MODELS OF Q-ALGEBRA REPRESENTATIONS: THE GROUP OF PLANE MOTIONS

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1. Introduction. This paper continues the study of function space models of irreducible representations of q-algebras. These algebras and models are motivated by recurrence relations satisfied by q-hypergeometric functions. Here, we consider the irreducible representations of the Lie algebra $m(2)$ of the group $M(2)$ of plane motions. We replace the usual exponential function mapping from the Lie algebra to the Lie group by the q-exponential mappings E_q and e_q. In place of the usual matrix elements on the group (arising from an irreducible representation) which are expressible in terms of Bessel functions of integer order, we find four types of matrix elements expressible in terms of the Jackson and the Hahn-Exton q-Bessel functions. These q-matrix elements do not satisfy group homomorphism properties, so they do not lead to addition theorems in the usual sense. However, they do satisfy orthogonality relations. (This was shown earlier by Koornwinder and Swarttouw [15].) Furthermore, in analogy with true group representation theory we can show that each of the four families of matrix elements determines a two-variable model for irreducible representations of $m(2)$. By q-exponentiating these models we get q-analogs of addition theorems for Bessel functions.

In §3 we use the definition of $m(2)$ as a true quantum algebra and take the tensor product of two infinite dimensional unitary irreducible representations of this quantum algebra. The tensor product decomposes into a direct sum of irreducible representations (rather than a direct integral as in the $q = 1$ case) and the decomposition is nonunique. Focussing our attention on the two simplest decompositions of the infinite-parameter family of choices, we compute the corresponding Clebsch-Gordan coefficients. Special cases of the unitarity relations for the CG coefficients are the q-Hankel orthogonality relations. Moreover, expressing the matrix elements of the "group" operators in the tensor product basis as a linear combination of the matrix elements in the basis adapted to the direct sum decomposition, via the CG coefficients, we obtain a generalization of Koelink's addition theorem for Hahn-Exton q-Bessel functions.

Our approach to the derivation and understanding of q-series identities is based on the study of q-algebras as q-analogs of Lie algebras, [7]. Essentially, we are

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attempting to find q-analogs of the theory relating Lie algebra and local Lie transformation groups. A similar approach has been adopted by Floreanini and Vinet [3-5]. This is an alternative to the elegant papers [10-17] which are based primarily on the theory of quantum groups. The main justification of the "local" approach is that it is more general; it applies to more general families of special functions than does the quantum group approach.

The notation used for q-series in this paper follows that of Gasper and Rahman [6]. We wish to thank the referee for suggestions that considerably improved the exposition and accuracy of §2.

One of the authors (W. M.) has worked with Dick Askey and Frank Olver, in an editorial capacity and as a friend and colleague, for some 25 years and respectfully dedicates this paper to the two of them.

2. Matrix elements of $m(2)$ representations. The three dimensional Lie algebra $m(2)$ is determined by its generators H, E_+, E_- which obey the commutation relations

$$\begin{aligned} &[H, E_+] = E_+, \quad [H, E_-] = -E_-, \\ &[E_+, E_-] = 0. \end{aligned} \tag{2.1}$$

The irreducible representations $Q(\omega, m_0)$ are characterized by the complex numbers ω and m_0 with $\omega \neq 0$ and $0 \leq \Re\ m_0 < 1$. The spectrum of H corresponding to $Q(\omega, m_0)$ is the set $S = \{m_0 + n : n \in Z\}$ and the complex representation space has basis vectors f_m, $m \in S$, such that

$$E_\pm f_m = \omega f_{m\pm 1},\ Hf_m = mf_m,\ E_+E_-f_m = \omega^2 f_m, \tag{2.2}$$

where $C \equiv E_+E_-$ is an invariant operator. Note that the representations $Q(\omega, m_0)$ and $Q(-\omega, m_0)$ are equivalent.

A simple realization of $Q(\omega, m_0)$ is given by the operators

$$H = m_0 + z\frac{d}{dz},\ E_+ = \omega z,\ E_- = \frac{\omega}{z} \tag{2.3}$$

acting on the space of all linear combinations of the functions z^n, z a complex variable, $n \in Z$, with basis vectors $f_m(z) = z^n$, where $m = m_0 + n$.

The representations $(\omega) \cong Q(\omega, 0)$ with $\omega > 0$ are of special interest. In this case we can introduce an inner product such that $< f_n, f_{n'} >= \delta_{nn'}$, $n, n' \in Z$. On the dense subspace $\mathcal{K}$ of all finite linear combinations of the basis vectors we have

$$< E_+f, f' > = < f, E_-f' >, \quad < Hf, f' > = < f, Hf' >, \tag{2.4}$$

for all $f, f' \in \mathcal{K}$, so $H = H^*$ and $E_+^* = E_-$. In terms of the operators (2.3) we can obtain a realization of (ω) and its Hilbert space structure by setting $z = e^{i\theta}$:

$$\begin{aligned} &H = -i\frac{d}{d\theta},\ E_+ = \omega e^{i\theta},\ E_- = \omega e^{-i\theta}, \\ &f_n(z) = e^{in\theta},\ < f, f' >= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{f'(e^{i\theta})}\ d\theta. \end{aligned} \tag{2.5}$$

Matrix elements $T_{m'm}$ of the complex motion group in the representation $Q(\omega, m_o)$ are typically defined by the expansions

$$e^{\beta E_+}e^{\alpha E_-}e^{\tau H}f_m = \sum_{m'=-\infty}^{\infty} T_{m'm}(\alpha,\beta,\tau)f_{m'} \tag{2.6}$$

with $m, m' \in S$, [3, 5, 19]. The group multiplication property of the operators on the left-hand side of (2.6) leads to addition theorems for the matrix elements. For convenience in the computations to follow we shall limit ourselves to the case where $\tau = 0$ and ω is real. For this case we have $T_{m'm}(\alpha,\beta) \equiv T_{m'm}(\alpha,\beta,0) \equiv T_{n'n}(\alpha,\beta)$ where $m = m_0 + n$, $m' = m_0 + n'$, i.e., the representations $Q(\omega, m_0)$ and (ω) have exactly the same matrix elements. (The extension of the matrix elements to complex ω will usually be obvious.)

