Variable separation in mathematical physics: From intuitive concept to computational tool

E. G. Kalnins
Department of Mathematics and Statistics,
University of Waikato,
Hamilton, New Zealand.
W. Miller, Jr.
School of Mathematics, University of Minnesota,
Minneapolis, Minnesota, 55455, U.S.A.

March 15, 2006

Abstract We review the basic intuitive definition of separation of variables (additive, multiplicative and functional) for the partial differential equations of mathematical physics and show how it leads to constructive methods for finding separable systems and their solutions. In many important cases there are deep connections with Lie theory and integrable systems, but there are interesting applications where no such link seems to exist. We survey the basic results relating orthogonal separation and R-separation for Hamilton-Jacobi equations and Schrödinger eigenvalue equations on n-dimensional Riemannian manifolds to the symmetries of these equations, and we present methods for classifying the possible separable coordinate systems. We conclude with a survey of results and challenging problems involving superintegrable systems, typically multiseparable systems in which the eigenvalues and eigenfunctions of the various separable systems can be determined by algebraic methods.

1 Some approaches to separability

In this section we will provide a brief review of some approaches to classical separation of variables methods for partial differential equations, from intuitive to technical. Here we will focus on separation for a single equation, although although questions of separability for systems of equations (particularly the spinor equations of mathematical physics where frames as well as separable coordinates must be specified) are of great current interest [1, 2].

We begin with the intuitive concept. Suppose we have a finite order partial differential equation in N independent variables $x^1, \cdots, x^N$ and one independent variable u:

$$H(x^i, u, u_i, u_{ij}, u_{ijk}, \cdots) = E \qquad 1 \leq i, j, k, \cdots \leq N \tag{1}$$

Here, $u_i = \partial_{x^i} u$, $u_{ij} = \partial_{x^i}\partial_{x^j} u$, etc., and E is a constant. A solution of (1) taking the form $u^{(1)} = \sum_{i=1}^N S^{(i)}(x^i)$ is an additively separable solution. Similarly a solution of the form $u^{(2)} = \prod_{i=1}^N T^{(i)}(x^i)$ is multiplicatively separable. Note that if we make the substitution $u = \exp v$ in (1) then the multiplicatively separable solution $u^{(2)}$ becomes an additively separable solution v of the modified PDE

$$\tilde{H}(x^i, v, v_i, v_{ij}, v_{ijk}, \cdots) = E \qquad 1 \leq i, j, k, \cdots \leq N.$$

Similarly we can regard any solution $u^{(3)} = f\left(\sum_{i=1}^N S^{(i)}(x^i)\right)$ of (1), where f is a given fixed function, as a separable solution in the coordinates x^i. By substituting the form of the separable solution into the PDE, we can always recast the problem as one in which we are searching for an additively separable solution $u = f^{-1}(u^{(3)})$ of an equation of the form (1).

This definition is general, but it gives little guidance in computing separable solutions and determining which systems x^i permit separation. Intuitive concepts need to be linked to computing mechanisms to make separation of variables a practical tool.

The following naive approach to a computing mechanism can be found in many textbooks and research papers. Since we are looking for additive separable solutions, we can assume no cross partials appear in the PDE':

$$H(x^i, u, u_i, u_{ii}, u_{iii}, \cdots) = E \qquad 1 \leq i \leq N. \tag{2}$$

Example 1: Suppose we can write (2) in the form

$$K^{(1)}(x^1, u_1, u_{11}, \cdots) = K(E, x^j, u_j, u_{jj}, \cdots), \qquad 2 \leq j \leq N. \tag{3}$$

Then we can split off the variable x^1 and find a solution $u = S^{(1)}(x^1) + T(x^2, \cdots, x^N)$. Indeed, substituting this solution into (3 we see that the left-hand side depends only on the variable x^1, whereas the right-hand side depends only on the variables, $x^2, \cdots, x^N$. Thus we have the ordinary differential equation $K^{(1)}(x^1, S^{(1)\prime}, S^{(1)\prime\prime}, \cdots) = c_1$ for $S^{(1)}(x^1)$ and the reduced PDE $K(E, x^j, T_j, T_{jj}, \cdots) = c_1$ where c_1 is the separation constant. If the reduced PDE allows another variable to be split off, we can continue the process, and sometimes obtain a separable solution $u = \sum_i S^i(x^i)$ where each S^i is the solution of an explicit ODE and depends on N separation constants $c_1, c_2, \cdots, c_N = E$. We call this approach naive, even though it leads to a procedure for computing the separated solutions, because many important separable systems cannot be so obtained. Indeed the ellipsoidal and paraboloidal coordinates for Euclidean N-space Helmholtz equations with $N \geq 3$ do not permit separation one variable at a time.

The next approach we present, basically due to Stäckel [3, 4], is technical rather than intuitive. It involves postulating separation equations and then combining these ODEs to form the desired PDE.

Example 2: Orthogonal separation for the Hamilton-Jacobi equation.

$$\sum_{i=1}^{N} H_i^{-2} u_i^2 + V(\mathbf{x}) = E. \tag{4}$$

Here, the metric in the coordinates x^i is $ds^2 = \sum_{i=1}^N H_i^2 (dx^i)^2$. We want to obtain additive separation, so that $\partial_j u_i = \partial_j \partial_i u = 0$ for $i \neq j$. We assume existence of separation equations in the form

$$u_i^2 + v_i(x^i) + \sum_{j=1}^{N} s_{ij}(x^i)\lambda^j = 0, \qquad i = 1, \cdots, N, \quad \lambda^1 = -E. \tag{5}$$

Here $\partial_k s_{ij}(x^i) = 0$ for $k \neq i$ and $\det(s_{ij}) \neq 0$. We say that $S = (s_{ij})$ is a *Stäckel matrix.*

