Tensor products of q-superalgebra representations and q-series identities.

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Abstract

We work out examples of tensor products for distinct q-generalizations of euclidean, oscillator and $s\ell(2)$ type superalgebras, in cases where the method of highest weight vectors will not apply. In particular, we use the three-term recurrence relations for Askey-Wilson polynomials to decompose the tensor product of representations from the positive discrete series and representations from the negative discrete series. We show that various q-analogs of the exponential function can be used to mimic the exponential mapping from a Lie algebra to its Lie group and we compute the corresponding matrix elements of the "group operators" on these representation spaces. We show

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1991 Mathematics Subject Classification: 33D55, 33D45, 17B37, 81R50 that the matrix elements themselves transform irreducibly under the action of the quantum superalgebra. The most important q-series identities derived here are interpreted as the expansion of the matrix elements of a "group operator" (via the exponential mapping) in a tensor product basis in terms of the matrix elements in a reduced basis. They involve q-hypergeometric series with base $-q,\ 0 < q < 1$.

1 Introduction

Zhedanov and others have introduced a product of generalized $s\ell_q(2)$ algebras that allows one to take tensor products of representations corresponding to two distinct algebras, [28, 8, 7]. Their generalization is an algebra (v, u) with generators H, E_+, E_- which obey the commutation relations

$$[H, E_+] = E_+, \quad [H, E_-] = -E_-, \quad [E_+, E_-] = -uq^{-H} - vq^H. \tag{1}$$

Here, u and v are real numbers and $0 < q < 1$. For $uv \neq 0$ this algebra is isomorphic to one of the true $s\ell_q(2)$ type algebras, for $uv = 0,\ u^2 + v^2 > 0$ it is isomorphic to a special realization of the q-oscillator algebra, and for $u = v = 0$ it is isomorphic to the Euclidean Lie algebra $m(2)$, [12]. This algebra has an invariant element

$$C = E_+E_- + \frac{vq^H - uq^{1-H}}{1-q}. \tag{2}$$

As pointed out by Zhedanov and others [28, 8, 7], the family of algebras admits a multiplication $(v, u) \otimes (-u, t) \cong (v, t)$, defined by

$$\begin{aligned} F_+ &= \Delta(E_+) = E_+ \otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H} \otimes E_+, \\ F_- &= \Delta(E_-) = E_- \otimes q^{\frac{1}{2}H} + q^{-\frac{1}{2}H} \otimes E_-, \\ L &= \Delta(H) = H \otimes I + I \otimes H. \end{aligned} \tag{3}$$

The operators $F_\pm$, L satisfy the commutation relations (1). Using (3) we can easily define the tensor product $\rho \otimes \mu$ of a representation ρ of (v, u) and the representation μ of $(-u, t)$, thereby obtaining a representation of (v, t). This construction yields a convenient generalization of the tensor product computations in, for example, [13, 14, 15]. We follow this idea to study generalizations of the $osp_q(1/2)$ algebra, [3, 21, 23].

In this paper the q-superalgebra $[v,u]$ is defined by the generators $H, V_\pm$ and relations

$$[H, V_\pm] = \pm\frac{1}{2}V_\pm, \quad \{V_+, V_-\} = -uq^{-2H} - vq^{2H}, \tag{4}$$

where $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$. Here, u, v are parameters. The "Casimir" operator C for $[v, u]$ is

$$C = V_+V_- + \frac{v}{1+q}q^{2H} + \frac{u}{1+q^{-1}}q^{-2H} \tag{5}$$

and satisfies the relations

$$[H, C] = 0, \quad \{V_\pm, C\} = 0. \tag{6}$$

Note that C^2 is an invariant operator, i.e., C^2 commutes with all elements of the $[v, u]$ algebra.

We define the coproduct

$$\begin{aligned} L &= \Delta(H) = H \otimes I + I \otimes H, \\ F_\pm &= \Delta(V_\pm) = V_\pm \otimes q^H + q^{-H} \otimes V_\pm, \end{aligned} \tag{7}$$

where

$$(A \otimes B)(C \otimes D) = (-1)^{p(B)p(C)} AC \otimes BD. \tag{8}$$

Here $p(A)$ is the *parity* of the operator A. In this case the bosonic variables H, q^H have parity 0 and the fermionic variables $V_\pm$ have parity 1, [2, 21]. It follows that this algebra admits a multiplication of the form

$$[v, u] \otimes [-u, t] = [v, t], \tag{9}$$

where the first factor corresponds to the algebra $[v, u]$ and the second factor to algebra $[-u, t]$. Indeed the relations

$$[L, F_\pm] = \pm\frac{1}{2}F_\pm, \quad \{F_+, F_-\} = -tq^{-2L} - vq^{2L}, \tag{10}$$

are satisfied. The only nontrivial part of the proof is

$$\{F_-, F_+\} =$$

$$(-uq^{-2H} - vq^{2H}) \otimes q^{2H} + q^{-2H} \otimes (-tq^{-2H} + uq^{2H}) =$$
$$-tq^{-2H} \otimes q^{-2H} - vq^{2H} \otimes q^{2H} = -tq^{-2L} - vq^{2L}.$$

In §2 and §3 we study irreducible representations of the "Euclidean" algebra $[0,0]$ and work out the (nonunique) tensor product decomposition for

$$[0,0] \otimes [0,0] \cong [0,0].$$

We compute the Clebsch-Gordan coefficients for the expansion and use them to derive q-series identities for the special functions that appear naturally in the theory. They are interpreted here as expansions of the matrix elements of a "group operator" in a tensor product basis in terms of the matrix elements in a reduced basis. The q-series are base $-q$, as follows naturally from expressions (7). In §4 we carry out the analogous constructions for positive discrete series representations of the general algebra $[v,u]$ for $v^2 + u^2 > 0$ tensored with negative discrete series representations. Here our methods lead naturally to a three-term recurrence relation for the Clebsch-Gordan coefficients that can be solved through comparison with the recurrence relation with Askey-Wilson polynomials. This yields the measure (not necessarily positive) determining the decomposition of the tensor product into irreducible components.

The notation used for q-series and q-integrals in this paper follows that of Gasper and Rahman [6].

2 Euclidean q-superalgebra representations

The three dimensional q-supersymmetric Lie algebra $[0,0]$ is determined by its generators H, V_+, V_- which obey the relations

$$[H, V_\pm] = \pm V_\pm, \quad \{V_+, V_-\} = 0. \tag{11}$$

We consider an analogy (ω) of the infinite dimensional irreducible representations of the Euclidean Lie algebra, characterized by the nonzero complex number ω The spectrum of H corresponding to (ω) is the set $S = \{n/2 : n \in Z\}$ and the complex representation space has basis vectors f_m, $m \in S$, such that

$$V_+ f_m = \omega f_{m+1}, \ V_- f_m = (-1)^m \omega f_{m-1}, \ H f_m = \frac{m}{2} f_m, \tag{12}$$

where $C \equiv V_+V_-$ is the "Casimir" operator, with $Cf_m = (-1)^m\omega^2 f_m$. Note that the representations (ω) and $(-\omega)$ are equivalent.

A simple realization of (ω) is given by the operators

$$H = \frac{1}{2}z\frac{d}{dz},\ V_+ = \omega z,\ V_- = \frac{\omega}{z}R_z \tag{13}$$

acting on the space of all linear combinations of the functions z^n, z a complex variable, $n \in Z$, with basis vectors $f_m(z) = z^m$. Here, the operatorR_z acts on functions $f(z)$ to give $R_z f(z) = f(-z)$. Note that

$$q^H = T_z^{1/2}, \quad \text{where} \quad T_z^\alpha f(z) = f(q^\alpha z).$$

We can introduce an inner product on the dense subspace of all finite linear combinations of the basis vectors, such that $< f_n, f_{n'} >= \delta_{nn'}$, $n, n' \in Z$. This induces the representation $(\overline{\omega})^*$ of $[0,0]$, adjoint to (ω), defined by the operators $V'_\pm = V^*_\mp$, $H' = H = H^*$ such that

$$V'_+ f_m = (-1)^{m+1}\overline{\omega} f_{m+1},\ V'_- f_m = \overline{\omega} f_{m-1},\ H' f_m = \frac{m}{2} f_m. \tag{14}$$

For ω real, this means $V'_+ = (-1)^{2H}V_+$, $V'_- = V_-(-1)^{2H}$.

In terms of the operators (13) we can obtain a realization of (ω) and its Hilbert space structure by setting $z = e^{i\theta}$:

$$H = -\frac{i}{2}\frac{d}{d\theta},\ V_+ = \omega e^{i\theta},\ V_- = \omega e^{-i\theta}R_z, \tag{15}$$

$$f_n(z) = e^{in\theta},\ < f, f' >= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{f'(e^{i\theta})}\ d\theta.$$