With the q-analogs of the exponential function

$$\begin{aligned} e_q(x) &= \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}, \ |x|<1, \\ E_q(x) &= \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}x^k}{(q;q)_k} = (-x;q)_\infty, \end{aligned} \tag{2.7}$$

we employ the model (2.3) to define the following q-analogs of matrix elements of (ω):

$$\begin{aligned} &a)\ e_q(\beta E_+)e_q(\alpha E_-)f_n = \sum_{n'=-\infty}^{\infty} T^{(e,e)}_{n'n}(\alpha,\beta)f_{n'},\ |\omega\alpha|,|\omega\beta|<1 \\ &b)\ e_q(\beta E_+)E_q(\alpha E_-)f_n = \sum_{n'=-\infty}^{\infty} T^{(e,E)}_{n'n}(\alpha,\beta)f_{n'},\ |\omega\beta|<1 \\ &c)\ E_q(\beta E_+)e_q(\alpha E_-)f_n = \sum_{n'=-\infty}^{\infty} T^{(E,e)}_{n'n}(\alpha,\beta)f_{n'},\ |\omega\alpha|<1 \\ &d)\ E_q(\beta E_+)E_q(\alpha E_-)f_n = \sum_{n'=-\infty}^{\infty} T^{(E,E)}_{n'n}(\alpha,\beta)f_{n'}. \end{aligned} \tag{2.8}$$

Here, $0<q<1$ and $\alpha, \beta \in \mathbb{C}$. Since $E_+^* = E_-$ we have

$$\begin{aligned} T^{(e,e)}_{n'n}(\alpha,\beta) &= < e_q(\beta E_+)e_q(\alpha E_-)f_n,\ f_{n'} > \\ &= < f_n,\ e_q(\overline{\alpha}E_+)e_q(\overline{\beta}E_-)f_{n'} > \\ &= \overline{T^{(e,e)}_{nn'}(\overline{\beta},\overline{\alpha})}, \end{aligned} \tag{2.9}$$

$$T^{(e,E)}_{n'n}(\alpha,\beta) = \overline{T^{(E,e)}_{nn'}(\overline{\beta},\overline{\alpha})}, \tag{2.10}$$

$$T^{(E,E)}_{n'n}(\alpha,\beta) = \overline{T^{(E,E)}_{nn'}(\overline{\beta},\overline{\alpha})}. \tag{2.11}$$

Furthermore, since $e_q(x)E_q(-x) = 1$, we have the identities

$$(2.12)\qquad \sum_{\ell=-\infty}^{\infty} T^{(e,e)}_{n\ell}(\alpha,\beta)T^{(E,E)}_{\ell n'}(-\alpha,-\beta) = \delta_{nn'},\ |\omega\alpha|,|\omega\beta|<1$$

$$(2.13)\qquad \sum_{\ell=-\infty}^{\infty} T^{(e,E)}_{n\ell}(\alpha,\beta)T^{(E,e)}_{\ell n'}(-\alpha,-\beta) = \delta_{nn'},\ |\omega\alpha|,|\omega\beta|<1.$$

(Note that our operator derivations of these formulas and of many formulas to follow lead automatically to formal power series identities in the 'group parameters'. These identities must then be examined case by case to determine when the series are convergent as analytic functions of the group parameters.) Using the model (2.3) to treat (2.8) as generating functions for the matrix elements and computing the coefficients of $z^{n'}$ in the resulting expressions we obtain the explicit results:

$$(2.14)\qquad T^{(e,e)}_{n'n}(\alpha,\beta) = \frac{(q^{n-n'+1};q)_\infty(\alpha\omega)^{n-n'}}{(q;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}0, & 0\\ q^{n-n'+1} & \end{matrix};q,\alpha\beta\omega^2\right)$$

$$= \frac{(q^{n-n'+1};q)_\infty(\alpha\omega)^{n-n'}}{(q;q)_\infty(\alpha\beta\omega^2;q)_\infty}\,{}_0\phi_1\left(\begin{matrix}-\\ q^{n-n'+1}\end{matrix};q,\alpha\beta\omega^2 q^{n-n'+1}\right),$$

(2.15)

$$T^{(e,E)}_{n'n}(\alpha,\beta) = \frac{(q^{n-n'+1};q)_\infty(\alpha\omega)^{n-n'}}{(q;q)_\infty}q^{(n-n')(n-n'-1)/2}\,{}_1\phi_1\left(\begin{matrix}0\\ q^{n-n'+1}\end{matrix};q,-\alpha\beta\omega^2 q^{n-n'}\right)$$

$$= \frac{(q^{n'-n+1};q)_\infty(\beta\omega)^{n'-n}}{(q;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}0\\ q^{n'-n+1}\end{matrix};q,-\alpha\beta\omega^2\right),$$

$$(2.16)\qquad T^{(E,e)}_{n'n}(\alpha,\beta) = T^{(e,E)}_{nn'}(\beta,\alpha),$$

(2.17)

$$T^{(E,E)}_{n'n}(\alpha,\beta) = \frac{(q^{n-n'+1};q)_\infty(\alpha\omega)^{n-n'}}{(q;q)_\infty}q^{(n-n')(n-n'-1)/2}\,{}_0\phi_1\left(\begin{matrix}-\\ q^{n-n'+1}\end{matrix};q,\alpha\beta\omega^2 q^{n-n'}\right).$$

If $\alpha\beta \neq 0$ we can express these results in terms of the Jackson q-Bessel functions [6, page 25]

$$J^{(1)}_\nu(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(\frac{z}{2})^\nu\,{}_2\phi_1\left(\begin{matrix}0, & 0\\ q^{\nu+1} & \end{matrix};q,-\frac{z^2}{4}\right),$$

$$(2.18)\qquad J^{(2)}_\nu(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(\frac{z}{2})^\nu\,{}_0\phi_1\left(\begin{matrix}-\\ q^{\nu+1}\end{matrix};q,-\frac{z^2q^{\nu+1}}{4}\right),$$

$$J^{(2)}_\nu(z;q) = (-z^2/4;q)_\infty J^{(1)}_\nu(z;q),$$

and the Hahn-Exton q-Bessel function [15]

$$J_\nu(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} z^\nu {}_1\phi_1\left(\begin{matrix}0\\ q^{\nu+1}\end{matrix};q,qz^2\right). \tag{2.19}$$

Indeed, setting $\alpha = ire^{i\psi}$, $\beta = ire^{-i\psi}$, we see that in terms of the new complex coordinates $[r, e^{i\psi}]$ we have

$$\begin{aligned} T^{(e,e)}_{n'n}(\alpha,\beta) \equiv T^{(e,e)}_{n'n}[r,e^{i\psi}] &= \frac{e^{i(\frac{\pi}{2}+\psi)(n-n')}}{(-r^2\omega^2;q)_\infty} J^{(2)}_{n-n'}(2r\omega;q) \\ T^{(e,E)}_{n'n}[r,e^{i\psi}] &= e^{i(\frac{\pi}{2}-\psi)(n'-n)} q^{(n'-n)/2} J_{n'-n}(r\omega q^{-\frac{1}{2}};q) \\ T^{(E,e)}_{n'n}[r,e^{i\psi}] &= e^{i(\frac{\pi}{2}+\psi)(n-n')} q^{(n-n')/2} J_{n-n'}(r\omega q^{-\frac{1}{2}};q) \\ T^{(E,E)}_{n'n}[r,e^{i\psi}] &= e^{i(\frac{\pi}{2}+\psi)(n-n')} q^{(n-n')^2/2} J^{(2)}_{n-n'}(2r\omega q^{-\frac{1}{2}};q). \end{aligned} \tag{2.20}$$

(Note that $J_{-n}(z;q) = (-1)^n q^{n/2} J_n(zq^{n/2};q)$, $J^{(2)}_{-n}(z;q) = (-1)^n J^{(2)}_n(z;q)$ for integer n.)