Then (4) can be recovered from (5) provided $H_j^{-2} = (S^{-1})^{1j}$ and $V = \sum_j v_j (S^{-1})^{1j}$. The quadratic forms $\mathcal{H}^\ell = \sum_{i=1}^N (S^{-1})^{\ell j}(u_j^2 + v_j)$ satisfy $\mathcal{H}^\ell = -\lambda^\ell$ for a separable solution. Furthermore, setting $u_i = p_i$, we have $\{\mathcal{H}^\ell, \mathcal{H}^j\} =$

$0, \quad \ell \neq j$ where $\{\mathcal{H}, \mathcal{K}\} = \sum_{i=1}^{N}(\partial_{x^i}\mathcal{H}\partial_{p_i}\mathcal{K} - \partial_{x^i}\mathcal{K}\partial_{p_i}\mathcal{H})$ is the Poisson Bracket. Thus the $\mathcal{H}^\ell, \quad 2 \leq \ell \leq N$, are *constants of the motion* for the *Hamiltonian* $\mathcal{H}^{(1)}$.

Similar constructions apply to obtain product separation for solutions of the 2nd order Helmholtz or time independent Schrödinger equations. They lead to 2nd order symmetry operators $\mathcal{H}^\ell$, i.e., 2nd order partial differential operators that map solutions of the PDE to solutions. This approach is powerful computationally and leads to methods of classifying separable systems for the equations of mathematical physics. However, the connection with the intuitive idea of separation is not so clear.

2 A compromise approach

Our approach to separation [1, 5], is a generalization of Levi-Civita's theory for separation of 1st order equations. Recall we can always assume that the separation is additive. If a PDE (1) permits additive separation in a given system $\{x^i\}$, then without loss of generality we can assume the equation takes the form (2). We look for solutions $u = \sum_{i=1}^{N} S^{(i)}(x^i)$. Now let $u_{i,1} \equiv u_1, \quad u_{i,j+1} \equiv \partial_{x^i}u_{i,j}, \quad j =\geq 1$. Let m_i be the largest integer ℓ such that $\partial_{u_{i,\ell}}H = H_{u_{i,\ell}} \not\equiv 0$ and let D_i be the total differentiation operator

$$D_i = \partial_{x^i} + u_{i,1}\partial_u + u_{i,2}\partial_{u_{i,1}} + \cdots + u_{i,m_i+1}\partial_{u_{i,m_i}} + \cdots$$

Then the equation $D_iH(x,u) = 0$ implies

$$u_{i,m_i+1} = -\frac{\tilde{D}_iH}{H_{u_{i,m_i}}}, \qquad i = 1, 2, \cdots, N,$$

where

$$\tilde{D}_i = \partial_{x^i} + u_{i,1}\partial_u + u_{i,2}\partial_{u_{i,1}} + \cdots + u_{i,m_i}\partial_{u_{i,m_i-1}}.$$

It follows that u satisfies the integrability conditions $D_ju_{i,m_i+1} = 0$ for $j \neq i$, or

$$\begin{aligned} H_{u_{i,m_i}}H_{u_{j,m_j}} \quad & (\tilde{D}_i \quad \tilde{D}_jH) + H_{u_{i,m_i}u_{j,m_j}}(\tilde{D}_iH)(\tilde{D}_jH) \\ = \quad & H_{u_{j,m_j}}(\tilde{D}_iH)(\tilde{D}_jH_{u_{i,m_i}}) + H_{u_{i,m_i}}(\tilde{D}_jH)(\tilde{D}_iH_{u_{j,m_j}}). \end{aligned} \tag{6}$$

Theorem 1 *[1, 5] If conditions (6) are satisfied identically in the dependent variables $u, u_{k,\ell}$, then the partial differential equation $H = E$ admits a $\sum_{i=1}^{N} m_i + 1$ parameter family of separable solutions.*

If conditions (6) are satisfied identically the separation is *regular*. Then u and all its derivatives up to the maximum orders present in the equation $H = E$ can be prescribed arbitrarily. If the conditions are not satisfied identically, the separation is *nonregular* and any separable solutions depend on strictly fewer than the maximal number of parameters. Most cases in the literature are regular, though separation occurring via the method of symmetry-adapted solutions (ignorable variables) is typically nonregular.

Example 3: $H = (x_1+x_2)(u_{11}+u_{22})-2(u_1+u_2)$. Equations (6) are satisfied identically so $\{x_1, x_2\}$ is a regular separable system. The general separable solution depends on five parameters and is given by

$$u = (\alpha x_1^3 + \beta x_1^2 + \gamma x_1 - \frac{1}{2}E x_1) + (-\alpha x_2^3 + \beta x_2^2 - \gamma x_2 + \delta).$$

Here the appropriate separation equations are

$$\begin{array}{llllll} \partial_1 u & +E/2 & -\gamma & -2\beta x_1 & -3\alpha x_1^2 = 0, \\ \partial_{11} u & & & -2\beta & -6\alpha x_1 = 0, \\ \partial_2 u & & +\gamma & -2\beta x_2 & 3\alpha x_2^2 = 0, \\ \partial_{22} u & & & -2\beta & +6\alpha x_2 = 0. \end{array}$$

The associated matrix responsible for the separation

$$\begin{bmatrix} \frac{1}{2} & -1 & -2x_1 & -3x_1^2 \\ 0 & 0 & -2 & -6x_1 \\ 0 & 1 & -2x_2 & 3x_2^2 \\ 0 & 0 & -2 & 6x_2 \end{bmatrix},$$

is not a Stäckel matrix since more than one row depends on a given variable x_i; the second and fourth rows are the derivatives of the first and third rows, respectively. It is an example of a differential-Stäckel matrix [6]. All additive separation for n^{th} order linear equations is of this form.

