SYMMETRY TECHNIQUES AND ORTHOGONALITY FOR Q-SERIES

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ABSTRACT. We advocate the exploitation of symmetry (recurrence relation) techniques for the derivation of properties associated with families of basic hypergeometric functions, in analogy with the local Lie theory techniques for ordinary hypergeometric functions. Here these ideas are motivated from the theory of partial differential equations and then applied to the (continuous) Askey-Wilson polynomials, q-Hahn and big and little q-Jacobi polynomials to obtain strikingly simple *derivations* (rather than *verifications*) of their orthogonality relations and associated integrals. Some q-analogs of Barnes' First Lemma are also derived as examples. This expository paper is based on joint work with Ernie Kalnins. Similar methods have been used by Nikiforov, Suslov and Uvarov.

1. Introduction. The relationship between the classical orthogonal poynomials (Hermite, Laplace, Gegenbauer, Laguerre, Jacobi, etc.) and the linear partial differential equations of mathematical physics is deep and well understood, Bateman (1969), Miller (1977). These polynomials arise as special solutions of the partial differential equations obtained by separation of variables. The orthogonality relations, Rodrigues' formulae, and defining second order Sturm-Liouville eigenvalue equation are inherited from the original PDE. This relationship is particularly clear in the quantum mechanical approach to these PDE's, e.g., the factorization method of Schrödinger-Inoui-Infeld-Hull and its Lie algebraic interpretation by the author, Infeld and Hull (1951), Inoui (1948), Miller (1968). On the other hand, discrete and q-series analogs (Hahn (1949)) of the classical orthogonal polynomials have not yet been well understood in terms of a relationship to Lie algebraic ideas and partial difference or q-difference equations, although there has been considerable recent progress along these lines. (In particular, Jimbo (1985) has found a q-difference analogue of the universal enveloping algebra of $s\ell(2)$ and worked out the finite-dimensional irreducible representations of this algebra. The action of these representations on a natural basis in the representation space coincides with the recurrence relations for families of q-series ${}_2\phi_1$ and ${}_1\phi_1$, see Miller (1970). Woronowicz (1987) has constructed a pseudogroup $S_\mu U(2)$, a q-analog of $SU(2)$ and worked out its finite-dimensional irreducible representations, intimately related to Jimbo's results. Masuda, Mimachi, Nakagami, Noumi and Ueno (1988) and Koornwinder (1988) have shown that the matrix elements in a standard basis of the irreducible representations of $S_\mu U(2)$ can be expressed in terms of little q-Jacobi polynomials

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and have obtained an addition theorem for these polynomials from the pseudogroup. Koelink and Koornwinder (1989) have computed the Clebsch-Gordan coefficients for $S_\mu U(2)$, expressing them in terms of q-Hahn polynomials.) There are, of course, close connections between the representation theory of finite groups, e.g., Chevalley groups, and (discrete) orthogonal polynomials given by basic hypergeometric series with restricted values of the parameters: the polynomials are spherical functions for the groups, Stanton (1984). More generally, there are connections between orthogonal polynomials of this type and association schemes, Bannai and Ito (1984).

Wilson (1980) introduced a family of hypergeometric orthogonal polynomials that include as special or limiting cases the classical polynomials and the $6-j$ symbols of angular momentum. In their Memoir, Askey and Wilson (1985) introduced a still more general class of q-hypergeometric orthogonal polynomials, the most extensive generalization of classical orthogonal polynomials known. The orthogonality proofs given in these papers are highly technical and their motivation is not easily fathomed by other than experts in this subject area. However, I will demonstrate here that the orthogonality relations and various identities satisfied by these generalizations of the classical polynomials can be proved and understood within the same general quantum mechanical framework that was so enlightening for the classical polynomials. The essentials of this framework are as follows:

(1) The orthogonal polynomials arise as solutions of partial differential (difference, q-difference) equations via separation of variables in an appropriate coordinate system. The defining Sturm-Liouville eigenvalue equation follows from separation of variables in the original PDE.
(2) The local symmetries of the PDE lead to recurrence relations for the orthogonal functions.
(3) The recurrence relations and simple Sturm-Liouville techniques lead to the orthogonality and (frequently) to a simple derivation of the explicit normalization of the orthogonal functions.

The originality in this paper lies in the exposition rather than the explicit results, which are mostly contained in the papers Agrawal, Kalnins and Miller (1987), Miller (1987) and Kalnins and Miller (1988), (1989). The same symmetry-based and quantum mechanical techniques have been developed independently and extensively by Nikiforov, Suslov and Uvarov . However, these authors have not applied them to the explicit computation of the normalization for orthogonal functions given here.

In Section 2 we trace steps (1)-(3) above for a familiar family of classical polynomials – the Jacobi polynomials, and obtain the well-known orthogonality properties of these polynomials. In Section 3 we retrace steps (1)-(3) for a q-analog of the Jacobi polynomials – the little q-Jacobi polynomials introduced by Hahn (1949), see also Andrews and Askey (1985), and obtain a strikingly simple derivation (not just a verification) of their orthogonality properties. (In some respects this derivation is simpler than the derivation for the classical Jacobi polynomials.)

In Section 4 we use the same procedure to study Hahn and q-Hahn polynomials, obtaining the orthogonality properties of the big q-Jacobi polynomials and deriving Barnes' Lemma and several of its q-analogs as simple consequences of the general method. In Section 5 we treat the (general) continuous Askey-Wilson polynomials and we conclude with some remarks on the scope of the symmetry method described here.

2. Jacobi Polynomials. We start by considering the complex wave (or Laplace) equation in four-dimensional space which for convenience we write in the form

$$(\partial_{u_1}\partial_{u_2} - \partial_{u_3}\partial_{u_4})\Phi = 0 \tag{2.1}$$

where Φ is a complex valued function of the complex variables u_j. (A simple complex linear change of coordinates recasts (2.1) into the more familiar form

$$(\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3} + \partial^2_{x_4})\Phi = 0.)$$

The generalized hypergeometric function ${}_{r+1}F_r$ is defined by the series expansion, e.g., Bailey (1964)

$$ {}_{r+1}F_r\left(\begin{matrix}\alpha,\dots,\alpha_{r+1}\\ \beta_1,\dots,\beta_r\end{matrix}, z\right) = \sum_{m=0}^{\infty}\frac{(\alpha_1)_m\dots(\alpha_{r+1})_m z^m}{(\beta_1)_m\dots(\beta_r)_m m!}, \tag{2.2}$$

where $|z| < 1$ and

$$(\alpha)_m = \begin{cases} 1 & \text{if} \quad m=0,\\ \alpha(\alpha+1)\dots(\alpha+m-1) & \text{if} \quad m\geq 1.\end{cases}$$

(Here we assume that none of the β_j are negative integers.) It is straightforward to verify directly from this series definition that the functions

$$\Phi\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\right) \equiv {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}; \frac{u_3u_4}{u_1u_2}\right) u_1^{-\alpha}u_2^{-\beta}u_3^{\gamma-1} \tag{2.3}$$

are solutions of the complex wave equation (2.1) where ${}_2F_1$ is a Gaussian hypergeometric function. These solutions are easily characterized in terms of the local Lie symmetries of (2.1). It is evident that the dilation generators $D_j = u_j\partial_j$ are symmetries of (2.1) in the sense that if Ψ is any analytic solution of (2.1) then so is $D_j\Psi$ a solution. The special solutions (2.3) are characterized to within a constant factor by the requirements that they are analytic functions of the u_j in a neighborhood of $u_4 = 0$ and that they satisfy the eigenvalue equations Kalnins, Manocha and Miller (1980)

$$\begin{aligned}(D_1 + D_4)\Phi &= -\alpha\Phi\\ (D_2 + D_4)\Phi &= -\beta\Phi\\ (D_3 - D_4)\Phi &= (\gamma-1)\Phi.\end{aligned} \tag{2.4}$$

