Delta-Trigonometric and Spline-Trigonometric Methods using the Single-Layer Potential Representation

by

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ABSTRACT

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We study several numerical methods for solving the plane Dirichlet problem using a single-layer potential representation. We introduce the delta-trigonometric Petrov-Galerkin method by extending Arnold's spline-trigonometric Petrov-Galerkin method. In other words, we use summations of delta functions instead of splines as trial functions. For this new method, we extend his proof of exponential convergence of the approximate potentials on compact sets disjoint from the boundary and global algebraic convergence in a weighted Sobolev norm. We also show that the same types of convergence still hold when appropriate quadrature rules are used to compute the matrices involved. Next, we investigate an analogous method where the single-layer potential is placed on a fictitious boundary, that is, a closed curve which properly encloses the true domain. For circular domains, this method achieves exponential convergence of the approximate potentials on the entire interior domain and the boundary, even if quadrature rules are used. We conjecture that exponential convergence of the approximate potentials is obtained on general smooth domains with analytic boundaries. Finally, we discuss our implementation of these methods in the program SPLTRG which uses the fast Fourier transform to compute the discretization matrices, and using SPLTRG we compute various cases in order to confirm our theories and conjectures, and to examine the behaviors of the methods in cases where the theory doesn't apply due to lack of smoothness.

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1. Introduction

We study the numerical methods for solving the Dirichlet problem,

$$\Delta u = 0 \quad \text{on} \quad \mathbb{R}^2 \backslash \Gamma, \qquad u = g_* \quad \text{on} \quad \Gamma, \tag{1.1}$$

based on a single-layer potential representation where Γ is a simple closed analytic curve, g is an analytic function, and u is bounded at infinity. The single-layer potential representation is:

$$u(z) = \int_\Gamma \Phi(y) \log|z-y| \; d\sigma_y \quad \text{for} \quad z \in \mathbb{R}^2, \tag{1.2}$$

where Φ is the density. For any harmonic u, there exists a unique Φ satisfying the representation (1.2) if the conformal radius of Γ does not equal 1 (see appendix). The density Φ solves the boundary integral equation,

$$g_*(z) = \int_\Gamma \Phi(y) \log|z-y| \; d\sigma_y \quad \forall \; z \in \Gamma. \tag{1.3}$$

We consider several numerical methods to approximate the potential in equation (1.1) based on the representation (1.2). In these methods, Φ is approximated using equation (1.3) by an approximate density selected from a finite-dimensional space of trial functions on Γ. Then the potential is approximated by using the approximate density instead of Φ in equation (1.2). Such a method is specified by choosing (1) the spaces of trial functions and (2) the procedures to select the trial function. These methods usually require integrations over Γ and therefore we also study the effects of numerical integrations.

Two common choices of trial spaces are spline spaces and spaces of trigonometric polynomials. We also consider approximating the density by a summation of delta functions which we will call a spline of degree -1. In this case, the approximate potential is:

$$u_n(z) = \sum_{j=1}^{n} \alpha_j \log|z-y_j| \quad \text{for} \quad z \in \mathbb{R}^2, \tag{1.4}$$

where the y_j's are given points on the boundary and the α_j's are the unknown coefficients. An advantage of using a summation of delta functions instead of an ordinary spline function is that fewer numerical integrations are needed. For instance, if we perform the collocation method on equation (1.3) then we require no numerical integration instead of one numerical integration per matrix element. If we perform the Galerkin method on equation (1.3) then we require single numerical integration instead of double numerical integration per matrix element. Also, the approximate potential in equation (1.4) does not require any further approximation by quadrature rule after the trial function is found.

The most common numerical schemes to select the approximate density are collocation methods, least square methods, and Petrov-Galerkin methods. Spline-collocation methods (splines as trial functions and collocation of the boundary integral equation (1.3)) are known to give the optimal asymptotic convergence rates in certain Sobolev spaces, i.e.,

$$||\Phi - \Phi_n||_{H^t(\Gamma)} \leq Cn^{-s+t}||\Phi||_{H^s(\Gamma)} \tag{1.5}$$

for all $-1 \leq t \leq s \leq d+1$, $t < d+1/2$, and $d/2 \leq s$ where Φ_n is the approximate density due to n subintervals and d is the degree of the splines [3, 25 pg. 5, 28]. The approximate potential u_n satisfies:

$$||u - u_n||_{L^\infty(\Omega_K)} \leq Cn^{-s-1}||\Phi||_{H^s(\Gamma)}$$

for all $-d-1 \leq s \leq d+1$ where Ω_K is a compact set disjoint from Γ [25 pg. 18]. The optimal asymptotic convergence rates are also achieved for elliptic equations of other orders. For more details, see Arnold and Wendland [3, 4], Saranen and Wendland [28], Prossdorf and Schmidt [23, 24], Prossdorf and Rathsfeld [21, 22], and Schmidt [29].

The spline-spline Galerkin method obtains the same convergence rates as the spline-collocation method except with a lesser regularity requirement, i.e., equation (1.5) holds for $-d-2 \leq t \leq s \leq d+1$ and $t < d+1/2$. However, it is more costly to evaluate the double integrals numerically. For more details, see [14, 15, 25 pg. 5].

Ruotsalainen and Saranen [25] proved that the delta-spline Petrov-Galerkin method (summations of delta functions as trial functions and splines as test functions) achieves the optimal asymptotic convergence rates, i.e.,

$$||\Phi - \Phi_n||_{H^t(\Gamma)} \leq Cn^{-s+t}||\Phi||_{H^s(\Gamma)}$$

for all $-d^*-2 \leq t \leq s \leq 0$, $t < -1/2$, and $-d^*/2 - 1 \leq s$ where d^* is the degree of the splines [25 pg. 15]. The approximate potential u_n satisfies:

$$||u - u_n||_{L^\infty(\Omega_K)} \leq Cn^{-s-d^*-2}||\Phi||_{H^s(\Gamma)}$$

for all $-d^*/2 - 1 \leq s \leq 0$ where Ω_K is a compact set disjoint from Γ [25 pg. 16]. The advantages of their method compared to the spline-spline methods or the splines-collocation methods are that fewer numerical integrations are needed and a lesser regularity is required of the boundary data. Numerical results were presented by Lusikka, Ruotsalainen, and Saranen [18].

Arnold [2] showed that the approximate potentials produced by the spline-trigonometric method (splines as trial functions and trigonometric polynomials as test functions) converge (in the L^∞ norm) exponentially on compact sets disjoint from Γ and algebraically up to the boundary. He also showed that the condition numbers of the matrices produced by his method are linearly proportional to the numbers of subintervals. McLean [20] showed that the approximate potentials produced by the trigonometric-trigonometric Galerkin method converge exponentially in $L^\infty(\mathbb{R}^2)$. Neither Arnold nor McLean took into account the effect of quadrature errors which would occur on the computer.

In this paper, we show that the approximate potentials produced by the delta-trigonometric Petrov-Galerkin method (summations of delta functions as trial functions and trigonometric polynomials as test functions) converge (in the L^∞ norm) exponentially on compact sets disjoint from the boundary and algebraically in a weighted Sobolev norm. Then we show that the convergence rates do not change when we use the appropriate quadrature rules. This is significant since now we have a fully discretized method using the single-layer potential representation (1.2) which approximates the potential exponentially. We also show that the condition numbers of the matrices produced by the delta-trigonometric method without quadrature rules are bounded proportionally to the numbers of subintervals. Finally we present computer results which confirm our theoretical analyses. We also show results in which the approximate potentials produced by the spline-trigonometric method with numerical quadrature do not converge exponentially. The reason for this phenomenon is that the spline-trigonometric method involves numerical integrations of non-analytic splines in (1.3) while the delta-trigonometric method avoids numerical integrations of (1.3).

We also study the case where the single-layer potential is placed on a fictitious boundary Γ_o to solve the *interior* Dirichlet problem. Let Ω and Ω_o be simple open bounded domains with boundaries Γ and Γ_o, respectively, such that $\overline{\Omega}$ is strictly contained in Ω_o. We approximate the potential as:

$$u(z) \cong v(z) := \int_{\Gamma_o} \Phi(y) \log|z - y| \; d\sigma_y \quad \text{for} \quad z \in \Omega, \tag{1.6}$$

where Φ is a (fictitious density) function defined on the fictitious boundary. In general, given a harmonic u there does not exist a Φ such that equation (1.6) is exact. However if we set the condition: Γ_o is such that $|y-z| \neq 1$ for all $z \in \overline{\Omega}$ and $y \in \Gamma_o$, then we can find a Φ such that $||u-v||_{L^2(\Omega)}$ is arbitrary small. Consequently, this condition implies that the set

$$\{v \mid v(z) = \int_{\Gamma_o} \Phi(y) \log|z-y| \; d\sigma_y \quad \text{for} \quad z \in \Omega, \quad \Phi \in C^\infty(\Gamma_o)\}$$

is dense in the set

$$\{u \in H^s(\Omega) \mid \Delta u = 0 \text{ in } \Omega\}$$

for all $s \in \mathbb{R}$ [5, theorem 2.1].

Again we have several choices of (1) the finite-dimensional trial spaces and (2) the procedures to select the trial function. The most interesting trial space is the span of delta functions. The resulting method is called the fundamental solution method (e.g. Bogomolny [5], Fairweather and Johnston [11], Mathon and Johnston [19], Kupradze and Aleksidze [16], Freeden and Kersten [12]), i.e.,

$$u_n(z) = \sum_{j=1}^{n} \alpha_j \log|z-y_j| \quad \text{for} \quad z \in \Omega, \tag{1.7}$$

where the y_j's are points outside of $\overline{\Omega}$ and the α_j's are the unknown coefficients.

Kupradze and Aleksidze [16] showed that the functions,

$$\log|z-y_j|, \qquad j = 1, \ldots, n,$$

are independent and complete in $L^2(\Gamma)$ and $C(\Gamma)$. Therefore for any $\epsilon > 0$ there exists N such that for any $n > N$ there is a u_n of the form (1.7) satisfying

$$||u-u_n||_{L^\infty(\Omega)} < \epsilon.$$

Bogomolny [5] showed that any harmonic polynomial of degree $\leq n$ can be approximated by a u_n of the form (1.7) with an L^∞ error which decreases exponentially as n increases. Then he showed that the exact solution can be approximated by a u_n of the form (1.7) with an L^∞ error which decreases very rapidly as n increases.

Mathon and Johnston [19] showed that there exists a u_n of the form (1.7) which minimizes $||u-u_n||_{L^2(\Omega)}$. They used a least square method to find the coefficients of the delta functions and the locations of the singularities. The main drawback of their program is the nonlinear aspect which arises from allowing the singularities to vary. However, their method works well when u is of low continuity and for the three-dimensional Dirichlet problem. Bogomolny [5] investigated where these singularities should be placed and then used a least square method to find only the coefficients of the delta functions (In this case, the matrices are linear). He obtained theoretical results which suggest that the singularities should be placed far away from the boundary.

In this paper, we examine the delta-trigonometric and spline-trigonometric method using a fictitious boundary. In the special case where Γ and Γ_o are concentric circles, we show that the approximate potentials produced by the delta-trigonometric method converge exponentially, even if quadrature rules are used. We note that the trial functions may not converge even though the associated approximate potentials do.

We also note that the delta-trigonometric and spline-trigonometric methods with trapezoidal quadrature produce the same results as the delta-collocation method (summation of delta functions as trial functions and collocation of the boundary integral equation). Hence, we prove that the approximate potentials produced by the delta-collocation method converge exponentially in circular domains. Since the spline-trigonometric method with trapezoidal quadrature and delta-trigonometric method with trapezoidal quadrature are exactly the same we will provide convergence analysis for the delta-trigonometric method only. However, we present numerical results for both methods. We conjecture that both methods with and without numerical quadrature obtain exponential convergence for the approximate potentials on general smooth domains with analytic boundaries and present computer results which support this conjecture.

The delta-trigonometric and spline-trigonometric methods (with and without numerical quadrature) work quite well if we are seeking the potential on compact sets disjoint from the boundary. To compute the potential on the boundary, better results are obtained using a fictitious boundary. However, note that we have assumed that the boundary and the boundary data are analytic. Obviously this is not true in the real world. G. DeMey [10] investigated the delta-collocation method on a rectangular domain with mixed data (Dirichlet and Neumann data) using a fictitious circular boundary. Using $n = 40$, he obtained relative error of about 1 percent. He did not examine the errors for different n's, but for different circle radii. He found that it was best not to let the fictitious circle be near the corners of the rectangle or to be too far away from the rectangle. No theoretical proof was given.

2. Preliminaries

In this section, we define some of the norms and spaces that are used throughout this paper. First, we define $\mathbb{Z}^+$ to be the set of positive integers and $\mathbb{Z}^*$ to be the set of integers except zero. Next, we define the vector norm

$$||v|| := \sqrt{v_1^2 + \ldots + v_n^2}$$

and the matrix norm

$$||\mathbf{A}|| := \sup_{v \in \mathbb{R}^n} \frac{||\mathbf{A}v||}{||v||}.$$

Then we define the space of trigonometric polynomials with complex coefficients,

$$T := \operatorname{span}\{\exp(2\pi ikt), k \in \mathbb{Z}\}.$$

Any function f in this space can be represented as

$$f(t) = \sum_{k \in \mathbb{Z}} \widehat{f}(k) \exp(2\pi ikt)$$

where

$$\widehat{f}(k) := \int_0^1 f(t) \exp(-2\pi ikt) \ dt$$

are arbitrary complex numbers, all but finitely many zero.

For $f \in T$, $s \in \mathbb{R}$, and $\epsilon > 0$, we define the Fourier norm [2, section 3]

$$||f||_{s,\epsilon} := \sum_{k \in \mathbb{Z}} |\widehat{f}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s}$$

where

$$\underline{k} := \begin{cases} 1, & \text{if } k = 0, \\ 2\pi|k|, & \text{if } k \neq 0, \end{cases}$$

and the corresponding space $X_{s,\epsilon}$ to be the completion of the closure of T with respect to this norm. For further discussions about the properties of this Fourier norm and space see [2, section 3]. We also define the functional space $L(X, Y)$ as the set of bounded linear functions which map from X to Y. Finally, we say f_n is $O(n^m)$ if for all n there exists a constant C such that $|f_n| \leq Cn^m$.

3. Delta-Trigonometric and Spline-Trigonometric Methods

In this section, we define several operators, the trial spaces, and the test spaces. Next, we define the delta-trigonometric and spline-trigonometric Petrov-Galerkin methods without numerical quadratures. Then for the delta-trigonometric method, we derive the matrix equations with and without numerical quadratures. In section 3.1, we show exponential error bounds for the approximate potentials (away from the boundary) for both methods without numerical quadrature. In section 3.2, we show that the matrix condition numbers for both methods without numerical quadrature are proportionally bounded by the numbers of subintervals. Then in section 3.3, we derive exponential errors bounds for the approximate potentials (away from the boundary) for the delta-trigonometric method with numerical quadrature. Finally, we present numerical results for both methods in section 3.4.

First we define the transformation,

$$\Phi(x(t))\left|\frac{dx}{dt}(t)\right| = \phi(t) \quad \text{and} \quad g_*(x(t))\left|\frac{dx}{dt}(t)\right| = g(t),$$

where $x : \mathbb{R} \to \Gamma$ is a 1-periodic analytic function which parametrizes Γ and has nonvanishing derivatives. We continue to assume that the conformal radius of Γ is not equal to 1. Next, we define three integral operators in $L(X_{s,\epsilon}, X_{s+1,\epsilon})$. Let

$$A\phi(s) := \int_0^1 \phi(t) \log|x(s) - x(t)| \; dt, \tag{3.1}$$

$$V\phi(s) := \int_0^1 \phi(t) \log|2\sin(\pi(s-t))| \; dt, \tag{3.2}$$

and

$$B\phi(s) := A\phi(s) - V\phi(s) = \int_0^1 \phi(t) K(s,t) \; dt,$$

where $K : \mathbb{R}^2 \to \mathbb{R}$ is a smooth kernel defined by

$$K(s,t) := \begin{cases} \log|\frac{x(s)-x(t)}{2\sin\pi(s-t)}|, & \text{if } s-t \notin \mathbb{Z}, \\ \log|\frac{x'(s)}{2\pi}|, & \text{if } s-t \in \mathbb{Z}. \end{cases} \tag{3.3}$$

Then the single-layer potential representation (1.2) becomes

$$u(z) := \int_0^1 \phi(t) \log|z - x(t)| \; dt \quad \forall \; z \in \mathbb{R}$$

and our boundary integral equation (1.3) becomes

$$A\phi(s) = g(s) \quad \forall \; s \in [0,1].$$

Note that $A\phi = B\phi + V\phi$ where V is the principal part of A and the remainder B has a smooth kernel. The advantage of the splitting is that the Fourier transforms of $V\phi$ can be calculated analytically. This fact will be useful for proving the inf-sup condition for A in the finite-dimensional spaces and for numerical implementations.

REMARK: Christiansen [7] described our formulation as the scaling formulation. The limitation of the scaling formulation is that a unique solution does not necessarily exist when Γ has a conformal radius of 1 (see appendix). Another formulation, which Christiansen called the restriction formulation, works on domains of arbitrary conformal radii.

For the restriction formulation, we define three operators in $L(X_{s,\epsilon}, X_{s+1,\epsilon})$:

$$A_1\phi(s) := \int_0^1 [\phi(t) - \widehat{\phi}(0)] \log|x(s) - x(t)| \; dt - \pi\widehat{\phi}(0), \tag{3.4}$$

$$V_1\phi(s) := \int_0^1 \phi(t)\Big(\log|2\sin(\pi(t-s))| - \pi\Big) \; dt, \tag{3.5}$$

and

$$B_1\phi(s) := A_1\phi - V_1\phi := \int_0^1 [\phi(t) - \widehat{\phi}(0)] K(s,t) \; dt, \tag{3.6}$$

where

$$\widehat{\phi}(0) = \int_0^1 \phi(t) \; dt.$$

The corresponding single-layer potential representation is

$$u(z) := \int_0^1 [\phi(t) - \widehat{\phi}(0)] \log|z - x(t)| \; dt - \pi\widehat{\phi}(0) \quad \forall \; z \in \mathbb{R}$$

and the boundary integral equation is

$$\int_0^1 A_1\phi(s) \; ds = \int_0^1 g(s) \; ds \quad \forall \; s \in [0,1].$$

The theoretical results in the sections 3.1 to 3.3 hold using A_1, B_1, and V_1 instead of A, B, and V with minor modifications. Note that the restriction formulation allows the conformal radius of

Γ to be equal to 1 but requires more terms. Christiansen [7] compared the two formulations using a least square method and preferred the restriction formulation because the condition numbers of the matrices were better. We chose the scaling formulation because of its simplicity and because this formulation relates better to the case where a fictitious boundary is used (We will discuss this later). $\diamondsuit$

Let n be a positive odd number, d be an integer ≥ -1, and

$$\Lambda_n := \Big\{ k \in \mathbb{Z} \;\Big|\; |k| \leq \frac{n-1}{2} \Big\}.$$

For $d \geq -1$, we define the trial space

$$S_n^d := \{\rho \in H^d([0,1]) \;\Big|\; \widehat{\rho}(m) m^{d+1} = \widehat{\rho}(m+n)(-m-n)^{d+1} \quad \forall\; m \in \mathbb{Z}\}.$$

Note that S_n^{-1} is the span of the 1-periodic extension of the delta functions at the points j/n, $j = 1, \ldots, n$. For $d \geq 0$, S_n^d is the space of 1-periodic splines of degree d subordinate to the mesh $\{j/n \;\big|\; j \in \mathbb{Z}\}$ for $d = 1, 3, 5, \ldots$ and to the mesh $\{(j + 1/2)/n \;\big|\; j \in \mathbb{Z}\}$ for $d = 0, 2, 4, \ldots$ [2, section 2]. We also define

$$T_n := \;\mathrm{span}\{\exp(2\pi i k t) \;\Big|\; k \in \Lambda_n\}$$

to be the space of trigonometric polynomials with degree $\leq n$.

REMARK: Let $d = -1$ and $\phi_n(t) = \sum_{j=1}^{n} \alpha_j \delta(t - j/n)$. We wish to confirm (1) ϕ_n is in S_n^{-1} and (2) all functions in S_n^{-1} are of this form.