With the q-analogs of the exponential function

$$e_q(x) = \sum_{k=0}^{\infty}\frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty},\ |x| < 1,$$

$$E_q(x) = \sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}x^k}{(q;q)_k} = (-x;q)_\infty, \tag{16}$$

we employ the model (13) to define the following q-analogs of matrix elements of (ω):

$$a)\ e_{-q}(\beta V_+)E_{-q}(\alpha V_-)f_n = \sum_{n'=-\infty}^{\infty} T_{n'n}^{(e+,E-)}(\alpha,\beta)f_{n'},\ |\omega\beta| < 1 \tag{17}$$

$$b)\ E_{-q}(\beta V_+)e_{-q}(\alpha V_-)f_n = \sum_{n'=-\infty}^{\infty} T^{(E+,e-)}_{n'n}(\alpha,\beta)f_{n'},\ |\omega\alpha|<1. \tag{18}$$

$$c)\ e_{-q}(\beta V_-)E_{-q}(\alpha V_+)f_n = \sum_{n'=-\infty}^{\infty} T^{(e-,E+)}_{n'n}(\alpha,\beta)f_{n'},\ |\omega\beta|<1 \tag{19}$$

$$d)\ E_{-q}(\beta V_-)e_{-q}(\alpha V_+)f_n = \sum_{n'=-\infty}^{\infty} T^{(E-,e+)}_{n'n}(\alpha,\beta)f_{n'},\ |\omega\alpha|<1. \tag{20}$$

where $0<q<1$ and $\alpha,\beta\in\mathbb{C}$. (Here, α, β are *not* fermionic variables which is the usual choice in defining the action of supersymmetric groups [20, 25].) The motivation for the choice of base $-q$ comes from the relations

$$(V_\pm\otimes q^H)(q^{-H}\otimes V_\pm) = -q^{\pm1}(q^{-H}\otimes V_\pm)(V_\pm\otimes q^H) \tag{21}$$

and the well known property that if X and Y are linear operators such that $YX=pXY$ then, [6], page 28,

$$(Y+X)^k = \sum_{\ell=0}^{k}\frac{(p;p)_k}{(p;p)_\ell(p;p)_{k-\ell}}X^\ell Y^{k-\ell},$$

$$e_p(X+Y)=e_p(X)e_p(Y),\quad E_p(X+Y)=E_p(Y)E_p(X). \tag{22}$$

Since $V_+^*=V_-'$, $V_-^*=V_+'$ we have

$$T^{(e\pm,E\mp)}_{n'n}(\alpha,\beta)=\overline{T'^{(E\pm,e\mp)}_{nn'}(\overline{\beta},\overline{\alpha})}. \tag{23}$$

$$T^{(E\pm,e\mp)}_{n'n}(\alpha,\beta)=\overline{T'^{(e\pm,E\mp)}_{nn'}(\overline{\beta},\overline{\alpha})}. \tag{24}$$

Furthermore, since $e_{-q}(x)E_{-q}(-x)=1$, we have the identities

$$\sum_{\ell=-\infty}^{\infty} T^{(e-,E+)}_{n\ell}(\alpha,\beta)T^{(e+,E-)}_{\ell n'}(-\beta,-\alpha)=\delta_{nn'},\ |\omega\alpha|,|\omega\beta|<1, \tag{25}$$

$$\sum_{\ell=-\infty}^{\infty} T^{(E+,e-)}_{n\ell}(\alpha,\beta)T^{(E-,e+)}_{\ell n'}(-\beta,-\alpha)=\delta_{nn'},\ |\omega\alpha|,|\omega\beta|<1. \tag{26}$$

(Note that our operator derivations of these formulas and of some formulas to follow lead automatically to formal power series identities in the 'group parameters'. These identities must then be examined case by case to determine when the series are convergent as analytic functions of the group parameters.) Using the model (13) to treat (17,18,19,20) as generating functions for the matrix elements and computing the coefficients of $z^{n'}$ in the resulting expressions we obtain the explicit results ($p=-q$):

$$T^{(e+,E-)}_{n'n}(\alpha,\beta)=\frac{(\beta\omega)^{n'-n}}{(p;p)_{n'-n}}\Big[{}_1\phi_1\left(\begin{matrix}0\\ p^{n'-n+1}\end{matrix};p,(-1)^{n+1}i\alpha\beta\omega^2\right)\frac{1-i}{2} \tag{27}$$
$$+{}_1\phi_1\left(\begin{matrix}0\\ p^{n'-n+1}\end{matrix};p,(-1)^{n}i\alpha\beta\omega^2\right)\frac{1+i}{2}\Big].$$

$$\begin{aligned}T^{(E+,e-)}_{n'n}(\alpha,\beta) &= \frac{(\beta\omega)^{n'-n}p^{(n'-n)(n'-n-1)/2}}{(p;p)_{n'-n}} \\ &\Big[\ {}_1\phi_1\left(\begin{matrix}0\\ p^{n'-n+1}\end{matrix};p,(-1)^{n+1}p^{n'-n}i\alpha\beta\omega^2\right)\frac{1-i}{2} \\ &+\ {}_1\phi_1\left(\begin{matrix}0\\ p^{n'-n+1}\end{matrix};p,(-1)^{n}p^{n'-n}i\alpha\beta\omega^2\right)\frac{1+i}{2}\Big].\end{aligned} \tag{28}$$

$$\begin{aligned}T^{(e-,E+)}_{n'n}(\alpha,\beta) &= \frac{(\alpha\omega)^{n'-n}p^{(n'-n)(n'-n-1)/2}}{(p;p)_{n'-n}} \\ &\Big[\ {}_1\phi_1\left(\begin{matrix}0\\ p^{n'-n+1}\end{matrix};p,(-1)^{n'}p^{n'-n}i\alpha\beta\omega^2\right)\frac{1-i}{2} \\ &+\ {}_1\phi_1\left(\begin{matrix}0\\ p^{n'-n+1}\end{matrix};p,(-1)^{n'+1}p^{n'-n}i\alpha\beta\omega^2\right)\frac{1+i}{2}\Big].\end{aligned} \tag{29}$$

$$T^{(E-,e+)}_{n'n}(\alpha,\beta)=\frac{(\alpha\omega)^{n'-n}}{(p;p)_{n'-n}}\Big[{}_1\phi_1\left(\begin{matrix}0\\ p^{n'-n+1}\end{matrix};p,(-1)^{n'}i\alpha\beta\omega^2\right)\frac{1-i}{2} \tag{30}$$
$$+{}_1\phi_1\left(\begin{matrix}0\\ p^{n'-n+1}\end{matrix};p,(-1)^{n'+1}i\alpha\beta\omega^2\right)\frac{1+i}{2}\Big].$$

Here we have made use of the identity

$$(-1)^{n(n-1)/2}=\frac{1-i}{2}(i)^n+\frac{1+i}{2}(-i)^n. \tag{31}$$

These results make sense for $n > n'$ as well as $n' \geq n$. Indeed, for $k > 0$ we have

$$\lim_{m\to -k} \frac{1}{(p;p)_m} {}_1\phi_1\left(\begin{array}{c} 0 \\ p^{m+1} \end{array}; p, B\right) = \frac{p^{k(k-1)/2}(-B)^k}{(p;p)_k} {}_1\phi_1\left(\begin{array}{c} 0 \\ p^{k+1} \end{array}; p, p^k B\right). \tag{32}$$

The matrix elements $T_{n'n}(\alpha,\beta)$ themselves define models of the representations (ω). We can see this directly from (11). For example, it is a simple consequence of these relations and $e_p(x) = (x;p)_\infty^{-1}$, $E_p(x) = (-x;p)_\infty$ that

$$e_p(\beta V_+)E_p(\alpha V_-)V_+ = \frac{1}{\beta}(I - T_\beta)R_\alpha e_p(\beta V_+)E_p(\alpha V_-) \tag{33}$$

$$e_p(\beta V_+)E_p(\alpha V_-)V_- = \frac{p}{\alpha}(T_\alpha^{-1} - I)e_p(\beta V_+)E_p(\alpha V_-) \tag{34}$$

where I is the identity operator and $T_\beta g(\alpha,\beta) = g(\alpha,\beta p)$, $R_\alpha g(\alpha,\beta) = g(-\alpha,\beta)$ for a function $g(\alpha,\beta)$. Thus

$$\begin{aligned} \omega T^{(e+,E-)}_{n',n+1}(\alpha,\beta) &= < e_p(\beta V_+)E_p(\alpha V_-)V_+ f_n, f_{n'} > \\ &= \frac{1}{\beta}(I - T_\beta)R_\alpha T^{(e+,E-)}_{n'n}(\alpha,\beta), \end{aligned} \tag{35}$$

$$(-1)^n \omega T^{(e+,E-)}_{n',n-1}(\alpha,\beta) = \frac{p}{\alpha}(T_\alpha^{-1} - I)T^{(e+,E-)}_{n'n}(\alpha,\beta). \tag{36}$$

Furthermore, induction with respect to $k+\ell$ yields $[H, \alpha^\ell\beta^k V_+^k V_-^\ell] = \frac{1}{2}(k-\ell)\alpha^\ell\beta^k V_+^k V_-^\ell$ and this implies

$$[H, e_p(\beta V_+)E_p(\alpha V_-)] = \frac{1}{2}(\beta\partial_\beta - \alpha\partial_\alpha)e_p(\beta V_+)E_p(\alpha V_-),$$

so

$$\begin{aligned} \frac{1}{2}(n' \;-\; n)&T^{(e+,E-)}_{n'n}(\alpha,\beta) \\ &= < e_p(\beta V_+)E_p(\alpha V_-)f_n, Hf_{n'} > - < e_p(\beta V_+)E_p(\alpha V_-)Hf_n, f_{n'} > \\ &= < [H, e_p(\beta V_+)e_p(\alpha V_-)]f_n, f_{n'} > = \frac{1}{2}(\beta\partial_\beta - \alpha\partial_\alpha)T^{(e+,E-)}_{n'n}(\alpha,\beta). \end{aligned} \tag{37}$$

Thus, the operators

$$\tilde{V}_+ = \frac{1}{\beta}(I - T_\beta)R_\alpha, \quad \tilde{V}_- = \frac{p}{\alpha}(T_\alpha^{-1} - I), \quad \tilde{H} = \frac{1}{2}(\alpha\partial_\alpha - \beta\partial_\beta) + \frac{n'}{2}$$

and the basis functions $f_n = T^{(e+,E-)}_{n'n}$ define a two variable realization of relations (11), hence, a realization of the representation (ω). Similar considerations apply to all of the matrix elements $T^{(e\pm,E\mp)}_{n'n}$, $T^{(E\pm,e\mp)}_{n'n}$, as well as to $T^{(e\pm,e\mp)}_{n'n}$, $T^{(E\pm,E\mp)}_{n'n}$ [16].