Note that (2.3) is a separable solution of the wave equation in terms of the variables

$$z = \frac{u_3u_4}{u_1u_2}, \quad u_1, u_2, u_3.$$

It is evident that the operators ∂_{u_j}, $1 \leq j \leq 4$ are also symmetries of (2.1), i.e., they map solutions to solutions. In particular these operators map the basis functions $\Phi\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\right)$ into other solutions. By consideration of the comutation relations of the ∂_{u_j} with the dilation symmetries D_ℓ or by direct power series computation one easily obtains the relations

$$\begin{aligned} \partial_{u_1}\Phi\begin{pmatrix}\alpha,\beta\\ \gamma\end{pmatrix} &= \alpha\Phi\begin{pmatrix}\alpha+1,\beta\\ \gamma\end{pmatrix}\\ \partial_{u_2}\Phi &= \beta\Phi\begin{pmatrix}\alpha,\beta+1\\ \gamma\end{pmatrix}\\ \partial_{u_3}\Phi &= (\gamma-1)\Phi\begin{pmatrix}\alpha,\beta\\ \gamma-1\end{pmatrix}\\ \partial_{u_4}\Phi &= \frac{\alpha\beta}{\gamma}\Phi\begin{pmatrix}\alpha+1,\beta+1\\ \gamma+1\end{pmatrix}. \end{aligned} \tag{2.5}$$

Note that upon factoring out the dependence on u_1, u_2, u_3 in (2.5) we obtain well known differential recurrence relations for the functions ${}_2F_1\begin{pmatrix}\alpha,\beta\\ \gamma\end{pmatrix};z\Big)$. A notation which suggests the raising operator (lowering operator) nature of the ∂_{u_j} is $E^\alpha \equiv \partial_{u_1}$, $E^\beta \equiv \partial_{u_2}$, $E_\gamma \equiv \partial_{u_3}$, $E^{\alpha\beta\gamma} \equiv \partial_{u_4}$.

The conformal symmetries of the wave equation Bateman (1969), Miller (1977), lead to eight more symmetries which also have a recurrence relation interpretation: E_α, E_β, E^γ, $E^{\alpha\gamma}$, $E_{\alpha\gamma}$, $E^{\beta\gamma}$, $E_{\beta\gamma}$, $E_{\alpha\beta\gamma}$. In particular

$$E_{\alpha\beta\gamma}\Phi = (\gamma-1)\Phi\begin{pmatrix}\alpha-1,\beta-1\\ \gamma-1\end{pmatrix}. \tag{2.6}$$

If we make the change of variable $(1-x)/2 = u_3u_4/u_1u_2$ and factor out the dependence on u_1, u_2, u_3 in the recurrence relations for $E^{\alpha\beta\gamma}$ and $E_{\alpha\beta\gamma}$ we obtain a pair of well-known recurrence relations for the Jacobi polynomials $P_n^{(\alpha,\beta)}$:

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+n+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{2.7a}$$

$$[(x^2-1)\frac{d}{dx} + (\alpha-\beta+(\alpha+\beta+2)x)]P_{n-1}^{(\alpha+1,\beta+1)}(x) = nP_n^{(\alpha,\beta)}(x) \tag{2.7b}$$

where

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}{}_2F_1\begin{pmatrix}-n,\alpha+\beta+n+1\\ \alpha+1\end{pmatrix};\frac{1-x}{2}\Big) \qquad n=0,1,2,\cdots. \tag{2.8}$$

Here we have replaced α by $-n$, β by $\alpha+\beta+n+1$ and γ by $\alpha+1$, and renormalized by the factor $\binom{n+\alpha}{n}$ to conform to the standard definition of the Jacobi polynomials. (We now assume that α and β are real.)

Note that the polynomials of orders 0 and 1 are given by

$$P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = \frac{\alpha+\beta}{2} + \frac{1}{2}(\alpha+\beta+2)x. \tag{2.9}$$

Furthermore composition of the two recurrence relations (2.7) leads to the standard Sturm-Liouville eigenvalue equation:

$$(2.10)\quad (1-x^2)P_n^{(\alpha,\beta)\ ''} + [\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)\ '} = n(\alpha+\beta+n+1)P_n^{(\alpha,\beta)}.$$

(This is precisely the ordinary differential equation for the Jacobi polynomials that we would obtain by directly separating variables in the wave equation (2.1). However, here we have demonstrated that this equation can be "factored" in terms of the recurrences (2.7).)

Now we derive the orthogonality relations for the Jacobi polynomials through a variant of the usual Sturm-Liouville procedure that exploits the factorization (2.7). Let $S_{\alpha,\beta}$ be the space of all polynomials in x with complex inner product

$$(2.11)\qquad < g_1, g_2 >_{\alpha,\beta} = \int_{-1}^{1} g_1(x)g_2(x)\rho_{\alpha,\beta}(x)\ dx, \quad g_1, g_2 \in S_{\alpha,\beta}$$

(We will consider the polynomials $P_n^{(\alpha,\beta)}$ to belong to $S_{\alpha,\beta}$. The interval of integration is motivated by the singularities of (2.10). However, the procedure to follow will work for other choices of integration paths.) Motivated by the recurrence relations (2.7) we define maps

$$\begin{aligned} \tau^{(\alpha,\beta)} &= \frac{d}{dx} : S_{\alpha,\beta} \to S_{\alpha+1,\beta+1} \\ \tau^{*(\alpha+1,\beta+1)} &= (x^2-1)\frac{d}{dx} + (\alpha-\beta+(\alpha+\beta+2)x) : S_{\alpha+1,\beta+1} \to S_{\alpha,\beta}, \end{aligned}$$

and look for density functions $\rho_{\alpha,\beta}(x)$ such that

$$(2.12)\qquad < g, \tau^{(\alpha,\beta)} f >_{\alpha+1,\beta+1} = < \tau^{*(\alpha+1,\beta+1)} g, f >_{\alpha,\beta}, \quad \forall f \in S_{\alpha,\beta}, \quad g \in S_{\alpha+1,\beta+1}.$$

That is, we require that τ^* is the adjoint operator to τ. A straightforward integration by parts argument, using the fact that f and g are arbitrary polynomials, leads to the necessary and sufficient conditions:

$$\begin{aligned} \rho_{\alpha+1,\beta+1}(x) &= (1-x^2)\rho_{\alpha,\beta}(x), \\ \frac{d}{dx}\rho_{\alpha+1,\beta+1}(x) &= [\beta-\alpha-(\alpha+\beta+2)x]\rho_{\alpha,\beta}(x) \end{aligned}$$

with solution, unique up to a constant multiplicative factor,

$$(2.13)\qquad \rho_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$

This solution will provide a satisfactory weight function for $S_{\alpha,\beta}$ provided $\alpha > -1$, $\beta > -1$, which we now assume.

Since τ and τ^* are mutual adjoints, it follows immediately from (2.10) that

$$\tau^*\tau : S_{\alpha,\beta} \to S_{\alpha,\beta}$$

is a self-adjoint operator with eigenvalues

$$\lambda_n = n(\alpha+\beta+n-1), \quad n = 0, 1, \cdots$$

and eigenfunctions

$$g_n = P_n^{(\alpha,\beta)}(x).$$

It is easy to show that $\lambda_n = \lambda_m$ iff $n = m$. Using the well-known fact that eigenfunctions corresponding to distinct eigenvalues of a self-adjoint operator are orthogonal we obtain the orthogonality relations

$$\begin{aligned} < P_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)} >_{\alpha,\beta} &= \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)(1-x)^\alpha (1+x)^\beta \, dx \\ &= \delta_{nm} M_n(\alpha,\beta). \end{aligned} \tag{2.14}$$

The polynomials $\{P_n^{(\alpha,\beta)}\}$ could have been computed from a knowledge of the weight function $\rho_{\alpha,\beta}$ via the Gram-Schmidt process and are uniquely determined once the coefficient of x^n in $P_n^{(\alpha,\beta)}(x)$ is specified. Since the measure is invariant under the interchange $x \leftrightarrow -x$, $\alpha \leftrightarrow \beta$, it follows easily that

$$P_n^{(\alpha,\beta)}(-x) = P_n^{(\beta,\alpha)}(x),$$

a nontrivial identity.