The matrix elements $T_{n'n}(\alpha,\beta)$ themselves define models of the representations (ω). We can see this directly from the commutation relations (2.1). It is a simple consequence of these relations and $e_q(x) = (x;q)^{-1}_\infty$, $E_q(x) = (-x;q)_\infty$ that

$$\begin{aligned} e_q(\beta E_+)e_q(\alpha E_-)E_+ &= \frac{1}{\beta}(I - T_\beta)e_q(\beta E_+)e_q(\alpha E_-) \\ e_q(\beta E_+)e_q(\alpha E_-)E_- &= \frac{1}{\alpha}(I - T_\alpha)e_q(\beta E_+)e_q(\alpha E_-) \end{aligned} \tag{2.21}$$

where I is the identity operator and $T_\beta g(\alpha,\beta) = g(\alpha,\beta q)$ for a function $g(\alpha,\beta)$. Thus

$$\begin{aligned} \omega T^{(e,e)}_{n',n+1}(\alpha,\beta) &= < e_q(\beta E_+)e_q(\alpha E_-)E_+ f_n, f_{n'} > \\ &= \frac{1}{\beta}(I - T_\beta)T^{(e,e)}_{n'n}(\alpha,\beta), \\ \omega T^{(e,e)}_{n',n-1}(\alpha,\beta) &= \frac{1}{\alpha}(I - T_\alpha)T^{(e,e)}_{n'n}(\alpha,\beta). \end{aligned} \tag{2.22}$$

Similarly, the relations

$$\begin{aligned} e_q(\beta E_+)E_q(\alpha E_-)E_+ &= \frac{1}{\beta}(I - T_\beta)e_q(\beta E_+)E_q(\alpha E_-) \\ e_q(\beta E_+)E_q(\alpha E_-)E_- &= \frac{q}{\alpha}(T_\alpha^{-1} - I)e_q(\beta E_+)E_q(\alpha E_-) \end{aligned} \tag{2.23}$$

yield

$$\begin{aligned} \omega T^{(e,E)}_{n',n+1}(\alpha,\beta) &= \frac{1}{\beta}(I - T_\beta)T^{(e,E)}_{n'n}(\alpha,\beta), \\ \omega T^{(e,E)}_{n',n-1}(\alpha,\beta) &= \frac{q}{\alpha}(T_\alpha^{-1} - I)T^{(e,E)}_{n'n}(\alpha,\beta). \end{aligned} \tag{2.24}$$

In the same way we find

$$
\begin{aligned}
\omega T^{(E,e)}_{n',n+1}(\alpha,\beta) =& \frac{q}{\beta}(T_\beta^{-1}-I)T^{(E,e)}_{n'n}(\alpha,\beta),\\
\omega T^{(E,e)}_{n',n-1}(\alpha,\beta) =& \frac{1}{\alpha}(I-T_\alpha)T^{(E,e)}_{n'n}(\alpha,\beta),
\end{aligned}
\tag{2.25}
$$

and

$$
\begin{aligned}
\omega T^{(E,E)}_{n',n+1}(\alpha,\beta) =& \frac{q}{\beta}(T_\beta^{-1}-I)T^{(E,E)}_{n'n}(\alpha,\beta),\\
\omega T^{(E,E)}_{n',n-1}(\alpha,\beta) =& \frac{q}{\alpha}(T_\alpha^{-1}-I)T^{(E,E)}_{n'n}(\alpha,\beta).
\end{aligned}
\tag{2.26}
$$

Furthermore, induction with respect to $k+\ell$ yields $[H,\alpha^\ell\beta^k E_+^k E_-^\ell]=(k-\ell)\alpha^\ell\beta^k E_+^k E_-^\ell$ and this implies

$$[H,e_q(\beta E_+)e_q(\alpha E_-)]=(\beta\partial_\beta-\alpha\partial_\alpha)e_q(\beta E_+)e_q(\alpha E_-),$$

so

$$
\begin{aligned}
(n'-n)T^{(e,e)}_{n'n}(\alpha,\beta) =&< e_q(\beta E_+)e_q(\alpha E_-)f_n, Hf_{n'}> - < e_q(\beta E_+)e_q(\alpha E_-)Hf_n, f_{n'}>\\
=&< [H,e_q(\beta E_+)e_q(\alpha E_-)]f_n, f_{n'}> = (\beta\partial_\beta-\alpha\partial_\alpha)T^{(e,e)}_{n'n}(\alpha,\beta).
\end{aligned}
$$

Similarly,

$$(n'-n)T_{n'n}(\alpha,\beta)=(\beta\partial_\beta-\alpha\partial_\alpha)T_{n'n}(\alpha,\beta) \tag{2.27}$$

for all the remaining cases.

Thus, denoting operators

$$\tilde{E}^x=\frac{1}{x}(I-T_x),\quad \hat{E}^x=\frac{q}{x}(T_x^{-1}-I),\quad \tilde{H}=\alpha\partial_\alpha-\beta\partial_\beta \tag{2.28}$$

we see that the following sets of operators and basis functions each define a two variable realization of relations (2.2), hence a realization of the representation (ω):

$$
\begin{aligned}
&a)\quad \tilde{E}^\beta_+,\tilde{E}^\alpha_-,\tilde{H},\quad f_{-n'+n}=T^{(e,e)}_{n'n}(\alpha,\beta)\\
&b)\quad \tilde{E}^\beta_+,\hat{E}^\alpha_-,\tilde{H},\quad f_{-n'+n}=T^{(e,E)}_{n'n}(\alpha,\beta)\\
&c)\quad \hat{E}^\beta_+,\tilde{E}^\alpha_-,\tilde{H},\quad f_{-n'+n}=T^{(E,e)}_{n'n}(\alpha,\beta)\\
&d)\quad \hat{E}^\beta_+,\hat{E}^\alpha_-,\tilde{H},\quad f_{-n'+n}=T^{(E,E)}_{n'n}(\alpha,\beta).
\end{aligned}
\tag{2.29}
$$

Moreover, from the explicit expressions (2.14) for the matrix elements $T_{n'n}(\alpha,\beta)$ it is easy to verify that relations (2.2) remain valid for $-n'=m_0$ a complex number with $0\le\Re\, m_0<1$, ω a nonzero complex number and n an integer. Thus the four families (2.29) provide realizations of all the representations $Q(\omega,m_0)$, although for $m_0\neq 0$ the basis functions are no longer matrix elements. (Indeed, each of the families $a)-d)$ defines two realizations of $Q(\omega,m_0)$ for $m_0\neq 0$, one where the q-Bessel functions are expressed in terms of the subscript $n-n'$ and one where they are expressed in terms of $n'-n$. For $m_0=0$ these expressions coincide.)