Example 4: $H = u_{11}^2 + u_1 + u_{22}$. Here we have $u_{111} = -\frac{1}{2}$ (provided $u_{11} \neq 0$) and $u_{222} = 0$ so equations (6) are satisfied identically and $\{x_1, x_2\}$, is a regular separable system. The general separable solution depends on five parameters:

$$u = (-\frac{1}{12}x_1^3 + \alpha x_1^2 + \beta x_1) + \left(\frac{1}{2}(E - 4\alpha^2 - \beta)x_2^2 + \gamma x_2 + \delta\right).$$

Example 5: $H = x_2 u_{11} + x_1 u_{22} + u_1 + u_2$. Equations (6) reduce to the requirement $u_{11} + u_{22} = 0$. The general (nonregular) separable solution depends on four parameters:

$$u = (\alpha x_1^2 + \beta x_1) + (-\alpha x_2^2 + (E - \beta)x_2 + \gamma).$$

There is a similar theory of additive separation for PDEs with $E = 0$. We make the usual assumptions on H and take the PDE $H = 0$. In case the conditions (6) are identities in the sense that there exist functions $P_{i,j}(x_k, u, u_{k,\ell})$, polynomials in $u_{k,\ell}$ such that, for $i \neq j$,

$$\begin{aligned} \mathcal{F}_{ij} &\equiv H_{u_{i,m_i}} H_{u_{j,m_j}} (\tilde{D}_i \tilde{D}_j H) + H_{u_{i,m_i}, u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H) \\ &- H_{u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H_{u_{i,m_i}}) - H_{u_{i,m_i}} (\tilde{D}_j H)(\tilde{D}_i H_{u_{j,m_j}}) = P_{i,j} H, \end{aligned} \tag{7}$$

we say that $\{x_k\}$ is a *regular separable coordinate* system for $H = 0$.

Theorem 2 *[1, 5] If $\{x_k\}$ is a regular separable system for $H = 0$ then for every set of $m_1 + m_2 + \ldots + m_N$ constants $\{v^0, v^0_{i,j}\}$ with $H(x^0, v^0) = 0$ and $H_{u_{j,m_j}}(x^0, v^0) \neq 0$, there is a unique separable solution u of $H(x, u) = 0$ such that $u(x^0) = v^0$, $u_{i,j}(x^0) = v^0$, $u_{i,j}(x^0) = v^0_{i,j}$, $1 \leq i \leq N$, $1 \leq j \leq m_i$.*

If equations (7) are not satisfied identically, separable solutions may exist but will depend on fewer than $\sum_{i=1}^N m_i$ independent parameters. This is *nonregular* separation.

Example 6: $H = (x_2 - x_3)u_{11} + (x_3 - x_1)u_{22} + (x_1 - x_2)u_{33}$. Equations (7) are satisfied with $P_{i,j} \not\equiv 0$, so $\{x_k\}$ is a regular separable system for $H = 0$, though not for $H = E$. The general separable solution depends on six parameters and is given by

$$u = \frac{1}{6}\alpha(x_1^3 + x_2^3 + x_3^3) + \frac{1}{2}\beta(x_1^2 + x_2^2 + x_3^2) + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \delta.$$

Example 7: Orthogonal R-separation for the Schrödinger Equation. Here,

$$\Delta_N \Psi(x) + V(x)\Psi(x) = E\Psi(x), \tag{8}$$

where, Δ_N is the Laplacian on a pseudo-Riemannian manifold [4]. In an orthogonal coordinate system $\{x^i\}$ we have

$$\Delta_N = \frac{1}{\sqrt{H_1 \cdots H_N}} \sum_{i=1}^{N} \partial_{x^i}(H_1 \cdots H_N H_i^{-2} \partial_{x^i})$$

and $ds^2 = \sum_{i=1}^{N} H_i^2 (dx^i)^2$. We look for multiplicative R-separable solutions (where $R(x)$ is some fixed function to be chosen): $\Psi = \exp R(x) \prod_{i=1}^{N} \Psi_{(i)}(x^i)$. We set $u = R - \ln \Psi$ to convert (8) to a PDE with additive separation:

$$\sum_{i=1}^{N} [H_i^{-2}(u_{ii} + u_i^2) + (2H_i^{-2}\partial_i R + s_i)u_i + H_i^{-2}(\partial_{ii}R + (\partial_i R)^2 + s_i \partial_i R)] + V = E.$$

Here, $s_i = \partial_i(H_1 \cdots H_N H_i^{-2})/(H_1 \cdots H_N)$. Now we substitute this expression into (6) and equate the coefficients of $u_i^2,\ u_{ii},\ 1$ to zero.

1. Coeff. of u_i^2: The H_i^{-2} are in Stäckel form. The resulting identities are

   $$\partial_{jk} H_i^{-2} = \partial_j H_i^{-2} \partial_k \ln H_j^{-2} + \partial_k H_i^{-2} \partial_j \ln H_k^{-2}, \quad j \neq k.$$

   These are Levi-Civita separability conditions [4], necessary and sufficient for the existence of a Stäckel matrix that accomplishes the separation for the associated Hamilton-Jacobi Equation (4).

2. Coeff. of u_{ii}: Determines R. We find $R = -\frac{1}{2} \ln \frac{g}{S} + \sum_{i=1}^{n} L^{(i)}(x^i)$ where the functions $L^{(i)}$ are arbitrary.