Note that

$$\begin{aligned} \widehat{\phi}_n(m) &= \int_0^1 \sum_{j=1}^{n} \alpha_j \delta(t - j/n) \exp(2\pi i m t)\; dt \\ &= \sum_{j=1}^{n} \alpha_j \exp(2\pi i m j/n). \end{aligned}$$

for all $m \in \mathbb{Z}$. Also note that

$$\begin{aligned} \widehat{\phi}_n(m + qn) &= \sum_{j=1}^{n} \alpha_j \exp(2\pi i (m + qn) j/n) \\ &= \sum_{j=1}^{n} \alpha_j \exp(2\pi i m j/n) \\ &= \widehat{\phi}_n(m) \end{aligned}$$

for all $m, q \in \mathbb{Z}$. This proves (1).

For (2), note that $\dim S_n^d = n$ and that ϕ_n has n degrees of freedom. Therefore, all functions in S_n^d are of the form of ϕ_n. $\diamondsuit$

We now define our methods without numerical quadratures. We seek $\phi_n \in S_n^d$ such that

$$\int_0^1 A\phi_n(s)\psi(s)\ \ ds = \int_0^1 g(s)\psi(s)\ \ ds \quad \forall\ \ \psi \in T_n. \tag{3.7}$$

Then our approximate potential is

$$u_n(z) := \int_0^1 \phi_n(t) \log|z - x(t)|\ \ dt \quad \forall\ \ z \in \mathbb{R}. \tag{3.8}$$

We call the above procedure the delta-trigonometric Petrov-Galerkin method for $d = -1$ and the spline-trigonometric Petrov-Galerkin method for $d \geq 0$.

REMARK: For the restriction formulation, we seek $\phi_n \in S_n^d$ such that

$$\int_0^1 A_1\phi_n(s)\psi(s)\ \ ds = \int_0^1 g(s)\psi(s)\ \ ds \quad \forall\ \ \psi \in T_n.$$

Then the approximate potential is

$$u_n(z) := \int_0^1 [\phi_n(t) - \widehat{\phi}_n(0)] \log|z - x(t)|\ \ dt - \pi\widehat{\phi}_n(0) \quad \forall\ \ z \in \mathbb{R}.$$

$\diamondsuit$

We now define the matrix equations with and without numerical quadratures for the delta-trigonometric method only. For the remaining part of this section, we assume that $d = -1$.

We represent the approximate density (trial function) as

$$\phi_n(t) = \sum_{j=1}^{n} \alpha_j \delta(t - j/n) \tag{3.9}$$

where α_j's are the unknown coefficients.

We also define the basis for test space T_n as

$$\psi_k(s) := \exp(2\pi iks) \quad \text{for} \quad k \in \Lambda_n.$$

Let

$$\alpha := (\alpha_1, \ldots, \alpha_n)^T, \tag{3.10}$$

$$\mathbf{A}_{kj} := \int_0^1 \log \big|x(s) - x(j/n)\big| \psi_k(s) \;\; ds, \tag{3.11}$$

$$\mathbf{B}_{kj} := \int_0^1 K(s, j/n)\psi_k(s) \;\; ds,$$

$$\mathbf{V}_{kj} := \int_0^1 \log \big|2\sin(\pi(s - j/n))\big| \psi_k(s) \;\; ds,$$

and

$$\mathbf{g}_k := \int_0^1 g(s)\psi_k(s) \;\; ds$$

for all $k \in \Lambda_n$ and $j = 1, \ldots, n$. Also let $\mathbf{A} := (\mathbf{A}_{kj})$, $\mathbf{B} := (\mathbf{B}_{kj})$, $\mathbf{V} := (\mathbf{V}_{kj})$, and $\mathbf{g} := (\mathbf{g}_k)$. Then the matrix form of equation (3.7) is

$$\mathbf{A}\alpha = \mathbf{g}$$

and the approximate potential (given in (3.8)) because

$$u_n(z) = \sum_{j=1}^{n} \alpha_j \log |z - x(j/n)| \;\; dt \quad \forall \;\; z \in \mathbb{R}.$$

Fortunately, $\mathbf{V}_{kj}$ can be calculated exactly. In [2, section 4], Arnold showed that the Fourier transform of

$$G(\theta) := \frac{-1}{\pi} \log |2\sin(\pi\theta)| + 1 \tag{3.12}$$

is

$$\widehat{G}(k) = \frac{1}{\underline{k}} \;\; . \tag{3.13}$$

Therefore,

$$\begin{aligned}
\mathbf{V}_{kj} &= \int_0^1 \log |2\sin(\pi(s - j/n))| \;\; \psi_k(s) \;\; ds \\
&= \int_0^1 \log |2\sin(\pi\theta)| \;\; \psi_k(\theta + j/n) \;\; d\theta \\
&= \int_0^1 \log |2\sin(\pi\theta)| \;\; \psi_k(\theta) \;\; \psi_k(j/n) \;\; d\theta \\
&= \int_0^1 -\pi G(\theta)\psi_k(\theta) \;\; d\theta \;\; \psi_k(j/n) + \pi \int_0^1 \psi_k(\theta) \;\; d\theta \;\; \psi_k(j/n) \\
&= \frac{-\pi}{\underline{k}} \psi_k(j/n) + \pi \int_0^1 \psi_k(\theta) \;\; d\theta \;\; \psi_k(j/n).
\end{aligned}$$

Considering all cases for k, we get

$$\mathbf{V}_{kj} = \begin{cases} \frac{-1}{2|k|}\psi_k(j/n), & \text{if } k \neq 0, \\ 0, & \text{if } k = 0. \end{cases}$$

We now define the matrix equation for the delta-trigonometric method with numerical quadratures. Since the principal terms can be calculated exactly, only $\mathbf{B}$ and $\mathbf{g}$ need numerical quadratures. We assume that the trapezoidal quadrature is used. Define

$$\widetilde{\alpha} := (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n)^T,$$

$$\widetilde{\mathbf{B}}_{kj} := \frac{1}{n}\sum_{l=1}^{n} K(l/n, j/n)\psi_k(l/n),$$

and

$$\widetilde{\mathbf{g}}_k := \frac{1}{n}\sum_{l=1}^{n} g(l/n)\psi_k(l/n)$$

for all $k \in \Lambda_n$ and $j = 1, \ldots, n$. Also let $\widetilde{\mathbf{g}} := (\widetilde{\mathbf{g}}_k)$ and $\widetilde{\mathbf{B}} := (\widetilde{\mathbf{B}}_{kj})$.

The delta-trigonometric method with numerical quadratures is to seek

$$\widetilde{\phi}_n(t) := \sum_{j=1}^{n} \widetilde{\alpha}_j \delta(t - j/n)$$

such that

$$\widetilde{\mathbf{A}}\widetilde{\alpha} := \widetilde{\mathbf{B}}\widetilde{\alpha} + \mathbf{V}\widetilde{\alpha} = \widetilde{\mathbf{g}}.$$

The corresponding approximate potential is

$$\widetilde{u}_n(z) := \sum_{j=1}^{n} \widetilde{\alpha}_j \log|z - x(j/n)| \quad \forall \ z \in \mathbb{R}.$$

3.1 Convergence Analysis without Numerical Quadrature

In this section, we prove convergence for the approximate potentials produced by the delta-trigonometric method. The convergence analyses for the spline-trigonometric method (where $d \geq 0$) was given by Arnold [2, section 4 and 5] using the restriction method. We will extend his analyses to the case $d = -1$ using the scaling formulation. (Recall that the difference between the two formulations is whether the conformal radius of Γ can be 1.) In this section, we continue to assume that Γ is a simple closed analytic curve such that the conformal radius of Γ is not equal to 1. We will show that the operator A satisfies the inf-sup condition in the finite-dimensional spaces. Then we prove exponential convergence rates for the approximate densities using the Fourier norms. Afterward, we derive error bounds for the approximate potentials on compact sets disjoint from the boundary, and at infinity (using weighted Sobolev norms).

Since $\widehat{V\phi}(0)$ is zero whenever ϕ is a constant function, we need an additional term. Let

$$M\phi := \int_0^1 \phi(t)\ dt.$$

The first theorem proves the inf-sup condition for the operator $V_1 = V - \pi M$ (see (3.2) and (3.5)) in the finite-dimensional spaces. Later this fact is used to show the same for the operator A. Then we prove exponential convergence using the projection operator (defined in 3.1.5).

THEOREM 3.1.1 *Let $d \geq d_0 \geq -1$ and $s \leq s_0 < d_0 + 1/2$. Then there exists a constant C depending only on d_0 and s_0 such that*

$$\inf_{0 \neq \rho \in S_n^d} \sup_{0 \neq \sigma \in T_n} \frac{(V_1\rho, \sigma)}{||\rho||_{s,\epsilon} ||\sigma||_{-s-1,\epsilon^{-1}}} \geq C$$

for all $\epsilon \in (0, 1]$ and $n \in \mathbb{Z}^+$.

PROOF:

We first show that there exists a constant C_1 depending only on d_o and s_o such that

$$||\rho||_{s,\epsilon}^2 \leq C_1 \sum_{\rho \in \Lambda_n} |\widehat{\rho}(p)|^2 \epsilon^{2|p|} \underline{p}^{2s} \quad \forall\ \rho \in S_n^d. \tag{3.1.1}$$

Since Arnold [2, lemma 4.2] proved (3.1.1) for $d \geq 0$, it remains to prove (3.1.1) for $d = -1$. Let $\rho \in S_n^{-1} := \{\rho \in H^{-1}([0, 1])|\ \widehat{\rho}(m) = \widehat{\rho}(m + n),\ \forall\ m \in \mathbb{Z}\}$. Then

$$\begin{aligned} ||\rho||_{s,\epsilon}^2 &= \sum_{k \in \mathbb{Z}} |\widehat{\rho}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s} \\ &= \sum_{p \in \Lambda_n} \sum_{m \in \mathbb{Z}} |\widehat{\rho}(p + mn)|^2 \epsilon^{2|p+mn|} (\underline{p + mn})^{2s} \\ &= \sum_{p \in \Lambda_n} |\widehat{\rho}(p)|^2 \epsilon^{2|p|} \underline{p}^{2s} \Big\{ \sum_{m \in \mathbb{Z}} \epsilon^{2|p+mn|-2|p|} \Big(\frac{\underline{p + mn}}{\underline{p}} \Big)^{2s} \Big\}. \end{aligned}$$

Note that $2|p + mn| - 2|p| \geq 0$ and $\epsilon \in (0, 1]$ imply that $\epsilon^{2|p+mn|-2|p|} \leq 1$. In other words,

$$||\rho||_{s,\epsilon}^2 \leq \sum_{p \in \Lambda_n} |\widehat{\rho}(p)|^2 \epsilon^{2|p|} \underline{p}^{2s} \Big\{ \sum_{m \in \mathbb{Z}} \Big(\frac{\underline{p + mn}}{\underline{p}} \Big)^{2s} \Big\}. \tag{3.1.2}$$

It suffices to show that the sum in braces is bounded by a constant depending only on s_o. We consider two cases using the fact that $s \le s_o < -\frac{1}{2}$ and $p \in \Lambda_n$. If $p = 0$ then

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \Big(\frac{\underline{p+mn}}{\underline{p}}\Big)^{2s} &= \sum_{m\in\mathbb{Z}} \underline{mn}^{2s}\\ &\le \sum_{m\in\mathbb{Z}} \underline{mn}^{2s_o}\\ &\le \sum_{m\in\mathbb{Z}} \underline{m}^{2s_o}\\ &\le C_2.\end{aligned}$$

If $p \neq 0$ then we let $p > 0$ without loss of generality. Since $p \in \Lambda_n$ implies $|n/p| > 2$ we derive

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \Big(\frac{\underline{p+mn}}{\underline{p}}\Big)^{2s} &= \sum_{m\in\mathbb{Z}} \Big(\frac{|p+mn|}{|p|}\Big)^{2s}\\ &\le \sum_{m\in\mathbb{Z}} |1+mn/p|^{2s_o}\\ &= \sum_{m=0}^{\infty}(1+mn/p)^{2s_o} + \sum_{m=-\infty}^{-1}(-1-mn/p)^{2s_o}\\ &\le \sum_{m=0}^{\infty}(1+2m)^{2s_o} + \sum_{m=-\infty}^{-1}(-1-2m)^{2s_o}\\ &\le C_3.\end{aligned}$$

Therefore, the braced term in (3.1.2) is bounded. This proves (3.1.1) for $d = -1$.

To finish the theorem simply choose

$$\sigma(x) = -\sum_{k\in\Lambda_n} \overline{\widehat{\rho}(k)}\epsilon^{2|k|}\underline{k}^{2s+1}\exp(-2\pi i k x). \tag{3.1.3}$$

Then

$$||\sigma||^2_{-s-1,\epsilon^{-1}} = \sum_{k\in\Lambda_n} |\widehat{\rho}(k)|^2\epsilon^{2|k|}\underline{k}^{2s}. \tag{3.1.4}$$

Combining (3.12), (3.13), and (3.1.3), we derive

$$\begin{aligned}(V_1\rho,\sigma) &= \int_0^1\int_0^1 \Big\{\log|2\sin(\pi(s-t))| - \pi\Big\}\rho(t)\sigma(s)\ dt\ \ ds\\ &= \int_0^1\int_0^1 \Big\{\log|2\sin(\pi(s-t))| - \pi\Big\}\rho(t)(-1)\sum_{k\in\Lambda_n}\overline{\widehat{\rho}(k)}\epsilon^{2|k|}\underline{k}^{2s+1}\exp(-2\pi i k s)\ dt\ \ ds\\ &= \sum_{k\in\Lambda_n}\overline{\widehat{\rho}(k)}\epsilon^{2|k|}\underline{k}^{2s+1}\int_0^1\rho(t)\int_0^1\Big\{-\log|2\sin(\pi(s-t))| + \pi\Big\}\exp(-2\pi i k s)\ ds\ \ dt\\ &= \sum_{k\in\Lambda_n}\overline{\widehat{\rho}(k)}\epsilon^{2|k|}\underline{k}^{2s+1}\int_0^1\rho(t)\frac{\pi}{\underline{k}}\exp(-2\pi i k t)\ dt\\ &= \pi\sum_{k\in\Lambda_n}|\widehat{\rho}(k)|^2\epsilon^{2|k|}\underline{k}^{2s}.\end{aligned}$$

By (3.1.1) and (3.1.4),

$$
\begin{aligned}
(V_1\rho, \sigma) &= \pi \sqrt{\sum_{k \in \Lambda_n} |\widehat{\rho}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s}} \; ||\sigma||_{-s-1,\epsilon^{-1}} \\
&\geq \pi \sqrt{C_1} ||\rho||_{s,\epsilon} ||\sigma||_{-s-1,\epsilon^{-1}} \\
&\geq C ||\rho||_{s,\epsilon} ||\sigma||_{-s-1,\epsilon^{-1}}.
\end{aligned}
$$

This proves the theorem. Q.E.D.

The next two lemmas concern the exponential decays of the Fourier coefficients of an arbitrary *analytic* function. These results will be useful in showing exponential convergence for the approximate densities and potentials.

LEMMA 3.1.2 *Let f be a 1-periodic analytic function on $\overline{S_\delta}$ where $S_\delta = \{z \in \mathbb{C} \mid |Im(z)| < \delta\}$. Then*

$$
|\widehat{f}(m)| \leq e^{-2\pi\delta|m|} ||f||_{L^\infty(S_\delta)} , \qquad m \in \mathbb{Z}.
$$

PROOF: See P. Henrici [13, section 2.1]. Q.E.D.

LEMMA 3.1.3 *The kernel K defined in (3.3) is a real 1-periodic analytic function in each variable and extends analytically to $S_\delta \times S_\delta$ for some $\delta > 0$. Moreover, there exists constants C and $\epsilon_K \in (0, 1)$ such that*

$$
|\widehat{K}(p, q)| \leq C \epsilon_K^{|p|+|q|}, \qquad p, q \in \mathbb{Z}.
$$

PROOF: This is an easy consequence of lemma 3.1.2. Q.E.D.

By theorem 3.1.3, there exists $\beta > 0$ such that for all n and $\rho \in S_n^d$, there exists $\sigma \in T_n$ satisfying

$$
(A\rho, \sigma) \geq \beta ||\rho||_{s,\epsilon} ||\sigma||_{-s-1,\epsilon^{-1}} - (K\rho, \sigma)
$$

The next theorem states the inf-sup condition for the operator A. Analogous theorems were mentioned by Arnold [2] and Aziz and Kellogg [5]. The proof is similiar to the compactness argument given by Aziz and Kellogg [5] and is omitted.

THEOREM 3.1.4 *Let $d \geq d_0 \geq -1$ and $s \leq s_0 < d_0 + 1/2$. Then for sufficiently large n, there* *exists a constant C depending only on d_0, s_0, and Γ such that*

$$\inf_{0\neq\rho\in S_n^d}\sup_{0\neq\sigma\in T_n}\frac{(A\rho,\sigma)}{\|\rho\|_{s,\epsilon}\|\sigma\|_{-s-1,\epsilon^{-1}}}\geq C$$

REMARK: Note that the constant in the previous theorem blows up as the conformal radius of Γ approaches 1. For a circular domain of radius r, this constant behaves like $1/\log(r)$. $\diamondsuit$

Arnold showed that B_1 (the operator with a smooth kernel using the restriction formulation given in (3.6)) is a compact operator and A_1 (the single-layer operator using the restriction formulation given in (3.4)) is an isomorphism from $X_{s,\epsilon}$ to $X_{s+1,\epsilon}$. With minor modifications, we conclude that $B+\pi M$ is compact and that A is an isomorphism from $X_{s,\epsilon}$ to $X_{s+1,\epsilon}$ (as long as the conformal radius of Γ is not equal to 1). Arnold also stated a theorem which allows us to prove convergence using the projection operator. We will state an analogous theorem for $d\geq -1$ without proof since only minor modifications are needed. For more details, the reader may refer to [2, theorem 4.6 to 4.10].

THEOREM 3.1.5 *There exists a constant N, depending only on d and Γ, such that for all $n\geq N$ and $g\in\bigcup\{X_{s,\epsilon}|s\in\mathbb{R},\epsilon>0\}$ the delta-trigonometric and spline-trigonometric methods (3.7) obtain unique solutions, $\phi_n\in S_n^d$. Moreover, if $s\in(-\infty,d+1/2)$, $\epsilon\in(\epsilon_K,1]$ (ϵ_K is determined in lemma 3.1.3), $g\in X_{s+1,\epsilon}$ and $n\geq N$, then there exists a constant C, depending only on d, ϵ, s, and Γ such that*

$$\|\phi-\phi_n\|_{s,\epsilon}\leq C\inf_{\rho\in S_n^d}\|\phi-\rho\|_{s,\epsilon}\ .$$

For any $\phi\in X_{s,\epsilon}$ we define the function $P_n\phi\in S_n^d$ by

$$(P_n\phi,\psi)=(\phi,\psi)\quad\forall\ \psi\in T_n. \tag{3.1.5}$$

Equivalently, P_n is characterized by the equation,

$$\widehat{P_n\phi}(k)=\widehat{\phi}(k)\quad\forall\ k\in\Lambda_n.$$

We now show convergence using this projection operator. The next theorem states exponential error bounds for the approximate densities.

THEOREM 3.1.6 *Let $s < d+1/2$, $t \in [s, d+1]$, $n \geq N$, $\phi \in H^t$, and $\phi_n \in S_n^d$ where ϕ and ϕ_n are the exact and the approximate densities, respectively. Then for $\epsilon \in (\epsilon_K, 1]$ (ϵ_K is determined in lemma 3.1.3), there exists a constant C depending only on d, ϵ, s, and Γ such that*

$$||\phi - \phi_n||_{s,\epsilon} \leq C\epsilon^{n/2} n^{s-t} ||\phi - \widehat{\phi}(0)||_t, \quad \text{if} \quad d \geq 0,$$

and

$$||\phi - \phi_n||_{s,\epsilon} \leq C\epsilon^{n/2} n^{s-t} ||\phi||_t, \qquad \text{if} \quad d \geq -1.$$

PROOF:

By theorem 3.1.5, it suffices to show that

$$||\phi - P_n\phi||_{s,\epsilon} \leq C\epsilon^{n/2} (\pi n)^{s-t} ||\phi - \widehat{\phi}(0)||_t \quad \forall \ \phi \in H^t, \quad \text{if} \quad d \geq 0,$$

and

$$||\phi - P_n\phi||_{s,\epsilon} \leq C\epsilon^{n/2} (\pi n)^{s-t} ||\phi||_t \qquad \forall \ \phi \in H^t, \quad \text{if} \quad d \geq -1. \tag{3.1.6}$$

where C depends only on d, ϵ, s, and Γ.