Now we try to determine the normalization of the Jacobi polynomials, i.e., to compute $M_n(\alpha,\beta)$. Identity (2.12) with $g = P_{n-1}^{(\alpha+1,\beta+1)}$, $f = P_n^{(\alpha,\beta)}$ yields the recurrence

$$\frac{1}{2}(\alpha+\beta+n+1) M_{n-1}(\alpha+1,\beta+1) = -n M_n(\alpha,\beta),$$

so it is sufficient to compute

$$||1||_{\alpha,\beta} \equiv M_0(\alpha,\beta) = < 1, 1 >_{\alpha,\beta} = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \, dx. \tag{2.15}$$

This is the beta-integral and can be easily evaluated by a well-known trick, but for pedagogical purposes we shall determine what a knowledge of the recurrence symmetries and the connection with orthogonality alone will tell us about (2.15). Using the evident facts that

$$< P_1^{(\alpha,\beta)}, P_0^{(\alpha,\beta)} >_{\alpha,\beta} = 0, \qquad < (1+x), 1 >_{\alpha,\beta} = < 1, 1 >_{\alpha,\beta+1}$$

and the expressions (2.9), we obtain the recurrence relations

$$\begin{aligned} \frac{1}{2}(\alpha+\beta+2)||1||_{\alpha,\beta+1} &= (\beta+1)||1||_{\alpha,\beta} \\ \frac{1}{2}(\alpha+\beta+2)||1||_{\alpha+1,\beta} &= (\alpha+1)||1||_{\alpha,\beta}. \end{aligned}$$

These recurrence relations have the solution

$$||1||_{\alpha,\beta} = \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+2)} 2^{\alpha+\beta} h(\alpha,\beta) \tag{2.16}$$

where $h(\alpha+1,\beta) = h(\alpha,\beta+1) = h(\alpha,\beta)$. (Here we are using the property $\Gamma(z+1) = z\Gamma(z)$ of the Gamma function. If $f(z+1) = zf(z)$ then $f(z) = \Gamma(z)h(z)$ where $h(z+1) = h(z)$.) This is as far as we can go using recurrence relations alone, since the Gamma function isn't uniquely determined by its fundamental recurrence relation. We need additional facts to compute h.

One way to proceed is to replace α, β by $\alpha+k$, $\beta+k$ in (2.16), $k = 0, 1, 2, \cdots$, and write the resulting identity in the form

$$\int_{-1}^{1} \left[\left(\frac{1-x}{2}\right)^{\alpha+k} \left(\frac{1+x}{2}\right)^{\beta+k} \frac{\Gamma(\alpha+\beta+2k+2)}{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \right] dx = h(\alpha,\beta).$$

Letting $k \to +\infty$ we find from Stirling's formula, Whittaker and Watson (1958), that the integrand of the left-hand side converges to 1 and that $h(\alpha,\beta) \equiv 2$.

None of the steps in the foregoing development is very novel and the results are, of course, well known. What is remarkable, is the fact that the same development works for families of polynomials satisfying difference and q-difference equations. Indeed the derivation of the norm is frequently more straightforward than in the differential equations case.

3. Little q-Jacobi Polynomials. We proceed in exact analogy with the previous section. Consider the q-wave equation

$$(\Delta_1^- \Delta_2^- - \Delta_3^+ \Delta_4^+)\Phi = 0 \tag{3.1}$$

where Φ is a function of the complex variables $u_j, 1 \leq j \leq 4$. The q-difference operators $\Delta_j^{\pm}$ are defined by

$$\Delta_j^{\pm} f(u_j) = u_j^{-1}[f(u_j) - f(q^{\pm 1}u_j)]$$

where q is a real parameter such that $-1 < q < 1$. (Dividing (3.1) by $(1-q)^2$ and letting $q \to 1$ we obtain (2.1) in the limit.)

The q-hypergeometric series ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; x \right) = \sum_{m=0}^{\infty} \frac{(a_1)_m \ldots (a_{r+1})_m x^m}{(b_1)_m \ldots (b_r)_m (q)_m} \tag{3.2}$$

where $|x| < 1$ and

$$\begin{aligned} (a)_0 &= 1, \\ (a)_m &= (1-a)(1-aq)\ldots(1-aq^{m-1}), \quad m \geq 1. \end{aligned}$$

From expression (3.2) we can easily verify that

$$\Phi \begin{pmatrix} a, b \\ c \end{pmatrix} \equiv {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; \frac{u_3 u_4}{u_1 u_2} \right) u_1^{-\alpha} u_2^{-\beta} u_3^{\gamma-1} \tag{3.3}$$

where $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$, is a solution of the q-wave equation. (For convenience we assume $0 < aq < 1$, $bq < 1$.) Note that $\lim_{q\to 1-} \Phi \begin{pmatrix} a, b \\ c \end{pmatrix} = \Phi \begin{pmatrix} \alpha, \beta \\ \gamma \end{pmatrix}$.

Equation (3.1) admits symmetries that are q-analogs of the conformal symmetries of the wave equation. In particular Δ_1^-, Δ_2^-, Δ_3^+, Δ_4^+ are obvious symmetries of the q-wave equation (they map solutions to solutions) and, as one can readily verify, they determine recurrence relations for the ${}_2\phi_1$:

$$\begin{aligned}
\Delta_1^-\Phi &= (1-a)\Phi\begin{pmatrix} aq, b \\ c \end{pmatrix} \equiv E^{\alpha}\Phi \\
\Delta_2^-\Phi &= (1-b)\Phi\begin{pmatrix} a, bq \\ c \end{pmatrix} \equiv E^{\beta}\Phi \\
\Delta_3^+\Phi &= (1-c/q)\Phi\begin{pmatrix} a, b \\ cq \end{pmatrix} \equiv E_{\gamma}\Phi \\
\Delta_4^+\Phi &= \frac{(1-a)(1-b)}{(1-c)}\Phi\begin{pmatrix} aq, bq \\ cq \end{pmatrix} \equiv E^{\alpha\beta\gamma}\Phi.
\end{aligned}$$

A less obvious (conformal) symmetry is

$$E_{\alpha\beta\gamma} = -u_1u_2T_4^{-1}\Delta_3^+ + q^{-1}u_4T_4^{-1} - q^{-2}u_4T_1^{-1}T_2^{-1}T_4^{-1}$$

$$E_{\alpha\beta\gamma}\Phi = (c/q-1)\Phi\begin{pmatrix} a/q, b/q \\ c/q \end{pmatrix}$$

where $T_i^{-1}f(u_i) = f(q^{-1}u_i)$.

Making the change of variable $z = u_3u_4/u_1u_2$ and factoring out the dependence of u_1, u_2, u_3 in the recurrence relations $E^{\alpha\beta\gamma}$, $E_{\alpha\beta\gamma}$, we obtain the (ordinary) difference equation

$$\begin{aligned}
\tau^{(a,b)}p_n^{(\alpha,\beta)} &= -q^{1-n}(1-q^{n+1}ab)p_{n-1}^{(\alpha+1,\beta+1)} \\
\tau^{*(aq,bq)}p_{n-1}^{(\alpha+1,\beta+1)} &= -(1-q^n)p_n^{(\alpha,\beta)}
\end{aligned} \tag{3.4}$$

where the

$$p_n^{(\alpha,\beta)}(z) = \frac{(aq)_n}{(q)_n}{}_2\phi_1\begin{pmatrix} q^{-n}, q^{n+1}ab \\ aq \end{pmatrix}; qz\Big) \qquad n = 0, 1, 2, \cdots$$

are little q-Jacobi polynomials, Hahn (1949), Andrews and Askey (1985). Note that

$$\lim_{q\to 1} p_n^{(\alpha,\beta)}(z) = P_n^{(\alpha,\beta)}(1-2z).$$

(Recall that $q^\alpha = a$, $q^\beta = b$.) Here,

$$\tau^{(a,b)} = \Delta_z^+, \quad \tau^{*(aq,bq)} = (z-1)T_z^{-1} + (aq - abq^2z).$$

