Second order superintegrable systems in conformally flat spaces. 2: The classical 2D Stäckel transform.

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Abstract

This paper is one of a series that lays the groundwork for a structure and classification theory of second order superintegrable systems, both classical and quantum, in conformally flat spaces. Here we study the Staĉkel transform (or coupling constant metamorphosis) as an invertible mapping between classical superintegrable systems on different spaces. Through the use of this tool we derive and classify for the first time all 2D superintegrable systems. The underlying spaces are exactly those derived by Koenigs in his remarkable paper giving all 2D manifolds (with zero potential) that admit at least 3 second order symmetries. Our derivation is very simple and quite distinct. We also show that every superintegrable system is the Stäckel transform of a superintegrable system on a constant curvature space.

1 Introduction

This is a sequel to paper [1]. Our purpose is to lay the groundwork for a structure and classification theory of second order superintegrable systems, both classical and quantum, in complex conformally flat spaces. Real spaces are considered as restrictions of these to the various real forms. In [1] we have given examples, described the background as well as the interest and importance of these systems in mathematical physics and given many relevant references. Observed features of the systems are multiseparability, closure of the quadratic algebra of second order symmetries at order 6, use of representation theory of the quadratic algebra to derive spectral properties of the quantum Schrödinger operator, and a close relationship with exactly solvable and quasi-exactly solvable problems, [2, 3, 4, 5, 6, 7, 8, 9]. Our approach is, rather than focus on particular spaces and systems, to use a general theoretical method based on integrability conditions to derive structure common to all systems.

In this paper we study the Stäckel transform, or coupling constant metamorphosis, [10, 11], for 2D classical superintegrable systems. Recall that for a classical 2D system on a Riemannian manifold we can always choose local coordinates x, y, not unique, such that the Hamiltonian takes the form

$$H = \frac{p_1^2 + p_2^2}{\lambda(x,y)} + V(x,y).$$

This system is *second order superintegrable* with *nondegenerate* potential $V = V(x, y, \alpha, \beta, \gamma)$ if it admits 3 functionally independent quadratic constants of the motion

$$S_k = \sum_{ij} a^{ij}_{(k)} p_i p_j + W_{(k)}(x, y, \alpha, \beta, \gamma).$$

(We also refer to these constants of the motion as symmetries because; each leads to a conserved quantity for the associated physical system; their Poisson brackets with the Hamiltonian vanish, so that they are generalized symmetries in the Lie sense; and their quantizations lead to second order partial differential operators that commute with the Schrödinger operator, so are again generalized symmetries in the Lie sense.) The potential V is nondegenerate in the sense that at any regular point x_0, y_0 where the potential is defined and analytic and the S_k are functionally independent, we can prescribe the values of $V_1(x_0, y_0), V_2(x_0, y_0), V_{11}(x_0, y_0)$ arbitrarily by choosing appropriate values for the parameters α, β, γ. Here, $V_1 = \partial V/\partial x$, $V_2 = \partial V/\partial y$, etc. (Another way to look at this is to say that $V_1(x_0, y_0), V_2(x_0, y_0), V_{11}(x_0, y_0)$ are the parameters.) This is in addition to the trivial constant that we can always add to a potential. This requirement implies that the potential satisfies a system of coupled PDEs of the form

$$V_{22} = V_{11} + A^{22}(x,y)V_1 + B^{22}(x,y)V_2, \quad V_{12} = A^{12}(x,y)V_1 + B^{12}(x,y)V_2.$$

The Stäckel transform is a conformal transformation of a superintegrable system on one space to a superintegrable system on another space. We prove that all nondegenerate 2D superintegrable systems are Stäckel transforms of constant curvature systems and give a complete and simple classification of all 2D superintegrable systems. The following papers will extend these results to 3D systems and the quantum analogs of 2D and 3D classical systems.

2 The Stäckel transform for two-dimensional systems

The Stäckel transform [10] or coupling constant metamorphosis [11] plays a fundamental role in relating superintegrable systems on different manifolds. The basic idea behind this transform has long been observed in various important classical and quantum mechanical systems. One of the most familiar is the Hamilton-Jacobi equation for the classical Coulomb problem $H \equiv p_1^2 + p_2^2 + p_3^2 + Z/r = E$ where r is the radial coordinate and Z is the charge. Division of the equation by the potential term r^{-1} converts it into the pseudo-Coulomb problem $H' \equiv r(p_1^2 + p_2^3 + p_3^2) - Er = -Z$, much easier to solve from a group theoretic point of view, where the space has changed and the energy and charge have switched roles. In [11] it was pointed out that if $H + ZV(\mathbf{x}) = E$ is an integrable Hamiltonian system for some additive potential V and all values of the parameters Z, E, then the system $H/V - E/V = Z$ is also integrable, where the parameters E and Z have changed roles. This general transformation was called coupling constant metamorphosis. Independently in [10] it was observed that if the Hamilton-Jacobi equations $\sum g^{ij}p_1p_j + V(\mathbf{q}) = E$, $\sum g^{ij}p_1p_j + U(\mathbf{q}) = E$ each admit a complete integral via separation of variables in the orthogonal coordinates $\mathbf{q}$, where U is nonzero, then the system $U^{-1}\sum g^{ij}p_1p_j + U^{-1}V(\mathbf{q}) = E'$ also admits a complete integral via separation in the same coordinates, but on a different manifold. The second order constants of the motion that describe the separation and the corresponding Stäckel matrices are mapped into one another by the transformation. We called this the Stäckel transform since it preserved the Stäckel form of the separable system. All of these observations have straightforward extensions to n dimensions and to the corresponding quantum mechanical operators.

