Math 1571H, Fall 2005
Solution to Quiz 7 (November 17)

Please Answer The Following Three Questions, (Turn Over the Page.)

Question 1. [5 points] Find the volume of the solid of revolution generated when the area bounded by the given curves is revolved around the $x-$axis.

$$y = 2x,\ y = 2,\ x = 0.$$

1. **Cylindrical Shell Method:** Radius of sheel is y, thickness is dy and height is x-coordinate on $y = 2x$. Therefore, $dV = 2\pi y \frac{y}{2} dy$

$$V = \int_0^2 2\pi \frac{y}{2} y dy = \left[\pi \frac{y^3}{3}\right]_0^2 = \frac{8\pi}{3}.$$

2. **Annulus (Washer) Method:** Inner radius is y-coordinate on $y = 2x$, outer radius is y-coordinate on $y = 2$, and thickness is dx.

$$V = \int_0^1 \pi(4 - (2x)^2)dx = \pi\left[4x - \frac{4}{3}x^3\right]_0^1 = \frac{8\pi}{3}.$$

Question 2. [5 points] Sketch the curves and find the areas of the region they bound.

$$y = x^2 + 1,\ y = 3 - x^2,\ x = -3, x = 2.$$

$$A = \int_{-3}^{-1} \left[(x^2+1) - (3-x^2)\right] dx + \int_1^2 \left[(x^2+1) - (3-x^2)\right] dx + \int_{-1}^1 \left[(3-x^2) - (1+x^2)\right] dx = \frac{56}{3}.$$

Question 3. [2 points Bonus!] Prove the mean value theorem for integrals by using the mean value theorem for derivatives. That is, if $f(x)$ is a continuous function on the closed interval $[a, b]$, then there exists some c in the open interval (a, b) such that

$$\int_a^b f(t)dt = f(c)(b - a).$$

Let $F(x) = \int_a^x f(t)dt$. $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b), therefore by Mean Value Theorem for derivatives, there exists some c in (a, b) such that

$$F'(c) = \frac{F(b) - F(a)}{(b-a)}, \quad \text{where by FTC } F'(x) = f(x).$$

Thus,

$$f(c) = \frac{\int_a^b f(t)dt - \int_a^a f(t)dt}{(b-a)} \Rightarrow \int_a^b f(t)dt = f(c)(b-a).$$