A NOTE ON GROUP CONTRACTIONS AND RADAR AMBIGUITY FUNCTIONS

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ABSTRACT. The Heisenberg group is a contraction of $G_A \times R$, where G_A is the affine group and R is the additive group of the real numbers. We work out the details of the contraction procedure and use it to show how the narrowband cross-ambiguity function arises as a limit of the wideband cross-ambiguity function. We also demonstrate how the expansion formula for the target distribution function in the narrowband case arises as a limit of the corresponding expansion formula for the wideband distribution function via the contraction procedure.

1. Introduction. It is well known that the narrowband radar ambiguity and cross-ambiguity functions are approximations of the corresponding wideband functions. Some of the details of the approximations can be found in [1], [4], [20]. Here we shall employ the concept of group contraction to shed more light on the relationship between these functions. Further, we shall demonstrate how the expansion formula for the target distribution function in the narrowband case arises as a limit of the corresponding expansion formula for the wideband distribution function via the contraction procedure. Most of our computations will be formal, but the needed rigor can be supplied. We thank R. E. Blahut for suggesting this problem.

We review briefly the Doppler effect as it relates to radar and sonar. Consider a stationary transmitter/detector and a moving (point) target located at a distance $R(t)$ from the transmitter at time t. We assume that the distance from the transmitter to the target changes linearly with time,

$$R(t) = r + vt. \tag{1.1}$$

Now suppose that the transmitter emits an electromagnetic pulse $s(t)$. The pulse is transmitted at a constant speed c in the ambient medium (e.g., air or water) impinges the target, and is reflected back to the transmitter/receiver where it is detected as the echo $e_{r,v}(t)$. (We assume $c > |v|$.) It is not difficult to show that the echo $e_{r,v}(t)$ takes the form

$$e_{r,v}(t) = \sqrt{\frac{c-v}{c+v}}\, s\left(t\left[\frac{c-v}{c+v}\right] - \frac{2r}{c+v}\right), \tag{1.2}$$

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where the scale factor $\sqrt{\frac{c-v}{c+v}}$ is needed if we require that the energy of the pulse is conserved:

$$\int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} |e(t)|^2 dt.$$

Changing notation, we write the echo as

$$e_{x,y}(t) = \sqrt{y} s(y[t+x]) \tag{1.3}$$

where $y = (c-v)/(c+v)$ and $x = -2r/(c-v)$. Note that the transformation $(x,y) \to (r,v)$ is one-to-one. The result (1.2) is the exact (wideband) solution for the given assumptions, [16], [17]. We can generalize this analysis to allow a distribution of moving targets with density function $D(x,y)$, where the support of this function is contained in the set $\{(x,y) : \ y > 0\}$. Then the echo from the signal $s(t)$ is

$$e(t) = \int_0^{\infty} \int_{-\infty}^{\infty} \sqrt{y} s(y[t+x]) D(x,y) dx dy. \tag{1.4}$$

It is common to approximate the wideband solution under the assumption that $|v| \ll c$. (See [4] and [17] for careful analyses of the relationship between the wideband solution and the narrowband approximation.) One writes the signal in the form

$$s(t) = a(t) e^{2\pi i \omega_0 t}$$

where ω_0 is the mean frequency of the Fourier transform of s and $a(t)$ is assumed to be a slowly varying complex function of t (the *envelope* of the waveform) with respect to the exponential factor. In particular one requires that $a(y[t+x]) \approx a(t+x)$ over the (x,y) domain contributing to the integral (1.4).

Then we have

$$e_{x,y}(t) = \sqrt{y} a(y[t+x]) e^{2\pi i (y[t+x])\omega_0}$$

or, since $y \approx 1 - 2\beta, \quad yx \approx -(2r/c)(1-\beta)$ where $\beta = v/c$, we have

$$\begin{aligned} e_{x,y}(t) &\approx a(t - 2r/c) e^{2\pi i \omega_0 [t(1-2\beta) - 2(r/c)(1-\beta)]} \\ &\approx s(t - 2r/c) e^{-4\pi i \beta \omega_0 [t - 2r/c]}. \end{aligned}$$

This is called the *narrowband approximation* of the Doppler effect. (Generally speaking, the narrowband approximation is usually adequate for radar applications but is less often appropriate for sonar.) Similarly, we can derive an expression for the echo due to a distribution of moving targets in the narrowband approximation.

