Assignment 2

1. The Fourier series is $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt$. Note that the periodic version of t^2 on the interval $[-\pi, \pi]$ is continuous. The coefficients of the Fourier series decay as $\frac{1}{n^2}$. See Notes pp. 65. Also, note that if you let $t = \pi$, you can show that $\sum_{n=1}^{\infty} n^2 = \frac{\pi^2}{6}$. Compare this result with that of exercise 7.

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2. The Fourier series of the Box function (sometimes called Gate function) is $\frac{1}{2}+\frac{2}{n\pi}\sum_{n=1}^{\infty} \sin(\frac{n\pi}{2}) \cos nt$. Now the decay of the Fourier coefficient is as $\frac{1}{n}$, because the Box function is discontinuous at $\pm\pi$. You should observe the Gibbs phenomenon. Also, you should notice that in this problem we need more coefficients than that of problem 1 to approximate $f(t)$. See the textbook pp. 45-47, and Notes on pp. 54.

3.
   - Let $f_1(x) = \frac{\sin x}{x}$, and $f_2(x) = \left(\frac{\sin x}{x}\right)^2$. We have proved in the class that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-i\lambda x} dx = \begin{cases} \pi & |\lambda| \leq 1 \\ \pi/2 & |\lambda| = 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 e^{-i\lambda x} dx = \begin{cases} \pi(1 - |\lambda|/2) & |\lambda| < 2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

   Let $f(x) = f_1(x) * f_2(x)$. Then,

$$f(0) = \int_{-\infty}^{\infty} \frac{\sin t}{t} \left(\frac{\sin(-t)}{(-t)}\right)^2 dt \tag{3}$$

   In the frequency domain, $\hat{f}(\lambda) = \hat{f}_1(\lambda)\hat{f}_2(\lambda)$. Since $f(x) = \int_{\infty}^{-\infty} f(\lambda)e^{i\lambda x} d\lambda$, we can evaluate $f(0)$ as
   follows

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}_1(\lambda)\hat{f}_2(\lambda)d\lambda = \frac{1}{2\pi}\int_{-1}^{1} \pi^2(1 - \frac{|\lambda|}{2})d\lambda = \frac{3\pi}{4} \tag{4}$$

   By changing the variable $u = ax$, you get the desired result.
   - Let $f_1(x) = f_2(x) = (\sin(x)/x)^2$, and follow the same procedure as above. Or you can use the Plancherel Formula as shown in the next problem.

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4. Using the *Plancherel Formula*: $(f, g)_{L^2} = (\mathcal{F}f, \mathcal{F}g)_{\hat{L}^2}$ for any $f, g \in L^2[-\infty, \infty]$. We have proved that $\text{sinc}(x) \in L^2[-\infty, \infty]$. So,

$$\int_{-\infty}^{\infty} \mathrm{sinc}(x-m)\mathrm{sinc}(x-n)dx$$

$$\begin{aligned}
&= \langle \mathrm{sinc}(x-m), \mathrm{sinc}(x-n)\rangle \\
&= \langle \mathcal{F}\mathrm{sinc}(x-m), \mathcal{F}\mathrm{sinc}(x-n)\rangle \\
&= \langle e^{-im\lambda}\mathcal{F}\mathrm{sinc}(x), e^{-in\lambda}\mathcal{F}\mathrm{sinc}(x)\rangle \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\lambda}d\lambda \\
&= \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}
\end{aligned}$$

5.
- $\hat{f}(\lambda) = \frac{2}{\lambda}(\sin 2\lambda - \sin\lambda)$. This function is even, that is $\hat{f}(\lambda) = \hat{f}(-\lambda)$.
- Since $\hat{f}(\lambda) \in L^1[-\infty, \infty]$, $\mathcal{F}^*\hat{f}'(\lambda) = -ixf(x)$.
- $\mathcal{F}^*(\hat{f} * \hat{f})(\lambda) = 2\pi f^2(t)$. Its graph is the same as that of $f(t)$, but scale the vertical axis by 2π.
- $\mathcal{F}^*\hat{f}(\lambda/2) = 2f(2t)$. Recall the uncertainty principle, here you expand the function in the frequency domain, so it's compressed in the time domain.