The case $d \geq 0$ has been proven in [2, theorem 5.1]. We will prove (3.1.6) for the case $d = -1$. Note that

$$\begin{aligned} ||\phi - P_n\phi||_{s,\epsilon}^2 &= \sum_{k \notin \Lambda_n} |\widehat{\phi}(k) - \widehat{P_n\phi}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s} \\ &\leq 2 \sum_{k \notin \Lambda_n} \{|\widehat{\phi}(k)|^2 + |\widehat{P_n\phi}(k)|^2\} \epsilon^{2|k|} \underline{k}^{2s}. \end{aligned} \tag{3.1.7}$$

We will bound each part. For the first part, we use $t \geq s$ and $\epsilon \in (\epsilon_1, 1]$ to get

$$\begin{aligned} \sum_{k \notin \Lambda_n} |\widehat{\phi}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s} &\leq \epsilon^n \sum_{k \notin \Lambda_n} \underline{k}^{2s-2t} |\widehat{\phi}(k)|^2 \underline{k}^{2t} \\ &\leq \epsilon^n (\pi n)^{2s-2t} ||\phi||_t^2. \end{aligned} \tag{3.1.8}$$

For the second part, we use $P_n\phi \in S_n^{-1}$ to get

$$\begin{aligned} \sum_{k \notin \Lambda_n} |\widehat{P_n\phi}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s} &= \sum_{p \in \Lambda_n} \sum_{m \in \mathbb{Z}^*} |\widehat{P_n\phi}(p+mn)|^2 \epsilon^{2|p+mn|} (\underline{p+mn})^{2s} \\ &= \sum_{p \in \Lambda_n} \sum_{m \in \mathbb{Z}^*} |\widehat{P_n\phi}(p)|^2 \epsilon^{2|p+mn|} \Big(2\pi|p+mn|\Big)^{2s} \\ &= \sum_{p \in \Lambda_n} |\widehat{\phi}(p)|^2 \underline{p}^{2t} (\pi n/\underline{p})^{2t} (\pi n)^{-2t} \sum_{m \in \mathbb{Z}^*} \epsilon^{2|p+mn|} \frac{(2\pi|p+mn|)^{2s}}{(\pi n)^{2s}} (\pi n)^{2s} \\ &\leq (\pi n)^{2s-2t} \epsilon^n \sum_{p \in \Lambda_n} |\widehat{\phi}(p)|^2 \underline{p}^{2t} (\pi n/\underline{p})^{2t} \Big\{ \sum_{m \in \mathbb{Z}^*} \Big(\frac{2|p+mn|}{n}\Big)^{2s} \Big\}. \end{aligned} \tag{3.1.9}$$

The quantity $(\pi n/\underline{p})^{2t}$ is bounded by 1 since $t \leq 0$ and $p \in \Lambda_n$. For the braced term, we use $s < -1/2$ and $p \in \Lambda_n$ to get

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^*} \Big(\frac{2|p+mn|}{n}\Big)^{2s} &= \sum_{m=1}^{\infty} \Big(\frac{2p}{n} + 2m\Big)^{2s} + \sum_{m=-\infty}^{-1} \Big(-\frac{2p}{n} - 2m\Big)^{2s} \\
&\leq \sum_{m=1}^{\infty} (-1+2m)^{2s} + \sum_{m=-\infty}^{-1} (-1-2m)^{2s} \\
&\leq C_1.
\end{aligned}$$

Therefore we rewrite (3.1.9) to get

$$\sum_{k\notin\Lambda_n} |\widehat{P_n\phi}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s} \leq (\pi n)^{2s-2t} \epsilon^n \|\phi\|_t^2 C_1. \tag{3.1.10}$$

Putting (3.1.7), (3.1.8), and (3.1.10) together, we have proved (3.1.6). Q.E.D.

The next theorem states exponential convergence rates for the approximate potentials on compact sets disjoint from the boundary.

THEOREM 3.1.7 *Let $d \geq -1$, $n \geq N$, $\phi \in H^t$, and Ω_K be a compact set in $\mathbb{R}^2 \backslash \Gamma$. Then there exists constants C and $\epsilon \in (0,1)$ depending only on d, t, N, Ω_k, and Γ such that*

$$\|\partial^\beta(u-u_n)\|_{L^\infty(\Omega_K)} \leq C\epsilon^n \|\phi - \widehat{\phi}(0)\|_t, \quad \text{if} \quad d \geq 0,$$

and

$$\|\partial^\beta(u-u_n)\|_{L^\infty(\Omega_K)} \leq C\epsilon^n \|\phi\|_t, \qquad \text{if} \quad d \geq -1.$$

PROOF: The proof is similiar to [2, theorem 5.3]. Q.E.D.

We now extend one of Arnold's theorems which give approximate potential error bounds in a weighted Sobolev norm. Let Ω_c be the exterior domain. We define the weighted Sobolev norm as

$$\begin{aligned}
\|v\|^2_{W^k(\Omega_c)} := \int_{\Omega_c} \Bigg[&\frac{|v(z)|^2}{(1+r^2)(1+\frac{1}{2}\log(1+r^2))^2} \\
&+ \sum_{1\leq|\beta|\leq k} \frac{|\partial^\beta v(z)|^2}{(1+r^2)^{1-|\beta|}} \Bigg] \, dz
\end{aligned} \tag{3.1.11}$$

where $r = |z|$. The corresponding space $W^k(\Omega_c)$ is the set of all functions in which their norms are finite. Note that $W^k(\Omega_c)$ contains the constant functions.

THEOREM 3.1.8 *Let $k \leq d+1$, $d \geq -1$, $t \in [k-3/2, d+1]$, $n \geq N$, and $\phi \in H^t$. Then there exists a constant C depending only on d and Γ such that*

$$\|u - u_n\|_{H^k(\Omega)} + \|u - u_n\|_{W^k(\Omega_c)} \leq Cn^{k-t-3/2}\|\phi - \widehat{\phi}(0)\|_t, \quad \text{if} \quad d \geq 0,$$

and

$$\|u - u_n\|_{H^k(\Omega)} + \|u - u_n\|_{W^k(\Omega_c)} \leq Cn^{k-t-3/2}\|\phi\|_t, \qquad \text{if} \quad d \geq -1.$$

PROOF: See [2, theorem 5.5] and [17, theorem 1.1 and 1.3]. Q.E.D.

3.2 Condition numbers

For the spline-trigonometric method, Arnold [2] proved that the condition numbers of the matrices are linearly proportional to the numbers of subintervals. We will show a similiar result for the delta-trigonometric method.

Recall that A (defined in (3.1)) represents the single-layer potential operator and $\mathbf{A}$ (defined in (3.11)) represents the matrix arising from the delta-trigonometric method. In lemma 3.2.1, we prove a relationship between $\|\phi_n\|_{-1}$ and $\|\alpha\|$ defined in (3.9) and (3.10), respectively. Then in theorem 3.2.2, we prove bounds for $\|\mathbf{A}\|$ and $\|\mathbf{A}^{-1}\|$. Finally, in theorem 3.2.3, we state bound for the condition numbers of $\mathbf{A}$.

LEMMA 3.2.1 *Let $d = -1$, then there exists a constant C such that*

$$\|\phi_n\|_{-1} \leq C\sqrt{n}\|\alpha\| \tag{3.2.1}$$

and

$$\|\alpha\| \leq C\sqrt{n}\|\phi_n\|_{-1}. \tag{3.2.2}$$

PROOF:

For the first half, note that

$$
\begin{aligned}
\|\phi_n\|_{-1}^2 &= \sum_{k\in\mathbb{Z}} |\widehat{\phi}_n(k)|^2 \underline{k}^{-2} \\
&= \sum_{k\in\mathbb{Z}} |\sum_{j=1}^{n} \alpha_j \exp(2\pi i k j/n)|^2 \underline{k}^{-2} \\
&\le \sum_{k\in\mathbb{Z}} \Big(\sum_{j=1}^{n} |\alpha_j \exp(2\pi i k j/n)| \Big)^2 \underline{k}^{-2} \\
&= \sum_{k\in\mathbb{Z}} \underline{k}^{-2} \Big(\sum_{j=1}^{n} |\alpha_j| \Big)^2 \\
&= C^2 \Big(\sum_{j=1}^{n} |\alpha_j| \Big)^2 \\
&\le C^2 n \|\alpha\|^2.
\end{aligned}
$$

For the second half, we use $p \in \Lambda_n$ to derive

$$
\begin{aligned}
\|\phi_n\|_{-1}^2 &= \sum_{k\in\mathbb{Z}} |\sum_{j=1}^{n} \alpha_j \exp(2\pi i k j/n)|^2 \underline{k}^{-2} \\
&= \sum_{p\in\Lambda_n} \sum_{m\in\mathbb{Z}} (\underline{p+mn})^{-2} \Big| \sum_{j=1}^{n} \alpha_j \exp(2\pi i p j/n) \Big|^2 \\
&\ge \sum_{p\in\Lambda_n} \underline{p}^{-2} |\sum_{j=1}^{n} \alpha_j \exp(2\pi i p j/n)|^2 \\
&\ge \sum_{p\in\Lambda_n} (\pi n)^{-2} \Big\{ \sum_{j=1}^{n} |\alpha_j|^2 + \sum_{j=1}^{n} \sum_{l=j+1}^{n} 2\alpha_j\alpha_l \exp(2\pi i p(j-l)/n) \Big\}.
\end{aligned}
$$

Rearranging the summations, we get

$$
\begin{aligned}
\|\phi_n\|_{-1}^2 \ge& \sum_{p\in\Lambda_n} (\pi n)^{-2} \sum_{j=1}^{n} |\alpha_j|^2 \\
&+ (\pi n)^{-2} \sum_{j=1}^{n} \sum_{l=j+1}^{n} 2\alpha_j\alpha_l \sum_{p\in\Lambda_n} \exp(2\pi i p(j-l)/n).
\end{aligned}
$$

But $\sum_{p\in\Lambda_n} \exp(2\pi i p(j-l)/n) = 0$ since $l \ne j (\mathrm{mod1}\ n)$. Therefore,

$$
\begin{aligned}
\|\phi_n\|_{-1}^2 &\ge \sum_{p\in\Lambda_n} (\pi n)^{-2} \sum_{j=1}^{n} |\alpha_j|^2 \\
&= n(\pi n)^{-2} \sum_{j=1}^{n} |\alpha_j|^2 \\
&= C^2 n^{-1} \|\alpha\|^2.
\end{aligned}
$$

Thus, (3.2.2) holds. Q.E.D.

THEOREM 3.2.2 *Let $d = -1$, then there exists a constant C depending only on Γ such that*

$$||\mathbf{A}|| \le C\sqrt{n} \tag{3.2.3}$$

and

$$||\mathbf{A}^{-1}|| \le C\sqrt{n}. \tag{3.2.4}$$

PROOF:

In the appendix, we note that A is an isomorphism from H^{-1} onto L^2 whenever the conformal radius of Γ is not equal to 1. In other words, $||A||_{L(H^{-1},L^2)}$ and $||A^{-1}||_{L(L^2,H^{-1})}$ are bounded constants depending only on Γ. Let α be an arbitrary vector and define $\beta := \mathbf{A}\alpha$. Also, define $f := \sum_{k\in\Lambda_n} \beta_k \psi_k$ and $\phi := A^{-1}f$. Note that the ψ's are orthonormal, and therefore $||\beta|| = ||f||_{L^2}$ Finally, let ϕ_n be the approximate density for the Dirichlet problem with data f, i.e., $\phi_n = \sum_{j=1}^{n} \alpha_j \delta(t - j/n))$. Then f is the L^2 projection of $A\phi_n$ onto T_n. By (3.2.1),

$$\begin{aligned} ||\mathbf{A}\alpha|| &= ||\beta|| = ||f||_{L^2} \\ &\le ||A\phi_n||_{L^2} \le ||A||_{L(H^{-1},L^2)} ||\phi_n||_{-1} \\ &\le C_1\sqrt{n}||A||_{L(H^{-1},L^2)}||\alpha||. \end{aligned} \tag{3.2.9}$$

This proves (3.2.3).

Also,

$$\begin{aligned} ||\mathbf{A}\alpha|| &= ||f||_{L^2} \\ &\ge \frac{||A^{-1}f||_{-1}}{||A^{-1}||_{L(L^2,H^{-1})}} \\ &= \frac{||\phi||_{-1}}{||A^{-1}||_{L(L^2,H^{-1})}}. \end{aligned} \tag{3.2.5}$$

Using $\epsilon = 1$, $s = -1$, and $t = -1$ in theorem 3.1.5, we derive

$$\begin{aligned} ||\phi_n||_{-1} &\le ||\phi - \phi_n||_{-1} + ||\phi||_{-1} \\ &\le C_2||\phi||_{-1}. \end{aligned} \tag{3.2.6}$$

By (3.2.2) and (3.2.6), equation (3.2.5) becomes

$$\begin{aligned} ||\mathbf{A}\alpha|| &\ge \frac{C_3||\phi_n||_{-1}}{||A^{-1}||_{L(L^2,H^{-1})}} \\ &\ge C_4\frac{||\alpha||}{\sqrt{n}||A^{-1}||_{L(L^2,H^{-1})}}. \end{aligned}$$

This implies that $\mathbf{A}^{-1}$ exists and (3.2.4) holds. Q.E.D.

THEOREM 3.2.3 *Let $d \geq -1$ and let $\kappa(\mathbf{A})$ represents the condition number of the matrix $\mathbf{A}$. Then there exists a constant C depending only on Γ such that*

$$\kappa(\mathbf{A}) \leq Cn.$$

PROOF:

For the case $d \geq 0$ Arnold [2, section 6] defined a special set of basis function for S_n^d and T_n. Then he proved that the condition numbers of the matrices are linearly proportional to the numbers of subintervals. The case $d = -1$ is proven in theorem 3.2.2. Q.E.D.

3.3 Convergence Analysis with Numerical Quadrature

In this section, we show that the delta-trigonometric method with numerical quadratures calculates the approximate potentials with exponential convergence rates. First, we use the Euler-MacLaurin theorem to bound the errors in numerical integrations of a given periodic analytic function times any trigonometric polynomial of degree less than n. Then we prove exponential error bounds due to numerical integration for the matrix terms, the unknown coefficients, and the approximate potentials on compact sets disjoint from the boundary. Finally we give numerical integration error bounds in a weighted Sobolev norm defined in (3.1.11). We continue to assume that g is an analytic function and Γ is a simple closed analytic curve such that the conformal radius of Γ is not equal to 1.

We now recall the Euler-MacLaurin theorem which tells us that the error in numerical integration of a given periodic smooth function is less than $O(n^{-m})$ for any $m > 0$ (where n is the number of subintervals).

THEOREM 3.3.1 *Let f be any C^∞ 1–periodic function. Set*

$$F := \int_0^1 f(s)\ ds$$

and

$$\widetilde{F} := \frac{1}{n}\sum_{j=1}^{n} f(j/n).$$

Then

$$F - \widetilde{F} = \frac{B_{2m}}{(2m)!} n^{-2m} f^{(2m)}(w_m) \quad \forall\ m \in \mathbb{Z}^+,$$

where B_{2m}'s are the Bernoulli numbers and w_m's are numbers in $[0,1]$. Moreover,

$$\frac{B_{2m}}{(2m)!} = (-1)^{m-1} \sum_{j=1}^{\infty} \frac{2}{(2j\pi)^{2m}} \quad \forall \ m \in \mathbb{Z}^+.$$

PROOF: See Aktinson [1, section 5.5] and Davis and Rabinowitz [9, pg. 108]. Q.E.D.

REMARK: Suppose we use a P-point Gaussian quadrature rule, i.e.,

$$\widetilde{F}_P = \frac{1}{n} \sum_{j=1}^{n} \sum_{p=1}^{P} q_p^P f\left(\frac{\xi_p^P + j - 3/2}{n}\right)$$

where q_p^P's are quadrature weights and ξ_p^P's are the quadrature points on [0,1]. Then

$$\begin{aligned} F - \widetilde{F}_P &= \int_0^1 f(s) \ ds - \frac{1}{n} \sum_{j=1}^{n} \sum_{p=1}^{P} q_p^P f\left(\frac{\xi_p^P + j - 3/2}{n}\right) \\ &= \sum_{p=1}^{P} q_p^P \left\{ \int_0^1 f(s) \ ds - \frac{1}{n} \sum_{j=1}^{n} f\left(\frac{\xi_p^P + j - 3/2}{n}\right) \right\} \\ &= \sum_{p=1}^{P} q_p^P \frac{B_{2m}}{(2m)!} n^{-2m} f^{(2m)}(w_m^p), \quad \forall \ m \in \mathbb{Z}^+, \end{aligned}$$

where w_m^p's are numbers in [0,1]. In other words, a similiar result holds for the P-point Gaussian quadrature. $\diamondsuit$

We modify theorem 3.3.1 in two ways. First, we seek numerical integration error bounds for analytic functions instead of C^∞ functions. Second, we consider the analytic functions multiplied by arbitrary trigonometric polynomials of degree less than n.

THEOREM 3.3.2 *Let f be any analytic 1-periodic function and define*

$$f_k := \int_0^1 f(s) \exp(2\pi i k s) \ ds$$

and

$$\widetilde{f}_k := \frac{1}{n} \sum_{l=1}^{n} f(l/n) \exp(2\pi i k l/n) \quad \forall \ k \in \mathbb{Z}.$$

Then there exist constants C and $\epsilon \in (0,1)$ depending only on f such that

$$|f_k - \widetilde{f}_k| \le C\epsilon^n \quad \forall \ k \in \Lambda_n.$$

PROOF:

By theorem 3.3.1, we have,

$$
\begin{aligned}
f_k - \widetilde{f}_k &= \frac{B_{2m}}{(2m)!} n^{-2m} \frac{\partial^{2m}}{\partial s^{2m}} (f(s)\exp(2\pi i k s))\Big|_{s=w_m} \\
&= \frac{B_{2m}}{(2m)!} n^{-2m} \sum_{l=0}^{2m} \binom{2m}{l} f^{(l)}(w_m)(2\pi i k)^{2m-l} \exp(2\pi i k w_m), \quad \forall\ m \in \mathbb{Z}^+ \quad \text{and} \quad k \in \mathbb{Z},
\end{aligned}
\tag{3.3.1}
$$

where w_m's are numbers in [0,1]. First, we bound

$$
\begin{aligned}
\Big|\frac{B_{2m}}{(2m!)}\Big| &= \frac{1}{(2\pi)^{2m}} \sum_{j=1}^{\infty} \frac{2}{j^{2m}} \\
&\le \frac{1}{(2\pi)^{2m}} \sum_{j=1}^{\infty} \frac{2}{j^{2}} \\
&\le \frac{C_1}{(2\pi)^{2m}}, \quad \forall\ m \in \mathbb{Z}^+,
\end{aligned}
\tag{3.3.2}
$$

where C_1 is a constant independent of m.