Suppose $\{s_n\}$ is a basis for $L_2(R)$, the standard space of square integrable functions on the real line. Then we can consider the cross-correlation of s_m with the echo e_n from s_n:

$$A_{nm}(x,y) = \langle e_n, s_m \rangle = \sqrt{y} \int_{-\infty}^{\infty} s_n(y[t+x]) \overline{s}_m(t) dt. \tag{1.5}$$

We call A_{nm} the *cross-ambiguity function* of s_n and s_m. Similarly the *(narrowband)* cross-ambiguity function is

$$\psi_{nm}(u,w) = \int_{-\infty}^{\infty} e^{2\pi i t w} s_n\,(t+u)\,\overline{s}_m\,(t)\,dt \tag{1.6}$$

where $u = -2r/c$, $w = -2v\omega_0/c$ and we have ignored the phase factor $e^{8\pi i v r \omega_0/c^2}$. If $n = m$ these expressions are called *ambiguity functions.* The ambiguity and cross-ambiguity functions are of fundamental importance in the theory of radar and sonar. The use of the ambiguity function to estimate the range and velocity of point targets and, more generally, of the cross-ambiguity function to estimate target distribution functions in both the wideband and narrowband cases is basic to the theory, [3], [4], [14], [20].

There is an intimate relationship between the cross-ambiguity functions defined above and the theory of group representations. In particular the wideband cross-ambiguity functions can be interpreted as matrix elements of unitary irreducible representations of the two-dimensional affine group; the narrowband cross-ambiguity functions are matrix elements of unitary irreducible representations of the three-dimensional Heisenberg group. The basic properties of these functions thus emerge as consequences of analyses on affine and Heisenberg groups.

The affine group G_A is the matrix group with elements

$$A(a,b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a > 0.$$

As is easy to check, the operators $\mathbf{L}[a,b]$ define a unitary representation of G_A on the Hilbert space $L_2(R)$:

$$(\mathbf{L}[a,b]\boldsymbol{\varphi})(t) = a^{-\frac{1}{2}}\boldsymbol{\varphi}\left(\frac{t+b}{a}\right), \quad \boldsymbol{\varphi} \in L_2(R). \tag{1.7}$$

The representation $\mathbf{L}$ is reducible, [9], [18]. Indeed it decomposes into a direct sum of two irreducible representations: $\mathbf{R}_+$ (whose representation space consists of those $\boldsymbol{\varphi} \in L_2(R)$ whose Fourier transforms have support on the positive half line) and $\mathbf{R}_-$ (whose representation space consists of those $\boldsymbol{\varphi} \in L_2(R)$ whose Fourier transforms have support on the negative half line). Note that the matrix element $\langle \mathbf{L}[a,b]s_n, s_m\rangle$ coincides with the wideband cross-ambiguity function (1.5) with $y = a^{-1}, x = b/a$.

The decomposition of the right regular representation of G_A is of interest for radar and sonar theory, [6], [7], [9], [10]. Here the representation space consists of functions $f(a,b)$ on G_A, square integrable with respect to the right invariant measure $da\, db/a$, $(0 < a < \infty, -\infty < b < \infty)$ and the group acts on this space by right multiplication. The decomposition can be written in the form

$$f(a,b) \equiv D(X,Y) = \sum_{n=0}^{\infty}(\langle \tilde{e}_n^-, \mathbf{L}[a,b]s_n^-\rangle + \langle \tilde{e}_n^+, \mathbf{L}[a,b]s_n^+\rangle), \tag{1.8}$$

where $X = b$, $Y = a^{-1}$ and

$$\begin{aligned}
\tilde{e}_n(\tau) &= \int_{G_A} f(a',b')\mathbf{L}[a',b']\tilde{s}_n(\tau)d_\ell A \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} f(a',b')\frac{1}{\sqrt{a'}}\tilde{s}_n\left(\frac{\tau+b'}{a'}\right)\frac{da'db'}{(a')^2} \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} D(x,y)\sqrt{y}\tilde{s}_n(y[\tau+x])dydx \\
&= F'(f)\tilde{s}_n(\tau).
\end{aligned} \tag{1.9}$$