3 Tensor products of Euclidean q-superalgebra representations

We study the tensor product of two irreducible representations (ω_1) and (ω_2) of $[0,0]$ in which the coproduct is (7):

$$L = \Delta(H) = H\otimes I + I\otimes H, \quad F_\pm = \Delta(V_\pm) = V_\pm \otimes q^H + q^{-H}\otimes V_\pm .$$

The operators $F_\pm$, L satisfy the same relations as the operators $V_\pm$, H:

$$[L, F_\pm] = \pm\frac{1}{2}F_\pm, \quad \{F_+, F_-\} = 0. \tag{38}$$

Each irreducible representation (ω_1), (ω_2) is defined on $L_2[0, 2\pi]$ by the prescription (15). To make sense of the operators (7) on a dense subspace of the tensor product space $L[0,2\pi]\otimes L[0,2\pi]$ we proceed as follows. The Hilbert space inner product is

$$< f, g >= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(\theta_1,\theta_2)\overline{g(\theta_1,\theta_2)}\ d\theta_1 d\theta_2 .$$

An orthonormal basis is $\{f_{n_1 n_2} = e^{i(n_1\theta_1+n_2\theta_2)}, n_j = 0, \pm1, \pm2, \cdots\}$. Using (13) we define the coproduct operators as

$$\begin{aligned} F_+ &= \omega_1 z_1 T_2^{\frac{1}{2}} + \omega_2 z_2 T_1^{-\frac{1}{2}}, \quad F_- = \frac{\omega_1}{z_1}R_1 T_2^{\frac{1}{2}} + \frac{\omega_2}{z_2}R_2 T_1^{-\frac{1}{2}}, \\ L &= \frac{1}{2}z_1\frac{d}{dz_1} + \frac{1}{2}z_2\frac{d}{dz_2}, \end{aligned} \tag{39}$$

where $R_j = R_{z_j}$, $T_j = T_{z_j}$, and $z_j = e^{i\theta_j}$. In order to satisfy relations (38) we require that $z_1 z_2 = -z_2 z_1$, i.e., that the variables $z_1,\ z_2$ anticommute.

Thus, if f satisfies

$$F_+F_- f = \lambda f, \quad Lf = \frac{h}{2}f \tag{40}$$

it should have the form $f[z,t] = t^h g_h[z]$ where

$$(\omega_1 + \omega_2 z q^{1/2}(-q^{-\frac{1}{2}})^h)(\omega_1(-1)^h + \frac{\omega_2(-q^{-\frac{1}{2}})^h)}{zq^{1/2}})g_h[-zq] = \lambda g_h[z],$$

and $z = z_2 z_1^{-1},\ t = z_1$. Note that $zt = -tz$.

Just as in [16], there is an infinite parameter family of bases associated with the eigenvalue problem. We shall focus on those two that are the simplest in structure. The first corresponds to the basis of eigenvectors $f^{(1)}_{kh} = t^h g^{(1)}_h[z]$,

$$g^{(1)}_h[z] = i^{hk}\frac{((-1)^{h+1}q^{(1-h)/2}\frac{\omega_2 z}{\omega_1}; -q)_\infty}{(q^{(1-h)/2}\frac{\omega_2}{\omega_1 z}; -q)_\infty} z^k, \quad k = 0, \pm 1, \cdots$$

with eigenvalues $\lambda^{(1)}_k = \omega_1^2(-q)^k(-1)^h$. The second has a basis of eigenvectors

$$g^{(2)}_h[z] = i^{hk}\frac{(-q^{(1+h)/2}\frac{\omega_1 z}{\omega_2}; -q)_\infty}{((-1)^h q^{(1+h)/2}\frac{\omega_1}{\omega_2 z}; -q)_\infty} z^{h+k}, \quad k = 0, \pm 1, \cdots$$

with eigenvalues $\lambda^{(2)}_h = \omega_2^2(-q)^k(-1)^h$. Setting $f^{(j)}_{kh}[z,t] = g^{(j)}_h[z]t^h$, $j = 1, 2$, we have

$$\begin{aligned} F_+ f^{(j)}_{kh} &= \omega_j(-iq^{1/2})^k f^{(j)}_{k,h+1} && (41)\\ F_- f^{(j)}_{kh} &= \omega_j(-iq^{1/2})^k(-1)^h f^{(j)}_{k,h-1} && (42)\\ L f^{(j)}_{kh} &= \frac{h}{2} f^{(j)}_{kh}, \quad C f^{(j)}_{kh} = \omega_j^2(-q)^k(-1)^h f^{(j)}_{kh}. && (43)\end{aligned}$$

Thus

$$(\omega_1) \otimes^j_q (\omega_2) \equiv \sum_{k=-\infty}^{\infty} \oplus(\omega_j(iq)^{k/2}),$$

i.e., in each case the tensor product decomposes into a direct sum of irreducible representations.

In a similar manner we can use the adjoint operators to (39) and compute the resolution of the tensor product representation of the factors:

$$\begin{aligned} F'_+ &= -\overline{\omega_1} z_1 R_1 T_2^{\frac{1}{2}} - \overline{\omega_2} z_2 R_2 T_1^{-\frac{1}{2}}, \quad F'_- = \frac{\overline{\omega_1}}{z_1} T_2^{\frac{1}{2}} + \frac{\overline{\omega_2}}{z_2} T_1^{-\frac{1}{2}},\\ L' &= \frac{1}{2} z_1 \frac{d}{dz_1} + \frac{1}{2} z_2 \frac{d}{dz_2}. && (44)\end{aligned}$$

If f' satisfies

$$F'_+F'_-f' = \lambda' f', \quad L'f' = \frac{h}{2}f' \tag{45}$$

it should have the form $f'[z,t] = t^h g'_h[z]$ where

$$(\overline{\omega_1} - \overline{\omega_2} z q^{1/2}(-q^{-\frac{1}{2}})^h)(\overline{\omega_1}(-1)^h - \frac{\overline{\omega_2}(q^{-\frac{1}{2}})^h)}{zq^{1/2}})g'_h[-zq] = \lambda g'_h[z],$$

and $z = z_2 z_1^{-1}$, $t = z_1$.

Again there is an infinite parameter family of bases associated with the eigenvalue problem, but two have the simplest structure. The first corresponds to the basis of eigenvectors $f'^{(1)}_{kh} = t^h g'^{(1)}_h[z]$,

$$g'^{(1)}_h[z] = i^{hk}\frac{(q^{(1-h)/2}\frac{\overline{\omega_2}z}{\overline{\omega_1}}; -q)_\infty}{((-1)^{h+1}q^{(1-h)/2}\frac{\overline{\omega_2}}{\overline{\omega_1}z}; -q)_\infty} z^k, \quad k = 0, \pm 1, \cdots$$

with eigenvalues $\lambda'^{(1)}_k = \overline{\omega_1}^2(-q)^k(-1)^h$. The second has a basis of eigenvectors

$$g'^{(2)}_h[z] = i^{hk}\frac{((-1)^{h+1}q^{(1+h)/2}\frac{\overline{\omega_1}z}{\overline{\omega_2}}; -q)_\infty}{(q^{(1+h)/2}\frac{\overline{\omega_1}}{\overline{\omega_2}z}; -q)_\infty} z^{h+k}, \quad k = 0, \pm 1, \cdots$$

with eigenvalues $\lambda'^{(2)}_h = \overline{\omega_2}^2(-q)^k(-1)^h$. Setting $f'^{(j)}_{kh}[z,t] = g'^{(j)}_h[z]t^h$, $j = 1, 2$, we have

$$F'_+ f'^{(j)}_{kh} = \overline{\omega_j}(iq^{1/2})^k(-1)^{h+1} f'^{(j)}_{k,h+1} \tag{46}$$

$$F'_- f'^{(j)}_{kh} = \overline{\omega_j}(iq^{1/2})^k f'^{(j)}_{k,h-1} \tag{47}$$

$$L' f'^{(j)}_{kh} = \frac{h}{2} f'^{(j)}_{kh}, \quad C f'^{(j)}_{kh} = \overline{\omega_j}^2(-q)^k(-1)^h f'^{(j)}_{kh}. \tag{48}$$

Thus

$$(\overline{\omega_1})' \otimes^j_q (\overline{\omega_2})' \equiv \sum_{k=-\infty}^{\infty} \oplus(\overline{\omega_j}(iq)^{k/2})'.$$

The functions $\{f'^{(j)}_{k'h'}\}$ form dual bases for the functions $\{f^{(j)}_{kh}\}$, $j = 1, 2$. In particular the biorthogonality relations

$$< f^{(j)}_{kh}, f'^{(j)}_{k'h'} >= \delta_{kk'}\delta_{hh'} \tag{49}$$

are satisfied.

The Clebsch-Gordan coefficients for the tensor product corresponding to the spectral resolution (3) are defined by

$$f_{kh}^{(1)}[z,t] = f_{kh}^{(1)}(\theta_1,\theta_2) = \sum_{n_1,n_2=-\infty}^{\infty} \begin{bmatrix} \omega_1 & \omega_2 & k \\ n_1 & n_2 & h \end{bmatrix}^{(1)} f_{n_1}^{\omega_1}(e^{i\theta_1}) \otimes_q^1 f_{n_2}^{\omega_2}(e^{i\theta_2}). \tag{50}$$

Clearly, these coefficients vanish unless $h = n_1 + n_2$. Note that the tensor product basis is $z_1^{n_1} z_2^{n_2} = (-1)^{n_2(n_2+1)/2} t^{n_1+n_2} z^{n_2}$. There is a similar expansion for the dual basis:

$$f'^{(1)}_{kh}[z,t] = f'^{(1)}_{kh}(\theta_1,\theta_2) = \sum_{n_1,n_2=-\infty}^{\infty} \begin{bmatrix} \overline{\omega_1} & \overline{\omega_2} & k \\ n_1 & n_2 & h \end{bmatrix}'^{(1)} f_{n_1}^{\overline{\omega_1}}(e^{i\theta_1}) \otimes_q^1 f_{n_2}^{\overline{\omega_2}}(e^{i\theta_2}). \tag{51}$$

The biorthogonality of the two bases implies the identities

$$\sum_{n_1,n_2} \begin{bmatrix} \omega_1 & \omega_2 & k \\ n_1 & n_2 & h \end{bmatrix}^{*(1)} \begin{bmatrix} \overline{\omega_1} & \overline{\omega_2} & k' \\ n_1 & n_2 & h \end{bmatrix}'^{(1)} = \delta_{kk'},$$

$$\sum_{k} \begin{bmatrix} \omega_1 & \omega_2 & k \\ n_1 & n_2 & h \end{bmatrix}^{*(1)} \begin{bmatrix} \overline{\omega_1} & \overline{\omega_2} & k \\ n_1' & n_2' & h \end{bmatrix}'^{(1)} = \delta_{n_1 n_1'}$$

where $n_1+n_2 = n_1'+n_2' = h$ and a^* is the complex conjugate of a. Explicitly,

$$\begin{bmatrix} \omega_1 & \omega_2 & k \\ n_1 & n_2 & h \end{bmatrix}^{(1)} (-1)^{n_2(n_2+1)/2}(-i)^{hk} =$$

$$\frac{((-1)^h a_k)^{n_2-k} p^{(n_2-k)(n_2-k-1)/2}}{(p;p)_{n_2-k}} {}_1\phi_1\left(\begin{matrix} 0 \\ p^{n_2-k+1} \end{matrix}; p; (-1)^{h+1} a_k^2 p^{n_2-k}\right) \tag{52}$$

$$-\sum_{\ell=0}^{s_h-1} \frac{(p^\ell a_h^2;p)_\infty p^{\ell(\ell-1)/2}(-q^\ell a_h)^{k-n_2}(-1)^\ell}{(p;p)_\infty (p;p)_\ell},$$

where $p = -q$,$h = n_1 + n_2$, $a_h = q^{(1-h)/2}\omega_2/\omega_1$ and $s_h \geq 0$ is the smallest integer such that $q^{s_h} a_h < 1$. (We assume that $a_h \neq q^n$ for any integer n.) Indeed the case $s_h = 0$ of (52) follows from (50) and this result can be written as a complex contour integral. The case $s_h > 0$ can be obtained from this result by shifting the contour.