Next, we bound $f^{(l)}$. We extend f to be an analytic function in the complex strip $\overline{S}_\delta$ for some $\delta > 0$. Moreover, this extension is 1-periodic. (This is possible because f is analytic in a complex neighborhood of the real line. Therefore f is analytic on $\Big\{(x,y) \in \mathbb{C} \ \Big|\ x \in [0,1] \quad \text{and} \quad |y| \le \delta\Big\}$ for some $\delta > 0$. Since f is 1-periodic, it is analytic on $\overline{S}_\delta$.) For each point $t \in [0,1]$,

$$
|f^{(l)}(t)| \le \|f\|_{L^\infty(\partial B(t,\delta))} \frac{l!}{\delta^l} \quad \forall\ l \in \mathbb{Z}^+
$$

where $B_t(\delta)$ is an open ball of radius δ centered at t [26, pg 154]. This implies that

$$
|f^{(l)}(t)| \le \|f\|_{L^\infty(\overline{S}_\delta)} \frac{l!}{\delta^l} \quad \forall\ t \in [0,1] \quad \text{and} \quad l \in \mathbb{Z}^+. \tag{3.3.3}
$$

Combining (3.3.1) to (3.3.3) we get:

$$
\begin{aligned}
|f_k - \widetilde{f}_k| &\le \frac{C_1}{(2\pi n)^{2m}} \Big| \sum_{l=0}^{2m} \binom{2m}{l} f^{(l)}(w_m)(2\pi i k)^{2m-l} \exp(2\pi i k w_m)\Big| \\
&\le \frac{C_1}{(2\pi n)^{2m}} \sum_{l=0}^{2m} \binom{2m}{l} \|f\|_{L^\infty(\overline{S}_\delta)} \frac{l!}{\delta^l} \ |2\pi k|^{2m-l} \\
&\le \frac{C_1 \|f\|_{L^\infty(\overline{S}_\delta)}}{(2\pi n)^{2m}} \sum_{l=0}^{2m} \frac{(2m)(2m-1)\dots(2m-l+1)|2\pi k|^{2m-l}}{\delta^l}, \\
&\quad \forall\ m \in \mathbb{Z}^+ \quad \text{and} \quad k \in \mathbb{Z}.
\end{aligned}
\tag{3.3.4}
$$

Since m is an arbitrary positive integer, we choose m such that $m \in (-1 + \delta\pi n/2, \delta\pi n/2]$. Using $m \leq \delta\pi n/2$ and $k \in \Lambda_n$, (3.3.3) implies

$$\begin{aligned} |f_k - \widetilde{f}_k| &\leq \frac{C_1 ||f||_{L^\infty(\overline{S}_\delta)}}{(2\pi n)^{2m}} \sum_{l=0}^{2m} (\pi n)^{2m} \\ &= \frac{C_1 ||f||_{L^\infty(\overline{S}_\delta)} (2m+1)}{2^{2m}} \\ &= C_1 ||f||_{L^\infty(\overline{S}_\delta)} \left(\frac{1}{4}\right)^m (2m+1) \\ &\leq C\epsilon^n, \quad \forall \ k \in \Lambda_n. \end{aligned}$$

Q.E.D.

In theorem 3.3.3, we use theorem 3.3.2 to bound the perturbations of the matrices and vectors due to numerical integration. Then we use theorem 3.3.3 to bound the approximate potential errors in theorem 3.3.4.

THEOREM 3.3.3 *Let $d = -1$. Then there exist constants C and $\epsilon \in (0,1)$ depending only on g and Γ such that*

$$||\mathbf{g} - \widetilde{\mathbf{g}}|| \leq C\epsilon^n$$

and

$$||\mathbf{B} - \widetilde{\mathbf{B}}|| \leq C\epsilon^n.$$

PROOF:

For the first half of this theorem, note that by theorem 3.3.2,

$$||\mathbf{g} - \widetilde{\mathbf{g}}|| \leq \sqrt{n} \max_{k \in \Lambda_n} |g_k - \widetilde{g}_k| \leq C\epsilon^n.$$

For the second half of this theorem, recall that K is an 1-periodic analytic function with respect to either variable (lemma 3.1.3). By theorem 3.3.2,

$$\begin{aligned} |\mathbf{B}_{kj} - \widetilde{\mathbf{B}}_{kj}| &= \Big| \int_0^1 K(s, j/n)\psi_k(s) \ ds - \frac{1}{n} \sum_{p=1}^{n} K(p/n, j/n)\psi_k(p/n) \Big| \\ &\leq C_3\epsilon^n, \quad \forall \ j = 1, \ldots, n \quad \text{and} \quad k \in \Lambda_n. \end{aligned}$$

Therefore,

$$||\mathbf{B} - \widetilde{\mathbf{B}}|| \leq n^2 \max_{j=1,\ldots,n} \max_{k \in \Lambda_n} |(\mathbf{B} - \widetilde{\mathbf{B}})_{kj}| \leq C\epsilon^n.$$

Q.E.D.

REMARK: Theorem 3.3.3 does not hold for $d \geq 0$. The reason is because we must apply numerical integration on an non-analytic function (i.e. spline trial functions). Exponential convergence may hold for the trigonometric-trigonometric method with numerical quadrature since the trigonometric functions are analytic. $\diamondsuit$

Finally we give exponential numerical integration bounds for the unknown coefficients and the approximate potentials.

THEOREM 3.3.4 *Let $d = -1$ and Ω_K be a compact set disjoint from the boundary. Then there exist constants C and $\epsilon \in (0,1)$ depending only on g and Γ such that for all $z \in \Omega_k$,*

$$||\alpha - \tilde{\alpha}|| \leq C\epsilon^n,$$

$$|u_n(z) - \tilde{u}_n(z)| \leq C\epsilon^n, \tag{3.3.5}$$

and

$$|u(z) - \tilde{u}_n(z)| \leq C\epsilon^n.$$

PROOF:

Note that

$$\begin{aligned} \alpha - \tilde{\alpha} &= \mathbf{A}^{-1}[\mathbf{g} - \tilde{\mathbf{g}} - (\mathbf{A} - \tilde{\mathbf{A}})\tilde{\alpha}] \\ &= \mathbf{A}^{-1}[\mathbf{g} - \tilde{\mathbf{g}} - (\mathbf{B} - \tilde{\mathbf{B}})\tilde{\alpha}]. \end{aligned}$$

Hence,

$$||\alpha - \tilde{\alpha}|| \leq ||\mathbf{A}^{-1}||\{||\mathbf{g} - \tilde{\mathbf{g}}|| + ||\mathbf{B} - \tilde{\mathbf{B}}||||\tilde{\alpha}||\}.$$

Using the fact that

$$||\tilde{\alpha}|| \leq ||\alpha - \tilde{\alpha}|| + ||\alpha|| \leq ||\alpha - \tilde{\alpha}|| + ||\mathbf{A}^{-1}||||\mathbf{g}||$$

we derive

$$||\alpha - \tilde{\alpha}|| \leq \frac{||\mathbf{A}^{-1}||\{||\mathbf{g} - \tilde{\mathbf{g}}|| + ||\mathbf{B} - \tilde{\mathbf{B}}||||\mathbf{A}^{-1}||||\mathbf{g}||\}}{\{1 - ||\mathbf{A}^{-1}||||\mathbf{B} - \tilde{\mathbf{B}}||\}} \quad . \tag{3.3.6}$$

By theorems 3.2.3 and 3.3.3,

$$\begin{aligned} ||\alpha - \tilde{\alpha}|| &\leq \frac{C_3\sqrt{n}\{C_1\epsilon_1^n + C_2\epsilon_2^n C_3\sqrt{n}||\mathbf{g}||\}}{1 - C_3\sqrt{n}C_2\epsilon_2^n)} \\ &\leq C_4\epsilon_3^n. \end{aligned}$$

To prove (3.3.5), we note that Ω_K is a compact set, and therefore, we bound the logarithmic term by a constant. Thus,

$$
\begin{aligned}
|u_n(z) - \widetilde{u}_n(z)| &= \Big| \int_0^1 \sum_{j=1}^{n} (\alpha_j - \widetilde{\alpha}_j) \log|z - x(j/n)| \; dt \Big| \\
&\leq \max_{j=1\dots,n} |\alpha_j - \widetilde{\alpha}_j| \Big| \int_0^1 \log|z - x(j/n)| \; dt \Big| \\
&\leq \sqrt{n} ||\alpha - \widetilde{\alpha}|| C_5 \\
&\leq C_6 \epsilon_4^n.
\end{aligned}
$$

Also, by theorem 3.1.7,

$$
\begin{aligned}
|u(z) - \widetilde{u}_n(z)| &\leq |u(z) - u_n(z)| + |u_n(z) - \widetilde{u}_n(z)| \\
&\leq C_7 \epsilon_5^n + C_6 \epsilon_4^n \\
&\leq C \epsilon^n.
\end{aligned}
$$

Q.E.D.

We also prove that the use of numerical quadratures does not affect the convergence rates in the weighted Sobolev norm defined in (3.1.11).

THEOREM 3.3.5 *Let $d = -1$, $k \leq 0$, $t \in [k - 3/2, 0]$, $n \geq N$, and $\phi \in H^t$. Then there exists a constant C depending only only g, k, and Γ such that*

$$
||u - \widetilde{u}_n||_{H^k(\Omega)} + ||u - \widetilde{u}_n||_{W^k(\Omega_c)} \leq C n^{k-t-3/2} ||\phi||_t. \tag{3.3.7}
$$

PROOF:

By theorem 3.1.7,

$$
\begin{aligned}
||u - u_n||_{H^k(\Omega)} + ||u - u_n||_{W^k(\Omega_c)} &\leq C_1 ||\phi - \phi_n||_{k-3/2} \\
&\leq C_2 n^{k-t-3/2} ||\phi||_t.
\end{aligned}
$$

Similiarly, by theorem 3.2.1,

$$
\begin{aligned}
||u_n - \widetilde{u}_n||_{H^k(\Omega)} + ||u_n - \widetilde{u}_n||_{W^k(\Omega_c)} &\leq C_1 ||\phi_n - \widetilde{\phi}_n||_{k-3/2} \\
&\leq C_3 || \sum_{j=1}^{n} (\alpha_j - \widetilde{\alpha}_j) \delta_j ||_{k-3/2} \\
&\leq C_3 n \max_{j \in [1,n]} |\alpha_j - \widetilde{\alpha}_j| \max_{j \in [1,n]} ||\delta_j||_{k-3/2} \\
&\leq C_4 n \sqrt{n} ||\alpha - \widetilde{\alpha}||.
\end{aligned}
$$

By theorem 3.3.4,

$$\|u_n - \widetilde{u}_n\|_{H^k(\Omega)} + \|u_n - \widetilde{u}_n\|_{W^k(\Omega_c)} \le C_5 \epsilon^n.$$

Since

$$\begin{aligned}\|u - \widetilde{u}_n\|_{H^k(\Omega)} + \|u - \widetilde{u}_n\|_{W^k(\Omega_c)} \le \|u - u_n\|_{H^k(\Omega)} + \|u - u_n\|_{W^k(\Omega_c)} \\ + \|u_n - \widetilde{u}_n\|_{H^k(\Omega)} + \|u_n - \widetilde{u}_n\|_{W^k(\Omega_c)},\end{aligned}$$

we conclude that (3.3.7) holds. Q.E.D.

3.4 Numerical Technique

The program SPLTRG implements the spline-trigonometric method with numerical quadratures using splines of degree 0 (piecewise constant splines) as trial functions. It can also be used to compute the delta-trigonometric method by selecting the 1-point quadrature rule for certain integrals. SPLTRG employs the fast Fourier transform to calculate the matrix entries. In this section, we show how the fast Fourier transform has been implemented in the program and give operation counts.

Assume that $d = 0$, n is odd, and let $\chi_{[\frac{j-3/2}{n}, \frac{j-1/2}{n}]}$ for $j = 1, \ldots, n$ be the basis for S_n^0 where $\chi_{[a,b]}$ denotes the characteristic function on the interval $[a, b]$. Then

$$\phi_n(t) := \sum_{j=1}^{n} \alpha_j \chi_{[\frac{j-3/2}{n}, \frac{j-1/2}{n}]}(t)$$

where α_j's are the unknown coefficients. Instead of complex test functions, we use real test functions. Let

$$\widetilde{\psi}_k(s) = \begin{cases} \sin(k\pi s), & \text{if } k = 2, 4, \ldots, n-1, \\ \cos((k-1)\pi s), & \text{if } k = 1, 3, \ldots, n. \end{cases}$$

We wish to perform numerical integrations on

$$\int_0^1 A\phi_n(s)\widetilde{\psi}_k(s) \; ds = \int_0^1 g(s)\widetilde{\psi}_k(s) \; ds \quad \forall \; k \in [1, n].$$

The above system can be rewritten as

$$\begin{aligned}\int_0^1 \int_0^1 \left\{ \sum_{j=1}^{n} \alpha_j \chi_{[\frac{j-3/2}{n}, \frac{j-1/2}{n}]}(t) \right\} \log\left|x(s) - x(t)\right| \widetilde{\psi}_k(s) \; dt \; ds \\ = \int_0^1 g(s)\widetilde{\psi}_k(s) \; ds \quad \forall \; k \in [1, n].\end{aligned}$$

The left hand side is split into two parts, the principal log term and the smooth remainder. Thus, we rewrite the last equation as

$$\begin{aligned}
&\int_0^1 \int_0^1 \left\{ \sum_{j=1}^{n} \alpha_j \chi_{[\frac{j-3/2}{n}, \frac{j-1/2}{n}]}(t) \right\} \log \left| \frac{x(s)-x(t)}{2\sin(\pi(s-t))} \right| \widetilde{\psi}_k(s) \ dt \ ds \\
&\quad + \int_0^1 \int_0^1 \left\{ \sum_{j=1}^{n} \alpha_j \chi_{[\frac{j-3/2}{n}, \frac{j-1/2}{n}]}(t) \right\} \log \left| 2\sin(\pi(s-t)) \right| \widetilde{\psi}_k(s) \ dt \ ds \\
&= \int_0^1 g(s) \widetilde{\psi}_k(s) \ ds \quad \forall \ k \in [1,n].
\end{aligned} \tag{3.4.1}$$

The equivalent matrix equation is

$$\mathbf{B}\alpha + \mathbf{V}\alpha = \mathbf{g}$$

where

$$\begin{aligned}
\mathbf{B}_{kj} &= \int_0^1 \int_0^1 \chi_{[\frac{j-3/2}{n}, \frac{j-1/2}{n}]}(t) \log \left| \frac{x(s)-x(t)}{2\sin(\pi(s-t))} \right| \widetilde{\psi}_k(s) \ dt \ ds, \\
\mathbf{V}_{kj} &= \int_0^1 \int_0^1 \chi_{[\frac{j-3/2}{n}, \frac{j-1/2}{n}]}(t) \log \left| 2\sin(\pi(s-t)) \right| \widetilde{\psi}_k(s) \ dt \ ds,
\end{aligned}$$

and

$$\widetilde{\mathbf{g}}_k = \int_0^1 g(s) \widetilde{\psi}_k(s) \ ds$$

for all $k, j = 1, \ldots, n$. An M-point Gaussian quadrature rule on n subintervals is applied on the right hand term to get

$$\widetilde{\mathbf{g}}_k = \frac{1}{n} \sum_{l=1}^{n} \sum_{m=1}^{M} q_m^M g\left(\frac{\xi_m^M + l - 3/2}{n}\right) \widetilde{\psi}_k\left(\frac{\xi_m^M + l - 3/2}{n}\right)$$

where q_m^M's are quadrature weights on [0,1] and ξ_m^M's are the quadrature points on [0,1]. For any even k, simple trigonometric identities imply

$$\widetilde{\psi}_k\left(\frac{\xi_m^M + l - 3/2}{n}\right) = \widetilde{\psi}_k\left(\frac{l-1}{n}\right) \widetilde{\psi}_{k+1}\left(\frac{\xi_m^M - 1/2}{n}\right) + \widetilde{\psi}_{k+1}\left(\frac{l-1}{n}\right) \widetilde{\psi}_k\left(\frac{\xi_m^M - 1/2}{n}\right) \tag{3.4.2}$$

and

$$\widetilde{\psi}_{k+1}\left(\frac{\xi_m^M + l - 3/2}{n}\right) = \widetilde{\psi}_{k+1}\left(\frac{l-1}{n}\right) \widetilde{\psi}_{k+1}\left(\frac{\xi_m^M - 1/2}{n}\right) - \widetilde{\psi}_k\left(\frac{l-1}{n}\right) \widetilde{\psi}_k\left(\frac{\xi_m^M - 1/2}{n}\right). \tag{3.4.3}$$

The sums,

$$\frac{1}{n} \sum_{l=1}^{n} q_m^M g\left(\frac{\xi_m^M + l - 3/2}{n}\right) \widetilde{\psi}_k\left(\frac{l-1}{n}\right) \quad \text{for} \quad m \in [1, M] \quad \text{and} \quad k \in [1, n]$$

are calculated using the fast Fourier transform in $O(nM\log n)$ operations. Then we calculate $\mathbf{g}$ using (3.4.2) and (3.4.3) in $O(nM)$ calculations. Thus, the total work to calculate $\widetilde{\mathbf{g}}$ is $O(nM\log n)$.

The smooth log matrix $\widetilde{\mathbf{B}}$ is calculated similiarly. Apply M_1-point and M_2-point quadrature rules on the inner and outer integrals respectively (i.e., the integrals with respect to t and s, respectively) to get

$$\begin{aligned}\widetilde{\mathbf{B}}_{kj} &= \frac{1}{n^2}\sum_{l=1}^{n}\sum_{m_2=1}^{M_2}\sum_{p=1}^{n}\sum_{m_1=1}^{M_1} q_{m_1}^{M_1} q_{m_2}^{M_2} \log\left|\frac{x\left(\frac{\xi_{m_2}^{M_2}+l-3/2}{n}\right) - x\left(\frac{\xi_{m_1}^{M_1}+p-3/2}{n}\right)}{2\sin\left(\pi\left(\frac{\xi_{m_2}^{M_2}+l-3/2}{n} - \frac{\xi_{m_1}^{M_1}+p-3/2}{n}\right)\right)}\right| \widetilde{\psi}_k\left(\frac{\xi_{m_2}^{M_2}+l-3/2}{n}\right)\\ &= \frac{1}{n^2}\sum_{p=1}^{n}\sum_{m_2=1}^{M_2}\left\{\sum_{l=1}^{n}\sum_{m_1=1}^{M_1} q_{m_1}^{M_1} q_{m_2}^{M_2} \log\left|\frac{x\left(\frac{\xi_{m_2}^{M_2}+l-3/2}{n}\right) - x\left(\frac{\xi_{m_1}^{M_1}+p-3/2}{n}\right)}{2\sin\left(\pi\left(\frac{\xi_{m_2}^{M_2}+l-3/2}{n} - \frac{\xi_{m_1}^{M_1}+p-3/2}{n}\right)\right)}\right| \widetilde{\psi}_k\left(\frac{\xi_{m_2}^{M_2}+l-3/2}{n}\right)\right\}.\end{aligned}$$

The double sum in the braces is calculated (for $k = 1, \ldots, n$) by the fast Fourier transform. Thus, the total time needed to calculate $\widetilde{\mathbf{B}}$ is $O(n^2 M_1 M_2 \log n)$.

The principal part can be integrated exactly. If $k = 1$, then $\mathbf{V}_{kj} = 0$. If $k \neq 1$ then we use the same idea as in (3.14) to get

$$\begin{aligned}\mathbf{V}_{kj} &= \int_0^1\int_0^1 \log\left|2\sin(\pi(s-t))\right|\chi_{[\frac{j-3/2}{n},\frac{j-1/2}{n}]}(t)\widetilde{\psi}_k(s)\ dt\ ds\\ &= \int_{\frac{j-3/2}{n}}^{\frac{j-1/2}{n}}\int_0^1 \log\left|2\sin(\pi(s-t))\right|\widetilde{\psi}_k(s)\ ds\ dt\\ &= \begin{cases}\frac{1}{k}\int_{\frac{j-3/2}{n}}^{\frac{j-1/2}{n}}\widetilde{\psi}_k(t)\ dt, & \text{if } k = 2, 4, \ldots, n-1,\\ \frac{1}{k-1}\int_{\frac{j-3/2}{n}}^{\frac{j-1/2}{n}}\widetilde{\psi}_k(t)\ dt, & \text{if } k = 3, 5, \ldots, n.\end{cases}\end{aligned} \tag{3.4.4}$$

We could easily integrate the trigonometric function in (3.4.4) if we desire to use 0^{th} degree splines in $O(n^2)$ calculations. However we found it better to integrate the outer integral analytically and perform trapezoidal quadratures on the inner integral. In other words, we use the delta-trigonometric method. The numerical errors due to approximating the non-analytic piecewise constant functions is terrible (see next section for numerical results). For either method we require $O(n^2)$ calculations to calculate $\mathbf{V}$.