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6. Given the recurrence $f(t) = f(2t) + f(2t-1)$, we prescribe f(t) on the interval $[0,1)$: $f(t) = g(t),\ 0 \leq t < 1$. Then for $0 \leq s < \frac{1}{2}$ we have $g(s) = g(2s) + f(2s-1)$, so $f(t) = g(\frac{t+1}{2}) - g(t+1)$ $-1 \leq t < \frac{1}{2}$. For $\frac{1}{2} \leq s < 1$ we have $g(s) = f(2s) + g(2s-1)$, so $f(t) = g(\frac{t}{2}) - g(t-1)$ for $1 \leq t < 2$. Continuing in this way (using mathematical induction) we can determine $f(t)$ for *all* t in terms of the function $g(t)$ on $0 \leq t < 1$. Unless $g(t)$ is a very special function, f will not have a Fourier integral at all. Now suppose $f(t)$ has a Fourier transform, then

$$\hat{f}(\lambda) = \frac{1}{2}\hat{f}(\lambda/2)(1 + e^{-i\lambda/2}) \tag{5}$$

We can define $\hat{h}(\lambda) = \frac{1}{2}(1 + e^{-i\lambda/2})$. Recursively, we can deduce that

$$\hat{f}(\lambda) = \prod_{j=1}^{J}\hat{h}(\lambda/2^j)\hat{f}(\lambda/2^J) \tag{6}$$

As $J \to \infty$,

$$\hat{f}(\lambda) = \prod_{j=1}^{\infty}\hat{h}(\lambda/2^j)\hat{f}(0) \tag{7}$$

Let $\alpha = e^{-i\lambda/2^J}$. Then, $\prod_{j=1}^{J}\hat{h}(\lambda/2^j) = 1/2^J(1+\alpha)(1+\alpha^2)(1+\alpha^4)\cdots(1+\alpha^{2^J-1})$. That is

$$\prod_{j=1}^{J} \hat{h}(\lambda/2^j)$$

$$\begin{aligned}
&= \frac{1}{2^J}(1+\alpha+\alpha^2+\alpha^3+\cdots+\alpha^{2^J-1}) \\
&= \frac{1}{2^J}\frac{1-\alpha^{2^J}}{1-\alpha} \\
&= \frac{1}{2^J}\frac{1-e^{-i\lambda}}{1-e^{-i\lambda/2^J}}
\end{aligned}$$

Consider the denominator of the last equation

$$2^J(1-e^{-i\lambda/2^J}) = 2^J\left(1-(1-i\lambda/2^J+(i\lambda/2^J)^2+\cdots)\right) = i\lambda + O(2^{-J}) \tag{8}$$

As $J \to \infty$,

$$\lim_{J\to\infty}\prod_{j=1}^{J}\hat{h}(\lambda/2^j) = \frac{1-e^{-i\lambda}}{i\lambda} \tag{9}$$

Substitute the final result into equation (7):

$$\hat{f}(\lambda) = \frac{1-e^{-i\lambda}}{i\lambda}\hat{f}(0) \tag{10}$$

Thus, if $\hat{f}(0)$ exists, then we know f to within a constant factor. It is $f(t) = 1$ for $0 \le t < 1$ and $f(t) = 0$ otherwise. There are a huge number of solutions of the original recurrence relation, but only one of those has a Fourier transform that is defined at $\lambda = 0$.

7.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi e^{-a|\lambda|}e^{i\lambda t}d\lambda \tag{11}$$

$$f(t) = \frac{1}{2}\int_{0}^{\infty}e^{-a\lambda+i\lambda t}d\lambda + \frac{1}{2}\int_{-\infty}^{0}e^{a\lambda+i\lambda t}d\lambda = \frac{a}{t^2+a^2} \tag{12}$$

Using Poisson Summation formula given by

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\hat{f}(n) \tag{13}$$

we get

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\frac{1+e^{-2\pi a}}{1-e^{-2\pi a}} \tag{14}$$

As $a \to +0$, both sides of the equation $\to \infty$. The left side because of the division by zero when $n = 0$, and the right side is obvious. Rearrange the terms in the equation so that you can take the limit as $a \to +0$.

$$2\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a}\frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}} - \frac{1}{a^2} = \frac{\frac{2\pi^3 a^3}{3} + \cdots}{2\pi a^3 - 2\pi^2 a^4 + \cdots}. \tag{15}$$

(You may use l'Hospital's theorem (3 times) , Taylor's theorem or Mathematica. It is only the terms of order a^3 in the numerator and denominator that contribute.).

$$2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} \tag{16}$$

Compare this result with that of problem 1.

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8. $\hat{h}(\lambda) = A/(\alpha + i\lambda)$. $|\hat{h}(\lambda)| = A/(\sqrt{\alpha^2 + \lambda^2})$, so $\lim_{\lambda\to\infty} |\hat{h}(\lambda)| = 0$. When $A = \alpha$, $|\hat{h}(\lambda)| = 1/(\sqrt{1 + (\lambda/\alpha)^2})$. α is now the cutoff frequency of the filter. As you increase α, the filtered signal becomes smoother by killing frequencies which are more than α. For example, $\alpha = 10$, the high frequency component at 40 will die out. You may plot these results with Mathematica. The convolution operator (*) is available in Mathematica.