Before passing on to a deeper consideration of these models we note that the identities (2.12) and (2.13) for $\alpha\beta \neq 0$ are essentially the following identities for q-Bessel functions [15, (2.14),(2.10)]:

$$\sum_{\ell=-\infty}^{\infty} J^{(2)}_{\ell-n}(r;q)q^{\ell^2/2}J^{(2)}_{\ell}(rq^{-\frac{1}{2}};q) = \delta_{n0}(-r^2/4;q)_\infty, \tag{2.30a}$$

$$\sum_{\ell=-\infty}^{\infty} q^\ell J_{\ell+n}(r;q)J_{\ell+n'}(r;q) = \delta_{nn'}q^{-n}, \quad |r| < q^{-\frac{1}{2}}. \tag{2.30b}$$

The identities (2.30a) can be further generalized by considering matrix element relations of the form

$$S_{n'n}(\beta,\gamma) = \sum_{\ell=-\infty}^{\infty} T^{(e,e)}_{n'\ell}(\alpha,\beta)T^{(E,E)}_{\ell n}(-\alpha,\gamma) \tag{2.12$'$}$$

where

$$\begin{aligned} S_{n'n}(\beta,\gamma) &=< E_q(\gamma E_+)e_q(\beta E_+)f_n, f_{n'} > \\ &= \begin{cases} 0 \quad \text{if } n'-n<0 \\ \frac{(\omega\beta)^{n'-n}}{(q;q)_{n'-n}}(-\frac{\gamma}{\beta};q)_{n'-n} \quad \text{otherwise.} \end{cases} \end{aligned}$$

(There are relations of a similar type for the E_- operator.) These identities yield a q-analog of the Hansen-Lommel orthogonality relations [10, 11]. Indeed, if we set $\alpha = \beta = ir, \gamma = -irq^{n'-n-1}$ in (2.12$'$) we see that the left-hand side of this equation vanishes unless $n = n'$. Furthermore with the choice $2r\omega = zq^{n'/2}, m = -\ell$ the expression becomes Koelink's formula

$$\sum_{m=-\infty}^{\infty} J^{(2)}_{m+n'}(zq^{n'/2};q)q^{(m+n')(m+n'-1)/4}J^{(2)}_{m+n}(zq^{n/2};q)q^{(m+n)(m+n-1)/4} = \delta_{nn'}(-z^2q^n/4;q)_\infty.$$

As additional examples the identities

$$T^{(E,E)}_{n'n}(\gamma,\alpha) = \sum_{\ell=-\infty}^{n'} S_{n'\ell}(-\beta,\alpha)T^{(E,E)}_{\ell n}(\gamma,\beta),\ T^{(e,E)}_{n'n}(\alpha,\beta) = \sum_{\ell=-\infty}^{n'} S_{n'\ell}(\beta,\gamma)T^{(e,E)}_{\ell n}(\alpha,-\gamma)$$

yield q-analogs of the Lommel relations:

$$\begin{aligned} x^{-m}J^{(2)}_m(xz;q) &= \sum_{j=0}^{\infty}(\frac{q^m z}{2})^j q^{j(j+1)/2}\frac{(x^2;q)_j}{(q;q)_j}J^{(2)}_{m+j}(z;q), \\ x^m J_m(\frac{z}{x}q^{m/2};q) &= \sum_{j=0}^{\infty}(-\frac{zq^{1/2}}{x^2})^j\frac{(x^2;q)_j}{(q;q)_j}J_{m+j}(zq^{(m+j)/2};q) \end{aligned}$$

where m is an integer.

We can use the relations (2.22), (2.24)-(2.26) to derive addition theorems for the basis functions $T_{n'n}(\alpha,\beta)$, including the cases where $n'=-m_0$ is a complex number. First of all, note that (for m,s complex numbers)

$$e_q(\gamma\hat{E}^x)x^m = x^m\frac{(-q^{-m+1}\gamma/x;q)_\infty}{(-q\gamma/x;q)_\infty}, \quad E_q(\xi\tilde{E}^x)x^s = x^s\frac{(-\xi/x;q)_\infty}{(-q^s\xi/x;q)_\infty}. \tag{2.31}$$

Now consider the operator $e_q(\gamma\hat{E}^\beta_+)E_q(\xi\tilde{E}^\alpha_-)$ applied to the basis function $f_{m_0}=T^{(E,e)}_{-m_0,0}(\alpha,\beta)=T^{(e,E)}_{0,-m_0}(\beta,\alpha)$:

$$e_q(\gamma\hat{E}^\beta_+)E_q(\xi\tilde{E}^\alpha_-)T^{(E,e)}_{-m_0,0}(\alpha,\beta) \tag{2.32}$$

$$= (\alpha\omega)^{m_0}\frac{(q^{m_0+1};q)_\infty}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}\omega^{2k}\alpha^k\beta^k(-q^{-k+1}\gamma/\beta;q)_\infty(-\xi/\alpha;q)_\infty}{(q^{m_0+1};q)_k(q;q)_k(-q\gamma/\beta;q)_\infty(-q^{m_0+k}\xi/\alpha;q)_\infty}$$

$$= (\alpha\omega)^{m_0}\frac{(q^{m_0+1};q)_\infty(-\xi/\alpha;q)_\infty}{(q;q)_\infty(-q^{m_0}\xi/\alpha;q)_\infty}{}_2\phi_1\left(\begin{matrix}-q^{m_0}\xi/\alpha, & -\beta/\gamma \\ q^{m_0+1} & \end{matrix};q,\alpha\gamma\omega^2\right),$$