Also, $\{s_n^+\}$ is an orthonormal basis for the subspace of $L_2(R)$ transforming according to $\mathbf{R}_+$ and $\{s_n^-\}$ is an orthonormal basis for the subspace transforming according to $\mathbf{R}_-$. Finally, $\tilde{S}_n^{\pm}(\omega) = \omega S_n^{\pm}(\omega)$ where $\tilde{S}_n^{\pm}$ and $S_n^{\pm}$ are the Fourier transforms of $\tilde{s}_n^{\pm}$ and $s_n^{\pm}$, respectively. If the $\{s_n\}$ belong to the Schwartz class then

$$\tilde{s}_n(t) = -\frac{i}{2\pi}\frac{ds_n}{dt}(t).$$

Note that (1.8) provides a scheme for determining the distribution $D(X,Y)$ experimentally: we can send out signals $\tilde{s}_n^{\pm}$, measure the echos $\tilde{e}_n^{\pm}$ and then cross-correlate these echos with the test signals $\mathbf{L}[a,b]s_n^{\pm}$ to reconstruct D.

It is not difficult to show [6], [10] that the operator $-\frac{i}{2\pi}\mathbf{L}^{-1}[a,b]F'(f)\frac{d}{dt}$ acting on Schwartz class functions is a Hilbert-Schmidt operator with finite trace. In particular, (1.8) is independent of the choice of orthonormal basis for $L_2(R)$, [5], [11], [12]. Choosing an arbitrary orthonormal basis $\{s_n\}$ in the Schwartz class, we obtain the more compact expansion

$$f(a,b) = \sum_{n=0}^{\infty}\langle \tilde{e}_n, \mathbf{L}[a,b]s_n\rangle. \tag{1.10}$$

The (three-dimensional real) Heisenberg group H_R is the group of matrices

$$B(x_1,x_2,x_3) = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \qquad x_j \in R.$$

For each real $\lambda \neq 0$, H_R admits a unitary irreducible representation by operators $\mathbf{T}^\lambda[x_1,x_2,x_3]$ acting on the Hilbert space $L_2(R)$:

$$\mathbf{T}^\lambda[x_1,x_2,x_3]\boldsymbol{\varphi}(t) = e^{2\pi i\lambda(x_3+tx_2)}\boldsymbol{\varphi}(t+x_1), \quad \boldsymbol{\varphi} \in L_2(R). \tag{1.11}$$

This is called the *Schrödinger representation* of the Heisenberg group, [9], [15].

It can be shown that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1dx_2\langle \mathbf{T}^\lambda(x_1,x_2)\ \boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2\rangle\langle \boldsymbol{\varphi}_4,\mathbf{T}^\lambda(x_1,x_2)\ \boldsymbol{\varphi}_3\rangle = \langle \boldsymbol{\varphi}_1,\boldsymbol{\varphi}_3\rangle\langle \boldsymbol{\varphi}_4,\boldsymbol{\varphi}_2\rangle \tag{1.12}$$

where

$$\langle \boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2\rangle = \int_{-\infty}^{\infty} \varphi_1(t)\ \overline{\varphi_2}(t)dt$$

is the usual inner product on $L_2(R)$ and $\mathbf{T}^\lambda(x_1,x_2) = \mathbf{T}^\lambda[x_1,x_2,0]$, so that

$$\langle \mathbf{T}^\lambda(x_1,x_2)\ \boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2\rangle = \int_{-\infty}^{\infty} e^{2\pi i\lambda t x_2}\boldsymbol{\varphi}_1(t+x_1)\ \overline{\boldsymbol{\varphi}_2}(t)dt. \tag{1.13}$$

If $\{s_n\}$ is an orthonormal basis for $L_2(R)$ then $\{\psi_{nm}(x_1,x_2) = \langle \mathbf{T}^\lambda(x_1,x_2)s_n, s_m\rangle\}$ is an orthonormal basis for $L_2(R^2)$, [2], [9], [14], [15]. It follows from (1.6) that narrowband cross-ambiguity functions are, essentially, matrix elements of the representation $\mathbf{T}^1$ of H_R.