Now let $S_{a,b}$ be the space of all polynomials in z with the complex inner product

$$(f_1, f_2)_{a,b} = \frac{1}{2\pi i}\oint_C f_1(z)f_2(z)w_{a,b}(z)\,\frac{dz}{z} \qquad f_1, f_2 \in S_{a,b} \tag{3.5}$$

where $C = \{z : |z| = 1+\epsilon, \quad \epsilon > 0\}$ and $w_{a,b}$ is a weight function to be determined. We consider τ and τ^* as mappings

$$\tau^{(a,b)} : S_{a,b} \to S_{aq,bq}$$
$$\tau^{*(aq,bq)} : S_{aq,bq} \to S_{a,b}$$

and look for a weight function so that τ^* is the adjoint of τ:

$$(\tau f, g)_{aq,bq} = (f, \tau^* g)_{a,b} \qquad \forall f \in S_{a,b}, \quad g \in S_{aq,bq}. \tag{3.6}$$

The q-analog of "integration by parts" in (3.6) yields the necessary and sufficient conditions

$$w_{aq,bq}(z/q) = \frac{z}{q}(1-z)w_{a,b}(z),$$
$$w_{aq,bq}(z) = aqz(1-bqz)w_{a,b}(z)$$

with the solution, unique to within a constant multiplier,

$$w_{a,b}(z) = \frac{(z/a)_\infty (qa/z)_\infty}{(zbq)_\infty (1/z)_\infty s(a,q)} \tag{3.7}$$

where

$$s(a,q) = (aq)_\infty (1/a)_\infty (-aq)_\infty (-1/a)_\infty$$

and

$$(z)_\infty = \lim_{n\to\infty} (z)_n.$$

(Note that $(zq)_\infty = (z)_\infty/(1-z)$. Also to carry out the "integration by parts" we have used the invariance of the contour integral under the change of variable $z \to zq$.)

It follows from (3.4) and (3.6) that

$$\tau^*\tau : S_{a,b} \to S_{a,b}$$

is self-adjoint and

$$\tau^*\tau p_n^{(\alpha,\beta)} = \lambda_n p_n^{(\alpha,\beta)}, \quad \lambda_n = -q(1-q^{-n})(1-q^{n+1}ab).$$

Clearly $\lambda_n \neq \lambda_m$ if $n \neq m$ so we must have

$$(p_n^{(\alpha,\beta)}, p_m^{(\alpha,\beta)})_{a,b} = 0 \quad \text{if } m \neq n.$$

We have proved orthogonality for the little q-Jacobi polynomials. It remains to compute the normalization of these functions. Setting $f = p_n^{(\alpha,\beta)}$, $g = p_{n-1}^{(\alpha+1,\beta+1)}$ in the adjoint relation (3.6) we obtain the recurrence

$$||p_n^{(\alpha,\beta)}||_{a,b}^2 = \frac{q(1-q^{n+1}ab)}{(1-q^n)} ||p_{n-1}^{(\alpha+1,\beta+1)}||_{aq,bq}^2.$$

Thus, to determine the norms it is sufficient to compute $||1||^2_{a,b} = (p_0^{(\alpha,\beta)}, p_0^{(\alpha,\beta)})_{a,b}$. From the relation $(p_1^{(\alpha,\beta)}, p_0^{(\alpha,\beta)})_{a,b} = 0$ and the explicit expressions for p_1, p_0 we find

$$||1||^2_{a,bq} = \frac{(1-bq)}{(1-abq^2)}||1||^2_{a,b}. \tag{3.8}$$

(Here we use the fact that $(1-zbq, 1)_{a,b} = ||1||^2_{a,bq}$.)

To completely determine the norm we use another conformal symmetry of (3.1), E^γ, together with E_γ. Their induced action on the basis $\{p_n^{(\alpha,\beta)}\}$ is

$$\begin{aligned} \mu^{(a,b)} p_n^{(\alpha,\beta)} &= (1-aq^n) p_n^{(\alpha-1,\beta+1)}, \\ \mu^{(aq^{-1},bq)} p_n^{(\alpha-1,\beta+1)} &= \frac{q^{-n}}{a}(1-bq^{n+1}) p_n^{(\alpha,\beta)} \end{aligned}$$

where

$$\mu^{(a,b)} = 1 - aT_z, \quad \mu^{(a,b)} : S_{a,b} \to S_{aq^{-1},bq},$$

$$\mu^{*(aq^{-1},bq)} = \frac{z-1}{az} T_z^{-1} + \frac{1-bqz}{az}, \quad \mu^{*(aq^{-1},bq)} : S_{aq^{-1},bq} \to S_{a,b}.$$

It is straightforward to verify that

$$\begin{aligned} (\mu f, g)_{aq^{-1},bq} &= (f, \mu^* g)_{a,b}, \\ &\forall f \in S_{a,b}, \quad g \in S_{aq^{-1},bq}, \end{aligned} \tag{3.9}$$

so μ^* is the adjoint of μ. Now set $f = p_0^{(\alpha,\beta)} = 1$, $g = p_0^{(\alpha-1,\beta+1)} = 1$ in (3.9) to obtain

$$||1||^2_{aq^{-1},bq} = \frac{(1-bq)}{a(1-a)}||1||^2_{a,b}. \tag{3.10}$$

The solution of the recurrences (3.8) and (3.10) is

$$||1||^2_{a,b} = \frac{(abq^2)_\infty K(q)}{(bq)_\infty (aq)_\infty (-1/a)_\infty (-aq)_\infty}.$$

Thus we have

$$\frac{1}{2\pi i} \oint_C \frac{(z/a)_\infty (qa/z)_\infty}{(1/a)_\infty (1/z)_\infty (zbq)_\infty}\, dz = \frac{(abq^2)_\infty K(q)}{(bq)_\infty}.$$

To compute $K(q)$ we make particular choices of a and b such that the integral becomes trivial:

$$a > 1 + \epsilon > 1 > qa, \quad abq = 1.$$

It follows easily that

$$K(q) = \frac{1}{(q)_\infty}.$$

Thus we have derived the normalization for the little q-Jacobi polynomials entirely by elementary techniques and proved complex orthogonality with respect to the inner product (3.5), (3.7).

To obtain the orthogonality of the $\{p_n^{(\alpha,\beta)}\}$ with respect to a real positive definite inner product we expand the integral expression for the complex inner product $(p_n^{(\alpha,\beta)}, p_m^{(\alpha,\beta)})_{a,b}$ by residues at the poles $z = q^k$, $k = 0, 1, 2, \cdots$ of the weight function. The straightforward computation yields

(3.11)
$$\sum_{k=0}^{\infty} \frac{(aq)^k (q^{k+1})_\infty}{(bq^{k+1})_\infty} p_n^{(\alpha,\beta)}(q^k) p_m^{(\alpha,\beta)}(q^k) = \frac{(aq)^n (abq^{n+1})_\infty (aq)_n (q)_\infty \delta_{m,n}}{(bq^{n+1})_\infty (aq)_\infty (q)_n (1 - abq^{2m+1})}.$$

That is, the little q-Jacobi polynomials are orthogonal with respect to a discrete measure concentrated at $z = q^k$, $k = 0, 1, 2, \cdots$. Setting $n = m = 0$ in (3.11) we get a q-analog of the beta integral.