convergent for $|q^{m_0}\xi/\alpha|<1$, $|\alpha\gamma\omega^2|<1$. Since

$$e_q(\gamma\hat{E}^\beta_+)E_q(\xi\tilde{E}^\alpha_-)T^{(E,e)}_{-m_0,0}(\alpha,\beta) = \sum_{n=-\infty}^{\infty}T^{(e,E)}_{n0}(\xi,\gamma)T^{(E,e)}_{-m_0,n}(\alpha,\beta) \tag{2.33}$$

from (2.8b), or

$$\frac{x^{m_0}(q^{m_0+1};q)_\infty(qz/xy;q)_\infty}{(q;q)_\infty(q^{m_0+1}z/xy;q)_\infty}{}_2\phi_1\left(\begin{matrix}q^{m_0+1}z/xy, & qx/yz \\ q^{m_0+1} & \end{matrix};q,xyz\right)$$

$$= \sum_{n=-\infty}^{\infty}y^nJ_n(z;q)J_{m_0+n}(x;q) \tag{2.33$'$}$$

$$|q^{m_0+1}z/xy|<1, \quad |xyz|<1,$$

this can be considered as an addition theorem for the $T^{(E,e)}_{-m_0,n}$ basis functions which generalizes the Lommel and Graf addition theorems for ordinary Bessel functions, [15 (4.5), 18, 20, 22]. In [15 (4.10)] (for integer m_0) and in [5] (for complex m_0) the addition theorem corresponding to

$$E_q(\gamma\tilde{E}^\beta_+)E_q(\xi\tilde{E}^\alpha_-)T^{(e,e)}_{-m_0,0}(\alpha,\beta) = \sum_{n=-\infty}^{\infty}T^{(E,E)}_{n0}(\xi,\gamma)T^{(e,e)}_{-m_0,n}(\alpha,\beta) \tag{2.34}$$

is worked out. The result is

$$(\frac{x}{2})^{m_0}\frac{(qz/xy;q)_\infty(q^{m_0+1};q)_\infty}{(q^{m_0+1}z/xy;q)_\infty(q;q)_\infty}{}_2\phi_1\left(\begin{matrix}q^{m_0+1}z/xy, & yz/x \\ q^{m_0+1} & \end{matrix};q,-\frac{x^2}{4}\right)$$

$$= \sum_{n=-\infty}^{\infty}q^{n(n-1)/2}y^nJ^{(2)}_n(z;q)J^{(1)}_{m_0+n}(x;q) \tag{2.34$'$}$$

$$|q^{m_0+1}z/xy| < 1, \quad |x^2| < 4.$$

To better understand, and to extend, identities (2.33) we note that, formally at least, the operators $e_q(\gamma\hat{E}_+^\beta)$, $E_q(\xi\tilde{E}_-^\alpha)$ and H are symmetry operators for the q-difference equation

$$\hat{E}_+^\beta \tilde{E}_-^\alpha \Psi(\alpha,\beta) = \ell^2\Psi(\alpha,\beta). \tag{2.35}$$

That is, these operators map solutions of the equations to new solutions. Thus if $\Psi(\alpha,\beta)$ is a solution then so are $e_q(\gamma\hat{E}_+^\beta)E_q(\xi\tilde{E}_-^\alpha)\Psi(\alpha,\beta) = \Psi'(\alpha,\beta)$ and $H\Psi(\alpha,\beta)$ solutions. It is easy to check that the only solution Ψ_λ of (2.35) and $H\Psi_\lambda = \lambda\Psi_\lambda$ for complex (noninteger) λ, such that $r^{-\lambda}\Psi_\lambda$ is analytic in r at $r=0$, is $\Psi_\lambda(\alpha,\beta) \equiv \Psi_\lambda[r,t] = t^\lambda J_\lambda(r\omega q^{-1/2};q)$, unique up to multiplication by a constant. Here, $\alpha = irt$, $\beta = ir/t$. (For $\lambda = n$ an integer, the only solution Ψ_λ analytic in r at $r=0$ is $\Psi_n = t^nJ_n(r\omega q^{-1/2};q)$.) Thus if $\Psi'_\lambda(\alpha,\beta)$ is analytic in the variables $R=r$, $S=rt$ at $R=0$ then it must have an expansion of the form, [9]

$$\Psi' = \sum_\lambda c_\lambda t^\lambda J_\lambda(r\omega q^{-1/2};q) \tag{2.36}$$

where the c_λ are complex constants. Expression (2.33) is a special case of this expansion.

It is easy to check from (2.31) that the operators $e_q(\gamma\hat{E}_+^\beta)$, $E_q(\xi\tilde{E}_-^\alpha)$ do commute with $\hat{E}_+^\beta$ and $\tilde{E}_-^\alpha$ when acting on monomials $\beta^m\alpha^s$ so, modulo convergence problems, they are symmetry operators mapping analytic solutions of $\hat{E}_+^\beta\tilde{E}_-^\alpha\Psi = \ell^2\Psi$ to analytic solutions. Now, instead of the definition $E_q(\xi\tilde{E}^x)x^s = x^s(-\xi/x;q)_\infty/(-q^s\xi/x;q)_\infty$ let us adopt the definition ($\xi \neq 0$)

$$E_q(\xi\tilde{E}_-^x)'x^s = \xi^s q^{s(s-1)/2}\frac{(-q^{-s+1}x/\xi;q)_\infty}{(-qx/\xi;q)_\infty}. \tag{2.37}$$

For s an integer these definitions coincide, but they are distinct for s complex. It is again easy to verify that

$$E_q(\xi\tilde{E}_-^\alpha)'\tilde{E}_-^\alpha\alpha^s = \tilde{E}_-^\alpha E_q(\xi\tilde{E}_-^\alpha)'\alpha^s$$

so $E_q(\xi\tilde{E}_-^\alpha)'$ is also a symmetry operator. As an example of the use of this observation consider

(2.38)

$$\begin{aligned}\Psi' =& E_q(\xi\tilde{E}_-^\alpha)'T^{(E,e)}_{-m_0,0}(\alpha,\beta)\\ =& \frac{(\xi\omega)^{m_0}(q^{m_0+1};q)_\infty q^{m_0(m_0-1)/2}(-q^{-m_0+1}\alpha/\xi;q)_\infty}{(q;q)_\infty(-q\alpha/\xi;q)_\infty}{}_1\phi_1\left(\begin{matrix}-q^{m_0}\xi/\alpha\\ q^{m_0+1}\end{matrix};q,-\alpha\beta\omega^2\right),\\ & |q\alpha/\xi| < 1.\end{aligned}$$

Thus we have

(2.39)

$$\frac{(-q^{-m_0+1}irt/\xi;q)_\infty}{(-qirt/\xi;q)_\infty}{}_1\phi_1\left(\begin{matrix}-q^{m_0}\xi/irt\\ q^{m_0+1}\end{matrix};q,r^2\omega^2\right) = \sum_{n=-\infty}^{\infty} c_n t^n J_n(r\omega q^{-1/2};q).$$