Similarly the Clebsch-Gordan coefficients for the tensor product corresponding to the first dual basis are given explicitly by (52), where now a_h is replaced by $a'_h = (-1)^{h+1} q^{(1-h)/2} \overline{\omega_2}/\overline{\omega_1}$.

With respect to the tensor product basis $\{f^{\omega_1}_{n_1} \otimes f^{\omega_2}_{n_2}\}$ the operator $e_q(\beta F_+) E_q(\alpha F_-)$ has matrix elements

$$\begin{aligned}
T_{m_1 m_2; n_1 n_2}(\alpha, \beta) &= \; < e_p(\beta \Delta(V_+)) E_p(\alpha \Delta(V_-)) f^{\omega_1}_{n_1} \otimes f^{\omega_2}_{n_2}, f^{\omega_1}_{m_1} \otimes f^{\omega_2}_{m_2} > \\
&= \frac{1}{2} [T^{(e+,E-)}_{m_1 n_1}(\alpha q^{m_2}, \beta q^{m_2})\, T^{(e+,E-)}_{m_2 n_2}\left(\alpha p^{-n_1}, \beta p^{-n_1}\right) \\
&+ T^{(e+,E-)}_{m_1 n_1}(\alpha q^{m_2}, \beta q^{m_2})\, T^{(e+,E-)}_{m_2 n_2}\left(\alpha p^{-n_1}, -\beta p^{-n_1}\right) \\
&+ T^{(e+,E-)}_{m_1 n_1}(-\alpha q^{m_2}, \beta q^{m_2})\, T^{(e+,E-)}_{m_2 n_2}\left(\alpha p^{-n_1}, \beta p^{-n_1}\right) \\
&- T^{(e+,E-)}_{m_1 n_1}(-\alpha q^{m_2}, \beta q^{m_2})\, T^{(e+,E-)}_{m_2 n_2}\left(\alpha p^{-n_1}, -\beta p^{-n_1}\right)].
\end{aligned} \tag{53}$$

Indeed,

$$\begin{aligned}
e_p(\beta F_+) E_p(\alpha F_-) &= e_p(\beta V_+ \otimes q^H) e_p(\beta q^{-H} \otimes V_+) E_p(\alpha V_- \otimes q^H) E_p(\alpha q^{-H} \otimes V_-) \\
&= \frac{1}{2} e_p(\beta V_+ \otimes q^H) \Big(E_p(B) e_p(A) + E_p(B) e_p(-A) \\
&+ E_p(-B) e_p(A) - E_p(-B) e_p(-A) \Big) E_p(\alpha q^{-H} \otimes V_-)
\end{aligned} \tag{54}$$

where $B = \alpha V_- \otimes q^H$, $A = \beta q^{-H} \otimes V_+$, $AB = -BA$, and we have used the identity

$$(-1)^{mn} = \frac{1}{2}(1 + (-1)^m + (-1)^n - (-1)^{m+n}). \tag{55}$$

From the definition of the Clebsch-Gordan coefficients we see immediately that the identities

$$\begin{aligned}
T_{m_1 m_2; n_1 n_2}(\alpha, \beta) &= \sum_{k=-\infty}^{\infty} \left[\begin{array}{ccc} \overline{\omega_1} & \overline{\omega_2} & k \\ n_1 & n_2 & n_1 + n_2 \end{array} \right]^{\prime (j)*} T^{(e+,E-),\omega_j (iq)^{k/2}}_{m_1+m_2, n_1+n_2}(\alpha, \beta) \\
&\times \left[\begin{array}{ccc} \omega_1 & \omega_2 & k \\ m_1 & m_2 & m_1 + m_2 \end{array} \right]^{(j)}, \quad j = 1, 2
\end{aligned} \tag{56}$$

must hold.

4 Representations of the q-supersymmetric algebras $[v, u]$

We consider the family of algebraically irreducible representations (bounded below) $\uparrow_\lambda$, of the q-superalgebra $[-u, t]$, where $t > 0, uq^{-4\lambda}, -uq^{-4\lambda}$, defined as follows [4]. A convenient orthonormal basis for the representation space is $\{e_n : n = 0, 1, \cdots\}$ where

$$\begin{aligned} V_- e_n &= \left[\frac{(1-p^n)(tq^{2\lambda-n+1} + u(-1)^n q^{-2\lambda})}{1+q}\right]^{\frac{1}{2}} e_{n-1} \\ V_+ e_n &= -\left[\frac{(1-p^{n+1})(tq^{2\lambda-n} - u(-1)^n q^{-2\lambda})}{1+q}\right]^{\frac{1}{2}} e_{n+1} \\ He_n &= (-\lambda + \frac{n}{2})e_n. \end{aligned} \tag{57}$$

We have $V_+ = -(V_-)^*$ and $H^* = H$. The "Casimir" operator (5) has the action

$$Ce_n = (-1)^n \mu e_n, \quad \mu = \frac{-uq^{2\lambda}}{1+q} + \frac{tq^{-2\lambda}}{1+q^{-1}}. \tag{58}$$

Note that $C^2 = \mu^2 I$ is a multiple of the identity operator

In analogy with reference [4], we define the following q-analog of matrix elements of $\uparrow_\lambda$:

$$e_{-q}(\beta V_-)E_{-q}(\alpha V_+)f_n = \sum_{n'=0}^{\infty} T_{n'n}^{(e-,E+)}(\alpha, \beta)f_{n'}, \ |q^{2\lambda-n}\alpha\beta t/(1+q)| < 1, \tag{59}$$

where $0 < q < 1$ and $\alpha, \beta \in \mathbb{C}$. Again the motivation for the choice of base $-q$ comes from the relations

$$(V_\pm \otimes q^H)(q^{-H} \otimes V_\pm) = -q^{\pm 1}(q^{-H} \otimes V_\pm)(V_\pm \otimes q^H),$$

relevant for the coproduct. Subject only to the conditions on (59), the left-hand side of this expression is an analytic function of z, convergent in a neighborhood of $z = 0$. Note that

$$< Te_n, e_{n'} >= T_{n'n}. \tag{60}$$

Since the "group" operators that we are considering are unbounded, it is not clear that relation (60) makes more than formal sense. However, by truncating the operator power series expressions for $e_p(\alpha V_-)$, $E_p(\beta V_+)$ at N terms, we see that (60) holds rigorously for the truncated series. The final result follows in the limit as $N \to \infty$.

Using the model in [4] to compute the matrix elements (which are model independent) we obtain the explicit result

$$\begin{aligned} T^{(e-,E+)}_{n'n}(\alpha,\beta) &= (-1)^{(n'-n)(n'-n+1)/2}\sqrt{\frac{(p;p)_{n'}(\frac{u}{t}q^{-4\lambda};p)_n}{(p;p)_n(\frac{u}{t}q^{-4\lambda};p)_{n'}}} \\ &\times\ (\sqrt{\frac{t}{1+q}}\alpha)^{n'-n}q^{(n'-n)(7\lambda+n'-3n-1)/4}\frac{(\frac{(-1)^n u}{t}q^{n-4\lambda};p)_{n'-n}}{(p;p)_{n'-n}} \\ &\times\ {}_2\phi_1\left(\begin{matrix} p^{n'+1}, & \frac{u}{t}(-1)^n q^{n'-4\lambda} \\ p^{n'-n+1} & \end{matrix} ;p;\frac{(-1)^{n-n'+1}\alpha\beta t}{q^{n-2\lambda}(1+q)}\right), \end{aligned} \tag{61}$$

where $|q^{2\lambda-n}\alpha\beta t/(1+q)| < 1$.

These results make sense for $n > n'$ as well as $n' \geq n$. Indeed, for $k > 0$ we have

$$\lim_{m\to -k}\frac{1}{(p;p)_m}{}_2\phi_1\left(\begin{matrix} A & B \\ p^{m+1} & \end{matrix};p,C\right) = \frac{(A,B;p)_k C^k}{(p;p)_k}{}_2\phi_1\left(\begin{matrix} Ap^k & Bp^k \\ p^{k+1} & \end{matrix};p,C\right). \tag{62}$$

We also consider a second family of algebraically irreducible representations (bounded above) $\downarrow_\xi$, of the q-superalgebra $[v,u]$, where $v > 0, uq^{4\xi-1}, -uq^{4\xi}$, defined as follows. A convenient orthonormal basis for the representation space is $\{j_m : m = 0,1,\cdots\}$ where

$$\begin{aligned} V_- j_m &= (-1)^{m+1}\left[\frac{(1-p^{m+1})(vq^{-2\xi-m}+u(-1)^m q^{2\xi})}{1+q}\right]^{\frac{1}{2}} j_{m+1} \\ V_+ j_m &= (-1)^{m+1}\left[\frac{(1-p^{m})(vq^{-2\xi-m+1}-u(-1)^m q^{2\xi})}{1+q}\right]^{\frac{1}{2}} j_{m-1} \\ H j_m &= (-\xi-\frac{m}{2})j_m. \end{aligned} \tag{63}$$

We have $V_+ = -(V_-)^*$ and $H^* = H$.