If $f(x_1, x_2)$ belongs to $L_2(R^2)$ we can expand it in terms of the basis of cross-ambiguity functions to obtain

$$\begin{aligned} f(x_1,x_2) &= \sum_{n,m} \int\int f(y_1,y_2)\langle \mathbf{T}^1(y_1,y_2)s_n, s_m\rangle dy_1 dy_2 \; \overline{\langle \mathbf{T}^1(x_1,x_2)s_n, s_m\rangle} \\ &= \sum_n \langle \int\int f(y_1,y_2)\mathbf{T}^1(y_1,y_2)s_n \; dy_1 dy_2, \mathbf{T}^1(x_1,x_2)s_n\rangle \\ &= \sum_n \langle e_n, \mathbf{T}^1(x_1,x_2)s_n\rangle, \end{aligned} \tag{1.14}$$

where

$$e_n(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1,y_2)\mathbf{T}^1(y_1,y_2)s_n(t) \; dy_1 dy_2$$

is the (narrowband) echo from the signal s_n due to the target distribution f.

We shall use group contractions to relate the expansion formulas (1.10) and (1.14).

2. The contraction procedure. We denote by $G_A \times R$ the direct product of the affine group G_A and the real line. A matrix realization of this group consists of the elements

$$A(a,b,c) = \begin{pmatrix} a & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^c \end{pmatrix}, \quad a > 0 \tag{2.1}$$

with multiplication rule

$$A(a,b,c)A(a',b',c') = A(aa', ab'+b, c+c').$$

Here a, b and c are real numbers. A basis for the Lie algebra $L(G_A \times R)$ consists of the matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

with commutation relations

$$[A_1, A_2] = A_2, \quad [A_1, A_3] = \Theta, \quad [A_2, A_3] = \Theta \tag{2.3}$$

where Θ is the zero matrix, [8], [18]. We can pass from the Lie algebra to the group by means of the relation

$$A(a,b,c) = \exp c \; A_3 \exp b \; A_2 \exp \ln a \; A_1, \tag{2.4}$$

where

$$\exp c \; A = \sum_{n=0}^{\infty} \frac{c^n}{n!} A^n,$$

as the reader can verify.

The Heisenberg group H_R has the matrix realization

$$B(x_1,x_2,x_3)=\begin{pmatrix}1 & x_1 & x_3\\ 0 & 1 & x_2\\ 0 & 0 & 1\end{pmatrix},\quad x_j\in R \tag{2.5}$$

with group product

$$B(x_1,x_2,x_3)B(y_1,y_2,y_3)=B(x_1+y_1,x_2+y_2,x_3+y_3+x_1y_2). \tag{2.6}$$

A basis for the Lie algebra $L(H_R)$ consists of the matrices

$$B_1=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad B_2=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix},\quad B_3=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{2.7}$$

with commutation relations

$$[B_1,B_2]=B_3,\quad [B_1,B_3]=\Theta,\quad [B_2,B_3]=\Theta. \tag{2.8}$$

Here,

$$B(x_1,x_2,x_3)=\exp x_3B_3\exp x_2B_2\exp x_1B_1. \tag{2.9}$$

The Lie algebra $L(H_R)$ is a *contraction* of $L(G_A\times R)$, [13], [19]. Rather than review the formal definition of contraction we shall say explicitly what we mean for this single example. Let $\epsilon\neq 0$ be a real parameter and define the elements $C_j(\epsilon)$ of $L(G_A\times R)$ by

$$C_1(\epsilon)=\epsilon\ A_1,\quad C_2(\epsilon)=A_2-A_3,\quad C_3(\epsilon)=\epsilon\ A_3. \tag{2.10}$$