Now set t=s/r and let $r \to 0$ in (2.39). We obtain $c_n = (-iq^{3/2}/\xi\omega)^n (q^{-m_0};q)_\infty/(q^{-m_0+n};q)_\infty$ for $n = 0, 1, \cdots$. Similarly, if we set $t = sr$ and let $r \to 0$ we find the same expression for c_n with $-n = 0, 1, \cdots$. Thus

$$\frac{(q^{-m_0}x;q)_\infty}{(x;q)_\infty}{}_1\phi_1\left(\begin{matrix} q^{m_0+1}/x \\ q^{m_0+1}\end{matrix};q,qz^2\right) = \sum_{n=-\infty}^{\infty}(\frac{qx}{z})^n \frac{(q^{-m_0};q)_\infty}{(q^{-m_0+n};q)_\infty}J_n(z;q),$$

$$0<|x|<1. \tag{2.40}$$

Similarly, applying the symmetry operator $E_q(\xi\tilde{E}_-^\alpha)'$ to the basis function $T^{(e,e)}_{-m_0,0}$ we obtain the identity

$$\frac{(q^{-m_0}ry;q)_\infty}{(ry;q)_\infty}{}_2\phi_1\left(\begin{matrix} q^{m_0+1}/ry, & 0 \\ q^{m_0+1} & \end{matrix};q,-r^2\right) = \sum_{n=-\infty}^{\infty} y^n \frac{(q^{-m_0};q)_\infty}{(q^{-m_0+n};q)_\infty}J^{(1)}_n(2r;q),$$

$$0<|y|, \quad |r^2|<1, \quad |ry|<1. \tag{2.41}$$

Also, we can apply the symmetry operator

$$e_q(\gamma\hat{E}_+^\beta)'\beta^m = \gamma^m q^{-m(m-1)/2}\frac{(-\beta/\gamma;q)_\infty}{(-q^m\beta/\gamma;q)_\infty},$$

rather than $e_q(\gamma\hat{E}_+^\beta)$, to obtain new identities.

3. Tensor products. In addition to the standard definition of the tensor product of two irreducible representations (ω_1) and (ω_2) of $m(2)$ there is a definition given by the nontrivial Hopf algebra structure in which the coproduct is [1, 4, 11]

$$\begin{aligned} F_+ =& \Delta(E_+) = E_+ \otimes q^{-\frac{1}{2}H} + q^{\frac{1}{2}H}\otimes E_+ \\ F_- =& \Delta(E_-) = E_- \otimes q^{-\frac{1}{2}H} + q^{\frac{1}{2}H}\otimes E_- \\ L =& \Delta(H) = H\otimes I + I\otimes H. \end{aligned} \tag{3.1}$$

The operators $F_\pm$, L satisfy the same commutation relations as the operators $E_\pm$, H:

$$[L, F_\pm] = \pm F_\pm, \quad [F_+, F_-] = 0. \tag{3.2}$$

The standard definition is obtained by letting $q \to 1$ in (3.1). Each irreducible representation (ω_1), (ω_2) is defined on $L_2[0, 2\pi]$ by the prescription (2.5). To make sense of the operators (3.1) on a dense subspace of the tensor product space $L[0, 2\pi] \otimes L[0, 2\pi]$ we proceed as follows. The Hilbert space inner product is

$$< f, g >= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(\theta_1,\theta_2)\overline{g(\theta_1,\theta_2)}\ d\theta_1 d\theta_2.$$

An orthonormal basis is $\{f_{n_1n_2} = e^{i(n_1\theta_1+n_2\theta_2)}, n_j = 0, \pm 1, \pm 2, \cdots\}$. In terms of this basis a general element f of the Hilbert space can be expanded as

$$f(\theta_1, \theta_2) \sim \sum_{n_1,n_2=-\infty}^{\infty} a_{n_1n_2} e^{i(n_1\theta_1+n_2\theta_2)} = \sum_{n,k=-\infty}^{\infty} A_{nk} e^{i(n\Theta+k\Phi)}, \tag{3.3}$$

where $\sum |a_{n_1n_2}|^2 < \infty$, $k = n_1 + n_2$, $n = n_1$, $\Theta = \theta_1 - \theta_2$, $\Phi = \theta_2$ and $A_{nk} = a_{n_1n_2}$. Now set $z = e^{i\Theta}$, $t = e^{i\Phi}$. Since $L = \Delta(H) = -i\partial_\Phi = t^{-1}\partial_t$ it is clear that the eigenvalues of L are $k_0 = 0, \pm 1, \pm 2, \cdots$ and that the corresponding eigenspace of $L[0, 2\pi] \otimes L[0, 2\pi]$ consists of those square integrable functions f for which $A_{nk} = 0$ if $k \neq k_0$. The operators (3.2) make sense on the dense subspace $\mathcal{K}$ of the Hilbert space consisting of finite sums of functions $f[z,t] = \sum f_k[z]t^k$, $(z,t) \in \mathcal{C}_2$, ($f[e^{i\Theta}, e^{i\Phi}] = f(\theta_1, \theta_2)$ belongs to the Hilbert space) such that each $f_k[z]$ is analytic except possibly for an essential singularity at $z = 0$ and a countable number of poles in the complex plane (not on the unit circle $|z| = 1$). Then

$$\begin{aligned} F_+ f[z,t] =& \omega_1 zt f[zq^{1/2}, tq^{-1/2}] + \omega_2 t f[zq^{1/2}, t] \\ F_- f[z,t] =& \frac{\omega_1}{zt} f[zq^{1/2}, tq^{-1/2}] + \frac{\omega_2}{t} f[zq^{1/2}, t] \\ L f[z,t] =& \frac{1}{t}\partial_t f[z,t]. \end{aligned} \tag{3.4}$$

We further require that $F_\pm f[e^{i\Theta}, e^{i\Phi}]$ and $F_+F_- f[e^{i\Theta}, e^{i\Phi}]$ are square integrable. Formally, the invariant operator $\Delta(E_+E_-) = F_+F_-$ takes the form

$$F_+F_- f[z,t] = \left(\omega_1^2 T_t^{-1} + \omega_1\omega_2(zq^{1/2} + \frac{1}{zq^{1/2}})T_t^{-1/2} + \omega_2^2 I\right) T_z f[z,t].$$