Suppose we have a superintegrable system

$$H = \frac{p_1^2 + p_2^2}{\lambda(x, y)} + V(x, y) \tag{1}$$

in local orthogonal coordinates, with nondegenerate potential $V(x, y)$:

$$\begin{array}{rcccl} V_{22} & = & V_{11} & + & A^{22}V_1 + B^{22}V_2, \\ V_{12} & = & & & A^{12}V_1 + B^{12}V_2 \end{array} \tag{2}$$

and suppose $U(x, y)$ is a particular solution of equations (2), nonzero in an open set. Then the transformed system

$$\tilde{H} = \frac{p_1^2 + p_2^2}{\tilde{\lambda}(x, y)} + \tilde{V}(x, y) \tag{3}$$

with nondegenerate potential $\tilde{V}(x, y)$:

$$\begin{array}{rcccl} \tilde{V}_{22} & = & \tilde{V}_{11} & + & \tilde{A}^{22}\tilde{V}_1 + \tilde{B}^{22}\tilde{V}_2, \\ \tilde{V}_{12} & = & & & \tilde{A}^{12}\tilde{V}_1 + \tilde{B}^{12}\tilde{V}_2 \end{array} \tag{4}$$

is also superintegrable, where

$$\tilde{\lambda} = \lambda U, \quad \tilde{V} = \frac{V}{U},$$

$$\tilde{A}^{12} = A^{12} - \frac{U_2}{U}, \; \tilde{A}^{22} = A^{22} + 2\frac{U_1}{U}, \; \tilde{B}^{12} = B^{12} - \frac{U_1}{U}, \; \tilde{B}^{22} = B^{22} - 2\frac{U_2}{U}.$$

Let $S = \sum a^{ij}p_ip_j + W = S_0 + W$ be a second order symmetry of H and $S_U = \sum a^{ij}p_ip_j + W_U = S_0 + W_U$ be the special case of this that is in involution with $p_1^2 + p_2^2/\lambda + U$. Then

$$\tilde{S} = S - \frac{W_U}{U}H$$

is the corresponding symmetry of $\tilde{H}$. Since one can always add a constant to a nondegenerate potential, it follows that $1/U$ defines an inverse Stäckel transform of $\tilde{H}$ to H. See [10, 12] for many examples of this transform. We say that two superintegrable systems are **Stäckel equivalent** if one can be obtained from the other by a Stäckel transform.

2.1 A Stäckel transform approach to the classification of nondegenerate superintegrable systems

Through the use the Stäckel transform we can develop a method for classifying 2D nondegenerate superintegrable systems that is differential equations based. (In particular it is distinct from the Koenigs analytic function approach to finding spaces that admit at least three second order Killing tensors.) Let

$$ds^2 = \lambda(x, y)(dx^2 + dy^2)$$

be a metric for a nondegenerate superintegrable system. We recall from Section 2 of [1] that necessary and sufficient conditions for a^{ij} to be a second order Killing tensor for λ are that

$$\Delta a^{12} = 0,\ \Delta(a^{11} - a^{22}) = 0, \quad \Delta = \partial_x^2 + \partial_y^2,$$

where

$$(a^{22} - a^{11})_2 = 2a^{12}_1, \qquad (a^{22} - a^{11})_1 = -2a^{12}_2,$$

and the a^{ij} satisfy the integrability condition

$$(\lambda_{22} - \lambda_{11})a^{12} - \lambda_{12}(a^{22} - a^{11}) = 3\lambda_1 a^{12}_1 - 3\lambda_2 a^{12}_2 + (a^{12}_{11} - a^{12}_{22})\lambda. \quad (5)$$

Since λ is nondegenerate superintegrable we have three independent symmetries of the form $\mathcal{S} = \sum a^{ij}p_i p_j + W$ and a nondegenerate potential V satisfying the Bertrand-Darboux equations

$$(V_{22}-V_{11})a^{12}+V_{12}(a^{11}-a^{22}) = \left[\frac{(\lambda a^{12})_1 - (\lambda a^{11})_2}{\lambda}\right] V_1+\left[\frac{(\lambda a^{22})_1 - (\lambda a^{12})_2}{\lambda}\right] V_2 \quad (6)$$

for all symmetries with quadratic terms a^{ij}.

For a superintegrable system we can always use the independent symmetries to solve equations (6) for $V_{22} - V_{11},\ V_{12}$ in the form (2). If these two equations are the only conditions on the potential function V then it will depend on four parameters, the maximum number possible. Thus we can prescribe the derivatives V_1, V_2, V_{11} and the value of V at a fixed point. This is the case of a nondegenerate potential. If, however, the equations (6) put additional conditions on the potential then there will be a restriction on the first derivatives and the potential will depend on fewer parameters than four. In this case the potential is degenerate. In [1] we showed that superintegrable systems with three and two parameter potentials were, essentially, just restrictions of the four parameter nondegenerate potentials. One parameter potentials (i.e., constant potentials) are different. They in general are not restrictions of nondegenerate potentials and, indeed, the quadratic algebra structure may not hold. See [14] for a counterexample.

Returning to our nondegeneracy assumption, the system of equations (6) has a four parameter family of solutions V, counting the addition of a scalar to V as a parameter. Also, every Stäckel transform of this system to a system with metric μ must be of the form $\hat{V} = \mu/\lambda$ where $V = \hat{V}$ is some particular solution of the equations (6). Thus it is of interest to determine the equations that characterize μ.

To simplify the computations to follow, we recall that we can choose our orthogonal coordinates x, y such that one of our symmetries takes the form $a^{12} \equiv 0$, $a^{22} - a^{11} = 1$. In this system the symmetry and (5) imply $\lambda_{12} = 0$, and, as we will see, $\mu_{12} = 0$. A second symmetry is defined by the Hamiltonian itself: $a^{11} = a^{22} = 1/\lambda$, $a^{12} = 0$, which clearly always satisfies equations (5) and (6). Due to nondegeneracy, for the third symmetry we must have $a^{12} \neq 0$ and it is on this third symmetry that we will focus our attention in the following. Now the fundamental integrability conditions can be rewritten as

$$\lambda_{12} = 0, \qquad \lambda_{22} - \lambda_{11} = 3\lambda_1 A_1 - 3\lambda_2 A_2 + (A_{11} + A_1^2 - A_{22} - A_2^2)\lambda, \quad (7)$$

where $A = \ln a^{12}$ and the subscripts denote differentiation. Similarly, using this result and (6) we find that the equations characterizing μ are

$$\mu_{12} = 0, \qquad \mu_{22} - \mu_{11} = 3\mu_1 A_1 - 3\mu_2 A_2 + (A_{11} + A_1^2 - A_{22} - A_2^2)\mu. \quad (8)$$

Note that these two equations appear identical. However they have different interpretations. The fixed metric λ satisfies (7) and is a special solution of (8). Here μ designates a 4-parameter family of solutions, of which λ is a particular special case. It follows that A satisfies the integrability conditions for this system.