4. q-Hahn Polynomials and q-analogues of Barnes' Lemma. We are concerned with the q-Hahn polynomials

$$\Phi_n^{a,b,c,d}(z) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, & q^{n-1}abcd, & az \\ ac, & ad & \end{matrix} ; q \right), \tag{4.1}$$

$$n = 0, 1, 2, \ldots.$$

Initially we require $0 < |q| < 1$, $acd \neq 0$, and $|a|, |b|, |c|, |d| < 1$, but some of these conditions can be relaxed later. The polynomials obey the fundamental recurrence relations

$$\mu^{(a,b,c,d)} \Phi_n^{(a,b,c,d)} = \frac{q^{\frac{1}{2}}}{d} \left(1 - \frac{ad}{q} \right) \Phi_n^{(aq^{-\frac{1}{2}}, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{-\frac{1}{2}})}, \tag{4.2A}$$

$$\tau^{(a,b,c,d)} \Phi_n^{(a,b,c,d)} = \frac{a(1 - q^{-n})(1 - q^{n-1}abcd)}{q^{-\frac{1}{2}}(1 - ad)(1 - ac)} \Phi_{n-1}^{(aq^{\frac{1}{2}}, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{\frac{1}{2}})} \tag{4.2B}$$

where

$$\mu^{(a,b,c,d)} = \frac{1}{z} \left[(1 - azq^{-\frac{1}{2}}) T_z^{\frac{1}{2}} - \left(1 - \frac{zq^{\frac{1}{2}}}{d} \right) T_z^{-\frac{1}{2}} \right],$$
$$\tau^{(a,b,c,d)} = \frac{1}{z} [T_z^{\frac{1}{2}} - T_z^{-\frac{1}{2}}]$$

and $T_z^\alpha f(z) = f(q^\alpha z)$. These relations follow from

$$\mu(za)_n = \frac{q^{\frac{1}{2}}}{d} (1 - adq^{n-1})(zaq^{\frac{1}{2}})_n,$$
$$\tau(za)_n = \frac{a}{q^{\frac{1}{2}}} (1 - q^n)(zaq^{-\frac{1}{2}})_{n-1}.$$

The existence of μ suggests the existence of a recurrence taking $\Phi_n^{(aq^{-\frac{1}{2}}, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{-\frac{1}{2}})}$ to $\Phi_n^{(a,b,c,d)}$. Indeed, the appropriate operator is

$$\mu^* = -\frac{q^{\frac{1}{2}}}{\rho} \mu^{(bq^{\frac{1}{2}}, aq^{-\frac{1}{2}}, dq^{-\frac{1}{2}}, cq^{\frac{1}{2}})},$$

(4.2C)
$$\mu^* \Phi_n^{(aq^{-\frac{1}{2}}, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{-\frac{1}{2}})}$$
$$= \frac{q^{\frac{1}{2}}}{\rho} \frac{(\frac{ad}{q} - q^{-n})(1 - bcq^n)}{c(1 - \frac{ad}{q})} \Phi_n^{(a,b,c,d)}, \quad \rho \neq 0,$$

which follows from

$$\frac{\rho}{q^{\frac{1}{2}}}\mu^*(azq^{-\frac{1}{2}})_n = (az)_n(bcq^n-1)\frac{q^{-n}}{c} + (az)_{n-1}(acq^{n-1}-1)(q^n-1)\frac{q^{-n}}{c}.$$

Let $w_{a,b,c,d}(z)$ be a (complex-valued) weight function and $S_{a,b,c,d}$ be the indefinite inner product space of polynomials $f(z)$ with respect to the inner product

$$(f_1,f_2)_{a,b,c,d} = \frac{1}{2\pi i}\oint_C f_1(z)f_2(z)w_{a,b,c,d}(z)\frac{dz}{z} \tag{4.3}$$

where C is a deformation of the unit circle $|z|=1$. Now consider $\mu=\mu^{(a,b,c,d)}$ and $\mu^* = -\frac{q^{\frac{1}{2}}}{\rho}\mu^{(bq^{\frac{1}{2}},aq^{-\frac{1}{2}},dq^{-\frac{1}{2}},cq^{-\frac{1}{2}})}$ as mappings

$$\begin{aligned}\mu : S_{a,b,c,d} &\to S_{aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}}},\\ \mu^* : S_{aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}}} &\to S_{a,b,c,d}\end{aligned} \tag{4.4}$$

and determine $w_{a,b,c,d}$ so that

$$(\mu f,g)_{aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}}} = (f,\mu^* g)_{a,b,c,d} \tag{4.5}$$

for all $f\in S_{a,b,c,d}$, $g\in S_{aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}}}$. Condition (4.5) yields a q-difference equation for $w_{a,b,c,d}$ with solution

$$w_{a,b,c,d}(z) = \frac{(\rho z/d)_\infty(qd/z\rho)_\infty(\rho c/z)_\infty(qz/c\rho)_\infty}{(az)_\infty(bz)_\infty(c/z)_\infty(d/z)_\infty}. \tag{4.6}$$

This result is a consequence of the invariance of the contour integral under the changes of variable $z\to q^{\pm\frac{1}{2}}z$, and the property $h(qz)=-h(z)/z$ where $h(z)=(z)_\infty(q/z)_\infty$.

It follows immediately that $\mu^*\mu$ is a self-adjoint operator

$$\mu^*\mu : S_{a,b,c,d}\to S_{a,b,c,d}$$

and from the recurrence relations (4.2) we have

$$\mu^*\mu\Phi_n^{(a,b,c,d)} = \lambda_n\Phi_n^{(a,b,c,d)},\quad \lambda_n = \frac{q}{cd\rho}\left(\frac{ad}{q}-q^{-n}\right)(1-bcq^n). \tag{4.7}$$

Since eigenfunctions corresponding to distinct eigenvalues are orthogonal we have

$$(\Phi_n^{(a,b,c,d)},\Phi_m^{(a,b,c,d)})_{a,b,c,d} = 0 \quad \text{for } m\neq n. \tag{4.8}$$

Relation (4.5) for $f=g\equiv 1$ yields

$$||1||^2_{aq^{-\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{-\frac{1}{2}}} = -\frac{d}{\rho c}\frac{1-bc}{\left(1-\frac{ad}{q}\right)}||1||^2_{a,b,c,d} \tag{4.9}$$

where
$$\|1\|^2_{a,b,c,d} = (1,1)_{a,b,c,d}.$$
The symmetry of the weight function in (a,b) yields an additional relation of the form (4.9). Furthermore the obvious relation
$$(\Phi_1^{(a,b,c,d)}, \Phi_0^{(a,b,c,d)})_{a,b,c,d} = 0,$$
the explicit expression (4.1), and the property $(1,p_n)_{a,b,c,d} = \|1\|^2_{aq^n,b,c,d}$ for $p_n(z) = (az)_n$, yield the relation

$$\|1\|^2_{aq,b,c,d} = \frac{(1-ad)(1-ac)}{(1-abcd)}\|1\|^2_{a,b,c,d}. \tag{4.10}$$

Again, the symmetry in (a,b) gives an additional relation. The solution of these q-difference equations is

$$\|1\|^2_{a,b,c,d} = \frac{(abcd)_\infty (c\rho/d)_\infty (qd/c\rho)_\infty}{(ad)_\infty (ac)_\infty (ac)_\infty (bc)_\infty (bd)_\infty}\mathcal{K}(\rho,q) \tag{4.11}$$

where $\mathcal{K}(\rho,q)$ is to be determined. In the special case $a=\rho/d$, $b=q/c\rho$ we can compute the (trivial) integral directly: $\|1\|^2_{\rho/d,q/c\rho,c,d} = 1$. Hence $\mathcal{K}(\rho,q) = (\rho)_\infty (q/\rho)_\infty/(q)_\infty$ and we have

$$\begin{aligned}\|1\|^2_{a,b,c,d} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d)_\infty (qde^{-i\theta}/\rho)_\infty (\rho c e^{i\theta})_\infty}{(ae^{i\theta})_\infty (be^{i\theta})_\infty (ce^{i\theta})_\infty} \\ &\quad\cdot \frac{(qe^{i\theta}/c\rho)_\infty}{(de^{-i\theta})_\infty} d\theta \\ &= \frac{(abcd)_\infty (c\rho/d)_\infty (qd/c\rho)_\infty (\rho)_\infty (q/\rho)_\infty}{(ad)_\infty (ac)_\infty (bc)_\infty (bd)_\infty (q)_\infty}\end{aligned} \tag{4.12}$$

in agreement with Askey and Roy (1986). As shown in Kalnins and Miller (1988) we can use the τ recurrence to express the norms of each of the q-Hahn polynomials in terms of the single norm (4.12). Thus the $\{\Phi_n^{(a,b,c,d)}\}$ are uniquely characterized by their orthogonality with respect to the complex weight function $w_{a,b,c,d}$.