Thus, if f satisfies

$$F_+F_- f = \lambda f, \quad Lf = kf \tag{3.5}$$

it should have the form $f[z,t] = g_k[z]t^k$ where

$$(\omega_1 q^{-k/2} + \omega_2 z q^{1/2})(\omega_1 q^{-k/2} + \frac{\omega_2}{zq^{1/2}}) g_k[zq] = \lambda g_k[z]. \tag{3.6}$$

In order to decompose the tensor product representation into irreducible representations it is sensible to require that the domain of definition of the above operators should be adjusted so that on this domain F_+ is the adjoint of F_- and L, $K = F_+F_-$ are symmetric. Then we extend the domain to make K self-adjoint. Let us consider the restriction K_k of K to the subspace of functions of the form $f_k[z,t] = g_k[z]t^k$, so that $Lf_k = kf_k$. Then

$$K_k = (\omega_1 q^{-k/2} + \omega_2 z q^{1/2})(\omega_1 q^{-k/2} + \frac{\omega_2}{zq^{1/2}}) T_z \tag{3.7}$$

and the induced Hilbert space inner product is

$$(g, g') = \frac{1}{2\pi i} \oint_{|z|=1} g[z]\overline{g}'[z^{-1}] \; \frac{dz}{z} \tag{3.8}$$

where $\overline{g[z]} = \overline{g}[\overline{z}]$. It follows that

$$(K_k g, g') = (g, K_k g') + R(g, g') \tag{3.9}$$

where $R(g,g')$ is the sum of residues of the function $z^{-1}(\omega_1 q^{-k/2}+\omega_2 z q^{-1/2})(\omega_1 q^{-k/2}+\omega_2 q^{1/2}/z)g[z]\overline{g}'[q/z]$ in the annulus $q < |z| < 1$. Thus K_k will be symmetric on a domain $\mathcal{D}$ such that $R(g,g') = 0$ for all $g, g' \in \mathcal{D}$. This shows that the role of boundary conditions for symmetric operators is played here by the placement of zeros and poles for the elements of $\mathcal{D}$.

There is an infinite parameter family of self-adjoint operators associated with the symbol (3.7). We shall focus on those two of the self-adjoint operators (and their spectral resolutions) that are the simplest in structure. The first corresponds to the orthonormal basis of eigenvectors

$$h_p^{(1)}[z] = \frac{(-q^{(1+k)/2}\omega_2 z/\omega_1; q)_\infty}{(-q^{(1+k)/2}\omega_2/\omega_1 z; q)_\infty} z^{p+k}, \quad p = 0, \pm 1, \cdots \tag{3.10}$$

with eigenvalues $\lambda_p^{(1)} = \omega_1^2 q^p$. The second has an orthonormal basis of eigenvectors

$$h_p^{(2)}[z] = \frac{(-q^{(1-k)/2}\omega_1 z/\omega_2; q)_\infty}{(-q^{(1-k)/2}\omega_1/\omega_2 z; q)_\infty} z^{p}, \quad p = 0, \pm 1, \cdots \tag{3.11}$$

with eigenvalues $\lambda_p^{(2)} = \omega_2^2 q^p$. Setting $f_{pk}^{(j)}[z,t] = h_p^{(j)}[z]t^k$, $j = 1, 2$, we have

$$\begin{aligned} F_\pm f_{pk}^{(j)} &= \omega_j q^{p/2} f_{p,k\pm 1}^{(j)} \\ L f_{pk}^{(j)} &= k f_{pk}^{(j)}, \quad K f_{pk}^{(j)} = \omega_j^2 q^p f_{pk}^{(j)}. \end{aligned} \tag{3.12}$$

Thus

$$(\omega_1) \otimes_q^j (\omega_2) \equiv \sum_{p=-\infty}^{\infty} \oplus (\omega_j q^{p/2}), \tag{3.13}$$

i.e., in each case the tensor product decomposes into a direct sum of irreducible representations. (This is dramatically different from the case $q = 1$ where the tensor product representation is unique and the spectrum of K is continuous, [22, page 224].)

As an indication of the multiplicity of possible self-adjoint operators and resulting spectral decompositions, consider the function

$$S_\lambda[z] = \frac{(-q^{1/2}\lambda/z; q)_\infty(-q^{1/2}z/\lambda; q)_\infty}{(-q^{1/2}\lambda z; q)_\infty(-q^{1/2}/\lambda z; q)_\infty} \tag{3.14}$$

for λ a nonzero real constant. Note that $|S_\lambda[z]| = 1$ for $|z| = 1$. Now $S_\lambda[zq] = \lambda^2 S_\lambda[z]$ so the sets $\{S_\lambda[z] f_{pk}^{(j)}\}$, $j = 1, 2$, are each orthonormal bases of $L_2[0, 2\pi] \otimes L_2[0, 2\pi]$ consisting of eigenvectors of K with eigenvalues $\lambda^2\omega_j^2 q^p$. Furthermore, the sets $\{f_{pk}^{(j)\prime} = S_\lambda[z^2] f_{pk}^{(j)}\}$ satisfy relations (3.12) with ω_j replaced by $\omega_j\lambda^2$.

The Clebsch-Gordan coefficients for the tensor product corresponding to the spectral resolution (3.10) are defined by

$$(3.15)\quad f^{(1)}_{pk}[z,t] = f^{(1)}_{pk}(\theta_1,\theta_2) = \sum_{n_1,n_2=-\infty}^{\infty} \begin{bmatrix} \omega_1 & \omega_2 & p \\ n_1 & n_2 & k \end{bmatrix}^{(1)} f^{\omega_1}_{n_1}(e^{i\theta_1}) \otimes^1_q f^{\omega_2}_{n_2}(e^{i\theta_2}).$$

Clearly, these coefficients vanish unless $k = n_1 + n_2$. The orthogonality of the two bases implies the identities

$$(3.16)\qquad \sum_{n_1,n_2} \begin{bmatrix} \omega_1 & \omega_2 & p \\ n_1 & n_2 & k \end{bmatrix}^{(1)} \begin{bmatrix} \omega_1 & \omega_2 & p' \\ n_1 & n_2 & k \end{bmatrix}^{(1)} = \delta_{pp'},$$

$$\sum_{p} \begin{bmatrix} \omega_1 & \omega_2 & p \\ n_1 & n_2 & k \end{bmatrix}^{(1)} \begin{bmatrix} \omega_1 & \omega_2 & p \\ n'_1 & n'_2 & k \end{bmatrix}^{(1)} = \delta_{n_1 n'_1}$$

where $n_1 + n_2 = n'_1 + n'_2 = k$. Explicitly,

$$\begin{bmatrix} \omega_1 & \omega_2 & p \\ n_1 & n_2 & k \end{bmatrix}^{(1)} = (-1)^{p+n_2} q^{(p+n_2)/2} J_{p+n_2}(q^{-1/2} b_k; q)$$

$$(3.17)\qquad - \sum_{\ell=0}^{s_k-1} \frac{(q^\ell b_k^2; q)_\infty q^{\ell(\ell+1)/2} (-q^\ell b_k)^{p+n_2} (-1)^\ell}{(q;q)_\infty (q;q)_\ell},$$

where $k = n_1 + n_2$, $b_k = q^{(1+k)/2}\omega_2/\omega_1$ and $s_k \geq 0$ is the smallest integer such that $q^{s_k} b_k < 1$. (We assume that $b_k \neq q^n$ for any integer n.) Indeed the case $s_k = 0$ of (3.17) follows from (3.15) and this result can be written as a complex contour integral. The case $s_k > 0$ can be obtained from this result by shifting the contour.