Let us apply ∂_{12} to both sides of (8). The result, using $\mu_{12} = 0$ and $\triangle a^{12} = 0$, is

$$0 = 3A_{12}(\mu_{11} - \mu_{22}) + (3A_{112} + 2[A_{11} + A_1^2]_2)\mu_1 + (-3A_{122} + 2[A_{11} + A_1^2]_1)\mu_2 \quad (9)$$
$$+ 2\mu(A_{11} + A_1^2)_{12}.$$

There are two possibilities here.

1. Case I: $A_{12} = 0$. Then every term in the preceding equation vanishes identically. We conclude that a^{12} factors as $a^{12} = X(x)Y(y)$, where $\Delta a^{12} = 0$. Thus there is a constant α such that

$$X'' = \alpha^2 X, \qquad Y'' = -\alpha^2 Y.$$

We have solutions

$$X(x) = \beta_1 e^{\alpha x} + \beta_2 e^{-\alpha_2 x}, \quad Y(y) = \gamma_1 e^{i\alpha y} + \gamma_1 e^{-i\alpha y}.$$

Variables separate in the equations for μ into two ODEs. Thus for every choice of a^{12} we can find all solutions μ explicitly.

2. Case II: $A_{12} \neq 0$. Now the coefficients of μ_{11}, μ_{22} in (9) are nonvanishing. The equation can be rewritten as

$$\mu_{22} - \mu_{11} = \mu_1 \left[\frac{3A_{112} + 2(A_{11} + A_1^2)_2}{3A_{12}} \right] + \mu_2 \left[\frac{-3A_{122} + 2(A_{11} + A_1^2)_1}{3A_{12}} \right]$$
$$+2\mu(A_{11} + A_1^2)_{12}.$$

Since μ is a 4-parameter solution, the coefficients of μ_1, μ_2 and μ can be equated. Thus we have three new identities, which together with $\triangle a^{12} = 0$ give

$$i)\ 9A_1A_{12} = 3A_{112} + 2(A_{11} + A_1^2)_2, \quad ii)\ 9A_2A_{12} = 3A_{122} + 2(A_{22} + A_2^2)_1, \tag{10}$$
$$iii)\ 3(A_{11} + A_1^2)A_{12} = (A_{11} + A_1^2)_{12}, \quad iv)\ A_{11} + A_1^2 + A_{22} + A_2^2 = 0.$$

The first two identities imply $A_{12} = Ce^A$ for some nonzero constant C. This is the Liouville equation with general solution

$$a^{12} = e^A = \frac{2X'(x)Y'(y)}{C(X(x) + Y(y))^2},$$

where $X(x)$ and $Y(y)$ are functions such that $X'(x)Y'(y) \neq 0$. At this point it is convenient to use X, Y as new coordinates. Thus there are functions $F(X), G(Y)$ such that

$$(X')^2 = F(X), \quad X'' = \frac{1}{2}F'(X), \quad (Y')^2 = G(Y), \quad Y'' = \frac{1}{2}G'(Y).$$

Substituting these expressions into the identities i)-iv) we obtain a system of functional differential equations for F, G with the general solution

$$F(X) = \frac{\alpha}{24}X^4 + \frac{\gamma_1}{6}X^3 + \frac{\gamma_2}{2}X^2 + \gamma_3 X + \gamma_4,$$
$$G(Y) = -\frac{\alpha}{24}Y^4 + \frac{\gamma_1}{6}Y^3 - \frac{\gamma_2}{2}Y^2 + \gamma_3 Y - \gamma_4,$$

where α, γ_j are constants. Note that the equations for x, y in terms of X, Y take the form of elliptic integrals

$$x = \int \frac{dX}{\sqrt{\frac{\alpha}{24}X^4 + \frac{\gamma_1}{6}X^3 + \frac{\gamma_2}{2}X^2 + \gamma_3 X + \gamma_4}},$$

$$y=\int \frac{dY}{\sqrt{\frac{-\alpha}{24}Y^4+\frac{\gamma_1}{6}Y^3-\frac{\gamma_2}{2}Y^2+\gamma_3 Y-\gamma_4}}.$$

Again, variables separate into two ODEs in the equations for μ . Thus for every choice of a^{12} we can find all solutions μ explicitly.

Theorem 1 *If $ds^2=\lambda(dx^2+dy^2)$ is the metric of a nondegenerate superintegrable system (expressed in coordinates x, y such that $\lambda_{12}=0$) then $\lambda=\mu$ is a solution of the system*

$$\mu_{12}=0, \qquad \mu_{22}-\mu_{11}=3\mu_1(\ln a^{12})_1-3\mu_2(\ln a^{12})_2+(\frac{a^{12}_{11}-a^{12}_{22}}{a^{12}})\mu, \quad (11)$$

where either

$$I) \quad a^{12}=X(x)Y(y), \quad X''=\alpha^2 X, \quad Y''=-\alpha^2 Y,$$

or

$$II) \quad a^{12}=\frac{2X'(x)Y'(y)}{C(X(x)+Y(y))^2},$$

$$(X')^2=F(X), \quad X''=\frac{1}{2}F'(X), \quad (Y')^2=G(Y), \quad Y''=\frac{1}{2}G'(Y)$$

where

$$F(X)=\frac{\alpha}{24}X^4+\frac{\gamma_1}{6}X^3+\frac{\gamma_2}{2}X^2+\gamma_3 X+\gamma_4,$$
$$G(Y)=-\frac{\alpha}{24}Y^4+\frac{\gamma_1}{6}Y^3-\frac{\gamma_2}{2}Y^2+\gamma_3 Y-\gamma_4.$$

Conversely, every solution λ of one of these systems defines a nondegenerate superintegrable system. If λ is a solution then the remaining solutions μ are exactly the nondegenerate superintegrable systems that are Stäckel equivalent to λ.

This result provides the basis for a simple classification of all nondegenerate superintegrable systems. In fact the spaces that arise correspond 1-1 with Koenigs' tables of 2D spaces that admit at least three second order symmetries. Indeed, from the fact that $F(X)$ and $G(Y)$ are fourth order polynomials we can determine which solutions of the functions $X(x)$ and $Y(y)$ yield the lists drawn up by Koenigs in his two Tableau. (We give the details of these tableau in §2.2.)