3. Coeff. of 1: Generalized Robertson conditions for the potential $\tilde{V}$:

   $$\tilde{V} = V - \frac{1}{2} \sum_{i=1}^{n} H_i^{-2}(\partial_i \ell_i + \frac{1}{2}\ell_i^2),$$

   where $\ell_i = \partial_i \ln(\sqrt{g} H_i^{-2}) = \partial_i \ln \frac{\sqrt{g}}{S}$. The conditions are

   $$\partial_{ik}\tilde{V} - \partial_k \ln H_j^{-2} \partial_j \tilde{V} - \partial_j \ln H_k^{-2} \partial_k \tilde{V} = 0, \quad j \neq k.$$

   This means precisely [1] that the potential function can be expressed in the form $\tilde{V} = \sum_{i=1}^{n} f^{(i)}(x^i) H_i^{-2}$.

The above conditions are necessary and sufficient for R-separation. We see that all multiplicative R-separable solutions of (8) follow from the Stäckel construction. Every orthogonal coordinate system permitting product separation of the Helmholtz equation corresponds to a Stäckel form; hence it permits additive separation of the Hamilton-Jacobi equation. Eisenhart [4] has shown that the Robertson condition for product separation in the zero potential case is equivalent to the requirement $R_{ij} = 0$ for $i \neq j$, where R_{ij} is the Ricci tensor expressed in the Stäckel coordinates $\{x^i\}$. It follows that the Robertson condition is automatically satisfied in Euclidean space, a space of constant curvature or any Einstein space.

The question arises whether nontrivial R-separation necessarily occurs. Using Eisenhart's formulation of Robertson's condition as $R_{ij} = 0$, $i \neq j$, we see that there is only trivial orthogonal R-separation in an Einstein space. However, nontrivial R-separation occurs, even in conformally flat spaces. An example is [7]

$$\begin{aligned} ds^2 &= (x+y+z)[(x-y)(x-z)dx^2 + (y-z)(y-x)dy^2 + (z-x)(z-y)dz^2], \\ e^R &= (x+y+z)^{-\frac{1}{4}}. \end{aligned} \tag{9}$$

Brief summary of for the scalar equations of mathematical physics [8]:

Equation	Type of Separation
Hamilton-Jacobi	additive separation
Helmholtz or Klein-Gordon	multiplicative R-separation
Laplace or wave	multiplicative R-separation
heat/time-dependent Schrödinger	multiplicative R-separation

All regular separation of these equations is determined via the Stäckel procedure and separation can be characterized via the symmetry operators for the equation. This approach generalizes to N-dimensional pseudo-Riemannian manifolds and to both orthogonal and non-orthogonal separation.

There is still a question as to the possible mechanisms of separation. So far the only major mechanisms known (for regular separation) are Stäckel form and differential Stäckel form. Are there other major mechanisms for solutions of equations (6) and some PDEs?

3 Intrinsic characterization of variable separation

In general, variable separation for a PDE is a coordinate dependent phenomenon has no direct relationship to symmetries or other intrinsic properties of the equation. However, for the equations of mathematical physics listed previously, separation has an intrinsic characterization in terms of symmetries.

Theorem 3 *[1, 5, 9] Necessary and sufficient conditions for the existence of an orthogonal additive separable coordinate system $\{x^i\}$ for the Hamilton-Jacobi equation $\mathcal{H}^1 = E$ on an N-dimensional pseudo-Riemannian manifold are that there exist N quadratic forms $\mathcal{H}^k = \sum_{i,j=1}^{N} H_{ij}^{(k)} p_i p_j$ on the manifold such that:*

1. $\{\mathcal{H}^k, \mathcal{H}^\ell\} = 0, \quad 1 \le k, i \le N,$
2. *The set $\{\mathcal{H}^k\}$ is linearly independent (as N quadratic forms).*
3. *There is a basis $\{\omega_{(j)} : 1 \le j \le N\}$ of simultaneous eigenforms for the $\{\mathcal{H}^k\}$.*

If conditions (1)-(3) are satisfied then there exist functions $g^i(x)$ such that: $\omega_{(j)} = g^j dx^j, \quad j = 1, \cdots, N.$

Theorem 4 *Necessary and sufficient conditions for the existence of a multiplicative orthogonal R-separable coordinate system $\{x^i\}$ for the Schrödinger equation $(\Delta_N + V)\Psi = E\Psi$ on an N-dimensional pseudo-Riemannian manifold are that there exists a linearly independent set $\{A_1 = \Delta_N, A_2, \cdots, A_N\}$ of second-order differential operators on the manifold such that:*

1. $[A_k, A_\ell] = 0, \quad 1 \le k, \ell \le N,$
2. *Each A_k is in self-adjoint form,*
3. *There is a basis $\{\omega_{(j)} : 1 \le j \le N\}$ of simultaneous eigenforms for the $\{A_k\}$.*

If conditions (1)-(3) are satisfied then there exist functions $g^i(x)$ such that: $\omega_{(j)} = g^j dx^j, \quad j = 1, \cdots, N.$

The main point is that, under the required hypotheses, the eigenforms ω^ℓ of the quadratic forms a^{ij} are normalizable, i.e., up to multiplication by a nonzero function, ω^ℓ is the differential of a coordinate. This fact permits us to compute the coordinates directly from a knowledge of the symmetries.

As an example, consider the Hamilton-Jacobi equation for two dimensional Minkowski space with $V = 0$: $\mathcal{H} \equiv u_x^2 - u_t^2 = E$. The vector space of all linear symmetries $\mathcal{L} = a(x,t)u_x + b(x,t)u_y$ is closed under the bracket $\{\cdot,\cdot\}$; hence the symmetries form a Lie algebra. Furthermore $\{\mathcal{H},\mathcal{L}\} \equiv 0$ for each linear symmetry. The Lie algebra is three dimensional, with basis

$$\begin{array}{lll} \mathcal{L}_1 = u_x, & \mathcal{L}_2 = u_t, & \mathcal{L}_3 = tu_x + xu_t, \\ \{\mathcal{L}_1,\mathcal{L}_2\} = 0, & \{\mathcal{L}_3,\mathcal{L}_1\} = \mathcal{L}_2, & \{\mathcal{L}_3,\mathcal{L}_2\} = \mathcal{L}_1. \end{array} \tag{10}$$

Every symmetry which is quadratic in the first derivatives of u (second order Killing tensor) is a polynomial in the linear symmetries $\mathcal{L}_i$ (true for all spaces of constant curvature). Thus all candidates for separation can be built from the basis symmetries (10).