Then $\{C_1,C_2,C_3\}$ is a new basis for $L(G_A\times R)$ with commutation relations

$$[C_1,C_2]=\epsilon C_2+C_3,\quad [C_1,C_3]=\Theta,\quad [C_2,C_3]=\Theta. \tag{2.11}$$

Thus $[C_j(\epsilon),C_k(\epsilon)]=\sum_{\ell=1}^{3}c^{\ell}_{jk}(\epsilon)C_\ell(\epsilon)$ where the $c^{\ell}_{jk}(\epsilon)$ are the *structure constants* of the Lie algebra $L(G_A\times R)$ with respect to this basis. Now pass to the limit in (2.11) as $\epsilon\to 0$. We see from (2.10) that in the limit the $\{C_j\}$ no longer form a basis. However the structure constants on the right side of (2.11) have a well defined limit and, since they must still satisfy the antisymmetry and Jacobi identity conditions, [10], in the limit, the coefficients $c^{\ell}_{jk}=\lim_{\epsilon\to 0}c^{\ell}_{jk}(\epsilon)$ determine a Lie algebra, not necessarily isomorphic to $L(G_A\times R)$. It follows immediately from (2.11) that this new Lie algebra has commutation relations

$$[C_1(0),C_2(0)]=C_3(0),\quad [C_1(0),C_3(0)]=\Theta,\quad [C_2(0),C_3(0)]=\Theta. \tag{2.12}$$

Comparing (2.12) and (2.8) we see that with the identification $C_1(0)\equiv -B_2$, $C_2(0)\equiv B_1$, $C_3(0)\equiv B_3$, $L(H_R)$ is a contraction of $L(G_A\times R)$.

Now let us consider this contraction at the group level. For $\epsilon\neq 0$ an alternate parametrization of $G_A\times R$ is provided by the group elements

$$A[a,\beta,c]=\exp(c-a\beta)C_3(\epsilon)\exp aC_2(\epsilon)\exp(-\beta)C_1(\epsilon)$$

$$
\begin{aligned}
(2.13) \qquad &= \exp \epsilon(c - a\beta)A_3 \exp a(A_2 - A_3) \exp(-\epsilon\beta)A_1 \\
&= \begin{pmatrix} e^{-\epsilon\beta} & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\epsilon(c-a\beta)-a} \end{pmatrix} = A\left(e^{-\epsilon\beta}, a, \epsilon(c - a\beta) - a\right).
\end{aligned}
$$

In terms of the group parameters $[a, \beta, c]$ the group multiplication law for $G_A \times R$ is

$$
\begin{aligned}
&[a, \beta, c][a', \beta', c'] = [a + a'e^{-\epsilon\beta}, \beta + \beta', \\
(2.14) \qquad &c + c' + a\beta' + \frac{a'}{\epsilon}(e^{-\epsilon\beta} - 1) + a'e^{-\epsilon\beta}\beta + a'\beta'(e^{-\epsilon\beta} - 1)].
\end{aligned}
$$

In the limit as $\epsilon \to 0$, (2.14) becomes

$$
(2.15) \qquad [a, \beta, c][a', \beta', c'] = [a + a', \beta + \beta', c + c' + a\beta'],
$$

which is the multiplication law for H_R with $x_1 = a$, $x_2 = \beta$, $x_3 = c$.

The action of $G_A \times R$ associated with the wideband ambiguity function is the unitary representation $\mathbf{L}^\lambda(a, b, c)$ acting on the Hilbert space $L_2(R)$ of Lebesgue square integrable functions f on the real line:

$$
(2.16) \qquad \mathbf{L}^\lambda(a, b, c)f(t) = a^{-\frac{1}{2}} e^{2\pi i\lambda c} f\left(\frac{t + b}{a}\right).
$$

Here λ is a real parameter, (a, b, c) are the coordinates (2.1) on $G_A \times R$ and the operators $\mathbf{L}^\lambda$ satisfy the group representation property

$$
(2.17) \qquad \mathbf{L}^\lambda(a, b, c)\mathbf{L}^\lambda(a', b', c') = \mathbf{L}^\lambda(aa', ab' + b, c + c').
$$

