and Schöberl's equilibrated residual error estimator to the case of edge elements of higher and possibly non-uniform polynomial order. This opens several possibilities for future research in computational science, even though many basic questions still remain. For example, not many computational studies of this error estimator are presently available in the literature. An interesting question is how to generalize the results of [34] from the case of edge elements in two and three dimensions to the full framework of finite element exterior calculus.

This thesis has investigated the foundations of finite element exterior calculus. We have contributed several extensions to the theoretical framework, which have already stimulated successive research activities. The elaboration of technical details has provided new qualitative insights and has driven the development of new techniques for future research in numerical analysis.

Revisiting the foundations of a mathematical theory can be demanding as much as it can be rewarding. The great book of mathematics is constantly being rewritten and annotated. I hope that this work will be useful both to present and future mathematicians, and that it will encourage other researchers to contribute their ideas to the greater good.

A. Appendix

This appendix outlines notational conventions, definitions, and results that are assumed to be known throughout this thesis and that may be used without any further explanation.

Basic Conventions and Combinatorics

We follow the tradition of Dedekind and Piano and let $\mathbb{N}$ denote the set of natural numbers, i.e. the positive integers, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We let $\mathbb{Z}$ denote the set of integers and let $\mathbb{R}$ denote the set of real numbers. We write $\mathbb{R}^+$ for the positive real numbers and $\mathbb{R}_0^+$ for the non-negative real numbers. For every set A we let $\# A$ denote its cardinality.

For every real number $s \in \mathbb{R}$ we let $\lceil s \rceil \in \mathbb{Z}$ denote the smallest integer that is not smaller than s. Moreover, we define

$$\operatorname{sgn}(s) = \begin{cases} -1 & \text{if } s < 0, \\ 0 & \text{if } s = 0, \\ 1 & \text{if } s > 0. \end{cases}$$

The Kronecker delta δ_{ij} for $i, j \in \mathbb{Z}$ is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{A.1}$$

We let $[a : b] = \{a, \dots, b\}$ for $a, b \in \mathbb{Z}$. Note that $[a : b] = \emptyset$ if $b < a$. We let $\operatorname{Perm}(a : b)$ denote the group of permutations acting on the set $[a : b]$. The signum of $\pi \in \operatorname{Perm}(a : b)$ is written $\operatorname{sgn}(\pi)$.

Given integers $m, n \in \mathbb{Z}$ with $m \leq n$, we let $A(m : n)$ be the set of functions from $[m : n]$ to $\mathbb{N}_0$. The members of $A(m : n)$ are called *multiindices* over $[m : n]$. The absolute value of $\alpha \in A(m : n)$ is defined as $|\alpha| := \alpha(m) + \dots + \alpha(n)$. We let $A(r, m : n)$ be the set of all multiindices over the index set $[m : n]$ with absolute value $r \in \mathbb{Z}$. We may abbreviate $A(r, n) := A(r, 0 : n)$. Whenever $\alpha \in A(m : n)$, we write

$$[\alpha] := \{\, i \in [m : n] \mid \alpha(i) > 0 \,\}, \tag{A.2}$$

and we write $\lfloor \alpha \rfloor$ for the minimal element of $[\alpha]$, provided that $[\alpha]$ is not empty. The sum $\alpha + \beta$ of multiindices $\alpha, \beta \in A(m : n)$ is defined in the obvious manner. If $\alpha \in A(m : n)$ and $p \in [m : n]$, then we let $\alpha + p \in A(m : n)$ be identical to α except for $(\alpha + p)(p) = \alpha(p) + 1$, and if $q \in [\alpha]$ then we let $\alpha - q \in A(m : n)$ be identical to α except for $(\alpha - q)(q) = \alpha(q) - 1$.