In summary, the program requires a total time of $O(M_1 M_2 n^2 \log n)$ to calculate the matrix. The LU decomposition requires $O(n^3/3)$ calculations. Therefore when n is sufficiently large, SPLTRG uses $O(n^3/3)$ time. Computer analysis show that the LU decomposition requires less than a third of the total time for n as large as 81. In other words, it is important to use the fast Fourier transform since the matrix formations require a significant amount of time.

3.5 Numerical Results

Program SPLTRG was implemented to test the delta-trigonometric and spline-trigonometric methods with numerical quadratures. In this section, we present several sample problems and their numerical results. The first problem is an ideal problem, that is, the boundary and boundary data are real analytic. Then we look at some problems where the boundary and/or boundary data are not so ideal.

Program SPLTRG calculates the approximate solutions and derivatives. If the user provides the exact answer, then SPLTRG calculates the exact numerical errors. Otherwise, SPLTRG calculates the approximate numerical errors.

There are seven integrals to be evaluated. Five of them come from equation (3.4.1). The other two integrals result from finding the approximate potentials and their normal derivatives. For each integral, program SLPTRG allows the user to pick the number of quadrature points per subinterval. Some of the integrals can be calculated exactly, in particular, the principal term.

For the best result (given a fixed n) using piecewise constant splines as trial functions, the user should calculate the principal term exactly and the rest by 8-points Guassian quadrature, the best quadrature rule available in SPLTRG. For the best result (given a fixed n) using summations of delta functions as trial functions, the user should use trapezoidal quadrature on all inner integrals and the approximate potential integral, and 8-points Gaussian quadrature rule on the other integrals.

For the following tables, we let

****	:=	no answer due to overflow or underflow
u	:=	the exact potential
ue_n	:=	the error for the approximate potential using n subintervals
r_{n-m}	:=	the convergence rate from n subintervals to m subintervals
delta 1-pt	:=	delta trial functions with 1-point quadrature
delta 3-pts	:=	delta trial functions with 3-points quadrature
delta 8-pts	:=	delta trial functions with 8-points quadrature
p.c. 1-pt	:=	piecewise constant trial functions with 1-point quadrature
p.c. 3-pts	:=	piecewise constant trial functions with 3-points quadrature
p.c. 8-pts	:=	piecewise constant trial functions with 8-points quadrature

We define the relative error to be the absolute error divided by the exact solution. In cases where the exact solution is near zero, SPLTRG will give the absolute errors. All calculations are done in double precision. Consequently, we can not expect the relative errors to be much smaller than 1.0E−14.

EXAMPLE 3.5.1 Ellipse with analytic data

The first example involves an elliptic boundary (an analytic boundary) with analytic boundary data. In this example, we examine the effects of using different trial functions and quadrature rules.

Boundary: $x^2/4 + y^2 = 1/25$

Data: $g = 5x/2$

Exact solution:

$$u = \begin{cases} 5x/2, & \text{if } (x,y) \in \text{ellipse}, \\ 5x - w, & \text{if } (x,y) \notin \text{ellipse and } x \geq 0, \\ 5x + w, & \text{if } (x,y) \notin \text{ellipse and } x \leq 0, \end{cases}$$

where

$$w = \sqrt{\frac{25(x^2-y^2) - 3 + \sqrt{(25(x^2-y^2)-3)^2 + 2500x^2y^2}}{2}} \ .$$

For table 1A and 1B, we pick a typical interior point and present the approximate potentials relative errors and convergence rates, respectively, using different trial functions and quadrature rules. The numerical results for other points away from the boundary are similiar. When delta trial functions are used, the approximate potentials converge very fast, i.e., relative errors are about 10^{-14} for $n = 81$. There are very little error differences when using different quadrature rules. Note that the convergence rates appear to be exponential in table 1B. For the piecewise constant trial functions, we found it necessary to use a high quadrature rule in order to obtain fast convergence. For 1-point and 2-points quadrature rules, the convergence rates approach 2.0 and 4.0, respectively. For higher quadrature rules, the convergence rates initially appear to increase and do not show any slowdown until after the roundoff errors become significant.

TABLE 1A: relative errors at (0.10, 0.05)

	$\lvert ue_3\rvert$	$\lvert ue_9\rvert$	$\lvert ue_{27}\rvert$	$\lvert ue_{81}\rvert$	$\lvert ue_{243}\rvert$
delta 1-pt	7.41E−01	5.89E−03	1.52E−06	3.78E−15	2.44E−15
delta 3-pts	7.30E−01	5.88E−03	1.52E−06	5.33E−15	1.55E−15
delta 8-pts	7.29E−01	5.88E−03	1.52E−06	4.66E−15	****
p.c. 1-pt	7.04E−01	9.39E−03	1.69E−03	1.88E−04	2.09E−05
p.c. 2-pts	5.48E−01	1.89E−03	1.94E−07	6.29E−09	7.76E−11
p.c. 3-pts	4.78E−01	1.93E−03	6.23E−08	8.82E−14	2.13E−14
p.c. 8-pts	4.83E−01	1.97E−03	3.10E−08	2.22E−15	****

TABLE 1B: convergence rates at (0.10, 0.05)

	r_{3-9}	r_{9-27}	r_{27-81}	r_{81-243}
delta 1-pt	4.40	7.52	18.04	0.10
delta 3-pts	4.39	7.52	17.72	1.12
delta 8-pts	4.39	7.52	17.80	****
p.c. 1-pt	3.93	1.56	2.00	2.00
p.c. 2-pts	5.16	8.36	3.12	4.00
p.c. 3-pts	5.02	9.42	12.26	1.29
p.c. 8-pts	5.01	10.07	14.97	****

We also examine the approximate potentials errors on the boundary. Note that the approximate potential in (3.15) has a logarithmic singularity at the quadrature points. Therefore we evaluate the maximum relative errors at the mesh points and present these results in table 1C. Table 1C shows that there are only small improvements in the errors when higher quadrature rules are used, and therefore, it is best to use a low quadrature rule with either trial space.

TABLE 1C: maximum relative errors in between subintervals on the boundary

	$\lvert ue_3 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{27} \rvert$	$\lvert ue_{81} \rvert$	$\lvert ue_{243} \rvert$
delta 1-pt	1.05E+01	8.94E−01	1.08E−01	1.28E−02	4.28E−03
delta 3-pts	1.10E+01	8.94E−01	1.08E−01	1.28E−02	4.28E−03
delta 8-pts	1.10E+01	8.94E−01	1.08E−01	1.28E−02	****
p.c. 1-pt	1.21E+01	8.92E−01	1.10E−01	1.26E−02	4.26E−03
p.c. 2-pt	1.66E+01	2.54E+00	4.16E−02	3.87E−03	1.29E−03
p.c. 3-pts	****	****	2.95E−02	1.94E−03	****
p.c. 8-pts	****	1.09E+00	****	****	****

In table 1D, we present the matrix condition numbers for different trial functions and quadrature rules. Note that the condition numbers grow proportionally slower than the numbers of subintervals.

TABLE 1D: matrix condition numbers

	$\lvert ue_3 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{27} \rvert$	$\lvert ue_{81} \rvert$	$\lvert ue_{243} \rvert$
delta 1-pt	0.59E+01	0.11E+02	0.21E+02	0.55E+02	0.92E+02
delta 3-pts	0.52E+01	0.11E+02	0.21E+02	0.55E+02	0.92E+02
delta 8-pts	0.52E+01	0.11E+02	0.21E+02	0.55E+02	****
p.c. 1-pt	0.81E+01	0.21E+02	0.28E+02	0.63E+02	0.16E+03
p.c. 2-pts	0.58E+01	0.21E+02	0.28E+02	0.62E+02	0.16E+03
p.c. 3-pts	0.59E+01	0.21E+02	0.28E+02	0.62E+02	0.16E+03
p.c. 8-pts	0.59E+01	0.21E+02	0.28E+02	0.62E+02	****

Table 1E shows the CPU time required for each run. From this table, we see that it is expensive to compute using a high quadrature rule. It is more efficient to use a low quadrature rule and more subintervals (larger n).

TABLE 1E: CPU time

	time_3	time_9	time_{27}	time_{81}	time_{243}
delta 1-pt	6.889	18.761	66.014	307.978	2749.478
delta 3-pts	6.997	20.893	85.215	487.562	4562.625
delta 8-pts	8.106	26.741	132.652	953.534	****
p.c. 1-pt	4.105	9.141	32.629	158.226	1472.793
p.c. 2-pts	7.010	18.696	68.012	325.431	2472.231
p.c. 3-pts	9.930	28.420	107.010	526.649	3880.172
p.c. 8-pts	55.953	190.065	811.078	4656.722	****

We also examine the relative errors on a sample line. Graph 1A, 1B, and 1C show the relative errors on the line $x = 2y$ for different values of n using delta trial functions with trapezoidal quadrature, piecewise constant trial functions with trapezoidal quadrature, and piecewise constant trial functions with 8-points Guassian quadrature, respectively. Note that the relative errors are worst when the line crosses the boundary (about (x,y)=(0.283,0.141)).

For this example, we conclude that very fast convergence is obtained for the approximate potentials on compact sets disjoint from the boundary using the delta-trigonometric method with numerical quadrature. In fact, the convergence rates appear to be exponential. For the spline-trigonometric method with numerical quadrature, the convergence rates does not appear to be exponential.

EXAMPLE 3.5.2 Ellipse with data of varying smoothness

This example involves the same elliptic boundary but with boundary data of different degrees of smoothness.

Boundary: $x^2/4 + y^2 = 1/25$

Data:

$$g = \begin{cases} 1.0, & \text{if } x \leq 0, \\ 1.0 + x^s, & \text{if } x \geq 0, \end{cases} \quad \text{for} \quad s = 0, 1, 2, 3, 4, 5, \quad \text{and} \quad 6.$$

The exact potential is not known, and therefore, the approximate relative errors are computed by using the approximate potentials for $n = 243$. For this problem, we present results using only the delta trial functions with trapezoidal quadrature. Table 2A compares the approximate relative errors at a typical interior point for difference data smoothness. We see that the smoothness of the data affects the convergence rates significantly.

TABLE 2A: approximate convergence rates and relative errors at (0.10, 0.05) using the delta trial functions with trapezoidal quadrature

s	r_{3-9}	r_{9-27}	r_{27-81}	$\lvert ue_3 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{27} \rvert$	$\lvert ue_{81} \rvert$
0	1.20	1.26	0.60	2.86E-01	7.66E-02	1.92E-02	9.96E-03
1	3.14	2.12	3.01	7.89E-02	2.52E-03	2.46E-04	8.98E-06
2	6.68	0.79	2.90	6.98E-02	4.54E-05	1.90E-05	7.90E-07
3	3.63	6.70	6.58	8.18E-02	1.52E-03	9.62E-07	6.98E-10
4	3.70	7.04	7.25	8.94E-02	1.54E-03	6.75E-07	2.36E-10
5	3.80	7.02	12.21	9.31E-02	1.43E-03	6.39E-07	9.56E-13
6	3.85	6.99	13.74	9.47E-02	1.38E-03	6.41E-07	1.79E-13

Graph 2A show the approximate relative errors on the line $x = 2y$ using $n = 243$ for s equal 0, 1, 2, 3, 4, 5, and 6. It is interesting to note that the errors are about the same as the line crosses the boundary. We did not study the errors where the boundary data is not smooth.

For this example, we conclude that the boundary data lack of smoothness affects the errors greatly. These results do not contradict our theoretical results, since we required the boundary and boundary data to be analytic in our proofs with and without numerical quadrature. Note that we did obtain fair results at points away from the boundary for $s \geq 1$. The condition numbers depend only on the geometry of the domain and are exactly the same as in table 1D (example 3.5.1).

EXAMPLE 3.5.3 Rectangle with linear data

The third example involves a boundary with corners, but the boundary data is linear.

Domain: $(-0.1, 0.1) \times (-0.1, 0.1)$

Data: $g = 5x/2$

The exact solution is known in the interior region only and coincides with the formula given for g. As in example 3.5.1, we examine the effects of using different trial functions and quadrature rules. Table 3A and 3B shows the exact relative errors and exact convergence rates, respectively, at a sample interior point. Note that there are only little differences in the error when different trial functions and quadrature rules are used. In other words, the corners of the rectangle affect the errors significantly.

TABLE 3A: exact relative errors at (0.05, 0.05)

	$\lvert ue_3 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{27} \rvert$	$\lvert ue_{81} \rvert$	$\lvert ue_{243} \rvert$
delta 1-pt	3.29E−01	4.41E−02	1.53E−03	8.46E−05	4.43E−06
delta 3-pts	3.14E−01	7.74E−02	1.91E−03	1.17E−04	6.20E−06
p.c. 1-pt	1.42E−01	6.98E−02	2.32E−03	9.74E−05	1.14E−06
p.c. 3-pts	1.78E−01	3.12E−02	7.25E−04	4.97E−05	2.64E−06

TABLE 3B: exact convergence rates at (0.05, 0.05)

	r_{3-9}	r_{9-27}	r_{27-81}	r_{81-243}
delta 1-pt	1.83	3.06	2.63	2.69
delta 3-pts	1.27	3.37	2.54	2.67
p.c. 1-pt	0.65	3.10	2.89	4.05
p.c. 3-pts	1.58	3.42	2.44	2.67

Table 3C shows the exact maximum relative errors for points in between the subintervals on the boundary (i.e. points which are not quadrature points). Again, there are little differences in the errors when different trial functions and quadrature rules are used.

TABLE 3C: maximum relative errors in between subintervals on the boundary

	$\lvert ue_3 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{27} \rvert$	$\lvert ue_{81} \rvert$	$\lvert ue_{243} \rvert$
delta 1-pt	2.29E+01	3.65E−01	1.55E−01	7.44E−02	3.58E−02
delta 3-pts	2.49E+01	1.77E+00	1.40E−01	6.73E−02	3.23E−02
p.c. 1-pt	2.74E+01	6.99E−01	1.49E−01	7.20E−02	3.47E−02
p.c. 3-pts	****	****	2.22E−01	3.86E−02	****

In table 3D, we present the matrix condition numbers for different trial functions and quadrature rules. As in example 3.4.1, the condition numbers grow proportionally slower than the numbers of subintervals.

TABLE 3D: matrix condition numbers

	$\lvert ue_3 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{27} \rvert$	$\lvert ue_{81} \rvert$	$\lvert ue_{243} \rvert$
delta 1-pt	0.58E+01	0.13E+02	0.27E+02	0.55E+02	0.11E+03
delta 3-pts	0.56E+01	0.13E+02	0.27E+02	0.59E+02	0.12E+03
delta 8-pts	0.56E+01	0.13E+02	0.27E+02	0.59E+02	****
p.c. 1-pt	0.69E+01	0.17E+02	0.35E+02	0.84E+02	0.18E+03
p.c. 3-pts	0.67E+01	0.16E+02	0.34E+02	0.88E+02	0.20E+03
p.c. 8-pts	0.67E+01	0.16E+02	0.34E+02	0.88E+02	****

Table 3E shows the CPU time required for each run. Again, we see that it is more efficient to use a low quadrature rule and more subintervals.

TABLE 3E: CPU time

	$time_3$	$time_9$	$time_{27}$	$time_{81}$	$time_{243}$
delta 1-pt	3.918	8.709	36.346	215.810	2885.201
delta 3-pts	4.311	10.423	51.849	374.263	4430.722
p.c. 1-pt	4.043	9.056	38.307	235.171	3044.403
p.c. 3-pts	9.538	28.770	130.968	801.703	7401.127

Graph 3A and 3B show the exact relative errors (for different n) on a sample line from the origin to a corner of the rectangle using delta trial functions with trapezoidal quadrature and piecewise constant trial functions with trapezoidal quadrature, respectively. We see that the errors become worse as the line approaches the boundary.

For this example, it is best to use trapezoidal quadrature with either trial function. The lack of boundary smoothness affects the errors significantly.

EXAMPLE 3.5.4 Wedge with analytic data

The last example involves an wedge problem in which the potential possesses a singularity at the corner of the domain.

Interior Domain: (in polar coordinates) $0 < r < 1$, $0 < \theta < 2\pi/5$

Data: $g = 1.0 + r^{5/2}\sin(5\theta/2)$

The exact solution is known in the interior region only and coincides with the formula given for g. Table 4A shows the exact relative errors using different quadrature rules at a typical interior point. The results show that there are little differences in the errors when using different quadrature rules. The convergence rates behave a little wildly, but we did obtain errors of order 10^{-4} for $n = 81$ regardless of which quadrature rules were used.

TABLE 4A: exact convergence rates and relative errors at (0.10, 0.05)

	r_{9-27}	r_{27-81}	r_{81-243}	$\lvert ue_9 \rvert$	$\lvert ue_{27} \rvert$	$\lvert ue_{81} \rvert$	$\lvert ue_{243} \rvert$
delta 1-pt	0.21	2.88	3.05	4.22E−03	3.36E−03	1.41E−04	4.98E−06
delta 3-pts	3.06	−0.09	3.26	7.22E−03	2.50E−04	2.76E−04	7.65E−06
delta 8-pts	3.54	−0.51	****	7.66E−03	1.57E−04	2.76E−04	****
p.c. 1-pt	0.95	2.48	2.26	2.71E−02	9.52E−03	6.23E−04	5.23E−05
p.c. 3-pts	3.85	0.84	3.45	4.11E−02	5.97E−04	2.37E−04	5.39E−06
p.c. 8-pts	4.13	0.79	****	5.42E−02	5.78E−04	2.43E−04	****

Table 4B shows the maximum relative errors in between the subintervals on the boundary. Again, all the results are similiar when using different trial functions and quadrature rules.

TABLE 4B: Maximum relative errors in between the subintervals on the boundary

	$\lvert ue_3 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{27} \rvert$	$\lvert ue_{81} \rvert$	$\lvert ue_{243} \rvert$
delta 1-pt	1.35E+00	3.55E-01	4.89E-02	1.89E-02	8.40E-03
delta 3-pts	1.94E+00	3.28E-01	4.88E-02	1.88E-02	8.34E-03
p.c. 1-pt	1.38E+00	4.30E-01	6.70E-02	1.85E-02	9.31E-03
p.c. 3-pts	1.45E+00	1.00E+04	3.14E-02	9.68E-03	2.79E-03

In table 4C, we observe that the matrix condition numbers grow proportionally less than the numbers of subintervals.

TABLE 4C: matrix condition numbers

	$\|ue_3\|$	$\|ue_9\|$	$\|ue_{27}\|$	$\|ue_{81}\|$	$\|ue_{243}\|$
delta 1-pt	0.21E+01	0.10E+02	0.26E+02	0.60E+02	0.96E+02
delta 3-pts	0.23E+01	0.10E+02	0.26E+02	0.55E+02	0.89E+02
p.c. 1-pt	0.23E+01	0.99E+01	0.25E+02	0.71E+02	0.16E+03
p.c. 3-pts	0.29E+01	0.89E+01	0.25E+02	0.71E+02	0.16E+03

Table 4D shows the CPU time required for each run. Again, we see that it is more efficient to use a low quadrature rule and more subintervals.

TABLE 4D: CPU time

	$time_3$	$time_9$	$time_{27}$	$time_{81}$	$time_{243}$
delta 1-pt	0.21E+01	0.10E+02	0.26E+02	0.60E+02	0.96E+02
delta 3-pts	0.23E+01	0.10E+02	0.26E+02	0.55E+02	0.89E+02
p.c. 1-pt	0.23E+01	0.99E+01	0.25E+02	0.71E+02	0.16E+03
p.c. 3-pts	0.29E+01	0.89E+01	0.25E+02	0.71E+02	0.16E+03

For this example, it is best to use trapezoidal quadrature. We did note that SPLTRG works slightly better using delta trial functions than using piecewise constant trial functions. The corners and the data singularity at the origin affect the errors significantly.

Considering all four examples together, we recommend using delta trial functions with trapezoidal quadrature. If the boundary and boundary data are analytic, then the delta-trigonometric method with trapezoidal quadrature appears to obtain exponential convergence for the approximate potentials at points away from the boundary. Hence, our numerical results confirm the theory. In examples, where the boundary and/or boundary data are not smooth, our results are only fair and do not invalidate the theory. In all examples, using delta trial functions with trapezoidal quadrature works as well as using a higher quadrature rule and/or using piecewise constant trial functions.