To understand more clearly the significance of cases I and II in the preceding theorem, we make use of the symmetry of equations (8), first exploited by Koenigs. We write the system in the form

$$a^{12}_{11}+a^{12}_{22}=0, \qquad \mu_{12}=0, \qquad a^{12}(\mu_{11}-\mu_{22})+3\mu_1 a^{12}_1-3\mu_2 a^{12}_2+(a^{12}_{11}-a^{12}_{22})\mu=0, \quad (12)$$

Lemma 1 *Suppose* $\mu = \lambda(x,y)$, $a^{12} = a(x,y)$ *satisfy (12). Then* $\mu = \tilde{a}(x,y)$, $a^{12} = \tilde{\lambda}(x,y)$ *also satisfy (12) where*

$$\tilde{a}(x,y) = a(x+y, ix-iy), \quad \tilde{\lambda}(x,y) = \lambda(x-iy, y-ix).$$

This transformation is invertible.

PROOF: It is straightforward to check that $\tilde{a}_{12} = 0$, $\tilde{\lambda}_{11} + \tilde{\lambda}_{22} = 0$. The symmetry of the third equation under this invertible transform is obvious. Q.E.D.

Theorem 2 *System (12) characterizes a nondegenerate superintegrable system if and only if the metric* $\tilde{a}^{12}(x,y)$ *is of constant curvature. Equivalently, the system (12) characterizes a nondegenerate superintegrable system if and only if the symmetry* a^{12} *is the image* $a^{12} = \tilde{\lambda}$ *where the metric* λ *(with* $\lambda_{12} = 0$ *is of constant curvature.*

PROOF: System (12) characterizes a nondegenerate superintegrable system if and only if the symmetry a^{12} satisfies the Liouville equation $(\ln a^{12})_{12} = Ca^{12}$ for some constant C. (If $C = 0$ we have Case I, and if $C \neq 0$ we have Case II.) It is straightforward to check that this means that

$$\frac{\tilde{a}^{12}_{11} + \tilde{a}^{12}_{22}}{(\tilde{a}^{12})^2} - \frac{(\tilde{a}^{12}{}_1)^2 + (\tilde{a}^{12}_2)^2}{(\tilde{a}^{12})^3} = 4iC,$$

so the scalar curvature of metric $\tilde{a}^{12}(dx^2 + dy^2)$ is constant. Similarly, if λ is of constant curvature then $\tilde{\lambda}$ satisfies Liouville's equation. Q.E.D.

Theorem 3 *Every nondegenerate superintegrable 2D system is Stäckel equivalent to a nondegenerate superintegrable system on a constant curvature space.*

PROOF: Every nondegenerate superintegrable 2D system with metric $\lambda(dx^2 + dy^2)$ corresponds to a function a^{12}_0 and a system of equations (12) (with $a^{12} = a^{12}_0$) where $\mu = \lambda$ is a solution and the integrabilty conditions are satisfied identically, so that the space of solutions μ is 4-dimensional. From Theorem 1 we see that a^{12}_0 must satisfy the Liouville equation, so by Theorem 2 the metric $\xi = \tilde{a}^{12}_0$ is of constant curvature. Recall that the space of second order symmetries of a constant curvature space is 6-dimensional. Consider the possible symmetries a^{12} such that standard equations

$$a^{12}_{11} + a^{12}_{22} = 0, \quad a^{12}(\xi_{11} - \xi_{22}) + 3\xi_1 a^{12}_1 - 3\xi_2 a^{12}_2 + (a^{12}_{11} - a^{12}_{22})\xi = 0$$

are satisfied. One constant curvature space symmetry with $a^{12} = 0$ determines the separable coordinates $\{x, y\}$ and one symmetry is the Hamiltonian $(p_1^2 + p_2^2)/\lambda$. A basis for the remaining symmetries consists of 4 linearly independent symmetries with a^{12} harmonic and nonzero. It is clear that the Koenig duality mapping $\tilde{\mu}$ for μ a solution of system (12) maps the 4-dimensional space of solutions μ (except $\mu = 0$) 1-1 onto the constant curvature space symmetries with a^{12} harmonic and nonzero. For constant curvature spaces we know that there are symmetries a^{12} that define nondegenerate superintegrable systems (the systems on flat space and the 2-sphere.) Let $a^{12} = b^{12}$ be one such symmetry. By Theorem 1 b^{12} satisfies the Liouville equation. Since the Koenigs duality map is onto, there must exist a solution $\mu = \nu$ of system (12) such that $\tilde{\nu} = b^{12}$. By Theorem 2 ν is the metric of a constant curvature space. This means that the system with metric λ is Stäckel equivalent to the constant curvature system with metric ν. Q.E.D.

2.2 Examples and relationship with the Koenigs Tableau

In a tour de force, Koenigs [13] has classified all 2D manifolds that admit exactly 3 second order Killing tensors and listed them in two tables: Tableau VI and Tableau VII.

In each case Koenigs gave the terms that give rise to the leading coefficients of the additional quadratic constant of the motion not implicitly defined by the Liouville form of the metric. We have given these metrics in a symmetric orthogonal form.

We can now reproduce the Tableau via the duality between separable coordinate systems on spaces of constant curvature and the form of the Killing tensors admitted in these particular coordinate systems.

For example, taking $\alpha = 1$ in case I, a solution for a^{12} is

$$X(x) = \sin x\,, \quad Y(y) = \sinh y \quad \Rightarrow \quad a^{12} = \sin x \sinh y\,.$$

Now $\mu_{12} = 0 \quad \Rightarrow \quad \mu = f(x) + g(y)$ and so equation (11) for μ becomes

$$g'' - f'' = 3f' \cot x - 3g' \coth x - 2(f + g)$$

which separates into a pair of ordinary differential equations,

$$g'' + 3 \coth y g' + 2g = K\,, \qquad f'' + \cot x f' - 2f = K\,,$$

for some separation constant K. These equations have solutions

$$f(x) = \frac{c_1 \cos x + c_2}{\sin^2 x} - \frac{1}{2}K\,, \qquad g(y) = \frac{c_3 \cosh y + c_4}{\sinh^2 y} + \frac{1}{2}K$$

and so

$$\mu = \frac{c_1 \cos x + c_2}{\sin^2 x} + \frac{c_3 \cosh y + c_4}{\sinh^2 y}\,. \tag{13}$$

In the preceding, we have used coordinates in which the metric was a multiple of $dx^2 + dy^2$, while Koenigs used coordinates in which the metric was a multiple of $dx\ dy$. To bridge this gap, we make the change of coordinates $x \to a$, $y \to ib$ to obtain (with a trivial redefinition of the parameters c_i) the first metric in Tableau VI.