Consider, for example, the quadratic symmetry $\mathcal{A}^2 = 2\mathcal{L}_3\mathcal{L}_1$. With respect to Cartesian coordinates, the corresponding symmetric quadratic forms are

$$\mathcal{A}^2 \sim \begin{pmatrix} 2t & x \\ x & 0 \end{pmatrix}, \quad \mathcal{A}^1 = \mathcal{H} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly, $\mathcal{A}^2$ has roots $\rho = t \pm \sqrt{t^2 - x^2}$ (assuming $t > |x|$) with a basis of eigenforms $\omega_1 = (t + \sqrt{t^2 - x^2})dx - xdt$, $\omega_2 = (t - \sqrt{t^2 - x^2})dx - xdt$. By the Theorem, $\mathcal{A}^2$ does define a regular separable coordinate system $\{\xi^1, \xi^2\}$ for the Hamilton-Jacobi equation and there exist functions f_i such that $d\xi^i = f_i\omega_i$, $i = 1, 2$. We find $f_1 = [\xi^2((\xi^2)^2 - (\xi^1)^2)]^{-1}$, $f_2 = -[\xi^1((\xi^2)^2 - \xi^1)^2)]^{-1}$, $t = \frac{1}{2}((\xi^1)^2 + (\xi^2)^2)$, $\quad x = \xi^1\xi^2$. On the other hand the symmetry $\mathcal{A} = 2\mathcal{L}_3(\mathcal{L}_1 - \mathcal{L}_2)$ has two equal roots and only one eigenform. Thus $\mathcal{A}$ cannot determine a separable system. For manifolds of dimension $N \geq 3$ a system of $N - 1$ commuting symmetries may fail to determine separable coordinates even if each quadratic symmetry determines a basis of eigenforms, for there may be no simultaneous basis.

4 Construction of separable coordinate systems for constant curvature spaces

A complete construction of separable coordinate systems on the real N-sphere and on real Euclidean N-space, and a graphical method for constructing these systems has been worked out by the authors [10, 11]. A complete construction of separable coordinate systems on the real N sphere and on real Euclidean N-space and a graphical method for constructing these systems has been given by the authors [10,11].The methods used to establish the completeness of the coordinates systems found involve differential geometry and Lie theory and can be described by a graphical calculus.The generic elliptical coordinate system on the N-sphere is denoted by the graph $[e_0|e_1|\cdots|e_N]$ representing the coordinates $X_\ell^2 = \Pi_{i=1}^N (s_i - e_\ell)/\Pi_{i\neq\ell}(e_i - e_\ell)$, $\ell = 0, \cdots, N$ where $\sum_{\ell=0}^N X_\ell^2 = 1$ and $e_0 < s_1 < e_1 < s_2 < \cdots < s_N < e_n$.

All the separable coordinate systems on the N sphere can be obtained by nesting the generic elliptic coordinate systems of k-spheres. For example we can obtain a separable system on the N sphere by starting with generic elliptic coordinates on the $(N-k)$-sphere and embedding in it a k-sphere. We could do this via the choice of coordinates

$$(X_0, X_1, \cdots, X_N) = (U_0 V_0, \cdots, U_0 V_k, U_1, \cdots, U_{N-k})$$

where $(V_0, \cdots, V_k)$ and $(U_0, \cdots, U_{N-k})$ are generic elliptic coordinates on the k-sphere and $(N-k)$-sphere, respectively:

$$V_\ell^2 = \frac{\Pi_{i=1}^k (v_i - f_\ell)}{\Pi_{i\neq\ell}(f_i - f_\ell)},\ \ell = 0, \cdots, k \quad U_j^2 = \frac{\Pi_{i=1}^k (v_i - e_j)}{\Pi_{i\neq j}(e_i - e_j)},\ j = 0, \cdots, N-k.$$

This coordinate system has the diagrammatic representation

$$\begin{array}{cccccccc} & [& e_0 & | & e_1 & | & \cdots & | & e_{N-k} &] \\ & & \downarrow & & & & & & & \\ [& f_0 & | & \cdots & | & f_k &]. & & & \end{array}$$

In general all coordinate systems are constructed from tree diagrams using the general branching law

$$\begin{array}{ccccccccc} [& e_1 & | & e_2 & | & \cdots & | & e_n &] \\ & \downarrow & & \downarrow & & \cdots & & \downarrow & \\ & S_{p_1} & & S_{p_2} & & \cdots & & S_{p_n} & \end{array}$$

where S_{p_j} is the diagrammatic representation of some separable coordinate system on the p_j sphere. If $p_i = 0$ then nothing is attached.