4. Delta-Trigonometric and Spline-Trigonometric Petrov-Galerkin Methods using a Fictitious Boundary

We now investigate a formulation where the single-layer potential is concentrated on a fictitious boundary. We analyze convergence for only the interior Dirichlet problem with analytic boundary and boundary data. Consequently, we choose a fictitious domain which strictly contains the true domain. If we wished to solve the exterior Dirichlet problem, we would choose a fictitious domain which is strictly contained by the true domain. First, we redefine the operator A and the corresponding approximate potential. After reviewing some properties of this *fictitious* single-layer potential representation, we define the delta-trigonometric and the spline-trigonometric methods using a fictitious boundary without numerical quadrature. Finally, we define the matrix equations for the delta-trigonometric method with and without numerical quadrature. In section 4.1 and 4.2, we show that the delta-trigonometric method with and without numerical quadrature (respectively) obtains unique approximate potentials on circular domains with exponential convergence if we use the canonical parameterization.

Let Ω and Ω_o be open interior domains with boundaries Γ and Γ_o, respectively, such that $\overline{\Omega} \subset \Omega_o$. We redefine the operator A as

$$A\phi(s) := \int_0^1 \phi(t) \log |x(s) - x_o(t)| \ dt \quad \forall \ s \in [0,1],$$

where $x : [0,1] \to \Gamma$ and $x_o : [0,1] \to \Gamma_o$. Here x and x_o are 1-periodic analytic functions that parameterize Γ and Γ_o, respectively, and have nonvanishing derivatives. We approximate the potential as

$$u(z) \cong v(z) := \int_0^1 \phi(t) \log |z - x_o(t)| \ dt \quad \forall \ z \in \Omega. \tag{4.1}$$

A natural question to ask is how well can the potential be approximated by this *fictitious* single-layer representation? Given a potential u, there does not generally exist a ϕ such that (4.1) is exact. However, Bogomolny [5] showed that if we require $|x(s) - x_o(t)| \neq 1$ for all $s, t \in [0,1]$, then there exists a ϕ such that $||u - v||_{L^2(\Omega)}$ is arbitrary small. For instance, this is so if $\mathrm{diam}(\Omega_o) < 1$ or if Γ_o is placed far from $\overline{\Omega}$. In this case, the set

$$\left\{ v \ \Big| \ v(z) = \int_0^1 \phi(t) \log |z - x(t)| \ dt \quad \text{for} \quad z \in \Omega, \quad \phi \in C^\infty([0,1]) \right\}$$

is dense in the set

$$\{ u \in H^s(\Omega) \ \big| \ \Delta u = 0 \text{ in } \Omega \}$$

for all $s \in \mathbb{R}$ [5, theorem 2.1].

We have several choices of (1) the finite-dimensional trial spaces and (2) the procedures to select the trial function. The most interesting trial space is the span of delta functions. The resulting method is called the fundamental solution method (e.g. Bogomolny [5], Fairweather and Johnston [11], Mathon and Johnston [19], Kupradze and Aleksidze [16], Freeden and Kersten [12]), i.e.,

$$u_n(z) = \sum_{j=1}^{n} \alpha_j \log|z - y_j| \quad \text{for} \quad z \in \Omega, \tag{4.2}$$

where the y_j's are points outside of $\overline{\Omega}$ and the α_j's are the unknown coefficients.

We need to consider the following question. How well can the potential be approximated by using delta trial functions (i.e. a summation of logarithmic functions) and what is the optimal convergence? Kupradze and Aleksidze [16] showed that the functions,

$$\log|z - y_j|, \qquad j = 1, \ldots, n,$$

are independent and complete in $L^2(\Gamma)$ and $C(\Gamma)$. Therefore for any $\epsilon > 0$ there exists N such that for any $n > N$ there is a u_n of the form (4.2) satisfying

$$||u - u_n||_{L^\infty(\Omega)} < \epsilon.$$

Bogomolny [5] showed that any harmonic polynomial of degree $\leq n$ can be approximated by a u_n of the form (4.2) with an L^∞ error which decreases exponentially as n increases. Then he showed that the exact solution can be approximated by a u_n of the form (4.2) with an L^∞ error which decreases very rapidly as n increases.

Mathon and Johnston [19] showed that there exists a u_n of the form (4.2) which minimizes $||u - u_n||_{L^2(\Omega)}$. They used a least square method to find the coefficients of the delta functions and the locations of the singularities. The main drawback of their program is the nonlinear aspect which arises from allowing the singularities to vary. However, their method works well when u is of low continuity and for the three-dimensional Dirichlet problem. Bogomolny [5] studied where the singularities should be placed and then used a least square method to find only the coefficients of the delta functions (in this case, the matrices are linear). He obtained theoretical results which suggest that the singularities should be placed far away from the boundary.

No method has been developed to obtain exponential convergence for the approximate potentials using a *fictitious* boundary (or fundamental solution method). For circular domains, the delta-trigonometric method with and without numerical quadrature obtains a u_n of the form (4.2) with exponential convergence. We conjecture that exponential convergence results also hold for arbitrary analytic boundaries.

As a numerical method, we seek $\phi_n \in S_n^d$ such that

$$\int_0^1 A\phi_n(s)\psi(s)\ ds = \int_0^1 g(s)\psi(s)\ ds \quad \forall\ \psi \in T_n.$$

Then our approximate potential is

$$u_n(z) := \int_0^1 \phi_n(t) \log|z - x_o(t)|\ dt \quad \forall\ z \in \mathbb{R}.$$

We call the above procedure the delta-trigonometric Petrov-Galerkin method using a fictitious boundary for $d = -1$ and the spline-trigonometric Petrov-Galerkin method using a fictitious boundary for $d \geq 0$. Since the logarithmic functions are independent and complete [16], we know that the delta-trigonmetric method obtains unique solutions. We conjecture the same for the spline-trigonometric method. We note that the delta-trigonometric method with trapezoidal quadrature and spline-trigonometric method with trapezoidal quadrature are the same. In this report, we give a convergence analysis for the delta-trigonometric method only.

We now define the matrix equation for the delta-trigonometric method with trapezoidal quadrature. For the remaining part of this section, we assume that $d = -1$. Again, write

$$\phi_n(t) = \sum_{j=1}^{n} \alpha_j \delta(t - j/n)$$

where α_j's are the unknown coefficients. We redefine $\mathbf{A} := (\mathbf{A}_{kj})$ where

$$\mathbf{A}_{kj} := \int_0^1 \log\big|x(s) - x_o(j/n)\big|\psi_k(s)\ ds$$

and recall that $\mathbf{g} := (\mathbf{g}_k)$ where

$$\mathbf{g}_k := \int_0^1 g(s)\psi_k(s)\ ds$$

for all $j = 1, \ldots, n$ and $k \in \Lambda_n$. Note that the kernel of A is nonsingular, and therefore, no splitting of A is needed. Our matrix equation is

$$\mathbf{A}\alpha = \mathbf{g}$$

and our approximate potential is

$$u_n(z) = \sum_{j=1}^{n} \alpha_j \log|z - x_o(j/n)| \quad \forall \ z \in \mathbb{R}.$$

We now define the matrix equation for the delta-trigonometric method with trapezoidal quadrature. Let

$$\widetilde{\phi}_n(t) = \sum_{j=1}^{n} \widetilde{\alpha}_j \delta(t - j/n)$$

where $\widetilde{\alpha}_j$'s are the unknown coefficients. Redefine $\widetilde{\mathbf{A}} := (\widetilde{\mathbf{A}}_{kj})$ where

$$\widetilde{\mathbf{A}}_{kj} := \frac{1}{n} \sum_{p=1}^{n} \log \left| x(p/n) - x_o(j/n) \right| \psi_k(p/n)$$

and recall that $\widetilde{\mathbf{g}} := (\widetilde{\mathbf{g}}_k)$ where

$$\widetilde{\mathbf{g}}_k := \frac{1}{n} \sum_{p=1}^{n} g(p/n) \psi_k(p/n)$$

for all $j = 1, \ldots, n$ and $k \in \Lambda_n$. Then our matrix equation with numerical quadrature is

$$\widetilde{\mathbf{A}} \widetilde{\alpha} = \widetilde{\mathbf{g}} \tag{4.3}$$

and our approximate potential with numerical quadrature is

$$\widetilde{u}_n(z) = \sum_{j=1}^{n} \widetilde{\alpha}_j \log|z - x_o(j/n)| \quad \forall \ z \in \mathbb{R}.$$

4.1 Convergence Analysis on a Circular Domain without Numerical Quadrature

In this section, we show that the approximate potentials produced by the delta-trigonometric method without numerical quadrature converge exponentially on a circular domain if we use the canonical parameterization and if the fictitious circular domain is sufficiently large. The restriction to a circular domain enables us to analyze convergence simply with Fourier series. We conjecture that the spline-trigonometric and delta-trigonometric methods without numerical quadrature obtains exponential convergence for the approximate potentials on arbitrary boundaries, but it is not clear whether special parametrizations are needed.

We continue to assume that g is an analytic function and that Γ and Γ_o are circles. First, we prove a simple lemma which will be used to bound $||A\phi_n||_{L^\infty[0,1]}$ independently of n in theorem

4.1.2. Then we prove exponential error bounds for the approximate potentials on the boundary using lemma 3.1.2 (exponential decays of the Fourier coefficients of analytic functions). By the maximum principle we obtain exponential error bounds for the approximate potentials on the entire circular domain.

LEMMA 4.1.1 *Let $y, z \in \mathbb{C}$ be such that $|z| < |y|$. Then*

$$\log|y-z| = \log|y| + Re\Big\{\sum_{\gamma=1}^{\infty} \frac{-1}{\gamma}\Big(\frac{z}{y}\Big)^{\gamma}\Big\}.$$

PROOF:

Define the complex function,

$$f(x) = \log(1-x),$$

where $x \in \mathbb{C}$ and $|x| < 1$. Then the Taylor series expansion for $f(x)$ is

$$f(x) = \sum_{\gamma=1}^{\infty} \frac{-1}{\gamma} x^{\gamma}.$$

Also recall that

$$\log|x| = \mathrm{Re}\{\log x\}.$$

Using $x = z/y$, we derive

$$\begin{aligned} \log|y-z| &= \log|y| + \log|1 - z/y| \\ &= \log|y| + \mathrm{Re}\Big\{\sum_{\gamma=1}^{\infty} \frac{-1}{\gamma}\Big(\frac{z}{y}\Big)^{\gamma}\Big\}. \end{aligned}$$

Q.E.D.

Our next goal is to bound $\|A\phi_n\|_{L^\infty[0,1]}$ independently of n. We show that there is a nice relationship between the Fourier coefficients of $A\phi_n$. Then we represent $A\phi_n$ as a Fourier series and use this relationship to represent $A\phi_n$ in terms of $\widehat{A\phi}(k)$ for $k \in \Lambda_n$. Afterward, we use the definition of the delta-trigonometric method and lemma 3.1.2 to bound $A\phi_n$ in the complex strip $\overline{S}_\delta$. (Recall that $S_\delta := \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \delta\}$.)

THEOREM 4.1.2 *Assume that $d = -1$, Γ and Γ_o are circles with radii r and R parametrized by $x(s) = r\exp(2\pi i s)$ and $x_o(t) = R\exp(2\pi i t)$, respectively, and that g is a 1-periodic real analytic* *function. Let $\overline{\delta}$ be such that g extends analytically to the closed strip $\overline{S_{\overline{\delta}}}$ and let $0 < \delta < \overline{\delta}$. Then for R sufficiently large (depending only on r and δ), the delta-trigonometric method obtains unique solutions. Moreover, there exists a positive constant C depending only on δ and $\overline{\delta}$ such that*

$$|A\phi_n(s)| \leq C\|g\|_{L^\infty(S_{\overline{\delta}})} \quad \forall \ s \in \overline{S_\delta} \quad \textit{and} \quad n \in \mathbb{Z}^+.$$

PROOF:

We first prove a nice relationship between the Fourier coefficients of $A\phi_n$. By lemma 4.1.1,

$$\begin{aligned}
\widehat{A\phi}_n(k) &= \int_0^1 A\phi_n(s)\exp(-2\pi iks) \ ds \\
&= \int_0^1 \sum_{j=1}^n \alpha_j \log|x(s) - x_o(j/n)|\exp(-2\pi iks) \ ds \\
&= \int_0^1 \sum_{j=1}^n \alpha_j \log|r\exp(2\pi is) - R\exp(2\pi ij/n)|\exp(-2\pi iks) \ ds \\
&= \int_0^1 \sum_{j=1}^n \alpha_j \left(\log R + \mathrm{Re}\left\{\sum_{\gamma=1}^\infty \frac{-1}{\gamma}\Big(\frac{r\exp(2\pi is)}{R\exp(2\pi ij/n)}\Big)^\gamma\right\}\right)\exp(-2\pi iks) \ ds \\
&= \int_0^1 \sum_{j=1}^n \alpha_j \left(\log R + \sum_{\gamma\in\mathbb{Z}^*} \frac{-1}{2|\gamma|}\Big(\frac{r}{R}\Big)^{|\gamma|}\exp(2\pi i\gamma(s - j/n))\right)\exp(-2\pi iks) \ ds,
\end{aligned}$$

for all $k \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. We integrate with respect to s (using the orthogonality of the exponential functions) to get

$$\widehat{A\phi}_n(k) = \begin{cases} \sum_{j=1}^n \alpha_j \log R, & \text{if } k = 0, \\ \frac{-1}{2|k|}\left(\frac{r}{R}\right)^{|k|}\sum_{j=1}^n \alpha_j \exp(-2\pi ikj/n), & \text{if } k \neq 0. \end{cases} \tag{4.1.1}$$

Define

$$k_* = \begin{cases} |k|, & \text{if } k \neq 0, \\ \frac{-1}{2\log R}, & \text{if } k = 0. \end{cases}$$

Then (4.1.1) becomes

$$\widehat{A\phi}_n(k) = \frac{-1}{2k_*}\Big(\frac{r}{R}\Big)^{|k|}\sum_{j=1}^n \alpha_j \exp(-2\pi ikj/n) \quad \forall \ k \in \mathbb{Z} \quad \text{and} \quad n \in \mathbb{Z}^+. \tag{4.1.2}$$

Consequently,

$$\widehat{A\phi}_n(k + qn) = \frac{-1}{2(k+qn)_*}\Big(\frac{r}{R}\Big)^{|k+qn|}\sum_{j=1}^n \alpha_j \exp(-2\pi i(k+qn)j/n) \quad \forall \ k, q \in \mathbb{Z} \quad \text{and} \quad n \in \mathbb{Z}^+. \tag{4.1.3}$$

Putting (4.1.2) and (4.1.3) together, we get

$$\widehat{A\phi}_n(k+qn) = \frac{k_*}{(k+qn)_*}\Big(\frac{r}{R}\Big)^{|k+qn|-|k|}\widehat{A\phi}_n(k) \quad \forall\ k,q \in \mathbb{Z} \quad \text{and} \quad n \in \mathbb{Z}^+.$$

Next, we seek to bound $A\phi_n(s)$ in $\overline{S}_\delta$ for all $n \in \mathbb{Z}^+$. Note that

$$\begin{aligned}
|A\phi_n(s)| &= |\sum_{k\in\mathbb{Z}} \widehat{A\phi}_n(k)\exp(2\pi i k s)| \\
&\le \sum_{k\in\mathbb{Z}} |\widehat{A\phi}_n(k)||\exp(2\pi i k s)| \\
&= \sum_{p\in\Lambda_n}\sum_{q\in\mathbb{Z}} |\widehat{A\phi}_n(p+qn)||\exp(2\pi i(p+qn)s)| \\
&= \sum_{p\in\Lambda_n}\sum_{q\in\mathbb{Z}} |\widehat{A\phi}_n(p)|\Big|\frac{p_*}{(p+qn)_*}\Big|\Big(\frac{r}{R}\Big)^{|p+qn|-|p|}|\exp(2\pi i(p+qn)s)|, \quad \forall\ s \in \mathbb{R}.
\end{aligned} \tag{4.1.4}$$

Later on, we will choose R to satisfy $2\log R \ge 1$ so that if $k = 0$ then $0 < |k_*| \le 1$. This implies that $\Big|\frac{p_*}{(p+qn)_*}\Big| \le 1$ for all $p \in \Lambda_n$, $q \in \mathbb{Z}$, and $n \in \mathbb{Z}^+$. Therefore, for all $s \in \overline{S}_\delta$,

$$\begin{aligned}
|A\phi_n(s)| &\le \sum_{p\in\Lambda_n}\sum_{q\in\mathbb{Z}} |\widehat{A\phi}_n(p)|\Big(\frac{r}{R}\Big)^{|p+qn|-|p|}|\exp(2\pi|p+qn|\delta)| \\
&= \sum_{p\in\Lambda_n} |\widehat{A\phi}_n(p)|\exp(2\pi|p|\delta)\Big\{\sum_{q\in\mathbb{Z}}\Big(\frac{r\exp(2\pi\delta)}{R}\Big)^{|p+qn|-|p|}\Big\}.
\end{aligned} \tag{4.1.5}$$

By definition of the method, $\widehat{A\phi}_n(p) = \widehat{g}(p)$ for all $p \in \Lambda_n$. Hence,

$$|A\phi_n(s)| \le \sum_{p\in\Lambda_n} |\widehat{g}(p)|\exp(2\pi|p|\delta)\Big\{\sum_{q\in\mathbb{Z}}\Big(\frac{r\exp(2\pi\delta)}{R}\Big)^{|p+qn|-|p|}\Big\} \quad \forall\ s \in \overline{S}_\delta. \tag{4.1.6}$$

We now choose R sufficiently large so that $\frac{r}{R}\exp(2\pi\delta) < 1/2$ (and satisfying $2\log R \ge 1$). Then

$$\begin{aligned}
\sum_{q\in\mathbb{Z}}\Big(\frac{r\exp(2\pi\delta)}{R}\Big)^{|p+qn|-|p|} &\le \sum_{q\in\mathbb{Z}}(1/2)^{|p+qn|-|p|} \\
&\le 1 + \frac{2}{1-(1/2)^n} \\
&\le 1 + \frac{2}{1-1/2} \\
&= 5, \quad \forall\ p \in \Lambda_n \quad \text{and} \quad n \in \mathbb{Z}^+.
\end{aligned} \tag{4.1.7}$$

By lemma 3.1.2, $|\widehat{g}(p)| \le \exp(-2\pi\overline{\delta}|p|)||g||_{L^\infty(S_{\overline{\delta}})}$ for all $p \in \mathbb{Z}$. Therefore, using $0 < \delta < \overline{\delta}$, we derive

$$\begin{aligned}
|A\phi_n(s)| &\le 5\sum_{p\in\Lambda_n} |\widehat{g}(p)|\exp(2\pi|p|\delta) \\
&\le 5||g||_{L^\infty(S_{\overline{\delta}})}\sum_{p\in\Lambda_n}\exp(2\pi|p|(\delta-\overline{\delta})) \\
&\le 5||g||_{L^\infty(S_{\overline{\delta}})}\sum_{p\in\mathbb{Z}}\exp(2\pi|p|(\delta-\overline{\delta})) \\
&\le C||g||_{L^\infty(S_{\overline{\delta}})}, \quad \forall\ s \in \overline{S}_\delta \quad \text{and} \quad n \in \mathbb{Z}^+,
\end{aligned} \tag{4.1.8}$$

where C depends only on δ and $\overline{\delta}$.

We now prove the uniqueness result. By (4.1.1),

$$\mathbf{A}_{kj} = \begin{cases} \log R, & \text{if } k = 0, \\ \frac{-1}{2|k|}\left(\frac{r}{R}\right)^{|k|} \exp(-2\pi ikj/n), & \text{if } k \neq 0. \end{cases}$$

It is well known that trigonometric vectors are linearly independent. This implies that $\mathbf{A}$ is non-singular and therefore the delta-trigonometric method obtains unique solutions. Q.E.D.

The next theorem states exponential error bounds for the approximate potentials on the boundary of a circular domain. Then we use the maximum principle in theorem 4.1.4 to show exponential error bounds for the approximate potentials on the entire domain.