For this representation we consider the matrix elements

$$(e-,E+):\quad e_p(\beta V_-)E_p(\alpha V_+)j_m=\sum_{m'}S^{(\xi)}_{m'm}(\alpha,\beta)j_{m'}.$$

$$\begin{aligned}
S^{(\xi)}_{m'm}(\alpha,\beta) &= \beta^{m'-m}q^{(m'-m)(-4\xi-m'-m+1)/4}\frac{(\frac{-u}{v}p^mq^{4\xi};p)_{m'-m}}{(p;p)_{m'-m}} \\
&\times\ (-1)^{(m'-m)(m'+m+1)/2}\sqrt{(\frac{v}{1+q})^{m'-m}\frac{(p;p)_{m'}(-\frac{u}{v}q^{4\xi};p)_m}{(p;p)_m(-\frac{u}{v}q^{4\xi};p)_{m'}}} \\
&\times\ \Big[\frac{1-i}{2}{}_2\phi_1\left(\begin{matrix}p^{-m}, & \frac{v}{u}q^{-4\xi+1}p^{-m} \\ p^{m'-m+1} & \end{matrix};p;\frac{-\alpha\beta iu}{q^{-2\xi-m}(1+q)}\right) \\
&+\ \frac{1+i}{2}{}_2\phi_1\left(\begin{matrix}p^{-m}, & \frac{v}{u}q^{-4\xi+1}p^{-m} \\ p^{m'-m+1} & \end{matrix};p;\frac{\alpha\beta iu}{q^{-2\xi-m}(1+q)}\right)\Big]
\end{aligned}\tag{64}$$

Now we form the tensor product representation

$$\downarrow_\xi[v,u]\otimes\uparrow_\lambda[-u,t]\cong[v,t]\tag{65}$$

of $[v,t]$. Rather than define the operators determining the tensor product in the form (7), we will use the equivalent definition

$$F_\pm=V_\pm\otimes q^H+(-1)^{2H'}q^{-H}\otimes V_\pm,\quad L=H\otimes I+I\otimes H,\tag{66}$$

where now

$$(A\otimes B)j_m\otimes e_n=Aj_m\otimes Be_n,\quad 2H'j_m=(-1)^mj_m.\tag{67}$$

With this definition the variables commute in our functional models. The invariant operator is

$$C=F_+F_-+\frac{v}{1+q}q^{2L}+\frac{t}{1+q^{-1}}q^{-2L}.$$

To decompose this representation we compute the common eigenfunctions of L and C. Clearly, eigenfunctions of L with eigenvalue $-\lambda-\xi+\alpha$, $\alpha\geq 0$ are just those linear combinations of the basis vectors $J^\alpha_m=j_m\otimes e_n$ where $n=2\alpha+m,\ m=0,1,\cdots$. For $\alpha<0$, they are linear combinations of the basis vectors $J^\alpha_m = j_m \otimes e_n$ where $n = 2\alpha + m,\ n = 0, 1, \cdots$. Taking the case $\alpha \geq 0$ and applying C to the ON set $\{J^\alpha_m\}$ we find

$$\begin{aligned}
-\frac{(1+q)}{\sqrt{tvq}} C J^\alpha_m &= \Big[(1-p^{m+1})(1-p^{2\alpha+m+1})(1+\frac{u}{v}p^m q^{4\xi}) \\
&\times\ (1-\frac{u}{t}(-1)^{2\alpha}q^{-4\lambda+2\alpha}p^m)\Big]^{\frac{1}{2}} J^\alpha_{m+1} \\
&+\ \Big[(1-p^m)(1-p^{2\alpha+m})(1+\frac{u}{v}p^{m-1}q^{4\xi}) \\
&\times\ (1+\frac{u}{t}p^{m+2\alpha}q^{-4\lambda-1})\Big]^{\frac{1}{2}} J^\alpha_{m-1} \\
&-\ \frac{1}{\sqrt{vt}}\Big\{tq^{2\lambda+2\xi-2\alpha-\frac{1}{2}} + vq^{-2\xi-2\lambda+2\alpha-\frac{1}{2}} \\
&-\ q^{-2\lambda+2\alpha-\frac{1}{2}}(1-p^{m+1})(vq^{-2\xi}+up^m q^{2\xi}) \\
&-\ q^{2\xi-\frac{1}{2}}(1-p^{m+2\alpha})(tq^{2\lambda-2\alpha}+u(-1)^{2\alpha}p^m q^{-2\lambda})\Big\} J^\alpha_m .
\end{aligned} \tag{68}$$

The operator C is self-adjoint. If we introduce the spectral transform of this operator so that C corresponds to multiplication by the transform variable x, then (68) takes the form of a three term recurrence relation for orthogonal polynomials $J^\alpha_m(x)$ of order m in x. Indeed, comparing (68) with the three term recurrence relation for the continuous Askey-Wilson polynomials

$$\begin{aligned}
p_m(x) &\equiv p_m(x; a, b, c, d|Q) \\
&= (ab, ac, ad; Q)_m a^{-m} \\
&\times\ {}_4\phi_3\left(\begin{matrix} Q^{-m}, & abcdQ^{m-1}, & ae^{i\theta}, & ae^{-i\theta} \\ ab, & ac, & ad & \end{matrix}; Q, Q\right),
\end{aligned} \tag{69}$$

[6], page 173, we get a match with $Q = p$,

$$a = \sqrt{\frac{vq}{t}}q^{-2\lambda-2\xi+2\alpha},\ b = -\sqrt{\frac{tq}{v}}q^{2\lambda+2\xi}(-1)^{2\alpha},\ c = \frac{u}{\sqrt{vtq}}q^{-2\lambda+2\xi}(-1)^{2\alpha},$$

$d = 0$, and $C \sim -2x\sqrt{vtq}/(1+q)$. Making the identification

$$J^\alpha_m(x, s) =$$

$$\frac{\left[\frac{(p^{m+1}, abp^m, acp^m, bcp^m; p)_\infty}{2\pi}\right]^{\frac{1}{2}} p_m(x; a, b, c, 0)s^{2\alpha}}{(\sqrt{\frac{v}{t}}q^{-2\lambda-2\xi+2\alpha+\frac{1}{2}}e^{i\theta}, -\sqrt{\frac{t}{v}}q^{2\lambda+2\xi+\frac{1}{2}}(-1)^{2\alpha}e^{i\theta}, \frac{u}{\sqrt{vt}}q^{-2\lambda+2\xi-\frac{1}{2}}(-1)^{2\alpha}e^{i\theta}; p)_\infty}$$

$$= K_m^\alpha(x) \tag{70}$$

where $s = e^{i\phi}$ and $x = \cos\theta$, we can verify that (68) holds, as well as the orthogonality relations

$$\begin{aligned}
< J_m^\alpha, J_{m'}^{\alpha'} > &= \delta_{m,m'}\delta_{\alpha,\alpha'}, \qquad (71)\\
< f, g > &= \frac{1}{2\pi}\int_{-1}^{1}\rho(x)\ dx \int_0^{2\pi} d\phi\ f(x,s)\overline{g(x,s)},\\
\rho(x) &= (1-x^2)^{-\frac{1}{2}}\\
&\times\ (e^{i\theta}, e^{-i\theta}, -e^{i\theta}, -e^{-i\theta}, p^{\frac{1}{2}}e^{i\theta}, p^{\frac{1}{2}}e^{-i\theta}, -p^{\frac{1}{2}}e^{i\theta}, -p^{\frac{1}{2}}e^{-i\theta}; p)_\infty.
\end{aligned}$$

For $\alpha = -\beta$, $\beta = 0, 1, \cdots$, the expression for $J_m^\alpha(x,t)$ can be obtained by analytic continuation and a standard limiting procedure, analogous to (62).

It is straightforward, though tedious, to verify that in terms of the new variables $r = e^{i\theta}, s$ the action of the operators $F_\pm, L$ is

$$\begin{aligned}
F_+ &= \frac{s}{\sqrt{1+q}}\left(-\sqrt{t}q^{\lambda+\xi}T_s^{-1/2} - \sqrt{v}rq^{-\xi-\lambda+\frac{1}{2}}T_s^{1/2}\right)R_r, \qquad (72)\\
F_- &= \frac{1}{s\sqrt{1+q}}\left(\sqrt{t}q^{\lambda+\xi+\frac{1}{2}}T_s^{-1/2} - \frac{\sqrt{v}}{r}q^{\xi-\lambda}T_s^{1/2}\right)R_r,\\
L &= -\xi-\lambda+\frac{1}{2}s\frac{\partial}{\partial s}, \quad C = -\frac{\sqrt{vtq}}{1+q}(r+\frac{1}{r}).
\end{aligned}$$

Here, $R_r g(r) = g(-r)$.

Applying these operators to the basis $f_h^{\pm x_0}(x,s) = \frac{s^h}{\sqrt{\rho(x)}}\delta(x \mp x_0)$, we see that the algebra action becomes

$$\begin{aligned}
F_+f_h^{\pm x_0} &= \frac{1}{\sqrt{1+q}}\left(-\sqrt{t}q^{\lambda+\xi-\frac{h}{2}} - \sqrt{v}rq^{-\xi-\lambda+\frac{h+1}{2}}\right)f_{h+1}^{\mp x_0}, \qquad (73)\\
F_-f_h^{\pm x_0} &= \frac{1}{\sqrt{1+q}}\left(\sqrt{t}q^{\lambda+\xi+\frac{1-h}{2}} - \frac{\sqrt{v}}{r}q^{\xi-\lambda+\frac{h}{2}}\right)f_{h-1}^{\mp x_0},\\
Lf_h^{\pm x_0} &= (-\xi-\lambda+\frac{h}{2})f_h^{\pm x_0}, \quad C = \mp 2x_0\frac{\sqrt{vtq}}{1+q}.
\end{aligned}$$

for $x_0 \geq 0$. Thus the algebra action preserves $|x|$ and decomposes the representation space for fixed $|x| = x_0$ into the direct sum of two irreducible representations. The basis vector f_h^x belongs to the space of the irreducible representation $(\lambda+\xi, |x|)$ if $(-1)^h x > 0$, and to the space of the irreducible representation $(\lambda+\xi, -|x|)$ if $(-1)^h x < 0$. (We ignore the measure 0 case $x = 0$).