Further, we have

$$
(2.18) \qquad \langle \mathbf{L}^\lambda(a, b, c)f_1, \mathbf{L}^\lambda(a, b, c)f_2\rangle = \langle f_1, f_2\rangle.
$$

The Lie algebra of generalized Lie derivatives associated with the representation (2.16) has the basis

$$
(2.19) \qquad A_1 = -t\frac{d}{dt} - \frac{1}{2}, \quad A_2 = \frac{d}{dt}, \quad A_3 = 2\pi i\lambda.
$$

Indeed

$$
(2.20) \qquad [A_1, A_2] = A_2, \quad [A_1, A_3] = [A_2, A_3] = O
$$

where O is the zero operator, and

$$
\mathbf{L}^\lambda(a, b, c) = \exp cA_3 \exp bA_2 \exp \ln aA_1.
$$

As stated earlier, the action of H_R associated with the narrowband ambiguity function is the unitary irreducible representation $\mathbf{T}^1$ acting on the Hilbert space $L_2(R)$, defined by the unitary operators $\mathbf{T}^1[x_1, x_2, x_3]$:

$$
(2.21) \qquad \mathbf{T}^1[x_1, x_2, x_3]g(t) = e^{2\pi i(x_3 + tx_2)}g(t + x_1), \quad g \in L_2(R).
$$

These operators satisfy the group property

$$\mathbf{T}^1[x_1,x_2,x_3]\mathbf{T}^1[y_1,y_2,y_3] = \mathbf{T}^1[x_1+y_1,x_2+y_2,x_3+y_3+x_1y_2]. \tag{2.22}$$

Moreover, the associated Lie algebra of generalized Lie derivatives has the basis

$$B_1 = \frac{d}{dt}, \quad B_2 = 2\pi i t, \quad B_3 = 2\pi i \tag{2.23}$$

with commutation relations

$$[B_1,B_2] = B_3, \quad [B_1,B_3] = [B_2,B_3] = O. \tag{2.24}$$

Now we will show how to relate the representations $\mathbf{L}^\lambda$ and $\mathbf{T}^1$ through the group contraction procedure. The method is most easily understood at the Lie algebra level. In the Hilbert space associated with $\mathbf{L}^\lambda$ we express each $f \in L_2(R)$ in the form $f(y) = e^{2\pi i\lambda t}g(t)$. Clearly $g \in L_2(R)$ and we can transfer the action of the operators $\mathbf{L}^\lambda(a,b,c)$ to the $g(t)$ via the unitarily equivalent action $\mathbf{K}^\lambda(a,b,c) = e^{-2\pi i\lambda t}\mathbf{L}^\lambda(a,b,c)e^{2\pi i\lambda t}$:

$$\mathbf{K}^\lambda(a,b,c)g(t) = a^{-\frac{1}{2}}e^{2\pi i\lambda c}e^{2\pi i\lambda[t(1-a)+b]/a}g\left(\frac{t+b}{a}\right). \tag{2.25}$$

The associated Lie algebra action is defined by the operators

$$A_1' = -t\frac{d}{dt} - \frac{1}{2} - 2\pi i\lambda t, \quad A_2' = \frac{d}{dt} + 2\pi i\lambda, \quad A_3' = 2\pi i\lambda, \tag{2.26}$$

which satisfy the commutation relations (2.20) again. Now, in analogy with (2.10), we construct a one-parameter family of bases $\{C_1,C_2,C_3\}$ for this Lie algebra:

$$\begin{aligned} C_1(\epsilon) &= \epsilon A_1' = -\epsilon t\frac{d}{dt} - \frac{\epsilon}{2} - 2\pi i\epsilon\lambda(\epsilon)t \\ C_2(\epsilon) &= A_2' - A_3' = \frac{d}{dt}, \quad C_3(\epsilon) = \epsilon A_3' = 2\pi i\epsilon\lambda(\epsilon). \end{aligned} \tag{2.27}$$

This family of bases satisfies the commutation relations (2.11). The only nonzero commutator is

$$[C_1(\epsilon),C_2(\epsilon)] = \epsilon C_2(\epsilon) + C_3(\epsilon).$$

Note that we have allowed the parameter λ to vary with ϵ. Indeed we choose $\lambda(\epsilon) = \epsilon^{-1}$. Then in the limit as $\epsilon \to 0$ we obtain the result

$$C_1(0) = -2\pi i t = -B_2, \quad C_2(0) = \frac{d}{dt} = B_1, \quad C_3(0) = 2\pi i = B_3, \tag{2.28}$$

i.e., the Lie algebra version of the representation $\mathbf{T}^1$ of H_R.