Since the weight function is symmetric in $\{a,b\}$ the orthogonal polynomials satisfy the transformation rule

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & q^{n-1}abcd, & bz \\ bc, & dc & \end{matrix};q\right) = \left(\frac{b}{a}\right)^n \frac{(ac)_n (ad)_n}{(bc)_n (bd)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, & q^{n-1}abcd, & az \\ ac, & ad & \end{matrix};q\right). \tag{4.13}$$

The complex orthogonality (4.8) for the q-Hahn polynomials leads to real discrete orthogonality for the big q-Jacobi polynomials (Andrews and Askey (1985)):

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, & q^{n-1+\alpha+\beta+\gamma+\delta}, & q^{x+1} \\ q^{\alpha+\gamma}, & q^{\alpha+\delta} & \end{matrix};q\right). \tag{4.14}$$

The polynomials (4.14) are orthogonal with respect to the discrete measure with mass points and corresponding weights (Ismail and Wilson (1982)):

$$\begin{aligned} x &= \alpha+\delta+k-1 \qquad \frac{(q^{\delta-\gamma+k+1})_\infty (q^{k+1})_\infty}{(q^{\alpha+\delta+k})_\infty (q^{\beta+\delta+k})_\infty} q^k q^{\alpha+\delta-1}(1-q), \\ x &= \alpha+\gamma+k-1 \qquad \frac{(q^{\gamma-\delta+k+1})_\infty (q^{k+1})_\infty}{(q^{\alpha+\gamma+k})_\infty (q^{\beta+\gamma+k})_\infty} q^k q^{\alpha+\gamma-1}(1-q),\end{aligned} \tag{4.15}$$

$k = 0, 1, 2, \cdots$. To obtain this result from (4.6), (4.8) we first set $a = q^\alpha, \cdots, d = q^\delta$. If $\mathrm{Re}(\alpha, \beta, \gamma, \delta) > 0$ and there are no double poles we can expand the integral $(\Phi_n, \Phi_m)_{a,b,c,d}$ by residues, using the simple poles of $w_{a,b,c,d}(z)$, (4.6), at $z = q^{\gamma+k}$, $z = q^{\delta+k}$. The result of this expansion is (4.15) with $z = q^x$. In particular, the dependence on ρ cancels out.

Our approach relating orthogonal polynomials to integrals of weight functions can be used to evaluate other important integrals. One class of such integrals can be conveniently studied through the change of variables given in the preceding paragraph:

$$a = q^\alpha, \quad b = q^\beta, \quad c = q^\gamma, \quad d = q^\delta, \quad z = q^x. \tag{4.16}$$

This change reduces q-difference equations for the weight function to ordinary difference equations. Indeed the operators (4.2) now take the form

$$\begin{aligned} \mu^{(\alpha,\beta,\gamma,\delta)} &= q^{-x}[(1 - q^{\alpha+x-\frac{1}{2}})\mathcal{T}_x^{\frac{1}{2}} - (1 - q^{x-\delta+\frac{1}{2}})\mathcal{T}_x^{-\frac{1}{2}}], \\ \tau^{(\alpha,\beta,\gamma,\delta)} &= q^{-x}[\mathcal{T}_x^{\frac{1}{2}} - \mathcal{T}_x^{-\frac{1}{2}}], \\ \mu* &= -q^{-x}[(1 - q^{\beta+x})\mathcal{T}_x^{\frac{1}{2}} - (1 - q^{x-\gamma})\mathcal{T}_x^{-\frac{1}{2}}] \end{aligned} \tag{4.17}$$

where $\mathcal{T}_x^s g(x) = g(x+s)$.

The orthogonal functions are now polynomials in q^x. We require that the inner product take the form

$$(f_1, f_2)_{\alpha,\beta,\gamma,\delta} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f_1(q^x) f_2(q^x) w_{\alpha,\beta,\gamma,\delta}(x)\ dx \tag{4.18}$$

where the contour in the complex x-plane will run from $-i\infty$ to $+i\infty$ so that decreasing sequences of poles for w lie on the left and increasing sequences of poles lie on the right. The condition that μ^* be the adjoint of μ now becomes

$$(\mu f, g)_{\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2}} = (f, \mu^* g)_{\alpha,\beta,\gamma,\delta} \tag{4.19}$$

with $f \in S_{\alpha,\beta,\gamma,\delta}$, $g \in S_{\alpha-\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2},\delta-\frac{1}{2}}$, where $S_{\alpha,\beta,\gamma,\delta}$ is the space of polynomials in q^x with inner product (4.18). The nonunique solution for w is

$$w_{\alpha,\beta,\gamma,\delta}(x) = \frac{(q^{x-\gamma+1})_\infty (q^{x-\delta+1})_\infty}{(q^{x+\alpha})_\infty (q^{x+\beta})_\infty} q^x H(\alpha, \beta, \gamma, \delta, x) \tag{4.20}$$

where H is an analytic function of its variables satisfying the periodicity properties

$$\begin{aligned} H(\alpha, \beta, \gamma, \delta, x) &= H(\alpha, \beta, \gamma, \delta, x+1), \\ H\left(\alpha - \frac{1}{2}, \beta - \frac{1}{2}, \gamma \pm \frac{1}{2}, \delta \mp \frac{1}{2}, x + \frac{1}{2}\right) & \\ = H\left(\alpha - \frac{1}{2}, \beta - \frac{1}{2}, \gamma \pm \frac{1}{2}, \delta \mp \frac{1}{2}, x + \frac{1}{2}\right) & \\ = H\left(\alpha - \frac{1}{2}, \beta + \frac{1}{2}, \gamma + \frac{1}{2}, \delta + \frac{1}{2}, x + \frac{1}{2}\right). & \end{aligned} \tag{4.21}$$

One solution of (4.21) is

$$H(\alpha,\beta,\gamma,\delta,x)=\frac{\sin\pi(\gamma-\delta)}{\sin\pi(\gamma-x)\sin\pi(\delta-x)}$$

so that the weight function becomes

$$w_{\alpha,\beta,\gamma,\delta}(x)=\frac{(q^{x-\gamma+1})_\infty(q^{x-\delta+1})_\infty q^x\sin\pi(\gamma-\delta)}{(q^{x+\alpha})_\infty(q^{x+\beta})_\infty\sin\pi(\gamma-x)\sin\pi(\delta-x)}. \tag{4.22}$$

Then the polynomials are

$$\Phi_n^{(\alpha,\beta,\gamma,\delta)}(q^x)={}_3\phi_2\left(\begin{matrix}q^{-n}, & q^{n+\alpha+\beta+\gamma+\delta-1}, & q^{\alpha+x}\\ q^{\alpha+\gamma}, & q^{\alpha+\delta} & \end{matrix};q\right),\qquad n=0,1,2,\ldots \tag{4.23}$$

and the eigenvalue equation is

$$\mu^*\mu\Phi_n^{(\alpha,\beta,\gamma,\delta)}=q^{\frac12-\gamma-\delta}(q^{\alpha+\delta-1}-q^{-n})(1-q^{\beta+\gamma+n})\Phi_n^{(\alpha,\beta,\gamma,\delta)}.$$

We have immediately

$$(\Phi_n^{(\alpha,\beta,\gamma,\delta)},\Phi_m^{(\alpha,\beta,\gamma,\delta)})_{\alpha,\beta,\gamma,\delta}=0,\qquad m\neq n$$

and

$$||1||^2_{\alpha-\frac12,\beta+\frac12,\gamma+\frac12,\delta-\frac12}=q^{-\frac12+\delta-\gamma}\frac{(1-q^{\beta+\gamma})}{(1-q^{\alpha+\delta-1})}||1||^2_{\alpha,\beta,\gamma,\delta}. \tag{4.24}$$