THEOREM 4.1.3 *Assume the same hypotheses as in theorem 4.1.2. Then for sufficiently large R (depending only on r and δ), there exist positive constants C and $\epsilon \in (0,1)$ depending only on δ and $\overline{\delta}$ such that*

$$|(A\phi_n - g)(s)| \leq C\epsilon^n \quad \forall\ s \in [0,1].$$

PROOF:

Note that

$$\begin{aligned}
|(A\phi_n - g)(s)| &= \Big|\sum_{k\in\mathbb{Z}} (\widehat{A\phi}_n - \widehat{g})(k)\exp(2\pi iks)\Big| \\
&= \Big|\sum_{k\notin\Lambda_n} (\widehat{A\phi}_n - \widehat{g})(k)\exp(2\pi iks)\Big| \\
&\leq \sum_{k\notin\Lambda_n} \Big|(\widehat{A\phi}_n - \widehat{g})(k)\Big| \\
&\leq \sum_{k\notin\Lambda_n} \Big\{|\widehat{A\phi}_n(k)| + |\widehat{g}(k)|\Big\} \\
&\leq \sum_{k\notin\Lambda_n} \Big\{\exp(-2\pi|k|\delta)\|A\phi_n\|_{L^\infty(S_\delta)} + \exp(-2\pi|k|\overline{\delta})\|g\|_{L^\infty(S_{\overline{\delta}})}\Big\}, \quad \forall\ s \in [0,1].
\end{aligned}$$

By theorem 4.1.2, (recall that $0 < \delta < \overline{\delta}$)

$$\begin{aligned}
|(A\phi_n - g)(s)| &\leq \sum_{k \notin \Lambda_n} \left\{ C_1 \exp(-2\pi|k|\delta) + \exp(-2\pi|k|\overline{\delta}) \right\} \|g\|_{L^\infty(S_{\overline{\delta}})} \\
&\leq (C_1 + 1)\|g\|_{L^\infty(S_{\overline{\delta}})} \sum_{k \notin \Lambda_n} \exp(-2\pi|k|\delta) \\
&= (C_1 + 1)\|g\|_{L^\infty(S_{\overline{\delta}})} \; 2 \sum_{k > n/2} \exp(-2\pi k\delta) \\
&= (C_1 + 1) \frac{2\exp(-\pi(n+1)\delta)}{1 - \exp(-2\pi\delta)} \|g\|_{L^\infty(S_{\overline{\delta}})} \\
&\leq C\epsilon^n, \quad \forall \; s \in [0, 1].
\end{aligned}$$

Q.E.D.

THEOREM 4.1.4 *Assume the same hypotheses as in theorem 4.1.2. Then for sufficiently large R (depending only on r and δ), there exist positive constants C and $\epsilon \in (0,1)$ depending only on δ and $\overline{\delta}$ such that*

$$|(u - u_n)(z)| \leq C\epsilon^n \quad \forall \; z \in \overline{\Omega}.$$

PROOF: Use theorem 4.1.3 and the maximum principle for harmonic functions. Q.E.D.

4.2 . Convergence Analysis on a Circular Domain with Numerical Quadrature

We now show that the approximate potentials produced by the delta-trigonometric method converge exponentially on a circular domain with numerical quadrature if the fictitious circular domain is sufficiently large. We note that the spline-trigonometric method with trapezoidal quadrature is the same as the delta-trigonometric method with trapezoidal quadrature, and therefore, we continue to analyze only the delta-trigonometric method. We conjecture that exponential convergence holds for both methods with numerical quadrature on arbitrary analytic boundaries. In lemma 4.2.1, we prove that $A\widetilde{\phi}_n$ is bounded independently of n. Then in theorem 4.2.2, we show exponential error bounds for the approximate potentials due to numerical quadrature.

REMARK: For the special case where both boundaries are circles, Christiansen and Lygung [8] showed that the condition numbers of the matrices produced by the collocation-discretization method (collocation of the boundary integral equation and discretization of the integral) can be calculated analytically. Their results showed that the matrices becomes ill-conditioned as n goes to infinity or as the (conformal) radius of the larger circle approaches 1. In section 4.3, we present results which shows that the matrix condition numbers in the delta-trigonometric method appear to grow exponentially as n increases. $\diamondsuit$

REMARK: For the case where a fictitious boundary is not used, we derived error bounds for the delta-trigonometric method without numerical quadrature (section 3.1). Then we bounded the perturbation of the matrices and vectors due to numerical integration and used this bound to derive error bounds for the delta-trigonometric method with numerical quadrature. Note that in section 3.1, we were able to bound the errors of the unknown coefficients due to numerical integration in (3.3.6) because $||\mathbf{A}||$ was proportionally bounded to $\sqrt{n}$, the square root of the numbers of subintervals (see section 3.2). However this idea fails when using a fictitious boundary because the condition numbers of $\mathbf{A}$ explode as n increases. In this section, we use Fourier analysis to bound $A\widetilde{\phi}_n$ as we did in section 4.1. $\diamondsuit$

Our next goal is to bound $||A\widetilde{\phi}_n||_{L^\infty[0,1]}$ independently of n. As in section 4.1, we note a nice relationship between the Fourier coefficients of $A\widetilde{\phi}$. Thus, we represent $A\widetilde{\phi}_n$ as a Fourier series and use this relationship to represent $A\widetilde{\phi}_n$ in terms of $\widehat{A\phi}(k)$ for $k \in \Lambda_n$. In section 4.1, we used $\widehat{A\phi}_n(k) = \widehat{g}(k)$ for all $k \in \Lambda_n$. This equation does not hold in the case of numerical quadrature. In this section, we show instead that $|\widehat{A\widetilde{\phi}_n}(k)| \leq |\widetilde{\mathbf{g}}_k|$ for all $k \in \Lambda_n$.

LEMMA 4.2.1 *Assume the same hypothesis as in theorem 4.1.2. Then for sufficiently large R (depending only on r and δ), there exist positive constants C and $\epsilon \in (0,1)$ depending only on δ and $\overline{\delta}$ such that*

$$|\widehat{A\widetilde{\phi}_n}(k)| \leq |\widetilde{\mathbf{g}}_k| \leq C\epsilon^k$$

for all $k \in \Lambda_n$. Moreover, $||A\widetilde{\phi}_n||_{L^\infty[0,1]}$ is bounded independently of n.

PROOF:

We first bound $\widehat{A\widetilde{\phi}_n}(k)$ in term of $\widetilde{\mathbf{g}}_k$. By equation (4.3) and lemma 4.1.1,

$$
\begin{aligned}
\widetilde{\mathbf{g}}_k &= (\widetilde{\mathbf{A}}\widetilde{\alpha})_k \\
&= \frac{1}{n}\sum_{p=1}^{n}\sum_{j=1}^{n}\widetilde{\alpha}_j \log|x(p/n)-x_o(j/n)|\psi_k(p/n) \\
&= \frac{1}{n}\sum_{p=1}^{n}\sum_{j=1}^{n}\widetilde{\alpha}_j \log|r\exp(2\pi ip/n)-R\exp(2\pi ij/n)|\exp(-2\pi ikp/n) \\
&= \frac{1}{n}\sum_{p=1}^{n}\sum_{j=1}^{n}\widetilde{\alpha}_j\Big(\log R+\mathrm{Re}\Big\{\sum_{\gamma\in\mathbb{Z}^+}\frac{-1}{\gamma_*}\Big(\frac{r}{R}\Big)^{|\gamma|}\exp(2\pi i\gamma(p-j)/n)\Big\}\Big)\exp(-2\pi ikp/n) \\
&= \frac{1}{n}\sum_{p=1}^{n}\sum_{j=1}^{n}\widetilde{\alpha}_j\sum_{\gamma\in\mathbb{Z}}\frac{-1}{2\gamma_*}\Big(\frac{r}{R}\Big)^{|\gamma|}\exp(2\pi i\gamma(p-j)/n)\exp(-2\pi ikp/n) \\
&= \frac{1}{n}\sum_{\gamma\in\mathbb{Z}}\sum_{j=1}^{n}\widetilde{\alpha}_j\frac{-1}{2\gamma_*}\Big(\frac{r}{R}\Big)^{|\gamma|}\exp(-2\pi i\gamma j)/n)\sum_{p=1}^{n}\exp(2\pi ip(\gamma-k)/n), \quad \forall\ k\in\Lambda_n.
\end{aligned}
$$

Now

$$
\sum_{p=1}^{n}\exp(2\pi ip(\gamma-k)/n)=\begin{cases} n, & \text{if } \gamma=k(\mathrm{mod1}\,n), \\ 0, & \text{otherwise.}\end{cases}
$$

Therefore,

$$
\widetilde{\mathbf{g}}_k=\sum_{\gamma=k\,(\mathrm{mod1}\,n)}\sum_{j=1}^{n}\widetilde{\alpha}_j\frac{-1}{2\gamma_*}\Big(\frac{r}{R}\Big)^{|\gamma|}\exp(-2\pi i\gamma j/n)\quad \forall\ k\in\Lambda_n.
$$

Also note that

$$
\begin{aligned}
\widehat{A\widetilde{\phi}_n}(\gamma) &= \int_0^1\sum_{j=1}^{n}\widetilde{\alpha}_j\log|r\exp(2\pi is)-R\exp(2\pi ij/n)|\exp(-2\pi i\gamma s)\ \ ds \\
&= \int_0^1\sum_{j=1}^{n}\widetilde{\alpha}_j\sum_{m\in\mathbb{Z}}\frac{-1}{2m_*}\Big(\frac{r}{R}\Big)^{|m|}\exp(2\pi im(s-j/n))\exp(-2\pi i\gamma s)\ \ ds \\
&= \sum_{j=1}^{n}\widetilde{\alpha}_j\frac{-1}{2\gamma_*}\Big(\frac{r}{R}\Big)^{|\gamma|}\exp(-2\pi i\gamma j/n),\quad \forall\ \gamma\in\mathbb{Z}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\widetilde{\mathbf{g}}_k &= \sum_{\gamma=k\,(\mathrm{mod1}\,n)}\widehat{A\widetilde{\phi}_n}(\gamma) \\
&= \Big\{\sum_{\gamma=k\,(\mathrm{mod1}\,n)}\frac{k_*}{\gamma_*}\Big(\frac{r}{R}\Big)^{|\gamma|-|k|}\Big\}\widehat{A\widetilde{\phi}_n}(k),\quad \forall\ k\in\Lambda_n.
\end{aligned}
$$

Since R will be chosen so that $2\log R\geq 1$, we know that $|k_*/\gamma_*|\leq 1$. By geometric series, the bracketed quantity in the last equation is bounded. Also note that the terms are all positive and the $\gamma = k$ term equal 1. Therefore the bracketed quantity in the last equation belongs to $[1, (1 + r/R)/(1 - r/R)]$ for all $k \in \Lambda_n$ and $n \in \mathbb{Z}^+$. Hence, $|\widehat{A\widetilde{\phi}_n}(k)| \le |\widetilde{\mathbf{g}}_k|$ for all $k \in \Lambda_n$ and $n \in \mathbb{Z}^+$. In view of the fact that $\widetilde{\mathbf{g}}_k$ is the same as $\widehat{g}(k)$ calculated with trapezoidal quadrature, we use theorem 3.3.2 to get

$$
\begin{aligned}
|\widehat{A\widetilde{\phi}_n}(k)| &\le C_1 |\widetilde{\mathbf{g}}_k| \\
&\le C_1 |\widetilde{\mathbf{g}}_k - \widehat{g}(k)| + C_1 |\widehat{g}(k)| \\
&\le C_1 \epsilon_1^n + C_2 \exp(-2\pi\overline{\delta}|k|) ||g||_{L^\infty(S_{\overline{\delta}})} \\
&\le C\epsilon^{|k|}, \quad \forall\ k \in \Lambda_n.
\end{aligned} \tag{4.2.1}
$$

To bound $A\widetilde{\phi}_n$ we represent it by a Fourier series. Then we use the relationship,

$$
\widehat{A\widetilde{\phi}_n}(k + qn) = \frac{k_*}{(k + qn)_*} \Big(\frac{r}{R}\Big)^{|k+qn|-|k|} \widehat{A\widetilde{\phi}_n}(k) \quad \forall\ k, q \in \mathbb{Z} \quad \text{and} \quad n \in \mathbb{Z}^+
$$

to express $A\widetilde{\phi}_n$ in term of $\widehat{A\widetilde{\phi}_n}(k)$ for $k \in \Lambda_n$. Afterward, we use (4.2.1) to bound $A\widetilde{\phi}_n$. (The idea is the same as in lemma 4.1.2, i.e, (4.1.4) to (4.1.8).) Q.E.D.

THEOREM 4.2.2 *Assume the same hypothesis as in theorem 4.1.2. Then for sufficiently large R (depending only on r and δ), there exist positive constants C and $\epsilon \in (0, 1)$ depending only on δ and $\overline{\delta}$ such that*

$$
|(A\widetilde{\phi}_n - g)(s)| \le C\epsilon^n \quad \forall\ s \in [0, 1].
$$

Moreover,

$$
|(u - \widetilde{u}_n)(z)| \le C\epsilon^n \quad \forall\ z \in \overline{\Omega}.
$$

PROOF:

Recall that

$$
\begin{aligned}
\widetilde{\mathbf{g}}_k &= (\widetilde{\mathbf{A}}\widetilde{\alpha})_k \\
&= \frac{1}{n} \sum_{j=1}^{n} \widetilde{\alpha}_j \sum_{p=1}^{n} \log|x(p/n) - x_o(j/n)| \exp(-2\pi i k p/n) \\
&= \frac{1}{n} \sum_{p=1}^{n} (A\widetilde{\phi}_n)(p/n) \exp(-2\pi i k p/n).
\end{aligned}
$$

Consequently, $\widetilde{\mathbf{g}}_k$ is equal to $\widehat{A\widetilde{\phi}_n}(k)$ with trapezoidal rule. By lemma 4.2.1, $A\widetilde{\phi}_n$ and g are bounded independently of n. Therefore by theorem 3.3.2,

$$
\sum_{k \in \Lambda_n} |\widehat{A\widetilde{\phi}_n}(k) - \widetilde{\mathbf{g}}_k| \le C_1 \epsilon_1^n \tag{4.2.2}
$$

and

$$\sum_{k\in\Lambda_n} |\tilde{g}_k - \hat{g}(k)| \le C_2\epsilon_2^n. \tag{4.2.3}$$

By lemma 3.1.2,

$$\sum_{k\notin\Lambda_n} \left\{ |\widehat{A\tilde{\phi}_n}(k)| + |\hat{g}(k)| \right\} \le C_3\epsilon_3^n. \tag{4.2.4}$$

Putting (4.2.2) to (4.2.4) together, we derive

$$\begin{aligned}
|(A\tilde{\phi}_n - g)(s)| &\le \sum_{k\in\Lambda_n} |\widehat{A\tilde{\phi}_n}(k) - \hat{g}(k)| + \sum_{k\notin\Lambda_n} |\widehat{A\tilde{\phi}_n}(k) - \hat{g}(k)| \\
&\le \sum_{k\in\Lambda_n} |\widehat{A\tilde{\phi}_n}(k) - \tilde{g}_k| + \sum_{k\in\Lambda_n} |\tilde{g}_k - \hat{g}(k)| + \sum_{k\notin\Lambda_n} \left\{ |\widehat{A\tilde{\phi}_n}(k)| + |\hat{g}(k)| \right\} \\
&\le C\epsilon^n, \quad \forall \ s \in [0,1].
\end{aligned}$$

By the maximum principle,

$$|(u - \tilde{u}_n)(z)| \le C\epsilon^n \quad \forall \ z \in \overline{\Omega}.$$

Q.E.D.

4.3 Numerical Results

Program SPLTRG is also able to perform the delta-trigonometric and the spline-trigonometric methods using a fictitious boundary. We present results for both methods using different quadrature rules. Recall that the spline-trigonometric method with trapezoidal quadrature is exactly the same as the delta-trigonometric method with trapezoidal quadrature. We also present results for the delta-trigonometric method with trapezoidal quadrature using different radii for the fictitious circle. We present the same four examples as in section 3.5 except we consider only the interior Dirichlet problem. In many of these examples, we study the errors on the boundary only. By the maximum principle, we know that the errors in the interior are no worse than the maximum errors on the boundary.

EXAMPLE 4.3.1 Ellipse with analytic data

The first example involves an elliptic boundary (an analytic boundary) with ideal boundary data.

Boundary: $x^2/4 + y^2 = 1/25$

Data: $g = 5x/2$

Exact solution:

$$u = \begin{cases} 5x/2, & \text{if } (x,y) \in \text{ellipse}, \\ 5x - w, & \text{if } (x,y) \notin \text{ellipse and } x \geq 0, \\ 5x + w, & \text{if } (x,y) \notin \text{ellipse and } x \leq 0, \end{cases}$$

where

$$w = \sqrt{\frac{25(x^2 - y^2) - 3 + \sqrt{(25(x^2 - y^2) - 3)^2 + 2500x^2y^2}}{2}} \;.$$

For the boundary, we examine the errors at the quadrature points and at the meshes points which are between the quadrature points. We first examine this problem for both methods using 1, 3, and 8 points quadrature. Table 1A shows the differences in the errors for both methods with different quadrature rules using $R = 2.0$. There are virtuely no differences in the errors when using different methods and quadrature rules. An analysis of table 1A indicates that very fast convergence rates are obtained for the approximate potentials on the boundary.

TABLE 1A: maximum relative errors on the boundary using $R = 2.0$

	$\lvert ue_1 \rvert$	$\lvert ue_3 \rvert$	$\lvert ue_5 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{15} \rvert$
delta 1-pt	7.80E+01	3.11E+00	2.96E−02	1.22E−05	3.50E−10
delta 3-pts	2.21E+00	3.23E+00	3.36E−02	1.44E−05	3.87E−10
delta 8-pts	1.00E+00	3.24E+00	3.37E−02	1.45E−05	3.88E−10
p.c. 3-pts	2.18E+00	1.62E+00	8.44E−03	1.88E−06	3.33E−11
p.c. 8-pts	1.00E+00	1.61E+00	8.19E−03	1.71E−06	2.72E−11

Table 1B shows the differences in CPU time for both methods with different quadrature rules using $R = 2.0$. It is most efficient to use delta trial functions with trapezoidal quadrature.

TABLE 1B: CPU time

	time_1	time_3	time_5	time_9	time_{15}
delta 1-pt	2.479	5.120	9.203	17.469	30.123
delta 3-pts	2.434	5.320	10.010	19.701	35.183
delta 8-pts	2.630	5.751	11.348	23.480	46.371
p.c. 3-pts	6.448	17.463	28.659	53.780	95.086
p.c. 8-pts	15.653	44.934	77.485	151.616	279.180

Table 1C shows the maximum relative errors on the boundary for the delta-trigonometric method with trapezoidal rule using different radii for the fictitious circle. Note that the errors are less as R increases. In fact, for $R \geq 50$, the errors are $O(10^{-12})$ for n as small as 9.

TABLE 1C: maximum relative errors on the boundary using delta trial function with trapezoidal quadrature

R	$\lvert ue_1 \rvert$	$\lvert ue_3 \rvert$	$\lvert ue_5 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{15} \rvert$
1.0	2.22E+00	2.53E+00	2.98E−01	2.08E−03	1.38E−06
1.2	4.58E+01	5.13E+00	1.29E−01	3.90E−04	2.37E−07
1.5	2.28E+02	4.10E+00	6.79E−02	8.62E−05	1.38E−08
2.0	7.80E+01	3.11E+00	2.96E−02	1.22E−05	3.50E−10
5.0	5.55E+01	1.28E+00	2.04E−03	2.23E−08	2.72E−12
10.0	5.35E+01	6.49E−01	2.61E−04	1.83E−10	2.72E−12
50.0	5.27E+01	1.31E−01	2.14E−06	4.93E−12	****
100.0	5.26E+01	6.56E−02	2.68E−07	8.99E−12	****

Table 1D show the matrix condition numbers for different R's. Note that the condition numbers are worse when the fictitious circle is too near the boundary or too far away. In fact, they increase almost as fast as the errors decrease. Thus, we have significant roundoff errors (as the errors decrease) and can not expect our errors to be better than $1.0E-12$.