For $a, b, c, d \in \mathbb{Z}$ we let $\Sigma(a : b, c : d)$ be the set of strictly ascending mappings from $[a : b]$ to $[c : d]$. If $a > b$, then this set contains only the empty function $\emptyset$, and hence $\Sigma(a : b, c : d) := \{\emptyset\}$ in that case. Whenever $\sigma \in \Sigma(a : b, c : d)$, we write

$$[\sigma] := \{\sigma(i) \mid i \in [a : b]\}, \tag{A.3}$$

and we write $\lfloor\sigma\rfloor$ for the minimal element of $[\sigma]$, provided that $[\sigma]$ is not empty.

Furthermore, if $q \in [c : d] \setminus [\sigma]$, then we write $\sigma + q$ for the unique element of $\Sigma(a : b + 1, c : d)$ with image $[\sigma] \cup \{q\}$. In that case, we also write $\epsilon(q, \sigma)$ for the signum of the permutation that brings the sequence $q, \sigma(a), \dots, \sigma(b)$ into ascending order, and we write $\epsilon(\sigma, q)$ for the signum of the permutation that brings the sequence $\sigma(a), \dots, \sigma(b), q$ into ascending order. Thus

$$\epsilon(q, \sigma) = (-1)^{\#\{\, p \in [\sigma] \,\mid\, q > p \,\}}, \quad \epsilon(\sigma, q) = (-1)^{\#\{\, p \in [\sigma] \,\mid\, q < p \,\}}.$$

Conversely, if $p \in [\sigma]$, then we write $\sigma - p$ for the unique element of $\Sigma(a : b-1, c : d)$ with image $[\sigma] \setminus \{p\}$.

When $\sigma \in \Sigma(1 : k, 0 : n)$ and $\rho \in \Sigma(0 : l, 0 : n)$ with $[\sigma] \cap [\rho] = \emptyset$, then we let

$$\sigma + \rho \in \Sigma(0 : k + l, 0 : n)$$

be the unique strictly ascending from $[0 : k + l]$ to $[0 : n]$ with image $[\sigma] \cup [\rho]$, and we let $\epsilon(\sigma, \rho)$ denote the signum of the permutation that orders the sequence $\sigma(1), \dots, \sigma(k)$, $\rho(0), \dots, \rho(l)$ in ascending order. In particular,

$$\epsilon(\sigma, \rho) = (-1)^{\#\{\, (p,q) \in [\sigma] \times [\rho] \,\mid\, q < p \,\}}.$$

If n is understood and $k, l \in [0 : n]$, then for any $\sigma \in \Sigma(1 : k, 0 : n)$ we define $\sigma^c \in \Sigma(0 : n - k, 0 : n)$ by the condition $[\sigma] \cup [\sigma^c] = [0 : n]$, and for any $\rho \in \Sigma(0 : l, 0 : n)$ we define $\rho^c \in \Sigma(1 : n - l, 0 : n)$ by the condition $[\rho] \cup [\rho^c] = [0 : n]$. In particular, $\sigma^{cc} = \sigma$ and $\rho^{cc} = \rho$. We emphasize that σ^c and ρ^c depend on n, which we choose to suppress in the notation.

Above we have introduced symbols for several signs that appear in combinatorial calculations. For $p, q \in \mathbb{Z}$ with $p \neq q$ we additionally introduce $\epsilon(p, q) := 1$ if $p < q$ and $\epsilon(p, q) = -1$ if $q < p$. Obviously we have $\epsilon(p, q) = -\epsilon(q, p)$ for $p, q \in \mathbb{Z}$ with $p \neq q$. With some combinatorial insight it is easily verified that

$$\epsilon(q, \sigma - p) = \epsilon(q, p)\epsilon(q, \sigma), \tag{A.4}$$

for $\sigma \in \Sigma(a : b, c : d)$, $p \in [\sigma]$ and $q \notin [\sigma]$, and that

$$\epsilon(p, \sigma + q - p) = \epsilon(p, q)\epsilon(p, \sigma - p) \tag{A.5}$$

for $\sigma \in \Sigma(a : b, c : d)$, $p \in [\sigma]$, and $q \notin [\sigma + p]$.

