SEPARABLE COORDINATES, INTEGRABILITY AND THE NIVEN EQUATIONS

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ABSTRACT. Finite (polynomial) solutions of Laplace's equation are investigated. The unifying features of this study are the so called Niven equations which yield the dimension of the space of such solutions. In carrying out this study complete sets of solutions are obtained on the n dimensional sphere in terms of ellipsoidal coordinates. This corresponds to an integrable system having all the integrals of motion given by quadratic orbits of the universal enveloping algebra of $O(n+1)$. We call this system the n dimensional Euler top. The spectrum of the integrals of motion has been recently computed for $n=3$ by Komarov and Kuznetsov using results originally due to Niven. These calculations are extended to arbitrary dimension.

1. Introduction. The Euler top on the 0(4) Lie algebra (or Manakov top) has been recently studied by Komarov and Kuznetsov [1]. It is one of the six integrable systems on the 0(4) Lie algebra that have integrals of motion that lie on quadratic orbits of the universal enveloping algebra [2]. Having observed the construction of conical harmonics as originally expounded by Niven [3] and summarised in the books of Hobson [4], and Whittaker and Watson [5], Komarov and Kuznetsov [6] showed that an analogous analysis can be performed for the Euler top on 0(4). In addition this work demonstrated that the eigenvalues of the quadratic first integrals can be calculated as algebraic expressions in terms of the zeros of the solutions and the parameters occuring in the defining elliptical coordinates. Komarov and Kuznetsov have also indicated how these results are equivalent to the two site $su(2)$ Gaudin magnet and the four site Gaudin magnet [6]. The problem of separation of variables on the real n dimensional sphere has been solved by Kalnins and Miller, [7],[8]. The general solution consists of nested ellipsoidal coordinates and can be described by a diagramatic calculus which extends that originally developed by Vilenkin [9] . In this article we demonstrate that the methods of Niven can be extended to general separable coordinates on the n dimensional sphere. In addition we give formulas for the eigenvalues in terms of the zeros of the corresponding polynomials. We show also that it is possible to analyze polynomial solutions of Laplace's equation for Euclidean space in a similar way. We introduce the generalised cyclidic finite solutions from which all other solutions can be obtained. Some examples of degenerate solutions are given.

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2. Separation of Variables on S_n. Let us recall the relevant details of the complete classification of separable coordinates on S_n. On the n sphere the generic separable coordinates are the ellipsoidal coordinates x^i, $i = 1, \cdots, n$. With natural coordinates s_i, $i = 1, \cdots, n+1$ such that $\sum_{i=1}^{n+1} s_i^2 = 1$, the coordinates ${}_es_j$ corresponding to x^i are given by

$$(2.1) \qquad {}_es_j^2 = \frac{\Pi_{i=1}^n (x^i - e_j)}{\Pi_{j\neq\ell}(e_\ell - e_j)}, \quad j = 1, \cdots, n+1$$

where

$$e_1 < x^1 < e_2 < x^2 < \cdots < e_n < x^n < e_{n+1}$$

and the subscript e can be taken as an abbreviation for the set $\{e_1, e_2, \cdots, e_{n+1}\}$. This ellipsoidal coordinate system is denoted by the box symbol

$$(2.2) \qquad [e_1|e_2|\cdots|e_{n+1}].$$

The infinitesmal distance is

$$ds^2 = -\frac{1}{4}\sum_{i=1}^n \frac{\Pi_{j\neq i}(x^i - x^j)}{\Pi_{k=1}^{n+1}(x^i - e_k)}(dx^i)^2.$$

Laplace's equation $\Delta\Psi = -\sigma(\sigma+n-1)\Psi$ for the eigenfunctions on the n sphere has, in these coordinates, the form

$$(2.3) \qquad \sum_{i=1}^n \frac{4}{\Pi_{j\neq i}(x^i - x^j)}[\sqrt{R}_i \frac{\partial}{\partial x^i}(\sqrt{R}_i \frac{\partial \Psi}{\partial x^i})] = -\sigma(\sigma+n-1)\Psi$$

where $R_i = \Pi_{j=1}^{n+1}(x^i - e_j)$. The separation equations have the form

$$(2.4) \qquad 4\sqrt{R}_i \frac{\partial}{\partial x^i}[\sqrt{R}_i \frac{\partial \Psi_i}{\partial x^i}] + [\sigma(\sigma+n-1)(x^i)^n + \sum_{j=2}^{n-1}\lambda_j (x^i)^{n-j}]\Psi_i = 0$$

where $\Psi = \Pi_{i=1}^n \Psi_i(x^i)$.