The remaining metrics in Tableau VI are obtained by similar calculations using the following particular solutions to the case I equations in Theorem 1.

[1] $\quad X = \sin x, \quad Y = \sinh y$

[2] $\quad X = \sinh x, \quad Y = e^{iy}$

[3] $\quad X = e^x, \quad Y = e^{iy}$

[4] $\quad X = x, \quad Y = y$

[5] $\quad X = x, \quad Y = 1$

[6] $\quad X = Y = 1$

The metrics in Tableau VII are obtained from particular solutions to the case II equations in Theorem 1 in the same way as described for Tableau VI.

$$
\begin{aligned}
[1]\ ds^2 &= \left[\frac{c_1\cos a + c_2}{\sin^2 a} + \frac{c_3 \cos b + c_4}{\sin^2 b}\right](da^2 - db^2)\\
[2]\ ds^2 &= \left[\frac{c_1\cosh a + c_2}{\sinh^2 a} + \frac{c_3 e^b + c_4}{e^{2b}}\right](da^2 - db^2)\\
[3]\ ds^2 &= \left[\frac{c_1 e^a + c_2}{e^{2a}} + \frac{c_3 e^b + c_4}{e^{2b}}\right](da^2 - db^2)\\
[4]\ ds^2 &= \left[c_1(a^2-b^2) + \frac{c_2}{a^2} + \frac{c_3}{b^2} + c_4\right](da^2 - db^2)\\
[5]\ ds^2 &= \left[c_1(a^2-b^2) + \frac{c_2}{a^2} + c_3 b + c_4\right](da^2 - db^2)\\
[6]\ ds^2 &= \left[c_1(a^2-b^2) + c_2 a + c_3 b + c_4\right](da^2 - db^2)
\end{aligned}
$$

Table 1: TABLEAU VI

[1] Both $F(X)$ and $G(Y)$ are general fourth order polynomials

[2] $$\left.\begin{aligned} 4F(X) &= 1 - X^2 \\ 4G(Y) &= Y^2 - 1 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} X &= -2\cos 2x \\ Y &= \cosh 2y \end{aligned}\right.$$

[3] $$\left.\begin{aligned} F(X) &= X^2(X-1)^2 \\ G(Y) &= -Y^2(Y+1)^2 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} X &= \frac{1}{1+e^x} \\ Y &= \frac{1}{-1+e^{iy}} \end{aligned}\right.$$

[4] $$\left.\begin{aligned} F(X) &= X^3(X-1) \\ G(Y) &= -Y^3(Y+1) \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} X &= \frac{1}{1-\frac{1}{4}x^2} \\ Y &= -\frac{1}{1+\frac{1}{4}y^2} \end{aligned}\right.$$

[5] $$\left.\begin{aligned} F(X) &= 1 \\ G(Y) &= 1 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} X &= x \\ Y &= y \end{aligned}\right.$$

There are clearly other choices possible for X and Y but they revert to various versions of the cases given in Koenigs' tables. Since a single space may have more than one nondegenerate potential, our classification may include a space more than once.

Next, we examine each of his spaces and show in detail what was proved in the last section: that every superintegrable system on the space can be obtained as the Stäckel transform of a constant curvature space with respect

$$
\begin{aligned}
[1]\ ds^2 &= \left[c_1\left(\frac{1}{\mathrm{sn}^2(a,k)}-\frac{1}{\mathrm{sn}^2(b,k)}\right)+c_2\left(\frac{1}{\mathrm{cn}^2(a,k)}-\frac{1}{\mathrm{cn}^2(b,k)}\right)\right.\\
&+ \left. c_3\left(\frac{1}{\mathrm{dn}^2(a,k)}-\frac{1}{\mathrm{dn}^2(b,k)}\right)+c_4\left(\mathrm{sn}^2(a,k)-\mathrm{sn}^2(b,k)\right)\right](da^2-db^2)\\
[2]\ ds^2 &= \left[c_1\left(\frac{1}{\sin^2 a}-\frac{1}{\sin^2 b}\right)+c_2\left(\frac{1}{\cos^2 a}-\frac{1}{\cos^2 b}\right)+c_3\left(\cos 2a-\cos 2b\right)\right.\\
&+ \left. c_4\left(\cos 4a-\cos 4b\right)\right](da^2-db^2)\\
[3]\ ds^2 &= [c_1(\sin 4a-\sin 4b)+c_2(\cos 4a-\cos 4b)+c_3(\sin 2a-\sin 2b)\\
&+ c_4(\cos 2a-\cos 2b)]\,(da^2-db^2)\\
[4]\ ds^2 &= \left[c_1\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+c_2(a^2-b^2)+c_3(a^4-b^4)+c_4(a^6-b^6)\right](da^2-db^2)\\
[5]\ ds^2 &= \left[c_1(a-b)+c_2(a^2-b^2)+c_3(a^3-b^3)+c_4(a^4-b^4)\right](da^2-db^2)
\end{aligned}
$$

Table 2: TABLEAU VII

to a nondegenerate superintegrable potential. In [14, 15, 16] the authors have computed all the nondegenerate (and degenerate) superintegrable potentials for complex 2D flat space, potentials [**E1**]-[**E20**], and nonzero constant curvature space, potentials [**S1**]-[**S9**], and we identify the relevant potentials on the list that is given in [16].

2.2.1 TABLEAU VI

[1] In this case the infinitesimal distance has the form

$$ds^2 = (\frac{c_1\cos a + c_2}{\sin^2 a} + \frac{c_3\cos b + c_4}{\sin^2 b})(da^2 - db^2).$$

If we rewrite the Hamilton-Jacobi equation on the sphere,

$$H = p_1^2 + p_2^2 + p_3^2 + \hat{c}_1 + \frac{i\hat{c}_2 s_3}{\sqrt{s_1^2+s_2^2}} + \frac{\hat{c}_3 s_2}{s_1^2\sqrt{s_1^2+s_2^2}} + \frac{\hat{c}_4}{s_1^2} = E,$$

using a variant of spherical coordinates

$$s_1 = \frac{\sin b}{\sin a},\ \ s_2 = \frac{\cos b}{\sin a},\ \ s_3 = -i\frac{\cos a}{\sin a}$$

we obtain the form

$$p_b^2 - p_a^2 - \frac{E+\hat{c}_1}{\sin^2 a} - \frac{\hat{c}_2\cos a}{\sin^2 a} - \frac{\hat{c}_3\cos b}{\sin^2 b} - \frac{\hat{c}_4}{\sin^2 b} = 0.$$

Thus the potential from which this metric has been derived via Stäckel transform is [**S7**].