TABLE 1D: matrix condition numbers

R	K_1	K_3	K_5	K_9	K_{15}
1.0	0.10E+01	0.28E+02	0.43E+04	0.15E+07	0.49E+09
1.2	0.10E+01	0.33E+01	0.17E+02	0.51E+03	0.64E+05
1.5	0.10E+01	0.67E+01	0.42E+02	0.20E+04	0.47E+06
2.0	0.10E+01	0.14E+02	0.11E+03	0.96E+04	0.57E+07
5.0	0.10E+01	0.77E+02	0.16E+04	0.86E+06	0.78E+10
10.0	0.10E+01	0.22E+03	0.91E+04	0.20E+08	0.15E+13
50.0	0.10E+01	0.19E+04	0.39E+06	0.22E+11	0.45E+18
100.0	0.10E+01	0.45E+04	0.18E+07	0.41E+12	0.25E+19

Graph 1A shows the relative errors on a line $x = 2y$. Note that there is no dramatic change in the errors as the line approaches the boundary (as it would be if a fictitious boundary is not used). For this interior Dirichlet problem, the results are significantly better using a fictitious boundary than not using a fictitious boundary since very fast convergence rates are obtained for the approximate potentials on the boundary. The most efficient way to solve this problem numerically is to use the delta-trigonometric method with trapezoidal quadrature and a very large fictitious circle (large R).

EXAMPLE 4.3.2 Ellipse with different data smoothness

This example involves the same elliptic boundary but with boundary data of different degrees of smoothness.

Boundary: $x^2/4 + y^2 = 1/25$

Data:

$$g = \begin{cases} 1.0, & \text{if } x \leq 0 \\ 1.0 + x^s, & \text{if } x \geq 0 \end{cases} \quad \text{for} \quad s = 0, 1, 2, 3, 4, 5, \quad \text{and} \quad 6$$

For this example, the exact potential is not known. Instead, the approximate relative errors are computed by using the approximate potentials for the largest n possible (before the condition numbers blow up). We present results using the delta-trigonometric method with trapezoidal quadrature only. (We note that using the spline-trigonometric method and/or higher quadrature rules do not change the errors significantly.) Table 2A and 2B show the maximum approximate relative errors on the boundary for different data smoothness using the delta-trigonometric method with trapezoidal quadrature with $R = 2.0$ and $R = 10.0$, respectively. The largest n such that the condition number is less than 10^{14} for $R = 10.0$ is 17 and for $R = 2.0$ is 31. At this largest n, we note that the results are better for $R = 2.0$. Thus, it is better not to use a fictitious circle too far from the true boundary.

TABLE 2A: maximum approximate relative errors on the boundary using delta trial functions with trapezoidal quadrature and $R = 2.0$

s	$\lvert ue_3 \rvert$	$\lvert ue_5 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{15} \rvert$
0	4.15E-01	1.01E+00	8.64E-01	1.91E-01
1	1.07E-01	4.70E-02	2.14E-02	1.17E-02
2	6.62E-02	1.91E-02	4.86E-03	1.07E-03
3	2.41E-02	8.61E-03	4.82E-04	1.09E-04
4	1.78E-02	5.53E-03	3.97E-04	2.78E-05
5	9.52E-03	3.03E-03	3.78E-04	1.12E-05
6	5.84E-03	1.50E-03	2.63E-04	4.19E-06

TABLE 2B: maximum approximate relative errors on the boundary using delta trial functions with trapezoidal quadrature and $R = 10.0$

s	$\lvert ue_1 \rvert$	$\lvert ue_3 \rvert$	$\lvert ue_5 \rvert$	$\lvert ue_9 \rvert$
0	1.47E+00	3.92E−01	9.74E−01	7.36E−01
1	4.54E−01	1.31E−01	3.32E−02	1.41E−02
2	2.02E−01	6.86E−02	2.52E−02	5.06E−03
3	1.02E−01	3.40E−02	1.38E−02	5.77E−04
4	6.19E−02	1.53E−02	5.95E−03	6.93E−04
5	4.60E−02	6.64E−03	3.15E−03	5.32E−04
6	3.96E−02	2.81E−03	1.51E−03	3.09E−04

The matrix condition numbers are the same as in example 4.3.1. Graph 2A and 2B shows the relative errors on the line $x = 2y$ for $s = 0, 1, \ldots, 6$. Again note that there is no significant change in the errors as the line approaches the boundary.

For this example, the results are slightly better using a smaller radius because the singularities in the potential is approximated better. For the interior points, it is better not to use a fictitious boundary. For the boundary points, it is better to use a fictitious boundary.

EXAMPLE 4.3.3 Rectangle with ideal data

The third example involves a boundary with corners, but the boundary data is linear.

Domain: $(-0.1, 0.1) \times (-0.1, 0.1)$

Data: $g = 5x/2$

The exact solution is known in the interior region only and coincides with the formula given for the data g. Table 3A shows the maximum relative errors on the boundary for both methods using different quadrature rules with $R = 2.0$. All the results are quite good, i.e., errors are $O(10^{-13})$ for $n = 15$. However, there are virtually no improvement in the errors when higher quadrature rules are used.

TABLE 3A: maximum relative errors on the boundary using $R = 2.0$

	$\|ue_1\|$	$\|ue_3\|$	$\|ue_5\|$	$\|ue_9\|$	$\|ue_{15}\|$
delta 1-pt	7.66E+01	1.11E+00	2.57E−03	1.94E−07	2.51E−13
delta 3-pts	9.84E+01	2.38E+00	3.42E−03	1.93E−07	1.79E−13
delta 8-pts	9.40E+01	2.38E+00	3.58E−03	1.93E−07	3.23E−13
p.c. 3-pts	1.07E+02	1.20E+00	8.79E−04	2.65E−08	1.12E−13
p.c. 8-pts	1.02E+02	1.19E+00	8.97E−04	2.41E−08	6.86E−14

The next table shows the maximum relative errors on the boundary for different size fictitious circles. Note that the errors are smaller as R increases.

TABLE 3B: maximum relative errors on the boundary using delta trial functions with trapezoidal quadrature

R	$\|ue_1\|$	$\|ue_3\|$	$\|ue_5\|$	$\|ue_9\|$	$\|ue_{15}\|$
1.0	4.06E+00	2.15E+00	1.46E-02	4.44E-06	****
1.2	4.06E+00	2.15E+00	1.46E-02	4.44E-06	****
1.5	7.08E+01	1.46E+00	6.07E-03	2.60E-07	2.66E-14
2.0	7.66E+01	1.11E+00	2.57E-03	1.94E-07	2.51E-13
5.0	8.10E+01	4.51E-01	1.65E-04	5.54E-11	8.99E-13
10.0	8.17E+01	2.27E-01	2.07E-05	1.04E-12	****
50.0	8.20E+01	4.56E-02	1.66E-07	3.23E-11	****
100.0	8.20E+01	2.28E-02	2.07E-08	2.29E-11	****

Table 3C shows the matrix condition numbers. As in example 3.1, the condition numbers grow almost as fast as the errors decrease.

TABLE 3C: matrix condition numbers using delta trial functions with trapezoidal quadrature

R	K_1	K_3	K_5	K_9	K_{15}
1.0	0.10E+01	0.22E+03	0.25E+05	0.42E+09	0.20E+15
1.2	0.10E+01	0.22E+03	0.25E+05	0.42E+09	0.20E+15
1.5	0.10E+01	0.11E+02	0.33E+03	0.11E+06	0.29E+09
2.0	0.10E+01	0.24E+02	0.92E+03	0.55E+06	0.35E+10
5.0	0.10E+01	0.13E+03	0.13E+05	0.46E+08	0.45E+13
10.0	0.10E+01	0.37E+03	0.73E+05	0.10E+10	0.81E+15
50.0	0.10E+01	0.32E+04	0.31E+07	0.11E+13	0.13E+18
100.0	0.10E+01	0.75E+04	0.15E+08	0.20E+14	0.16E+20

Graph 3A shows the relative errors on the line $x = y$. Note that the results are very nice and the errors remain small at the boundary.

For this problem, it is best to let R be large. It is remarkable that very fast convergence was obtained even though the boundary has corners. Thus, it is worthwhile using a fictitious boundary.

EXAMPLE 4.3.4 Wedge with analytic data

The last example involves an wedge-shaped domain. The potential possesses a singularity at the corner of the domain.

Interior Domain: (in polar coordinate) $0 < r < 1$, $0 < \theta < 2\pi/5$

Data: $g = 1.0 + r^{2.5}\sin(2.5\theta)$

The exact solution is known in the interior region only and coincides with the formula given for the data g.

Table 4A shows the maximum relative errors on the boundary for both methods using different quadrature rules and $R = 2.0$. Note that there are virtually no differences in the errors for different quadrature rules. Overall, the results are only fair.

TABLE 4A: maximum relative errors on the boundary using $R = 2.0$

	$\|ue_1\|$	$\|ue_3\|$	$\|ue_5\|$	$\|ue_9\|$	$\|ue_{15}\|$
delta 1-pt	9.99E-01	3.73E-01	3.14E-01	1.05E-02	1.99E-04
delta 3-pts	9.99E-01	5.50E-01	2.48E-01	4.73E-03	1.87E-04
delta 8-pts	9.99E-01	5.68E-01	2.48E-01	4.79E-03	1.90E-04
p.c. 3-pts	9.05E-01	5.38E-01	1.65E-01	1.57E-03	2.07E-04
p.c. 8-pts	6.18E-01	5.54E-01	1.65E-01	1.53E-03	1.93E-04

Table 4B shows the maximum relative errors on the boundary using the delta trial functions with trapezoidal quadrature for different radii for the fictitious circle. Note that the errors did not improve very much as R becomes large. Again the results are only fair.

TABLE 4B: maximum relative errors on the boundary using delta trial functions with trapezoidal quadrature

R	$\|ue_1\|$	$\|ue_3\|$	$\|ue_5\|$	$\|ue_9\|$	$\|ue_{15}\|$
1.2	9.43E+00	5.22E-01	2.00E-01	5.50E-01	1.74E-01
1.5	2.64E+00	3.27E-01	4.01E-01	8.67E-02	6.78E-03
2.0	9.99E-01	3.73E-01	3.14E-01	1.05E-02	1.99E-04
5.0	5.53E-01	3.79E-01	1.82E-01	3.50E-03	2.69E-04
10.0	5.18E-01	3.77E-01	1.51E-01	3.64E-03	3.65E-04
50.0	5.02E-01	3.76E-01	1.30E-01	3.24E-03	****
100.0	5.01E-01	3.76E-01	1.27E-01	3.17E-03	****

The last table shows the matrix condition numbers. Again, the condition numbers grow faster as n increases. We also noted that the condition numbers are worse if the fictitious circle is too close to the true boundary or too far away.

TABLE 4C: matrix condition numbers using delta trial functions with trapezoidal quadrature

R	K_1	K_3	K_5	K_9	K_{15}
1.2	0.10E+01	0.13E+02	0.54E+02	0.73E+05	0.88E+07
1.5	0.10E+01	0.81E+01	0.79E+02	0.14E+05	0.24E+07
2.0	0.10E+01	0.78E+01	0.13E+03	0.16E+05	0.57E+07
5.0	0.10E+01	0.26E+02	0.76E+03	0.22E+06	0.86E+09
10.0	0.10E+01	0.74E+02	0.38E+04	0.34E+07	0.68E+11
50.0	0.10E+01	0.64E+03	0.15E+06	0.30E+10	0.86E+16
100.0	0.10E+01	0.15E+04	0.72E+06	0.55E+11	0.12E+18

Graph 4A shows the relative errors on the line $x = 2y$. Note that there is no significant changes as the line approaches the boundary. For this example, the results are not so good because of the singularity of the data.

Considering both methods and different quadrature rules, we see that the delta-trigonometric method with trapezoidal quadrature works the best. We need to consider when a fictitious boundary should be used. If the data is smooth, then using a fictitious boundary obtains very rapid convergence on the boundary as well as in the interior. (Recall that in section 3.5, we obtain rapid convergence at points away from the boundary only). If the data is bad, then using a fictitious boundary obtains better results on the boundary, but worse results in the interior. It turns out that the fictitious boundary should not be too far from the true boundary. If the fictitious boundary is used, then there are no significant differences between the errors on the boundary and the errors in the interior. (In section 3.5, we saw significant changes in the errors as we approach the boundary.)

5. Appendix

In this section, we define the conformal radius and discuss some of its basic properties in relation to the single-layer potential representation. In particular, we explain why the conformal radius of Γ should not be equal to 1 in (1.2). We begin with the Riemann mapping theorem and a corollary. Then we present a theorem which states that solving the Dirichlet problem is equivalent to solving the single-layer potential problem using the restriction formulation (defined in section 3). Afterward, we present a theorem which states that if the conformal radius of Γ is not equal to 1 then solving the Dirichlet problem is equivalent to solving the single-layer potential problem using the scaling formulation (defined in section 3).

For the next two theorems, let $B_0(1)$ be the open unit ball centered at the origin and let Ω be an a simply-connected open set with analytic boundary Γ. Also let $B_0^c(1)$ and Ω^c be the corresponding open exterior regions. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

THEOREM 5.1 *(Riemann mapping theorem) Let $z_o \in \Omega$ be arbitrary. Then there exists a unique conformal mapping $S : \Omega \to B_0(1)$ such that $S(z_o) = 0$ and $S'(z_o) > 0$.*

PROOF: See for example, B. Choudhay [6, chapter 9 and appendix I]. Q.E.D.

COROLLARY 5.2 *There exists a unique positive number a and conformal mapping $T : \Omega^c \to B_0^c(1)$ such that $T(z) = a^{-1}z(1 + O(|z|^{-1}))$ as $|z| \to \infty$. The number a is called the conformal radius of Γ.*

PROOF:

Let $s(z) = z/|z|^2$ and define $\Omega^+ = \{s(z)\Big| z \in \Omega^c\} \cup \{0\}$ so that Ω^+ is a bounded simply-connected region in $\mathbb{R}^2$. Then by theorem 5.1, there exists a mapping $S : \Omega^+ \to B_0(1)$ such that $S(0) = 0$ and $S'(0) > 0$. Hence, $T(z) = s(S(s(z)))$ is a conformal mapping of Ω^c onto $B_0^c(1)$.

By Taylor theorem, $S(y) = S(0) + yS'(0) + O(|y^2|) = yS'(0) + O(|y^2|)$ as $y \to 0$. This implies that

$$
\begin{aligned}
T(z) &= \frac{S(s(z))}{|S(s(z))|^2} \\
&= \frac{s(z)S'(0) + O(|s(z)^2|)}{|s(z)|^2 S'(0)^2 + O(|s(z)^3|)} \\
&= \frac{(z/|z|^2)S'(0) + O(1/|z|^2)}{(1/|z|^2)S'(0)^2 + O(1/|z|^3)} \\
&= \frac{z + O(1)}{S'(0)}
\end{aligned}
$$

as $|z| \to \infty$. This shows that T has the desired form with $a = S'(0)$. Uniqueness follow from the uniqueness of the conformal map (theorem 5.1). Q.E.D.

REMARK: If Γ is a circle, then the conformal radius is the usual radius. If $\Gamma^\rho = \{\rho z \mid z \in \Gamma\}$ for some $\rho > 0$ then the conformal radius of Γ^ρ is ρ times the conformal radius of Γ.

We now state a uniqueness theorem for solving the single-layer potential problem using the restriction formulation (defined in section 3).

THEOREM 5.3 *Given* $(g, \lambda) \in (H^{1/2}(\Gamma), \mathbb{R})$*, there exists a unique pair* $(\Phi, c) \in (H^{-1/2}(\Gamma), \mathbb{R})$ *such that*

$$g(z) = \int_\Gamma \Phi(y) \log|z - y| \; d\sigma_y + c \qquad \forall \; z \in \Gamma$$

and

$$\lambda = \int_\Gamma \Phi(y) \; d\sigma_y.$$

Moreover, if Γ *is* C^1*, then the relation* $(g, \lambda) \to (\Phi, c)$ *is an isomorphism from* $(H^{1/2}(\Gamma), \mathbb{R})$ *to* $(H^{-1/2}(\Gamma), \mathbb{R})$.

PROOF: See M.N. LeRoux [17] Q.E.D.

We now define the operator

$$A^*\Phi(z) := \int_\Gamma \Phi(y) \log|z - y| \; d\sigma_y.$$

The next theorem explains an important property of A^*. In particular, we show that the conformal radius of Γ should not be 1 when we use the scaling formulation.

THEOREM 5.4 *Assume that Γ is C^1. The following are equivalent:*

1) $A^* : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is an isomorphism.

2) There does not exist a $\Phi_o \in H^{-1/2}(\Gamma)$ such that $A^*\Phi_o = 0$ on Γ and $\int_\Gamma \Phi_o \ d\sigma_y = 1$.

3) The conformal radius of Γ does not equal 1.

PROOF:

It is obvious that 1) implies 2). Suppose 1) is not true, i.e., A^* is not an isomorphism. Then there exists a nonzero Φ such that $A^*\Phi = 0$. There are two possibilities, $\int_\Gamma \Phi \ d\sigma_y = 0$ or $\int_\Gamma \Phi \ d\sigma_y \neq 0$. For the first case, $A^*\Phi = 0$ implies that $(\Phi, 0)$ solves (5.1) and (5.2) with data $(f, \lambda) = (0, 0)$. We also know that $(0, 0)$ solves (5.1) and (5.2) with data $(f, \lambda) = (0, 0)$. By the uniqueness result in theorem 5.3, this is a contradiction. For the second case, set $\xi = \Phi/(\int_\Gamma \Phi \ d\sigma_y)$ so that $\int_\Gamma \xi = 1$. This contradicts 2). Therefore 2) implies 1).

We now prove 3) implies 2). Suppose that 2) is not true, i.e., there exists a $\Phi_o \in H^{-1/2}(\Gamma)$ such that $A^*\Phi_o = 0$ on Γ and $\int_\Gamma \Phi_o = 1$. First, note that $u(z) = \log|z|$ solves the exterior homogeneous Dirchlet problem on $B_0^c(1)$, i.e.,

$$\Delta u = 0 \quad \text{on} \quad B_0^c(1),$$
$$u_1 = 0 \quad \text{on} \quad \partial B_0(1).$$

Let T be the 1-1, onto, conformal mapping as defined in theorem 5.2. Define $u_1(z) = \log|T(z)|$. Then

$$\Delta u_1 = 0 \quad \text{on} \quad \Omega^c,$$
$$u_1(z) = \log|z| - \log|a| + O(1) \quad \text{as} \quad |z| \to \infty,$$
$$u_1 = 0 \quad \text{on} \quad \Gamma.$$

Also define $u_2(z) = \int_\Gamma \log|z - y|\Phi_o(y) \ d\sigma_y$. Then $A^*\Phi_o = 0$ on Γ implies

$$\Delta u_2 = 0 \quad \text{on} \quad \Omega^c.$$
$$u_2(z) = \log|z| + O(1) \quad \text{as} \quad |z| \to \infty.$$
$$u_2 = 0 \quad \text{on} \quad \Gamma.$$

Therefore $u_1 - u_2$ is harmonic, bounded and vanishes on Γ. Hence $u_1 = u_2$, which implies $\log|a| = 0$, i.e., $a = 1$. This contradicts 3). This proves that 3) implies 2).

Finally suppose that 3) is not true, i.e., that $a = 1$. Then u_1 satisfies

$$\Delta u_1 = 0 \quad \text{on} \quad \Omega^c,$$
$$u_1(z) = \log|z| + O(1) \quad \text{as} \quad |z| \to \infty,$$
$$u_1 = 0 \quad \text{on} \quad \Gamma.$$

Now there exists (Φ, c) such that $A^*\Phi + c = 0$ and $\int_\Gamma \Phi \; d\sigma_y = 1$. Define

$$u_3(z) = \int_\Gamma \log|z - y|\Phi(y) \; d\sigma_y + c.$$

Then

$$\Delta u_3 = 0 \quad \text{on} \quad \Omega^c.$$
$$u_3(z) = \log|z| + c + O(|z|^{-1}) \quad \text{as} \quad |z| \to \infty.$$
$$u_3 = 0 \quad \text{on} \quad \Gamma.$$

Hence, $u_1 = u_3$ and therefore $c = 0$, which contradicts 2). Therefore, 2) implies 3). Q.E.D.

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