Similarly the Clebsch-Gordan coefficients for the tensor product corresponding to the spectral resolution (3.11) are defined by

$$f^{(2)}_{pk}[z,t] = f^{(2)}_{pk}(\theta_1,\theta_2) = \sum_{n_1,n_2=-\infty}^{\infty} \begin{bmatrix} \omega_1 & \omega_2 & p \\ n_1 & n_2 & k \end{bmatrix}^{(2)} f^{\omega_1}_{n_1}(e^{i\theta_1}) \otimes^2_q f^{\omega_2}_{n_2}(e^{i\theta_2}).$$

They satisfy orthogonality relations analogous to (3.16) and are given explicitly by (3.17), where now $b_k = q^{(1-k)/2}\omega_1/\omega_2$ and n_2 is replaced by $-n_1$.

With respect to the tensor product basis $\{f^{\omega_1}_{n_1} \otimes f^{\omega_2}_{n_2}\}$ the operator $e_q(\beta F_+)E_q(\alpha F_-)$ has matrix elements

$$\begin{aligned} T_{m_1 m_2; n_1 n_2}(\alpha,\beta) &= < e_q(\beta \Delta(E_+)) E_q(\alpha \Delta(E_-)) f^{\omega_1}_{n_1} \otimes f^{\omega_2}_{n_2}, f^{\omega_1}_{m_1} \otimes f^{\omega_2}_{m_2} > \\ &= (\beta\omega_1)^{m_1-n_1} (\beta\omega_2)^{m_2-n_2} \frac{(q^{m_1-n_1+1}; q)_\infty (q^{m_2-n_2+1}; q)_\infty}{(q;q)^2_\infty q^{(m_1 m_2 + n_1 n_2 - 2 n_1 m_2)/2}} \times \\ &\quad {}_1\phi_1\left(\begin{matrix} 0 \\ q^{m_1-n_1+1} \end{matrix} ; q, -\alpha\beta\omega_1^2 q^{-m_2} \right) {}_1\phi_1\left(\begin{matrix} 0 \\ q^{m_2-n_2+1} \end{matrix} ; q, -\alpha\beta\omega_2^2 q^{n_1} \right) \end{aligned}$$

or, in coordinates $[r,t]$:

$$(3.18)\qquad T_{m_1 m_2; n_1 n_2}[r,t] =$$

$$(-q^{-1/2}it)^{n_1+n_2-m_1-m_2}J_{m_1-n_1}(r\omega_1 q^{-(m_2+1)/2};q)J_{m_2-n_2}(r\omega_2 q^{(n_1-1)/2};q).$$

To see this most simply, recall that if X and Y are linear operators such that $YX=qXY$ then, [6, page 28],

$$(Y+X)^k=\sum_{\ell=0}^{k}\frac{(q;q)_k}{(q;q)_\ell(q;q)_{k-\ell}}X^\ell Y^{k-\ell},$$

$$e_q(X+Y)=e_q(X)e_q(Y),\quad E_q(X+Y)=E_q(Y)E_q(X).$$

Thus, using the facts that $E_+q^{\frac{1}{2}H}=q^{-\frac{1}{2}}q^{\frac{1}{2}H}E_+$, $E_-q^{\frac{1}{2}H}=q^{\frac{1}{2}}q^{\frac{1}{2}H}E_-$ for these representations, we have

$$\begin{aligned}&e_q(\beta F_+)E_q(\alpha F_-)=\\&=e_q(\beta E_+\otimes q^{-\frac{1}{2}H})E_q(\alpha E_-\otimes q^{-\frac{1}{2}H})e_q(\beta q^{\frac{1}{2}H}\otimes E_+)E_q(\alpha q^{\frac{1}{2}H}\otimes E_-),\end{aligned}$$

so the matrix elements with respect to the tensor product basis are given by

$$\begin{aligned}T_{m_1m_2;n_1n_2}(\alpha,\beta)=&<e_q(\beta q^{-m_2/2}E_+)E_q(\alpha q^{-m_2/2}E_-)f^{\omega_1}_{n_1},f^{\omega_1}_{m_1}>\\&\times<e_q(\beta q^{n_1/2}E_+)E_q(\alpha q^{n_1/2}E_-)f^{\omega_2}_{n_2},f^{\omega_2}_{m_2}>.\end{aligned}$$

From the definition of the Clebsch-Gordan coefficients we see immediately that the identities

(3.19)

$$\begin{aligned}T_{m_1m_2;n_1n_2}(\alpha,\beta)=&\sum_{p=-\infty}^{\infty}\begin{bmatrix}\omega_1&\omega_2&p\\n_1&n_2&n_1+n_2\end{bmatrix}^{(j)}T^{(e,E),\omega_j q^{p/2}}_{m_1+m_2,n_1+n_2}(\alpha,\beta)\\&\times\begin{bmatrix}\omega_1&\omega_2&p\\m_1&m_2&m_1+m_2\end{bmatrix}^{(j)},\quad j=1,2\end{aligned}$$

must hold. Choosing $j=1$ and taking the special case where $q^{(1+m_1+m_2)/2}\omega_2/\omega_1<1$ and $q^{(1+n_1+n_2)/2}\omega_2/\omega_1<1$ we obtain the identity

(3.20)

$$(-q)^nJ_{x-n}(Sq^{-y};q^2)J_n(RSq^n;q^2)=\sum_{k=-\infty}^{\infty}q^{2k}J_{k-n}(Rq^y;q^2)J_x(Sq^{k-y};q^2)J_k(Rq^{y+x};q^2),$$

valid for $0<Rq^{y+x+1}<1$, $0<Rq^{y+1}<1$, $0\le S$ and n,x,y integers. The case $S=q^z$ of this formula for z an integer is the addition theorem for Hahn-Exton q-Bessel functions derived by Koelink using the theory of quantum groups, [10, 21]. For an analytic proof of Koelink's formula see [20].

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