Remark A.0.1.
The notion of multiindex is commonplace, while the definition of $\Sigma(a : b, c : d)$ is not. The latter notion is a minor generalization of the sets $\Sigma(k, n)$ and $\Sigma_0(k, n)$ in several publications on finite element exterior calculus (e.g. [10, 11]). There does not seem to be an established name for them in natural languages. The author proposes *alternating index* as a spoken term.

Notions of Linear Algebra

We briefly summarize basic notions and notation from the fields of linear algebra, metric spaces, and analysis.

All vector spaces in this thesis are over the real numbers. If X is a vector space and A is a linear subspace, then we let X/A or $\frac{X}{A}$ denote the quotient space obtained from X by factoring out A. We let $\mathbb{R}^n$ be the canonical n-dimensional real vector space. If $p \in [1, \infty]$ and $x \in \mathbb{R}^n$ with entries $(x_i)_{1 \leq i \leq n}$, then the *p-norm* is given by

$$\|x\|_p := (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and by

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

for $p = \infty$.

If $T : X \to Y$ is a linear mapping from a vector space X into another vector space Y, then we let $\ker T \subseteq X$ denote the kernel of T and let $\operatorname{ran} T \subseteq Y$ denote the range of T.

Let $M \in \mathbb{R}^{n \times m}$ be a matrix with entries $(M_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$. If $m = n$ and M is invertible, then M^{-1} denotes the inverse of M. In any case, we let $M^\dagger \in \mathbb{R}^{m \times n}$ denote the Moore-Penrose pseudoinverse of M. For $p, q \in [1, \infty]$ the *operator norm* $\|M\|_{p,q}$ of M is given by

$$\|M\|_{p,q} := \sup_{x \in \mathbb{R}^m \setminus \{0\}} \frac{\|Mx\|_q}{\|x\|_p}.$$

Assume that $m \leq n$. We let the non-negative scalars $\sigma_1(M), \dots, \sigma_m(M)$ denote the *singular values* of M in ascending order. We also write $\sigma_{\min}(M) = \sigma_1(M)$ and $\sigma_{\max}(M) = \sigma_m(M)$ for the smallest and the largest singular value of M, respectively. If the singular values of M are all positive, then

$$\|M\|_{2,2} = \sigma_{\max}(M), \quad \|M^\dagger\|_{2,2} = \sigma_{\min}(M)^{-1}.$$

The *generalized condition number* $\kappa(M)$ of M is the quantity

$$\kappa(M) := \|M\|_{2,2} \|M^\dagger\|_{2,2}.$$

If $\sigma_{\min}(M) > 0$, then it can be expressed equivalently as $\kappa(M) = \sigma_{\max}(M)/\sigma_{\min}(M)$.

The determinant $\det(M)$ of a square matrix $M \in \mathbb{R}^{n \times n}$ can be estimated by

$$\det(M) \leq \prod_{i=1}^n \|M_i\|,$$

known as *Hadamard's inequality*, where $M_1, M_2, \dots, M_n$ denote the columns of M.

Notions of Metric Spaces

Whenever X is a topological space, taken to be understood, and $U \subseteq X$ is a subset, then $\overline{U}$ denotes the closure of U and U^c denotes the complement of U in the topological space X.

Throughout this thesis, and unless stated otherwise, we let finite-dimensional real vector spaces $\mathbb{R}^n$ and their subsets be equipped with the canonical Euclidean metric. We let $B_r(U)$ be the closed Euclidean r-neighborhood, $r > 0$, of any set $U \subseteq \mathbb{R}^n$, and we write $B_r(x) := B_r(\{x\})$ for the closed Euclidean ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$.

More generally, suppose that $\rho : U \to \mathbb{R}$ is a function over some set $U \subseteq \mathbb{R}^n$. For any subset $A \subseteq U$ we then write $\rho_{\inf}(A)$ and $\rho_{\sup}(A)$ for the infimum and the supremum, respectively, of ρ over A whenever these exist, and we write $\rho_{\min}(A)$ and $\rho_{\max}(A)$ for the minimum and the maximum, respectively, of ρ over A whenever these exist.