The symmetry of w in (α,β) leads to a similar recurrence. Also, the skew-symmetry of w in γ,δ yields a new recurrence. Finally, the orthogonality

$$(\Phi_1^{(\alpha,\beta,\gamma,\delta)},\Phi_0^{(\alpha,\beta,\gamma,\delta)})_{\alpha,\beta,\gamma,\delta}=0$$

leads to the recurrences

$$||1||^2_{\alpha+1,\beta,\gamma,\delta}=\frac{(1-q^{\alpha+\gamma})(1-q^{\alpha+\delta})}{(1-q^{\alpha+\beta+\gamma+\delta})}||1||^2_{\alpha,\beta,\gamma,\delta}$$

and

$$||1||^2_{\alpha,\beta,\gamma+1,\delta}=-\frac{q^{\delta-\gamma}(1-q^{\alpha+\gamma})(1-q^{\beta+\gamma})}{(1-q^{\alpha+\beta+\gamma+\delta})}||1||^2_{\alpha,\beta,\gamma,\delta}$$

with the solution

$$||1||^2_{\alpha,\beta,\gamma,\delta}=\frac{(q^{\alpha+\beta+\gamma+\delta})_\infty(q^{\delta-\gamma})_\infty(q^{1+\gamma-\delta})_\infty q^\gamma}{(q^{\alpha+\delta})_\infty(q^{\alpha+\gamma})_\infty(q^{\beta+\gamma})_\infty(q^{\beta+\delta})_\infty}M(\alpha,\beta,\gamma,\delta) \tag{4.25}$$

where M is an analytic function of its arguments, symmetric in (α,β) and in (γ,δ), satisfying the periodicity relations

$$\begin{aligned}M(\alpha,\beta,\gamma,\delta)&=M(\alpha+1,\beta,\gamma,\delta)=M(\alpha,\beta,\gamma+1,\delta)\\ &=M\left(\alpha-\frac12,\beta+\frac12,\gamma+\frac12,\delta-\frac12\right).\end{aligned} \tag{4.26}$$

To evaluate M we replace α by $\alpha+k$, k a positive integer, and rewrite the integral for $\|1\|^2_{\alpha+k,\beta,\gamma,\delta}$ in the form $(x=iy)$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}&\left[\frac{(q^{\alpha+k+\delta})_\infty(q^{\alpha+k+\delta})_\infty}{(q^{\alpha+\beta+\gamma+\delta+k})_\infty(q^{iy+\alpha+k})_\infty}\right]\frac{(q^{iy-\gamma+1})_\infty}{(q^{iy+\beta})_\infty}\\ &\cdot\frac{(q^{iy-\delta+1})_\infty q^{iy}\,dy}{\sin\pi(\gamma-iy)\sin\pi(\delta-iy)}\\ &=\frac{q^\gamma(q^{\delta-\gamma})_\infty(q^{1+\gamma-\delta})_\infty}{(q^{\beta+\gamma})_\infty(q^{\beta+\delta})_\infty}M(\alpha,\beta,\gamma,\delta).\end{aligned}\tag{4.27}$$

Notice that the right-hand side of (4.27) is independent of k and that the bracketed quantity on the left goes to 1 as $k\to+\infty$. From the Lebesgue dominated convergence theorem we conclude that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}&\frac{(q^{iy-\gamma+1})_\infty(q^{iy-\delta+1})_\infty q^{iy}}{(q^{iy+\beta})_\infty\sin\pi(\gamma-iy)\sin\pi(\delta-iy)}\\ &=\frac{q^\delta(q^{\delta-\gamma})_\infty(q^{1+\gamma-\delta})_\infty}{(q^{\beta+\gamma})_\infty(q^{\beta+\delta})_\infty}M(\alpha,\beta,\gamma,\delta).\end{aligned}$$

It follows immediately that M is independent of α and β. Now in (4.27) set $k=0$, $\alpha=1-\gamma$, $\beta=1-\delta$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{q^{iy}\sin\pi(\gamma-\delta)dy}{\sin\pi(\gamma-iy)\sin\pi(\delta-iy)}=\frac{M(\gamma,\delta)q^\gamma(1-q^{\delta-\gamma})}{(1-q)(q)_\infty}.\tag{4.28}$$

The rather elementary integral on the left-hand side of (4.28) can be easily evaluated by residues and the resulting geometric series summed to yield $M(\gamma,\delta)=(q)_\infty/\pi$. Thus

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}&\frac{(q^{iy-\gamma+1})_\infty(q^{iy-\delta+1})_\infty q^{iy}\sin\pi(\gamma-\delta)}{(q^{iy+\alpha})_\infty(q^{iy+\beta})_\infty\sin\pi(\gamma-iy)\sin\pi(\delta-iy)}\,dy\\ &=\frac{(q)_\infty q^\gamma(q^{\alpha+\beta+\gamma+\delta})_\infty(q^{\delta-\gamma})_\infty(q^{1+\gamma-\delta})_\infty}{\pi(q^{\alpha+\delta})_\infty(q^{\alpha+\gamma})_\infty(q^{\beta+\gamma})_\infty(q^{\beta+\delta})_\infty},\end{aligned}\tag{4.29}$$

which is Watson's q-analogue of Barnes' First Lemma (Watson (1910)).

By choosing other solutions H of relations (4.21) we can evaluate other integrals in the form (4.25), (4.26). However, in general one cannot evaluate M by the simple method we used for (4.29).

To see the relationship between these results and Barnes' Lemma we could, with care, let $q\to1-$ in expressions (4.12). However, it is more instructive to take the limit $q\to1-$ in expressions (4.1), (4.2) and then proceed step-by-step through the preceding argument. The details are given in Kalnins and Miller (1988). Making the change of variables (4.16) we see that (4.1) goes to the Hahn polynomials

$$\Phi_n^{(\alpha,\beta,\gamma,\delta)}(x)={}_3F_2\left(\begin{matrix}-n, & n+\alpha+\beta+\gamma+\delta-1, & \alpha+x\\ \alpha+\gamma, & \alpha+\delta & \end{matrix};1\right)\tag{4.30}$$

$$n=0,1,2,\ldots,\qquad \alpha,\beta,\gamma,\delta>0$$

in the limit. The q-difference recurrence operators (4.2), divided by $1-q$, go to ordinary difference operators. Following our usual procedure we find that the Hahn polynomials are orthogonal with respect to the complex inner product

$$(g_1, g_2)_{\alpha,\beta,\gamma,\delta} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} g_1(x)g_2(x)w_{\alpha,\beta,\gamma,\delta}(x)dx \tag{4.31}$$

with weight function

$$w_{\alpha,\beta,\gamma,\delta}(x) = \Gamma(\alpha+x)\Gamma(\beta+x)\Gamma(\gamma-x)\Gamma(\delta-x),$$

Suslov (1984), Atakishiyev and Suslov (1985). Using the difference operator recurrences obeyed by the Hahn polynomials, and Stirling's formula, we find

$$\begin{aligned}\|1\|^2_{\alpha,\beta,\gamma,\delta} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(\alpha+iy)\Gamma(\beta+iy)\Gamma(\gamma-iy)\Gamma(\delta-iy)dy\\ &= \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}.\end{aligned}$$

which is Barnes' First Lemma, Bailey (1964, p. 6), Slater (1966, p. 109).

Note that for the models considered in this section, the limits as $q \to 1$ are difference equations whereas in Section 3 they were differential equations. The same techniques are effective in both cases.