For the reduced tensor product we consider the matrix elements

$$(e-, E+): \quad e_p(\beta F_-)E_p(\alpha F_+)f_h^x = \sum_{h'} U_{h'h}^{(-1)^{h+h'}x}(\alpha,\beta) f_{h'}^{(-1)^{h+h'}x}, \tag{74}$$

$$\begin{aligned} U_{h'h}^x(\alpha,\beta) &= \Big(\frac{\beta\sqrt{v}}{r\sqrt{1+q}}\Big)^{h-h'} q^{(h-h')(h+h'+1)/4} \frac{(-1)^{(h-h')(h-h'+1)/2}}{q^{\mu(h-h')}(p;p)_{h-h'}} \\ \times & \ \Big((-1)^{1-h'}\sqrt{\frac{t}{v}} r q^{2\mu+\frac{1}{2}}p^{-h}; p\Big)_{h-h'} \\ \times & \ \Big[\frac{1-i}{2} {}_2\phi_1\left(\begin{matrix} \frac{(-1)^{h'}}{r}\sqrt{\frac{vq}{t}}q^{-2\mu}p^h, & (-1)^{h'}\sqrt{\frac{vq}{t}}rq^{-2\mu}p^h \\ p^{h-h'+1} & \end{matrix}; p; \frac{-\alpha\beta q^{2\mu} it}{q^h(1+q)}\right) \\ + & \ \frac{1+i}{2} {}_2\phi_1\left(\begin{matrix} \frac{(-1)^{h'}}{r}\sqrt{\frac{vq}{t}}q^{-2\mu}p^h, & (-1)^{h'}\sqrt{\frac{vq}{t}}rq^{-2\mu}p^h \\ p^{h-h'+1} & \end{matrix}; p; \frac{\alpha\beta q^{2\mu} it}{q^h(1+q)}\right)\Big], \end{aligned} \tag{75}$$

where $\mu = \lambda + \xi$ and $x = (r + r^{-1})/2$.

Comparing (66, 71, 73), we have proved the direct integral decomposition

$$\downarrow_\xi [v,u] \otimes \uparrow_\lambda [-u,t] \cong \int_0^1 \rho(x) \left\{ (\lambda+\xi, x) \oplus (\lambda+\xi, -x) \right\} \, dx. \tag{76}$$

The functions $J_m^{h/2}(x,s)$ are, essentially, the Clebsch-Gordan coefficients for this decomposition; the orthogonality and completeness relations for the corresponding Askey-Wilson polynomials are the unitarity conditions for the C-G coefficients.

The decomposition (76) can be used to obtain an identity relating the matrix elements (61), (64) and (75). We can compute the matrix element

$$T_{m'n',mn}(\alpha,\beta) = < e_p(\beta F_-)E_p(\alpha F_+) j_m \otimes e_n, j_{m'} \otimes e_{n'} >$$

in two diferent ways. On one hand we have the integral representation

$$\begin{aligned} T_{m'n',mn}(\alpha,\beta) &= \ < e_p(\beta F_-)E_p(\alpha F_+) J_m^{h/2}, J_{m'}^{h'/2} > \\ &= \ < U_{h'h}^{(\cdot)}(\alpha,\beta) K_m^{h/2}((-1)^{h+h'}\cdot), K_{m'}^{h'/2}(\cdot) >, \end{aligned} \tag{77}$$

where $J^{h/2}_m(x,s) = K^{h/2}_m(x)s^h$, and $n = h+m, n' = h'+m'$. On the other hand, from relations (7), (22) and (55) we have the formal identity

$$
\begin{aligned}
& e_p(\beta F_-)E_p(\alpha F_+) = \\
& e_p(\beta V_- \otimes q^H)e_p(\beta q^{-H} \otimes V_-)E_p(\alpha V_+ \otimes q^H)E_p(\alpha q^{-H} \otimes V_+) \\
= & \frac{1}{2}e_p(\beta V_- \otimes q^H)\Big(E_p(B)e_p(A) + E_p(B)e_p(-A) \\
& + E_p(-B)e_p(A) - E_p(-B)e_p(-A)\Big)E_p(\alpha q^{-H} \otimes V_+),
\end{aligned} \tag{78}
$$

where $B = \alpha V_+ \otimes q^H$, $A = \beta q^{-H} \otimes V_-$, $AB = -BA$. Thus,

$$
\begin{aligned}
& 2T_{m'n',mn}(\alpha,\beta) = \\
= \; & S^{(\xi)}_{m'm}(\alpha q^{-\lambda+\frac{n'}{2}}, \beta q^{-\lambda+\frac{n'}{2}})T^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}}, \beta(-1)^m q^{\xi+\frac{m}{2}}) \\
+ \; & S^{(\xi)}_{m'm}(\alpha q^{-\lambda+\frac{n'}{2}}, \beta q^{-\lambda+\frac{n'}{2}})T^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}}, -\beta(-1)^m q^{\xi+\frac{m}{2}}) \\
+ \; & S^{(\xi)}_{m'm}(-\alpha q^{-\lambda+\frac{n'}{2}}, \beta q^{-\lambda+\frac{n'}{2}})T^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}}, \beta(-1)^m q^{\xi+\frac{m}{2}}) \\
- \; & S^{(\xi)}_{m'm}(-\alpha q^{-\lambda+\frac{n'}{2}}, \beta q^{-\lambda+\frac{n'}{2}})T^{(e-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}}, -\beta(-1)^m q^{\xi+\frac{m}{2}})
\end{aligned} \tag{79}
$$

where the functions $T_{n'm',nm}(\alpha,\beta)$ are the matrix elements of the operator $e_p(\beta F_-)E_p(\alpha F_+)$ in the tensor product basis $\{j_m \otimes e_n\}$.

Next we consider a degenerate case of the tensor product with some special features, a representation of the q-supereuclidean algebra $[0,0]$ of the form $\downarrow'_\xi [0,u] \otimes \uparrow'_\lambda [-u,0]$. Here the family of algebraically irreducible representations (bounded below) $\uparrow'_\lambda$, of the q-superalgebra $[-u,0]$, where $u > 0$, is defined as follows. An orthogonal basis for the representation space is $\{e_n : n = 0,1,\cdots\}$ where

$$
\begin{aligned}
V_- e_n &= \left[\frac{(1-p^n)uq^{-2\lambda}}{1+q}\right]^{\frac{1}{2}} e_{n-1}, \quad He_n = (-\lambda + \frac{n}{2})e_n \\
V_+ e_n &= \left[\frac{(1-p^{n+1})uq^{-2\lambda}}{1+q}\right]^{\frac{1}{2}} (-1)^n e_{n+1}
\end{aligned} \tag{80}
$$

The representation space is a Hilbert space with indefinite inner product. Indeed

$$
< e_n, e_{n'} >= \delta_{nn'}(-1)^{n(n-1)/2}, \qquad n,n' = 0,1,\cdots \tag{81}
$$

and $V_+ = (V_-)^*$, $H^* = H$. We define $H' = \lambda I + H$.

The following matrix elements of $\uparrow'_\lambda$:

$$e_{-q}(\beta V_-)E_{-q}(\alpha V_+)e_n = \sum_{n'=0}^{\infty} T'^{(e-,E+)}_{n'n}(\alpha,\beta)e_{n'}, \tag{82}$$

are given by:

$$\begin{aligned} T'^{(e-,E+)}_{n'n}(\alpha,\beta) &= q^{\frac{1}{2}(n'-n)(n'+n-1)}\alpha^{n'-n}(\frac{uq^{-2\lambda}}{1+q})^{\frac{1}{2}(n'-n)}\frac{\sqrt{(p^{n+1};p)_{n'-n}}}{(p;p)_{n'-n}} \\ &\times \Big[\frac{1-i}{2}{}_1\phi_1\left(\begin{matrix} p^{n'+1} \\ p^{n'-n+1}\end{matrix};p;\frac{-iq^{n'}\alpha\beta u}{p^n q^{2\lambda}(1+q)}\right) \\ &+ \frac{1+i}{2}{}_1\phi_1\left(\begin{matrix} p^{n'+1} \\ p^{n'-n+1}\end{matrix};p;\frac{iq^{n'}\alpha\beta u}{p^n q^{2\lambda}(1+q)}\right)\Big]. \end{aligned} \tag{83}$$

Here,

$$< T'e_n, e_{n'} >= T'_{n'n}(-1)^{n(n-1)/2}. \tag{84}$$

We also consider a second family of algebraically irreducible representations (bounded above) $\downarrow'_\xi$, of the q-superalgebra $[0,u]$, where $u>0$, defined as follows. A convenient orthogonal basis for the representation space with indefinite inner product is $\{j_m : m = 0,1,\cdots\}$ where

$$< j_m, j_{m'} >= \delta_{mm'}(-1)^{m(m+1)/2}, \qquad m,m' = 0,1,\cdots \tag{85}$$

and

$$\begin{aligned} V_-j_m &= \left[\frac{(1-p^{m+1})uq^{2\xi}}{1+q}\right]^{\frac{1}{2}} j_{m+1}, \quad Hj_m = -(\xi+\frac{m}{2})j_m \\ V_+j_m &= \left[\frac{(1-p^{m})uq^{2\xi}}{1+q}\right]^{\frac{1}{2}} j_{m-1} \end{aligned} \tag{86}$$

We have $V_+ = (V_-)^*$ and $H^* = H$. We define $H' = \xi I + H$.