Suppose that $v_0, \dots, v_N \in \mathbb{R}^n$. We define the *convex hull* by

$$\operatorname{convex}\{v_0, \dots, v_N\} = \left\{ \sum_{i=0}^{N} a_i v_i \;\middle|\; a_0, \dots, a_N \in [0,1],\ \sum_{i=0}^{N} a_i = 1 \right\}.$$

The following important result is also known as Lebesgue's number lemma.

Lemma A.0.2.
Let $U \subseteq \mathbb{R}^n$ be compact and let $U_1, \dots, U_m$ be a finite covering of U by sets that are relatively open in U. Then there exists $\gamma > 0$ such that for all $x \in U$ there exists $1 \leq i \leq m$ satisfying $B_\gamma(x) \cap U \subseteq U_i$.

Notions of Analysis

If $m, n \in \mathbb{N}$ and $U \subseteq \mathbb{R}^n$ is an open set, and if $u : U \to \mathbb{R}^m$ is a differentiable function, then we let $\partial_1 u, \dots, \partial_n u$ denote the partial derivatives of u into the coordinate directions. We let $\mathrm{D}\, u : U \to \mathbb{R}^{m \times n}$ denote the Jacobian of u over U. If $\alpha \in A(1:n)$ is a multiindex in n variables, then we write

$$\partial^\alpha u := \partial_1^{\alpha(1)} \cdots \partial_n^{\alpha(n)} u$$

for the corresponding higher order derivative of u. The same notation is applied when the derivative only exist in the weak sense or in the sense of distributions.

The *standard mollifier* is the function

$$\mu : \mathbb{R}^n \to [0,1], \quad y \mapsto \begin{cases} C \exp\left(\frac{1}{\|y\|^2 - 1}\right) & \text{if} \quad \|y\| < 1, \\ 0 & \text{if} \quad \|y\| \geq 1, \end{cases}$$

where $C > 0$ is chosen such that μ has unit integral. The function μ is smooth and is supported in $B_1(0)$. For $\epsilon > 0$ we define the *scaled mollifiers*

$$\mu_\epsilon : \mathbb{R}^n \to [0,1], \quad y \mapsto \frac{1}{\epsilon^n}\mu\left(\frac{y}{\epsilon}\right).$$

In particular, $\mu = \mu_1$.

Differential Complexes

Suppose that $V = (V_i)_{i\in\mathbb{Z}}$ is a family of real vector spaces indexed over the integers and that $\partial = (\partial_i)_{i\in\mathbb{Z}}$ is a family of linear operators $\partial_i : V_i \to V_{i+1}$ indexed over the integers such that for all $i \in \mathbb{Z}$ we have $\partial_{i+1}\partial_i = 0$. Then the tuple (V, ∂) is called a *differential complex*. A differential complex can be written as a diagram

$$\dots \xrightarrow{\partial_{i-1}} V_i \xrightarrow{\partial_i} V_{i+1} \xrightarrow{\partial_{i+1}} \dots$$

We have the inclusion $\operatorname{ran}\partial_{i-1} \subseteq \ker\partial_i$, and the homology spaces can be seen as a measure in how far this inclusion is proper. The *i-th homology space* of (V, ∂) is defined as the factor space

$$\mathcal{H}_i := \frac{\ker\partial_i}{\operatorname{ran}\partial_{i-1}}.$$

We say that (V, ∂) is *exact at index i* if $\mathcal{H}_i = \{0\}$, and we say that (V, ∂) is *exact* if $\mathcal{H}_i = \{0\}$ for all $i \in \mathbb{Z}$.

Sometimes we use the notion of differential complex with different index conventions, where the indices of successive operators are not ascending but descending. Moreover, most differential complexes in this thesis have only finitely many non-zero terms, and we often display only those non-zero terms.

Remark A.0.3.
The terms *homology* and *cohomology* generally designate different concepts in homological algebra, but for the purposes of this thesis they are used interchangeably. Which of the terms we use depends on terminological conventions.

Alphabetical Index

List of Symbols

List of symbols in order of appearance.

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