5. Askey-Wilson Polynomials.. The (continuous) Askey-Wilson polynomials are defined by the series, Askey and Wilson (1985),

$$\begin{aligned}\Phi_n^{(a,b,c,d)}(x) &= {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, a/z \\ ab, ac, ad\end{matrix}; q\right)\\ n = 0,1,2,\cdots, \quad x &= \frac{(z+z^{-1})}{2},\\ z = e^{i\theta}, \quad &|a|,|b|,|c|,|d|,|q| < 1.\end{aligned} \tag{5.1}$$

It is easy to verify from this explicit expression that Φ_n is a polynomial of order n in x. As shown in Kalnins and Miller (1989), these polynomials satisfy the recurrence relations

$$\begin{aligned}\mu\Phi_n^{(a,b,c,d)} &= (1-\frac{ab}{q})\Phi_n^{(aq^{-\frac12},bq^{-\frac12},cq^{\frac12},dq^{\frac12})}\\ \mu^*\Phi_n^{(aq^{-\frac12},bq^{-\frac12},cq^{\frac12},dq^{\frac12})} &= \frac{q^{-n}(1-q^n cd)(1-abq^{n-1})}{(1-\frac{ab}{q})}\Phi_n^{(a,b,c,d)}\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}\mu &= \frac{1}{z-z^{-1}}\left[-\frac{1}{z}(1-\frac{az}{q^{\frac12}})(1-\frac{bz}{q^{\frac12}})T_z^{\frac12} + z(1-\frac{a}{q^{\frac12}z})(1-\frac{b}{q^{\frac12}z})T_z^{-\frac12}\right]\\ \mu &\equiv \mu^{(a,b,c,d)}\end{aligned}$$

and

$$\mu^* = \mu^{(cq^{\frac12},dq^{\frac12},aq^{-\frac12},bq^{-\frac12})}$$

From these relations one can construct a partial q-difference equation such that the recurrences μ and μ^* arise as symmetries and the polynomials (5.1) appear as solutions via separation of variables, but we shall not do this here.

Consider μ and μ^* as mappings

$$\begin{aligned}\mu :& S_{a,b,c,d} \to S_{aq^{-\frac{1}{2}},bq^{-\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}}\\ \mu^* :& S_{aq^{-\frac{1}{2}},bq^{-\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}} \to S_{a,b,c,d}\end{aligned}$$

where $S_{a,b,c,d}$ is the space of polynomials in x, equipped with a complex inner product of the form

$$\begin{aligned}(f_1,f_2)_{a,b,c,d} &= \frac{1}{2\pi i}\oint_{|z|=1} f_1(z)f_2(z)w_{a,b,c,d}(z)\ \frac{dz}{z},\\ &f_1,f_2 \in S_{a,b,c,d}.\end{aligned}$$

(Here we consider $\Phi_n^{(a,b,c,d)}$ as an element of $S_{a,b,c,d}$.) Now we require that μ and μ^* are mutually adjoint with respect to this inner product:

$$\begin{aligned}(f,\mu g)_{aq^{-\frac{1}{2}},bq^{-\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}} &= (\mu^* f, g)_{a,b,c,d}\\ \forall f \in S_{aq^{-\frac{1}{2}},bq^{-\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}},&\quad g\in S_{a,b,c,d}.\end{aligned} \tag{5.3}$$

The "integration by parts" yields the necessary and sufficient condition for the weight function

$$\frac{w_{a,b,c,d}(qz)}{w_{a,b,c,d}(z)} = \frac{(1-az)(1-bz)(1-cz)(1-dz)(1-1/qz^2)(1-1/q^2z^2)}{(1-a/qz)(1-b/qz)(1-c/qz)(1-d/qz)(1-z^2)(1-qz^2)}$$

with the solution, unique up to multiplication by a constant,

$$w_{a,b,c,d}(z) = \frac{(z^2)_\infty(z^{-2})_\infty}{(az)_\infty(a/z)_\infty(bz)_\infty(b/z)_\infty(cz)_\infty(c/z)_\infty(dz)_\infty(d/z)_\infty}. \tag{5.4}$$

Thus, $\mu^*\mu$ is a self-adjoint operator with respect to this inner product and from (5.2)

$$\mu^*\mu\Phi_n = \lambda_n\Phi_n,\quad \lambda_n = q^{-n}(1-q^n cd)(1-abq^{n-1}),$$

which implies

$$(\Phi_n^{(a,b,c,d)},\Phi_m^{(a,b,c,d)})_{a,b,c,d} = 0 \quad \text{if } m\neq n. \tag{5.5}$$

Since the operators $\mu^*\mu$ and the weight function are symmetric with respect to the interchange $a\leftrightarrow b$ the orthogonal polynomials, which could be generated up to a constant normalization factor by the Gram-Schmidt process, must also exhibit this symmetry. Thus there exists a constant K_n such that

$$\Phi_n^{(b,a,c,d)}(x) = K_n\Phi_n^{(a,b,c,d)}(x).$$

Equating coefficients of x^n on both sides of this expression to obtain K_n, we find the formula of Sears (1951) for a balanced ${}_4\phi_3$:

(5.6)

$${}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}bacd,bz,b/z\\ ba,bc,bd\end{matrix};q\right) = \frac{(b/a)^n(ac)_n(ad)_n}{(bc)_n(bd)_n}{}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}abcd,az,a/z\\ ab,ac,ad\end{matrix};q\right).$$

Just as in the preceding sections, we can find recurences for the polynomials (5.1) which reduce the problem of normalizing the polynomials to the computation of the single norm $||1||^2_{a,b,c,d}$, Askey and Wilson (1985), Kalnins and Miller (1989). Expression (5.3) gives a useful recurrence for this norm. Setting $f(x) = g(x) = 1$ in (5.3) and using (5.1), (5.2) we find

$$||1||^2_{aq^{-\frac{1}{2}},bq^{-\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}} = \frac{(1-cd)}{(1-\frac{ab}{q})}||1||^2_{a,b,c,d}. \tag{5.7}$$

Symmetry in a, b, c, d yields a total of 6 recurences of this type. They imply

$$||1||^2_{a,b,c,d} = \frac{h(abcd,q)}{(ab)_\infty(ac)_\infty(ad)_\infty(bc)_\infty(bd)_\infty(cd)_\infty} \tag{5.8}$$

for some analytic function h. The evident relation

$$(\Phi_1^{(a,b,c,d)}, \Phi_0^{(a,b,c,d)})_{a,b,c,d} = 0$$

yields, through substitution of the explicit expansion (5.1) and use of the property

$$((az)_m(a/z)_m, 1)_{a,b,c,d} = ||1||^2_{aq^m,b,c,d},$$

the recurrence

$$||1||^2_{a,b,c,d} = \frac{(1-abcd)}{(1-ab)(1-ac)(1-ad)}||1||^2_{aq,b,c,d}.$$

This result and (5.8) give

$$h(abcd,q) = K(q)(abcd)_\infty \tag{5.9}$$

where $K(q)$ is to be determined. In the special case

$$a = 1, b = q^{\frac{1}{2}}, c = -1, d = -q^{\frac{1}{2}},$$

suggested by Askey, the norm is easily computed directly:

$$||1||^2_{1,q^{\frac{1}{2}},-1,-q^{\frac{1}{2}}} = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1. \tag{5.10}$$

Comparing this result with (5.8) and (5.9) we obtain a simple derivation of the formula (Askey and Wilson (1985))

$$\frac{1}{2\pi i}\oint_C \frac{(z^2)_\infty(z^{-2})_\infty}{(az)_\infty(a/z)_\infty(bz)_\infty(b/z)_\infty(cz)_\infty(c/z)_\infty(dz)_\infty(d/z)_\infty}\frac{dz}{z}$$
$$= \frac{2(abcd)_\infty}{(ab)_\infty(ac)_\infty(ad)_\infty(bc)_\infty(bd)_\infty(cd)_\infty(q)_\infty}.$$

We conclude with a few remarks about the symmetry-based method presented in this paper.

1. The method is quite general and can be applied to any family of polynomials satisfying differential or difference recurrence relations, e.g., the families studied in Miller (1970).
2. The ansatz for the form of the inner product is by no means unique, so a variety of (in general complex) inner products and computations of norms can be associated with a single family of orthogonal polynomials.
3. The method is not restricted to polynomials; it also applies to families of rational biorthogonal functions, e.g., Kalnins and Miller (1988). Wilson (1977) introduced a very general class of rational biorthogonal functions, expressed in terms of a very well poised and balanced ${}_{10}\phi_9$. It will be of interest to see if this class can be studied by the method given here.

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