In the case of real Euclidean N-space the generic systems are the ellipsoidal coordinates $y_\ell^2 = \Pi_{i=1}^N (x_i - e_\ell)/\Pi_{i\neq\ell}(e_i - e_\ell)$, $\ell = 1, .., N$, $e_1 < x_1 < e_2 < \cdots < e_N < x_N$ with graphical representation

$$< e_1|e_2|\cdots|e_N >$$

and parabolic coordinates $y_1 = \frac{c}{2}(x_1 + \cdots + x_n + e_1 + \cdots + e_{N-1})$, $y_j^2 = -c^2\Pi_{i=1}^n (x_i - e_{j-1})/\Pi_{i\neq j}(e_i - e_{j-1})$, $j = 2, \cdots, N$, $x_1 < e_1 < x_2 < e_2 < \cdots < e_{N-1} < x_N$ with graphical representation

$$(e_1|e_2|\cdots|e_{N-1}).$$

Each separable system in Euclidean space consists of a finite number of disjoint tree graphs with one of these two types of generic graphs at its base subject to branching laws of the type encountered for the real sphere, i.e.,

$$\begin{array}{ccccccccc} \langle & e_1 & | & e_2 & | & \cdots & | & e_n & \rangle \\ & \downarrow & & \downarrow & & \cdots & & \downarrow & \\ & S_{p_1} & & S_{p_2} & & \cdots & & S_{p_n} & \end{array}$$

$$\begin{array}{ccccccccc} (& e_1 & | & e_2 & | & \cdots & | & e_{n-1} &) \\ & \downarrow & & \downarrow & & \cdots & & \downarrow & \\ & S_{p_1} & & S_{p_2} & & \cdots & & S_{p_{n-1}} & \end{array}$$

A similar graphical calculus describes all the separable coordinates on a positive definite hyperboloid and solutions of Laplace's equation in real N dimensional Euclidean space. The problem of calculating all the separable systems for the complex N-sphere and complex Euclidean N-space is more difficult and as yet unsolved. Complete lists of separable coordinates are known only for dimensions 2,3 and 4. This is a challenging problem.

5 Superintegrability and multiseparability

Hamilton-Jacobi and associated Schrödinger equations, separable in some coordinate system are *integrable*. Their solutions are expressible in terms of special functions that satisfy the separation equations. Superintegrable systems are even better. To explain this we consider an N-dimensional Riemannian manifold. In local coordinates $q_1, \cdots, q_N$, the metric tensor is $\left(g^{jk}(\mathbf{q})\right)$. Given a potential function $V(\mathbf{q})$, the Hamilton-Jacobi equation is $\mathcal{H}(\mathbf{q},\mathbf{p}) \equiv \sum_{j,k=1}^{n} g^{jk}(\mathbf{q})p_jp_k + V(\mathbf{q}) = E \sum_{j,k=1}^{n} g^{jk}(\mathbf{q})\frac{\partial S}{\partial q_j}\frac{\partial S}{\partial q_k} + V(\mathbf{q})$, where $p_j = \frac{\partial S}{\partial q_j}$ and $S(\mathbf{q})$ is the action function. The quantum analog of this is the Schrödinger equation $H\Psi(\mathbf{q}) = E\Psi(\mathbf{q})$ where $H = \Delta_n + V(\mathbf{q})$. A *second-order constant of the motion* or second-order symmetry for the classical Hamiltonian is a quadratic function $\mathcal{L} = \sum a^{jk}(\mathbf{q})p_jp_k + W(\mathbf{q})$ such that $\{\mathcal{L},\mathcal{H}\} = 0$. Note that $\mathcal{L}$ is constant along a classical trajectory: $\frac{d}{dt}\mathcal{L} = \{\mathcal{L},\mathcal{H}\} = 0$, where $\frac{d}{dt}\mathbf{q} = \partial_{\mathbf{p}}\mathcal{H}$, $\frac{d}{dt}\mathbf{p} = -\partial_{\mathbf{q}}\mathcal{H}$. The null space of the map $T: \quad df(\mathbf{q},\mathbf{p}) \to \{f,\mathcal{H}\}(\mathbf{q},\mathbf{p})$ is $2N-1$ dimensional, so there are $2N-1$ functionally independent constants of the motion (but not necessarily second-order).

Definition 1 *The classical system $\mathcal{H} = E$ is second-order superintegrable [12, 13, 14, 15] if there are $2N-1$ functionally independent second-order constants of the motion $\mathcal{L}_\ell$, $\quad \ell = 0, 1, \cdots, 2N-2$, $\quad \mathcal{L}_0 = \mathcal{H}$, i.e., $\{\mathcal{L}_\ell,\mathcal{H}\} = 0$. Similarly, the quantum system $H\Psi = E\Psi$ is second-order superintegrable if there are $2N-1$ linearly independent second-order partial differential symmetry operators: L_ℓ, $\quad \ell = 0, 1, \cdots, 2N-2$, $\quad L_0 = H$, $\quad [L_\ell, H] \equiv L_\ell H - HL_\ell = 0$.*

Let's compare the concepts of second-order superintegrability and separability for a classical Hamiltonian $\mathcal{H}$ on an N-dimensional pseudo-Riemannian manifold. Similar statements hold for the Schrödinger equation.

Superintegrable: There are $2N-1$ functionally independent 2nd order constants of the motion $\mathcal{L}_0 = \mathcal{H}, \mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_{2N-2}$: $\{\mathcal{H},\mathcal{L}_j\} = 0, \qquad j = 1, \cdots, 2N-2$.

Separable: There are N linearly independent 2nd order constants of the motion $\mathcal{L}_0 = \mathcal{H}, \mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_{N-1}$,: $\{\mathcal{L}_j,\mathcal{L}_k\} = 0, 0 \le j,k \le N-1$. The symmetries must also satisfy eigenform conditions.

One of the most effective methods of finding superintegrable systems is to search for systems that are *multiseparable*.