[2] In this case the metric is

$$ds^2 = \left(\frac{c_1 \cosh a + c_2}{\sinh^2 a} + c_3 e^{-b} + c_4 e^{-2b} \right) (da^2 - db^2).$$

Choosing Euclidean space coordinates of the form

$$x = \exp(-\frac{1}{2}u)\cosh(\frac{1}{2}v), \ \ y = i\exp(-\frac{1}{2}u)\sinh(\frac{1}{2}v)$$

and substituting into the Hamilton-Jacobi equation

$$H = p_x^2 + p_y^2 + \hat{c}_1(x^2+y^2) + \frac{\hat{c}_2}{x^2} + \frac{\hat{c}_3}{y^2} + \hat{c}_4 = E$$

we obtain the form

$$p_u^2 - p_v^2 + \frac{1}{4}\hat{c}_1 e^{-2u} + \mathcal{C}_2 \frac{\cosh v}{\sinh^2 v} + \mathcal{C}_3 \frac{1}{\sinh^2 v} + \frac{1}{4}(\hat{c}_4 - E)e^{-u}$$

where $\mathcal{C}_2 = \frac{1}{2}(\hat{c}_2 + \hat{c}_3)$ and $\mathcal{C}_3 = \frac{1}{2}(\hat{c}_3 - \hat{c}_2)$. From this it follows that the potential from which this metric is derived via Stäckel transform is [**E1**].

[3] In this case the infinitesimal distance has the form

$$ds^2 = (c_1 e^{-a} + c_2 e^{-2a} + c_3 e^{-b} + c_4 e^{-2b})(da^2 - db^2)$$

In the variables

$$x = e^{-a}\cosh b, \ \ y = -ie^{-a}\sinh b$$

this metric assumes the form

$$ds^2 = \left(\frac{c_1}{\sqrt{x^2+y^2}} + c_2 + \frac{c_3}{\sqrt{x^2+y^2}(x+iy)} + \frac{c_4}{(x+iy)^2} \right)(dx^2+dy^2).$$

We recognize this as arising via Stäckel transform from [**E17**]. Indeed note that if we write out the equation $H = E$ in suitable coordinates we obtain

$$p_1^2 + p_2^2 + \frac{\hat{c}_1}{\sqrt{x^2+y^2}} + \frac{\hat{c}_2}{(x+iy)^2} + \frac{\hat{c}_3}{\sqrt{x^2+y^2}(x+iy)} - E = 0$$

from which we can clearly see the identification.

[4] In this case the infinitesimal distance is

$$ds^2 = (c_1(a^2 - b^2) + \frac{c_2}{a^2} + \frac{c_3}{b^2} + c_4)(da^2 - db^2),$$

and by putting $a = x,\ b = iy$ this metric can be clearly related to a Stäckel transform from the potential [**E1**].

[5] Here

$$ds^2 = (c_1(a^2 - b^2) + \frac{c_2}{a^2} + c_3 b + c_4)(da^2 - db^2).$$

It is clear that this metric is derived by Stäckel transform from the potential

$$V = \hat{c}_1(x^2 + y^2) + \frac{\hat{c}_2}{x^2} + \hat{c}_3 y + \hat{c}_4$$

where $a = x,\ b = iy$. As we do not distinguish the use of Cartesian coordinates in any way it is always possible to rotate and translate them. If we do this then for the various choices of $\hat{c}_i$ we have the following potentials from our complete list.

(i) $\hat{c}_1 \neq 0$: We can translate with respect to y and make $\hat{c}_3 = 0$ to obtain a special case of [**E1**]. If further $\hat{c}_2 = 0$ then we obtain [**E3**].

(ii) $\hat{c}_1 = 0$: We have a special case of [**E2**] if $\hat{c}_2,\ \hat{c}_3 \neq 0$. If $\hat{c}_3 = 0$ we obtain [**E6**], and if $\hat{c}_2 = 0$ we obtain [**E5**].

[6] Here

$$ds^2 = (c_1(a^2 - b^2) + c_2 a + c_3 b + c_4)(da^2 - db^2)$$

and this is easily recognized to be in the form corresponding to the potential

$$V = \hat{c}_1(x^2 + y^2) + \hat{c}_2 x + \hat{c}_3 y + \hat{c}_4.$$

This can easily be interpreted. If $\hat{c}_1 \neq 0$ then we can take $\hat{c}_2$ and $\hat{c}_3 = 0$ by suitable translations and relate our system to a Stäckel transform of [**E3**]. If $\hat{c}_1 = 0$ then V can take one of the two forms

(i) $V = \alpha(x + iy) + \beta$ corresponding to [**E4**] or

(ii) $V = \alpha x$ corresponding to [**E6**].

2.2.2 TABLEAU VII

[1] Here the metric has the form

$$ds^2 = c_1(P(a)-P(b))+c_2(P(a+\omega_1)-P(b+\omega_1))+c_3(P(a+\omega_2)-P(b+\omega_2))+$$

$$c_4(P(a+\omega_3)-P(b+\omega_3))(da^2-db^2)$$

where $P(a)$ is the Weierstrass function, [17]. If we make the choice $e_1 = 1/k^2$, $e_2 = 1$ and $e_3 = 0$ in the standard formulas for these functions we can relate them directly to the Jacobi elliptic functions, [17], via the formulas

$$P(kz) = \frac{1}{k^2 \ \mathrm{sn}^2(z,k)}, \quad P(kz+\omega_1) = \frac{1}{k^2} - \frac{k'^2 \ \mathrm{sn}^2(z,k)}{k^2 \ \mathrm{cn}^2(z,k)},$$

$$P(kz+\omega_2) = \mathrm{sn}^2(z,k), \quad P(kz+\omega_3) = 1 - k'^2 \ \frac{\mathrm{sn}^2(z,k)}{\mathrm{cn}^2(z,k)}.$$

With these formulas the relationship to a constant curvature superintegrable system becomes clear. Indeed if we write the Hamilton-Jacobi equation

$$H = p_1^2 + p_2^2 + p_3^2 + \frac{\hat{c}_1}{s_1^2} + \frac{\hat{c}_2}{s_2^2} + \frac{\hat{c}_3}{s_3^2} + \hat{c}_4 = E$$

using conical coordinates in Jacobi elliptic function form, [17], viz.