At the group level, we see from (2.13) that for $\epsilon \neq 0$ we have the representation $\mathbf{K}^{\epsilon^{-1}}$ of $G_A \times R$:

$$\begin{aligned} \mathbf{K}^{\epsilon^{-1}}[a,\beta,c] &= \exp[c-a\beta]C_3(\epsilon)\exp aC_2(\epsilon)\exp -\beta C_1(\epsilon) \\ \mathbf{K}^{\epsilon^{-1}}[a,\beta,c]g(t) &= e^{2\pi i(c-a\beta+\epsilon\beta/2)}e^{\frac{2\pi i}{\epsilon}(e^{\epsilon\beta}-1)(t+a)}g\left((t+a)e^{\epsilon\beta}\right). \end{aligned} \tag{2.29}$$

For $\epsilon \neq 0$, $\mathbf{K}^{\epsilon^{-1}}$ is unitary equivalent to the representation $\mathbf{L}^{\epsilon^{-1}}$ of $G_A \times R$. In the limit $\epsilon \to 0$ it becomes

$$\mathbf{K}[a, \beta, c]g(t) = e^{2\pi i c} e^{2\pi i \beta t} g(t+a),$$

in agreement with the operator $\mathbf{T}^1[x_1, x_2, x_3]$, (2.21), associated with H_R for $x_1 = a$, $x_2 = \beta$, $x_3 = c$.

It follows from these results that the narrowband cross-ambiguity function can be obtained from the wideband cross-ambiguity function by taking the limit

$$\lim_{\epsilon \to 0} e^{-i(a\beta + a/\epsilon)} \langle \mathbf{L}(e^{-\epsilon\beta}, a) f_1^\epsilon, f_2^\epsilon \rangle \;=\; \langle \mathbf{T}^1(a, \beta) g_1, g_2 \rangle \tag{2.30}$$

where $\mathbf{L}$ and $\mathbf{T}^1$ are given by (1.7) and (1.11), respectively. Here, $f_j^\epsilon(t) = e^{2\pi i t/\epsilon} g_j(t)$. Moreover, for $\epsilon > 0$ sufficiently small we have

$$|\langle \mathbf{L}(e^{-\epsilon\beta}, a) f_1^\epsilon, f_2^\epsilon \rangle| \approx |\langle \mathbf{T}^1(a, \beta) g_1, g_2 \rangle| = |\langle \mathbf{T}^1(a, \beta) f_1^\epsilon, f_2^\epsilon \rangle|.$$

3. Decomposition of the regular representation. Recall that if $f(a, b)$ is a function on the group G_A, square integrable with respect to the right invariant measure $d_r A = \frac{da\ db}{a}$, then f can be expanded in the form

$$f(a, b) = \sum_{n=0}^{\infty} \langle \tilde{e}_n, \mathbf{L}(a, b) s_n \rangle \tag{3.1}$$

where $\{s_n\}$ is an orthonormal basis for $L_2(R)$. Further, if s_n is continuously differentiable,

$$\begin{aligned} \tilde{e}_n(t) &= \frac{-i}{2\pi} \int_{G_A} f(a', b') \mathbf{L}(a', b') \frac{ds_n}{dt}(t)\ d_\ell A \\ &= \frac{-i}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{f(a', b')}{\sqrt{a'}} \frac{ds_n}{dt} \left(\frac{t + b'}{a'} \right) \frac{da'\ db'}{(a')^2} \end{aligned} \tag{3.2}$$