Example 8: Real Euclidean 2-space. Here, $N = 2$, $2N-1 = 3$, so each separable system yields one new symmetry. Consider the Schrödinger equation with potential

$$V(x,y) = \frac{1}{2}\left(\omega^2(x^2+y^2) + \frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right),$$

i.e.,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi - \left(\omega^2(x^2+y^2) + \frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right)\Psi = -2E\Psi.$$

This equation separates in three systems: *Cartesian* coordinates (x,y); *polar* coordinates $x = r\cos\theta$, $y = r\sin\theta$, and *elliptical* coordinates

$$x^2 = c^2\frac{(u_1-e_1)(u_2-e_1)}{(e_1-e_2)}, \qquad y^2 = c^2\frac{(u_1-e_2)(u_2-e_2)}{(e_2-e_1)}.$$

The bound state energies are given by $E_n = \omega(2n+2+k_1+k_2)$ for integer n. The corresponding wave functions are 1) *Cartesian*:

$$\Psi_{n_1,n_2}(x,y) = 2\omega^{\frac{1}{2}(k_1+k_2+2)}\sqrt{\frac{n_1!n_2!}{\Gamma(n_1+k_1+1)\Gamma(n_2+k_2+1)}}\,x^{(k_1+\frac{1}{2})}y^{(k_2+\frac{1}{2})}$$

$$e^{-\frac{\omega}{2}(x^2+y^2)}L_{n_1}^{k_1}(\omega x^2)L_{n_2}^{k_2}(\omega y^2), \qquad n = n_1+n_2,$$

and the $L_n^k(x)$ are Laguerre polynomials. 2) *polar*:

$$\Psi(r,\theta) = \Phi_q^{(k_1,k_2)}(\theta)\omega^{\frac{1}{2}(2q+k_1+k_2+1)}\sqrt{\frac{2m!}{\Gamma(m+2q+k_1+k_2+1)}}$$

$$e^{(-\omega r^2/2)}r^{(2q+k_1+k_2+1)}L_m^{2q+k_1+k_2+1}(\omega r^2), \quad n = m+q,$$

$$\Phi_q^{(k_1,k_2)}(\theta) = \sqrt{2(2q+k_1+k_2+1)\frac{q!\Gamma(k_1+k_2+q+1)}{\Gamma(k_2+q+1)\Gamma(k_1+q+1)}}$$

$$\times(\cos\theta)^{k_1+(1/2)}(\sin\theta)^{k_2+(1/2)}P_q^{(k_1,k_2)}(\cos 2\theta),$$

and the $P_q^{(k_1,k_2)}(\cos 2\theta)$ are Jacobi polynomials . 3) *elliptical*:

$$\Psi = e^{-\omega(x^2+y^2)}x^{k_1+\frac{1}{2}}y^{k_2+\frac{1}{2}}\prod_{m=1}^{n}\left(\frac{x^2}{\theta_m-e_1} + \frac{y^2}{\theta_m-e_2} - c^2\right)$$

where

$$\frac{x^2}{\theta - e_1} + \frac{y^2}{\theta - e_2} - c^2 = -c^2 \frac{(u_1 - \theta)(u_2 - \theta)}{(\theta - e_1)(\theta - e_2)}.$$

A basis for the second order symmetry operators is

$$L_1 = \partial_x^2 + \frac{(\frac{1}{4} - k_1^2)}{x^2} - \omega^2 x^2, \qquad L_2 = \partial_y^2 + \frac{(\frac{1}{4} - k_2^2)}{y^2} - \omega^2 y^2$$

$$M^2 = (x\partial_y - y\partial_x)^2 + (\frac{1}{4} - k_1^2)\frac{y^2}{x^2} + (\frac{1}{4} - k_2^2)\frac{x^2}{y^2} - \frac{1}{2}.$$

(Note that $H = L_1 + L_2$.) The separable solutions are eigenfunctions of the symmetry operators L_1, M^2 and $M^2 + e_2 L_1 + e_1 L_2$ with eigenvalues

$$\lambda_c = -\omega(2n_1 + k_1 + 1), \qquad \lambda_p = (2q + k_1 + k_2 + 1)^2 + (1 + k_1^2 + k_2^2),$$

$$\lambda_e = 2(1 - k_1)(1 - k_2) - 2e_2\omega(k_1 + 1) - 2e_1\omega(k_2 + 1) - \omega^2 e_1 e_2 -$$

$$4\sum_{m=1}^{q} [e_2 \frac{k_1 + 1}{\theta_m - e_1} + e_1 \frac{k_2 + 1}{\theta_m - e_2}].$$

The algebra constructed by repeated commutators is

$$[L_1, M^2] = [M^2, L_2] \equiv R, \qquad [L_i, R] = -4\{L_i, L_j\} + 16\omega^2 M^2, \qquad i \neq j,$$

$$[M^2, R] = 4\{L_1, M^2\} - 4\{L_2, M^2\} + 8(1 - k_2^2)L_1 - 8(1 - k_1^2)L_2,$$

$$R^2 = \frac{8}{3}\{M^2, L_1, L_2\} + \frac{64}{3}\{L_1, L_2\} + 16\omega^2 M^4 - 16(1 - k_2^2)L_1^2$$

$$-16(1 - k_1^2)L_2^2 - \frac{128}{3}\omega^2 M^2 - 64\omega^2(1 - k_1^2)(1 - k_2^2).$$

Note that these relations are quadratic. Here $\{A, B\} = \frac{1}{2}(AB + BA)$ is a symmetrizer. The important fact to observe about the algebra generated by L_1, L_2, M^2, R is that it is *closed under commutation.*

In real Euclidean two-space there are precisely four potentials that have the multiseparation property [16]. The second potential is

$$V(x, y) = \omega^2(4x^2 + y^2) - \frac{(\frac{1}{4} - k_2^2)}{y^2}.$$

The corresponding Schrödinger equation is separable in *Cartesian* coordinates and *parabolic* coordinates $x = \frac{1}{2}(\xi^2 - \eta^2), y = \xi\eta$. The third potential is

$$V(x,y) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{B_1}{4}\frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}} + \frac{B_2}{4}\frac{\sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{x^2+y^2}},$$

separable in *parabolic* and *parabolic coordinates of the second type* $x = \mu\nu,\ \ y = \frac{1}{2}(\mu^2 - \nu^2)$. The fourth potential is

$$V(x,y) = -\frac{\alpha}{\sqrt{x^2+y^2}} + \frac{1}{4\sqrt{x^2+y^2}}\left(\frac{(k_1^2 - \frac{1}{4})}{\sqrt{x^2+y^2}+x} + \frac{(k_2^2 - \frac{1}{4})}{\sqrt{x^2+y^2}-x}\right),$$

separable in *polar*, *parabolic* and *modified elliptic* coordinates.