$$s_1 = k \ \mathrm{sn}(\alpha,k) \ \mathrm{sn}(\beta,k), \ s_2 = i\frac{k'}{k} \ \mathrm{cn}(\alpha,k) \ \mathrm{cn}(\beta,k),$$

$$s_3 = \frac{k'}{k} \ \mathrm{dn}(\alpha,k) \ \mathrm{dn}(\beta,k), \quad s_1^2 + s_2^2 + s_3^2 = 1,$$

then it becomes

$$p_\alpha^2 + p_\beta^2 + \frac{\hat{c}_1}{k^2}\left(\frac{1}{\mathrm{sn}^2(\alpha,k)} - \frac{1}{\mathrm{sn}^2(\beta,k)}\right) + \frac{\hat{c}_2 k'^2}{k^2}\left(\frac{1}{\mathrm{cn}^2(\alpha,k)} - \frac{1}{\mathrm{cn}^2(\beta,k)}\right)$$

$$+\frac{\hat{c}_3 k'^2}{k^2}\left(\frac{1}{\mathrm{dn}^2(\alpha,k)} - \frac{1}{\mathrm{dn}^2(\beta,k)}\right) + (\hat{c}_4 - E)\left(\mathrm{sn}^2(\alpha,k) - \mathrm{sn}^2(\beta,k)\right) = 0$$

which has the form we expect. This system is therefore related to [**S9**] on the sphere, via a Stäckel transform.

[2] In this case

$$ds^2 = \left(c_1\left(\frac{1}{\sin^2 a}-\frac{1}{\sin^2 b}\right)+c_2\left(\frac{1}{\cos^2 a}-\frac{1}{\cos^2 b}\right)\right.$$
$$\left.+\,c_3(\cos 2a-\cos 2b)+c_4(\cos 4a-\cos 4b)\right)(da^2-db^2).$$

If we write out the Hamilton-Jacobi equation

$$H = p_1^2+p_2^2+\hat{c}_1(x^2+y^2)+\frac{\hat{c}_2}{x^2}+\frac{\hat{c}_3}{y^2}+\hat{c}_4 = E$$

using coordinates $x=\cos a\cos b$, $y=i\sin a\sin b$ we obtain

$$p_a^2-p_b^2+\hat{c}_1(\cos^4 b-\cos^4 b)+\hat{c}_2\left(\frac{1}{\cos^2 a}-\frac{1}{\cos^2 b}\right)$$

$$+\hat{c}_3\left(\frac{1}{\sin^2 a}-\frac{1}{\sin^2 b}\right)+(\hat{c}_4-E)(\cos^2 b-\cos^2 a)=0.$$

The potential for this case arises from [**E1**] via the choice of elliptic coordinates. This is clear from the usual multiplication formulas

$$\cos 2x = 2\cos^2 x-1, \quad \cos 4x = 8\cos^4 x-8\cos^2 x+1.$$

[3] Here

$$ds^2 = (c_1(\sin 4a-\sin 4b)+c_2(\cos 4a-\cos 4b)+c_3(\sin 2a-\sin 2b)$$

$$+c_4(\cos 2a-\cos 2b))(da^2-db^2).$$

If we write the Hamilton-Jacobi equation

$$H = p_1^2+p_2^2+\hat{c}_1+\frac{\hat{c}_2(x-iy)}{\sqrt{(x-iy)^2+4}}+\frac{\hat{c}_3(x+iy)}{((x-iy)^2+4)(x-iy+\sqrt{(x-iy)^2+4})^2}$$

$$+\hat{c}_4(x^2+y^2)=E,$$

using the coordinates $x=2i\cos u\cos v$, $y=2\sin u\sin v$ we obtain

$$p_u^2-p_v^2+2(\hat{c}_1-2E)(\cos 2u-\cos 2v)+\hat{c}_2(\sin 2u-\sin 2v)$$

$$+\frac{1}{4}\hat{c}_3(\cos 4u+i\sin 4u-\cos 4v-i\sin 4v)+2\hat{c}_4(\cos 4v-\cos 4u)=0$$

which gives rise to a metric of this type. This corresponds to system [**E7**].

[4] Here

$$ds^2 = \left(c_1(\frac{1}{a^2} - \frac{1}{b^2}\right) + c_2(a^2 - b^2) + c_3(a^4 - b^4) + c_4(a^6 - b^6))(da^2 - db^2).$$

In the coordinates $x = \frac{1}{2}(\xi^2 + \eta^2)$,$y = i\xi\eta$ the Hamilton-Jacobi equation

$$p_1^2 + p_2^2 + \hat{c}_1(4x^2 + y^2) + \hat{c}_2 x + \frac{\hat{c}_3}{y^2} + \hat{c}_4 = E$$

is equivalent to

$$p_\xi^2 - p_\eta^2 + (\hat{c}_4 - E)(\xi^2 - \eta^2) + \hat{c}_1(\xi^6 - \eta^6) + \frac{1}{2}\hat{c}_2(\xi^4 - \eta^4) + \hat{c}_3\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right) = 0,$$

from which we see that this system is obtained from [**E2**].

[5] The infinitesimal distance has the form

$$ds^2 = (c_1(a^4 - b^4) + c_2(a^3 - b^3) + c_3(a^2 - b^2) + c_4(a - b))(da^2 - db^2).$$

Consider the Hamilton-Jacobi equation

$$H = p_z p_{\overline{z}} + \hat{c}_1 + \hat{c}_2 z + \hat{c}_3(\bar{z} - \frac{3}{8}iz^2) - \frac{i}{8}\hat{c}_4(z^3 + 8iz\bar{z}) = E$$

where $z = x + iy$, $\bar{z} = x - iy$. In coordinates $z = 4i(u+w)$, $\bar{z} = 2i(u-w)^2$ this equation is equivalent to

$$p_u^2 - p_w^2 + 16(\hat{c}_1 - E)(u - w) + 64i\hat{c}_2(u^2 - w^2) + 128i\hat{c}_3(u^3 - w^3) - 256\hat{c}_4(u^4 - w^4) = 0$$

from which we see that this system is Stäckel equivalent to [**E10**] with some minor corrections.

In the last section we gave a simple derivation of all 2D superintegrable systems with nondegenerate potential. Such systems must admit at least 3 second order Killing tensors. Koenigs solved a different and more general problem. He found all spaces that admit at least 3 second order Killing tensors. It is a remarkable fact that the lists are the same! Thus from our point of view the Koenigs derivation is a proof of the following result.