For this representation we consider the matrix elements

$$(e-,E+): \quad e_p(\beta V_-)E_p(\alpha V_+)j_m = \sum_{m'} S'^{(\xi)}_{m'm}(\alpha,\beta)j_{m'},$$

$$S'^{(\xi)}_{m'm}(\alpha,\beta) = (\beta)^{m'-m}(\frac{uq^{2\xi}}{1+q})^{\frac{1}{2}(m'-m)}\frac{\sqrt{(p^{m+1};p)_{m'-m}}}{(p;p)_{m'-m}} \tag{87}$$
$$\times\ {}_2\phi_1\left(\begin{array}{cc} p^{-m}, & 0 \\ p^{m'-m+1} & \end{array};p;\frac{-\alpha\beta p^m u}{q^{-2\xi}(1+q)}\right).$$

Now we form the tensor product representation

$$\downarrow'_\xi [0,u]\otimes \uparrow'_\lambda [-u,0]\cong[0,0] \tag{88}$$

of $[0,0]$ using the definition (66). The invariant operator is $C=F_+F_-$. Note that $F_+^*=F_-$, $L^*=L$ and that a basis for the indefinite inner product space is $\{j_m\otimes e_n,\quad m,n=0,1,\cdots\}$ where

$$<j_m\otimes e_n, j_{m'}\otimes e_{n'}>=<j_m,j_{m'}><e_n,e_{n'}>=\delta_{mm'}\delta_{nn'}(-1)^{\frac{1}{2}(m+n)(m-n+1)}. \tag{89}$$

To decompose this representation we compute the common eigenfunctions of L and C. Clearly, eigenfunctions of L with eigenvalue $-\lambda-\xi+\alpha$, $\alpha\geq 0$ are just those linear combinations of the basis vectors $J^\alpha_m=j_m\otimes e_n$ where $n=2\alpha+m,\ m=0,1,\cdots$. For $\alpha<0$, they are linear combinations of the basis vectors $J^\alpha_m=j_m\otimes e_n$ where $n=2\alpha+m,\ n=0,1,\cdots$. Note that

$$||J^\alpha_m||^2\equiv<J^\alpha_m,J^\alpha_m>=\frac{(-1)^{[\frac{\alpha}{2}]}}{2}\Big[1-(-1)^\alpha+(-1)^m+(-1)^{\alpha+m}\Big]. \tag{90}$$

Taking the case $\alpha\geq 0$ and applying C to the orthogonal set $\{J^\alpha_m\}$ we find

$$\begin{aligned}\frac{(1+q)q^{2(\lambda-\xi)}}{u}CJ^\alpha_m &= (-1)^\alpha p^m q^{\frac{\alpha}{2}+\frac{1}{2}}\Big[(1-p^{m+1})(1-p^{\alpha+m+1})\Big]^{\frac{1}{2}}J^\alpha_{m+1}\\ &+ p^m q^{\frac{\alpha}{2}-\frac{1}{2}}\Big[(1-p^m)(1-p^{\alpha+m})\Big]^{\frac{1}{2}}J^\alpha_{m-1}\\ &- p^m\Big[q^\alpha(i-p^{m+1})+(-1)^\alpha(1-p^{\alpha+m})\Big]J^\alpha_m.\end{aligned} \tag{91}$$

The operator C is self-adjoint. If we introduce the spectral transform of this operator so that C corresponds to multiplication by the transform variable x, then (91) takes the form of a three term recurrence relation for orthogonal polynomials $J^\alpha_m(x)$ of order m in x. Indeed, comparing (91) with the three term recurrence relation for the continuous Askey-Wilson polynomials we see that we get a match for a special case of the little q-Jacobi polynomials

$$p_n(x;a,b;Q)={}_2\phi_1\left(\begin{array}{cc} Q^{-n}, & abQ^{n+1} \\ aQ & \end{array};Q,xQ\right) \tag{92}$$

where $Q = p$, $a = p^\alpha$ and $C \sim xuq^{2(\xi-\lambda)}(-1)^{\alpha+1}/(1+q)$. Making the identification

$$J_m^\alpha(x,s) = \left[\frac{q^{-m(\alpha+1)}(p^{\alpha+1};p)_\infty(p^{\alpha+1};p)_m}{(p;p)_m}\right]^{\frac{1}{2}} p_m(x;p^\alpha,0;p)s^\alpha \qquad (93)$$

we have

$$< J_m^\alpha, J_m^{\alpha'} >= \delta_{\alpha\alpha'}\frac{(-1)^{[\frac{\alpha}{2}]}}{2}\left[1-(-1)^\alpha+(-1)^m+(-1)^{\alpha+m}\right] \qquad (94)$$

where

$$< f(x)s^\alpha, g(x)s^{\alpha'} >\equiv \delta_{\alpha\alpha'}(-1)^{\alpha(\alpha-1)/2}\sum_{y=0}^{\infty}\frac{p^{y(\alpha+1)}}{(p;p)_y}f(p^y)g(p^y). \qquad (95)$$

For fixed integer α odd, this defines a true inner product for little q-Jacobi polynomials; for α even it is just a bilinear product since the discrete measure can be negative. It is not dificult to verify that on this basis the superalgebra acts as follows:

$$\begin{aligned} F_+ &= q^{\xi-\lambda}\sqrt{\frac{u}{q+1}}sR_s, \quad F_- = q^{\xi-\lambda}\sqrt{\frac{u}{q+1}}\frac{x}{s}, \\ L &= -(\xi+\lambda)+\frac{1}{2}s\partial_s \end{aligned} \qquad (96)$$

where $R_sG(s) = G(-s)$. For fixed x this representation is equivalent to the representation $(\omega) = (q^{\xi-\lambda}\sqrt{\frac{xu}{q+1}})$ of $[0,0]$, as studied in §2. Thus we have derived the decomposition

$$\downarrow'_\xi [0,u]\otimes \uparrow'_\lambda [-u,0] \cong \oplus\sum_{y=0}^{\infty}(q^{\xi-\lambda}\sqrt{\frac{p^y u}{q+1}})[0,0] \qquad (97)$$

Again, the functions $J_m^h(x,s)$ are essentially the Clebsch-Gordan coefficients for this decomposition; the orthogonality and completeness relations for the corresponding little p-Jacobi polynomials are the unitarity conditions for the C-G coefficients. Also, the decomposition (97) can be used to obtain an identity relating the matrix elements (83), (87) and the adjoint of (30). We can compute the matrix element

$$T'_{m'n',mn}(\alpha,\beta) =< e_p(\beta F_-)E_p(\alpha F_+)j_m\otimes e_n, j_{m'}\otimes e_{n'} >$$

in two diferent ways. On one hand we have the representation

$$
\begin{aligned}
T'_{m'n',mn}(\alpha,\beta) &= \; < e_p(\beta F_-)E_p(\alpha F_+)J^h_m, J^{h'}_{m'} > \\
&= \; < U'^{(\cdot)}_{h'h}(\alpha,\beta)K^h_m(\cdot), K^{h'}_{m'}(\cdot) >, \qquad (98)
\end{aligned}
$$

where $J^h_m(x,s) = K^h_m(x)s^h$, $U'^{(\omega)}_{h'h}(\alpha,\beta) = x^{(h-h')/2}L'_{h'h}(\alpha,\beta)$, $n = h+m$, $n' = h'+m'$ and the L' are essentially the adoint matrix elements to (30):

$$
\begin{aligned}
L'^{(x)}_{h'h}(\alpha,\beta) &= \frac{(\beta q^{\xi-\lambda}\sqrt{\frac{ux}{1+q}})^{h-h'}}{(p;p)_{h-h'}} \qquad (99) \\
&\times \Big[{}_1\phi_1\left(\begin{matrix} 0 \\ p^{h-h'+1}\end{matrix}\; ;p, \frac{i\alpha\beta ux(-1)^{h+1}}{q^{2\lambda-2\xi}(1+q)}\right)\frac{1-i}{2} \\
&+ {}_1\phi_1\left(\begin{matrix} 0 \\ p^{h-h'+1}\end{matrix}\; ;p, (-1)^h\frac{i\alpha\beta q^{2\xi-2\lambda}ux}{1+q}\right)\frac{1+i}{2}\Big].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&2T'_{m'n',mn}(\alpha,\beta) = \qquad (100) \\
= \; & S'^{(\xi)}_{m'm}(\alpha q^{-\lambda+\frac{n'}{2}}, \beta q^{-\lambda+\frac{n'}{2}})T'^{(e_-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}}, \beta(-1)^m q^{\xi+\frac{m}{2}}) \\
+ \; & S'^{(\xi)}_{m'm}(\alpha q^{-\lambda+\frac{n'}{2}}, \beta q^{-\lambda+\frac{n'}{2}})T'^{(e_-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}}, -\beta(-1)^m q^{\xi+\frac{m}{2}}) \\
+ \; & S'^{(\xi)}_{m'm}(-\alpha q^{-\lambda+\frac{n'}{2}}, \beta q^{-\lambda+\frac{n'}{2}})T'^{(e_-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}}, \beta(-1)^m q^{\xi+\frac{m}{2}}) \\
- \; & S'^{(\xi)}_{m'm}(-\alpha q^{-\lambda+\frac{n'}{2}}, \beta q^{-\lambda+\frac{n'}{2}})T'^{(e_-,E+),\lambda}_{n'n}(\alpha(-1)^m q^{\xi+\frac{m}{2}}, -\beta(-1)^m q^{\xi+\frac{m}{2}}).
\end{aligned}
$$

For our last example we consider another degenerate case of the tensor product with some special features, a representation of the q-supereuclidean algebra $[0,-u]$ of the form $\omega'[0,0]\otimes\uparrow''_\lambda[0,-u]$. Here the family of algebraically irreducible representations (bounded below) $\uparrow''_\lambda$, of the q-superalgebra $[0,-u]$, where $u>0$, is defined as follows. An orthonormal basis for the representation space is $\{e_n : n = 0,1,\cdots\}$ where

$$
\begin{aligned}
V_-e_n &= \left[\frac{(1-p^n)uq^{2\lambda-n+1}}{1+q}\right]^{\frac{1}{2}} e_{n-1}, \quad He_n = (-\lambda+\frac{n}{2})e_n, \\
V_+e_n &= \left[\frac{(1-p^{n+1})uq^{2\lambda-n}}{1+q}\right]^{\frac{1}{2}} e_{n+1}. \qquad (101)
\end{aligned}
$$

This Hilbert space representation satisfies $V_+ = (V_-)^*$, $H^* = H$. We define $H' = \lambda I + H$.