What good is a quadratic algebra? Consider the third potential. There, a basis for the quadratic algebra is L_1, L_2 and H with defining relations

$$[R, L_1] = -4L_2H + B_1B_2, \qquad [R, L_2] = 4L_1H + \frac{1}{2}(B_1^2 - B_2^2)$$

$$R^2 = 4L_1^2H + 4L_2^2H - 16\alpha^2H + (B_2^2 - B_1^2)L_1 - 2B_1B_2L_2 - 2\alpha^2(B_1^2 + B_2^2)$$

with $R = [L_1, L_2]$. If we look for eigenfunctions of the operators L_1, L_2 respectively, we have $L_1\varphi_m = \lambda_m\varphi_m$, $L_2\psi_n = \rho_n\psi_n$. If we write $L_1\psi_n = \sum_\tau C_{n\tau}\psi_\tau$ then the quadratic algebra relations imply

$$[(\rho_n - \rho_\tau)^2 + 8E]C_{n\tau} = -[\frac{1}{2}(B_1^2 - B_2^2) - 16\alpha E]\delta_{n\tau}$$

$$\sum_\tau C_{n\tau}C_{\tau\sigma}(2\rho_\tau - \rho_n - \rho_\sigma) = (8E\rho_n + B_1B_2 + 16\alpha E)\delta_{n\sigma}.$$

These relations in turn imply that $C_{nn} = -[\frac{1}{2}(B_1^2 - B_2^2) + 16\alpha E]/8E$ and $C_{nn+1} = C^*_{n+1n}$ are the only nonzero coefficients. They can essentially be determined by the relation

$$4\sqrt{-2E}\,(|C_{n,n+1}|^2 - |C_{n-1,n}|^2) = 8E\rho_n + B_1B_2 + 16\alpha E$$

and we see that the eigenvalues λ_m and ρ_n are

$$\lambda_m = 2\alpha - \frac{B_1^2}{8E} - (2m+1)\sqrt{-2E}, \quad \rho_n = 2\alpha - \frac{(B_1+B_2)^2}{16E} - (2n+1)\sqrt{-2E}$$

and the quantization condition for E is $4\alpha - \frac{B_1^2+B_2^2}{8E} = -(q+2)\sqrt{-2E}$ for integer q. Thus the bound state spectral resolution of H, L_1, L_2 has been obtained from the structure of the quadratic algebra alone.

For $N=2$, a complete classification of all second-order superintegrable potentials (classical and quantum) has been given for real and complex Euclidean space and real and complex spheres [17]. The first major advance in a classification for 2D spaces with non-constant curvature is contained in [18, 19] where all Darboux spaces are treated, i.e., spaces that admit 2 constants of the motion, of which 1 is a perfect square. Now all $N=2$ manifolds and potentials are known [20, 21, 22]. There are many systems on spaces of non-constant curvature, but the major result is that all such systems are equivalent to constant curvature systems via the Stäckel transform [21].

All these cases share the same basic features:

1. Except for one degenerate case, the potential V permits separability of the Hamilton-Jacobi equation $\mathcal{H} = E$ and the Schrödinger equation $H\Psi = E\Psi$ in at least two coordinate systems, characterized by symmetry conditions $\mathcal{L}_1 = \lambda_1$, $\mathcal{L}_2 = \lambda_2$ in the first case and $L_1\Psi = \lambda_1\Psi$, $L_2\Psi = \lambda_2\Psi$ in the second. *Superintegrability implies multiseparability.*

2. One can obtain alternate spectral resolutions $\{\Psi_j^{(1)}\}$, $\{\Psi_k^{(2)}\}$ for the multiply-degenerate eigenspaces of H, $L_1\Psi_j^{(1)} = \lambda_1^{(1)}\Psi_j^{(1)}$, $L_2\Psi_k^{(2)} = \lambda_2^{(2)}\Psi_k^{(2)}$. These alternate resolutions resolve the bound state degeneracy problem.

3. The interbasis expansions $\Psi_k^{(2)} = \sum_j a_{jk}\Psi_j^{(1)}$ yield important special function identities.

4. The operators H, L_1, L_2 generate a quadratic algebra. With $R = [L_1, L_2]$, R^2 is a polynomial of order 3 in H, L_1, L_2, whereas $[L_1, R]$ and $[L_2, R]$ are of order 2 in H, L_1, L_2. Closure of the algebra **is a remarkable property**, and is false for general symmetries. The quadratic algebra structure can be used to compute the interbase expansion coefficients.

5. There are deep connections between the theory of quasi-exactly solvable problems (QES) for ODEs and the theory of superintegrable systems [23, 24].

For $N=3$ conformally flat spaces the authors and Kress have established a structure and classification theory of nondegenerate potentials [25, 26]. Many results have been shown to extend to specific superintegrable potentials in other contexts and in dimensions ≥ 3, [23, 27, 28], but as yet there have been few general theorems. This is a major challenge. These maximal symmetry systems possess many beautiful properties. Not only are they integrable, they are integrable in multiple ways and in comparing these alternate solutions one obtains further information.

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