Theorem 4 *Every 2D Riemannian space with at least 3 linearly independent second order Killing tensors admits a superintegrable system with nondegenerate potential.*

Corollary 1 *Necessary and sufficient conditions for a superintegrable system with nondegenerate potential on a 2D Riemannian manifold are that there are local orthogonal coordinates x, y such that the system takes the form $H/U(x,y)$ where*

$$H = \frac{p_x^2 + p_y^2}{\lambda(x,y)} + V(x,y)$$

is a superintegrable system on a constant curvature space with nondegenerate potential

$$V(x,y) = \alpha V^{(1)}(x,y) + \beta V^{(2)}(x,y) + \gamma V^{(3)}(x,y) + \delta$$

and

$$U(x,y) = \alpha_0 V^{(1)}(x,y) + \beta_0 V^{(2)}(x,y) + \gamma_0 V^{(3)}(x,y) + \delta_0.$$

Corollary 2 *Necessary and sufficient conditions for a 2D Riemannian manifold to admit a 3 dimensional space of second order Killing tensors are that there are local orthogonal coordinates x, y such that the metric takes the form $ds^2 = \lambda(x,y)U(x,y)(dx^2 + dy^2)$ where $\lambda(x,y)(dx^2 + dy^2)$ is a metric on a constant curvature space with nondegenerate potential*

$$V(x,y) = \alpha V^{(1)}(x,y) + \beta V^{(2)}(x,y) + \gamma V^{(3)}(x,y) + \delta$$

and

$$U(x,y) = \alpha_0 V^{(1)}(x,y) + \beta_0 V^{(2)}(x,y) + \gamma_0 V^{(3)}(x,y) + \delta_0.$$

3 Conclusions and further work

In this paper we have shown that every 2D nondegenerate superintegrable system is Stäckel equivalent (or equivalent via coupling constant metamorphosis) to a 2D nondegenerate superintegrable system on a constant curvature space. We found a simple derivation of all such spaces and potentials. We found that the list of spaces with nondegenerate potentials coincided with the Koenigs list of all 2D manifolds with 3 linearly independent second order Killing tensors. Thus any 2D space with 3 second order Killing tensors necessarily admits a nondegenerate potential.

In a forthcoming paper we will extend these results to 2D quantum systems, where the same spaces and potentials will occur. We will uncover the structure of the quantum quadratic algebra generated by the second order symmetry operators and show how to compute it in general.

Extension of our results to 3D systems is more challenging. Here the spaces we consider are conformally flat, since the Stäckel transform is conformal and the best known examples of superintegrable systems are in constant curvature spaces. Now for a superintegrable system we must have 5 functionally independent symmetries. Although several technical problems related to dimension must be overcome, we will be able to show that the structure theory for the quadratic algebras works in analogy to the 2D case. The extension to the quantum case is again more challenging, but the basic structure results for the quadratic algebra carry over for suitably modified potentials.

References

[1] E. G. Kalnins, J. M. Kress, and W. Miller, Jr. Second order superintegrable systems in conformally flat spaces. 1: 2D classical structure theory. (submitted) (2004).

[2] N. W. Evans. Superintegrability in Classical Mechanics; *Phys.Rev.* **A 41** (1990) 5666; Group Theory of the Smorodinsky-Winternitz System. *J. Math. Phys.* **32**, 3369 (1991).

[3] N. W. Evans. Super-Integrability of the Winternitz System. *Phys. Lett.* **A 147**, 483 (1990).

[4] S. Wojciechowski. Superintegrability of the Calogero-Moser System. *Phys. Lett.* **A 95**, 279 (1983);

[5] J. Friš, Ya. A. Smorodinskii, M. Uhlír and P. Winternitz. Symmetry Groups in Classical and Quantum Mechanics; *Sov. J. Nucl. Phys.* **4**, 444 (1967).

[6] A .A. Makarov, Ya. A. Smorodinsky, Kh. Valiev and P. Winternitz. A Systematic Search for Nonrelativistic Systems with Dynamical Symmetries. *Nuovo Cimento* **A 52**, 1061 (1967).

[7] F. Calogero. Solution of a Three-Body Problem in One Dimension. *J. Math. Phys.* **10**, 2191 (1969).

[8] P. Letourneau and L. Vinet. Superintegrable systems: Polynomial Algebras and Quasi-Exactly Solvable Hamiltonians. *Ann. Phys.* **243**, 144-168, (1995).

[9] M. F. Rañada. Superintegrable $n = 2$ systems, quadratic constants of motion, and potentials of Drach. *J. Math. Phys.* **38**, 4165, (1997).

[10] C. P. Boyer, E. G. Kalnins, and W. Miller. Stäckel - equivalent integrable Hamiltonian systems. *SIAM J. Math. Anal.* **17**, 778 (1986).

[11] J. Hietarinta, B. Grammaticos, B. Dorizzi and A. Ramani. Coupling-constant metamorphosis and duality between integrable Hamiltonian systems. *Phys. Rev. Lett.* **53**, 1707 (1984).

[12] E. G. Kalnins, J. M. Kress, W. Miller, Jr. and P. Winternitz. Superintegrable systems in Darboux spaces. *J. Math. Phys.* **44**, 5811–5848 (2003).

[13] G. Koenigs. Sur les géodésiques a intégrales quadratiques. A note appearing in "Lecons sur la théorie générale des surfaces". G. Darboux. Vol 4, 368-404, *Chelsea Publishing* 1972.

[14] E. G. Kalnins, W. Miller Jr. and G. S. Pogosyan. Completeness of multiseparable superintegrability in $E_{2,C}$. *J. Phys. A: Math Gen.* **33**, 4105 (2000).

[15] E. G. Kalnins, W. Miller Jr. and G. S. Pogosyan. Completeness of multiseparable superintegrability on the complex 2-sphere. *J. Phys. A: Math Gen.* **33**, 6791-6806 (2000).

[16] E. G. Kalnins, J. M. Kress, W. Miller, Jr. and G. S. Pogosyan. *Completeness of superintegrability in two-dimensional constant curvature spaces.* *J. Phys. A: Math Gen.* **34**, 4705 (2001).

[17] E. T. Whittaker and G. N. Watson. A Course of Modern Analysis. *Cambridge University Press*, Fourth Edition reprinted, 1958.