where $d_\ell A = \frac{da'\ db'}{(a')^2}$ is the left invariant measure on G_A. Using the concept of group contraction, we will show how (3.1) goes to the expansion formula (1.14) for the group H_R, in an appropriate limit. Strictly speaking, we should consider instead the expansion formula for functions $f(a^\circ, b^\circ, c^\circ)$ on the group $G_A \times R$. Since the unitary irreducible representations of R are one-dimensional, it is sufficient to consider functions of the form $f_\lambda[a^\circ, b^\circ, c^\circ] = f_\lambda(a^\circ, b^\circ) e^{2\pi i \lambda c^\circ}$ for λ a real parameter. The right invariant measure on $G_A \times R$ is $da^\circ\ db^\circ\ dc^\circ / a^\circ$, and the left invariant measure is $da^\circ\ db^\circ\ dc^\circ / (a^\circ)^2$. Introducing the coordinates $[a', \beta', c']$, (2.13)

$$a^\circ = e^{-\epsilon\beta'}, \quad b^\circ = a', \quad c^\circ = \epsilon(c' - a\beta') - a',$$

we find that the right and left invariant measures are, respectively,

$$\epsilon^2 d\beta'\ da'\ dc', \qquad \epsilon^2 e^{\epsilon\beta'} d\beta'\ da'\ dc'. \tag{3.3}$$

Performing the integration on c' from $-\infty$ to $+\infty$ we cancel a factor $\epsilon\ dc'$ in (3.3). Finally we set $s_n(t) \equiv s_n^{\lambda^{-1}}(t) = e^{2\pi i\lambda t} g_n(t)$ and $\lambda = \epsilon^{-1}$. Then $\tilde{e}_n(t)$ becomes

(3.4)
$$\tilde{e}_n^{(\epsilon)}(t) = \frac{-i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\epsilon^{-1}}(e^{-\epsilon\beta'}, a') e^{\epsilon\beta'/2} e^{-2\pi i(a'\beta' + a'/\epsilon)} \cdot \\ \cdot [\frac{2\pi i}{\epsilon} g_n\left(\frac{t+a'}{e^{-\epsilon\beta'}}\right) + \frac{dg_n}{dt}\left(\frac{t+a'}{e^{-\epsilon\beta'}}\right)] e^{\frac{2\pi i}{\epsilon}(t+a')e^{\epsilon\beta'}} \epsilon e^{2\epsilon\beta'}\ d\beta'\ da'$$

and

(3.5) $$\langle \tilde{e}_n^\epsilon, \mathbf{L}(e^{-\epsilon\beta}, a) s_n^\epsilon \rangle = \int_{-\infty}^{\infty} \tilde{e}_n^\epsilon(t) e^{\epsilon\beta/2} \overline{g}_n\left(\frac{t+a}{e^{-\epsilon\beta}}\right) e^{-\frac{2\pi i}{\epsilon}(t+a)e^{\epsilon\beta}}\, dt$$

with

(3.6) $$f_{\epsilon^{-1}}(e^{-\epsilon\beta}, a) e^{-2\pi i(a\beta + a/\epsilon)} = \sum_{n=0}^{\infty} \langle \tilde{e}_n^\epsilon, \mathbf{L}(e^{-\epsilon\beta}, a) s_n^\epsilon \rangle.$$

Now we multiply both sides of (3.6) by the factor $e^{2\pi i(a\beta + a/\epsilon)}$ and pass to the limit as $\epsilon \to 0$. With the assumption that $\lim_{\epsilon\to 0} f_{\epsilon^{-1}}(e^{-\epsilon\beta}, a) = F(a,\beta)$ where F is Lebesgue square integrable with respect to the measure $d\beta\ da$, we find

$$F(a,\beta) = \sum_{n=0}^{\infty} \langle e_n, \mathbf{T}^1(a,\beta) g_n \rangle$$

where
$$e_n(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(a',\beta') e^{2\pi i\beta' t} g_n(t+a')\ da'\ d\beta',$$

in agreement with the expansion formula (1.14).

We note that from the point of view of group representation theory, expansions (1.8), (1.10) and (1.14), can be expressed as traces of operators, i.e., in a basis independent fashion, [9]. We have followed Naparst's choice [10] of a signal basis for the wideband case in our derivation. However, this choice is not essential for the validity of the proofs.

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