The following matrix elements of $\uparrow''_\lambda$:

$$e_{-q}(\beta V_-)E_{-q}(\alpha V_+)e_n = \sum_{n'=0}^{\infty} T''^{(e-,E+)}_{n'n}(\alpha,\beta)e_{n'}, \tag{102}$$

are given by:

$$\begin{aligned} T''^{(e-,E+)}_{n'n}(\alpha,\beta) &= (-1)^{(n'-n)(n'-n+1)/2}\sqrt{\frac{(p;p)_{n'}}{(p;p)_n}} \\ &\times (-\sqrt{\frac{u}{1+q}}\alpha)^{n'-n}\frac{q^{(n'-n)(7\lambda+n'-3n-1)/4}}{(p;p)_{n'-n}} \\ &\times {}_2\phi_1\left(\begin{matrix} p^{n'+1}, & 0 \\ p^{n'-n+1} & \end{matrix}; p; \frac{(-1)^{n-n'}\alpha\beta u}{q^{n-2\lambda}(1+q)}\right), \end{aligned} \tag{103}$$

where $|q^{2\lambda-n}\alpha\beta u/(1+q)| < 1$.

The family of algebraically irreducible representations ω', of the q-superalgebra $[0,0]$, is defined by the action (12) on the basis $\{f_m\}$, except that here we have an indefinite inner product

$$< f_m, f_{m'} >= \delta_{mm'}(-1)^{m(m+1)/2}, \qquad m, m' = 0, \pm1, \cdots. \tag{104}$$

Thus $V_+ = (V_-)^*$ and $H^* = H$. We define $H' = H$. The matrix elements are given by (29).

Now we form the tensor product representation

$$\omega'[0,0] \otimes \uparrow''_\lambda[0,-u] \cong [0,-u] \tag{105}$$

of $[0,-u]$ using the definition (66). The invariant operator is $C = F_+F_- - \frac{uq^{-2L}}{1+q^{-1}}$. Note that $F_+^* = F_-$, $L^* = L$ and that a basis for the indefinite inner product space is $\{f_m \otimes e_n, \quad n, \pm m = 0, 1, \cdots\}$ where

$$< f_m \otimes e_n, f_{m'} \otimes e_{n'} >=< f_m, f_{m'} >< e_n, e_{n'} >= \delta_{mm'}\delta_{nn'}(-1)^{\frac{1}{2}m(m+1)}. \tag{106}$$

Again, to decompose this representation we compute the common eigenfunctions of L and C. Clearly, eigenfunctions of L with eigenvalue $-\lambda + \frac{\alpha}{2}$

are just those linear combinations of the basis vectors $J^\alpha_n = f_m \otimes e_n$ where $m = \alpha - n,\ n = 0, 1, \cdots$. Note that

$$||J^\alpha_n||^2 \equiv < J^\alpha_n, J^\alpha_n >= (-1)^{\frac{1}{2}(\alpha-n)(\alpha-n+1)}. \tag{107}$$

Applying C to the orthogonal set $\{J^\alpha_n\}$ we find

$$\begin{aligned} CJ^\alpha_n &= -q^{\frac{1}{2}(n-\alpha+1)}\omega\Big[\frac{(1-p^{n+1})u}{1+q}\Big]^{\frac{1}{2}} J^\alpha_{n+1} \\ &+ (-1)^{n+\alpha} q^{\frac{1}{2}(n-\alpha)}\omega\Big[\frac{(1-p^n)u}{1+q}\Big]^{\frac{1}{2}} J^\alpha_{n-1} \\ &+ p^n\Big[-q^{2\lambda-\alpha+1}u + (-1)^\alpha\omega^2 q^{-2\lambda}\Big] J^\alpha_n. \end{aligned} \tag{108}$$

The operator C is self-adjoint. If we introduce the spectral transform of this operator so that C corresponds to multiplication by the transform variable x, then (108) takes the form of a three term recurrence relation for orthogonal polynomials $J^\alpha_n(x)$ of order m in x. Indeed, comparing (108) with the three term recurrence relation for the polynomials

$$P_n(x,z) = {}_2\phi_1\left(\begin{array}{cc} Q^{-n}, & \frac{1}{x} \\ 0 & \end{array}; Q, -\frac{xQ}{z}\right), \tag{109}$$

$$xP_n = zQ^nP_{n+1} + (1-z)Q^nP_n + (1-Q^n)P_{n-1}, \tag{110}$$

we see that we get a match provided $Q = p,\ z_\alpha = q^{4\lambda+1}u/(1+q)\omega^2p^\alpha$ and $C \sim x\omega^2q^{-2\lambda}(-1)^\alpha$. Making the identification

$$J^\alpha_n(x,s) = \frac{q^{n(n-1)/4}}{\sqrt{(p;p)_n}}\left[\frac{q^{4\lambda+1}u}{(1+q)\omega^2q^\alpha}\right]^{\frac{n}{2}} P_n(x,z_\alpha)s^\alpha(-1)^{n(n+1)/2} \tag{111}$$

we have

$$< J^\alpha_n, J^{\alpha'}_{n'} >'= \delta_{\alpha\alpha'}\delta_{nn'}(-1)^{(\alpha-n)(\alpha-n+1)/2} \tag{112}$$

where

$$\begin{aligned} < f(x)s^\alpha, g(x)s^{\alpha'} >' \ \equiv\ & \delta_{\alpha\alpha'}\frac{(-1)^{\alpha(\alpha+1)/2}(1+z_\alpha)}{z_\alpha(p,-\frac{1}{z_\alpha},-z_\alpha;p)_\infty} \times \\ & \Big(\sum_{k=0,x=p^k}^{\infty} x(px;p)_\infty(-\frac{xp}{z_\alpha};p)_\infty f(x)g(x) - \\ & \sum_{k=0,x=-z_\alpha p^k}^{\infty} x(px;p)_\infty(-\frac{xp}{z_\alpha};p)_\infty f(x)g(x)\Big). \end{aligned} \tag{113}$$

This is just a bilinear product since the discrete measure can be negative. It is not dificult to verify that on this basis the superalgebra acts as follows:

$$
\begin{aligned}
F_{+} &= \omega q^{-\lambda} s(x - \frac{q^{4\lambda}u}{(1+q)\omega^2} T_s^{-1} R_s), \quad F_{-} = q^{-\lambda}\frac{\omega}{s} R_s, \\
L &= -\lambda + \frac{1}{2} s\partial_s
\end{aligned}
\tag{114}
$$

where $R_s G(s) = G(-s)$.

For fixed $x = z_0 p^{-\alpha_0}$ this representation is equivalent to the irreducible representation $\uparrow''_{\lambda-\frac{\alpha_0}{2}}$ of $[0,-u]$, as given in (101). For fixed $x = p^k$ the representation is equivalent to an irreducible representation $R_k(\omega^2, q^\lambda)$, unbounded above and below. Thus we have derived the decomposition

$$
\begin{aligned}
\omega'[0,0] \quad &\otimes \quad \uparrow''{}_{\lambda}[0,-u] \cong \oplus \sum_{\alpha_0=-\infty}^{\infty} \uparrow''_{\lambda-\frac{\alpha_0}{2}} [0,-u] \\
&\oplus \quad \sum_{k=0}^{\infty} R_k(\omega^2, q^\lambda)[0,-u].
\end{aligned}
\tag{115}
$$

Here the reduced basis corresponding to the representation $\uparrow''_{\lambda-\frac{\alpha_0}{2}}$, (101), is given by

$$
e_n^{(\alpha_0)}(x,s) = (-1)^{n(n+1)/2+\alpha_0 n} \frac{\delta(x+z_{\alpha_0})}{q^{n(n+1)/4}} \sqrt{(p;p)_n(-z_{\alpha_0}^{-1};p)_\infty |z_{\alpha_0}|^n} s^\alpha. \tag{116}
$$

We have

$$
< e_n^{(\alpha_0)}, e_{n'}^{(\alpha_0')} >'' = \delta_{nn'}\delta_{\alpha_0\alpha_0'}(-1)^{\alpha_0(\alpha_0+1)/2} \ \mathrm{sign}(-z_{\alpha_0}^{-1};p)_\infty.
$$

The reduced basis corresponding to representation $R_k(\omega^2, q^\lambda)$ is given by

$$
g_\alpha^{(k)}(x,s) = \frac{\delta(x-p^k) q^{k(k-1)/4}}{|z_0|^{k/2} q^{-\alpha k/2}} \sqrt{(p;p)_k |(-z_\alpha p^{-k};p)_\infty|}, \tag{117}
$$

where

$$
< g_\alpha^{(k)}, g_{\alpha'}^{(k')} >'' = \delta_{\alpha\alpha'}\delta_{kk'} \frac{(-1)^{k+(\alpha+k)(\alpha+k+1)/2}}{\mathrm{sign}(-z_\alpha p^{-k};p)_\infty}.
$$

The superalgebra action is

$$\begin{aligned} F_+ g_\alpha^{(k)} &= (-1)^k q^{k/2} \sqrt{|\omega^2 q^{-2\lambda} - \frac{q^{2\lambda} p^{-k-\alpha}}{1+q}|}\ \mathrm{sign}(\omega^2 - \frac{q^{4\lambda} p^{-k-\alpha}}{1+q})\ g_{\alpha+1}^{(k)}, \\ F_- g_\alpha^{(k)} &= (-1)^\alpha q^{k/2} \sqrt{|\omega^2 q^{-2\lambda} + \frac{q^{2\lambda+1} p^{-k-\alpha}}{1+q}|}\ g_{\alpha-1}^{(k)}, \\ L g_\alpha^{(k)} &= (-\lambda + \frac{\alpha}{2}) g_\alpha^{(k)}, \end{aligned} \tag{118}$$

where $k, \pm\alpha = 0, 1, \cdots$.

Again, the functions $J_n^\alpha(x, s)$ are the Clebsch-Gordan coefficients for this decomposition. As before, the decomposition (115) can be used to obtain an identity relating the matrix elements of the "group" operators with respect to the tensor product and the reduced bases.

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