We adopt the convention $\lambda_1 = \sigma(\sigma+n-1)$. The separation constants λ_j are eigenvalues of commuting second order symmetric operators in the enveloping algebra of $so(n)$ generated by $I_{ij} = s_i\frac{\partial}{\partial s_j} - s_j\frac{\partial}{\partial s_i}$, $i, j = 1, \cdots, n+1$. Specifically these operators are

$$(2.5) \qquad I_j^n = \sum_{i>j} S_{j-1}^{ij} I_{ij}^2, \quad j = 1, \cdots, n$$

where $S_p^{ij} = \frac{1}{p!}\left[\sum_{i_1 i_2\cdots i_p \neq} e_{i_1}\cdots e_{i_p}\right]$ and the summation extends over $i_1, \cdots, i_p \neq i, j$ and $i_r \neq i_s$ for $r \neq s$. The eigenfunctions satisfy the eigenvalue equations

$$(2.6) \qquad I_j^n \Psi = \lambda_j \Psi, \quad j = 1, \cdots, n.$$

The general construction of coordinates proceeds by imbedding ellipsoidal coordinates within ellipsoidal coordinates. This is represented diagramatically by a tree-like graph made out of components of the form

$$\begin{array}{ccccc} [\ e_1 & |\ e_2 & |\ e_3\ | & \cdots & |\ e_{n+1}\ \ ] \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ S_{p_1} & S_{p_2} & S_{p_3} & \cdots & S_{p_{n+1}} \end{array} \tag{2.7}$$

where S_{p_i} denotes coordinates on a sphere of dimension p_i, $p_i = 0, 1, 2, \cdots$. The arrows indicate that the coordinates attached to each e_i come from S_{p_i}. For the graph above considered as a single entity, a suitable choice of coordinates would typically look like

$$\begin{aligned} s_{P_1+1} &= ({}_e s_1)({}_1 s_1) \\ s_{P_1+2} &= ({}_e s_1)({}_1 s_2) \\ &\cdots \\ s_{P_1+p_1+1} &= ({}_e s_1)({}_1 s_{p_1+1}) \\ &\cdots \\ &\cdots \\ s_{P_{n+1}+1} &= ({}_e s_{n+1})({}_{n+1} s_1) \\ s_{P_{n+1}+2} &= ({}_e s_{n+1})({}_{n+1} s_2) \\ &\cdots \\ &\cdots \\ s_{P_{n+1}+p_{n+1}+1} &= ({}_e s_{n+1})({}_{n+1} s_{p_{n+1}+1}) \end{aligned} \tag{2.8}$$

where $P_1 = 0$, $P_i = \sum_{j=1}^{i-1}(p_j + 1)$ and $\sum_{m=1}^{p_k+1}({}_k s_m)^2 = 1$. (This coordinate system would be separable for the Laplace Beltrami eigenvalue equation in S_N where $N = P_{n+2}$) By applying these rules to tree graphs where blocks of ellipsoidal coordinates are joined by directed arrows that point in the direction of the branches, all separable cordinate systems on S_N are recovered.

To see how the calculations of Niven work in n dimensions let us first look at the case of generic ellipsoidal coordinates. If we seek solutions of Laplace's equation of the form

$$\Psi(s_i) = (\Pi_{j=1}^{k}\ {}_e s_{\alpha_j})\Pi_{i=1}^{M}\left(\sum_{p=1}^{n+1}\frac{{}_e s_p^2}{u_i - e_p}\right) \tag{2.9}$$

with $0 \le k \le n+1, 1 \le \alpha_j \le n+1, \alpha_j \ne \alpha_k$ if $j \ne k$ we will be successful provided the u_j satisfy the Niven equations

$$\begin{aligned} &N[e_1, \cdots, e_{n+1}; e_{\alpha_1}, \cdots, e_{\alpha_k}; u_1, \cdots, u_M] \\ &\equiv \sum_{j=1}^{n+1}\frac{1/4}{(u_i - e_j)} + \sum_{p=1}^{k}\frac{1/2}{(u_i - e_{\alpha_p})} + \sum_{r \ne i}\frac{1}{(u_i - u_r)} = 0 \end{aligned} \tag{2.10}$$

and $2M + k = \sigma$. These equations can, of course, be obtained by direct substitution of the ansatz into Laplace's equation. Also, as computed in each of the solutions, $\Psi_i(x^i)$ can be written in the form

$$\Psi_i(x^i) = \sqrt{\Pi_{j=1}^{k}(x^i - e_{\alpha_j})}\,\Pi_{r=1}^{M}(x^i - u_r). \tag{2.11}$$

Note that the polynomial solutions $\Psi(s_i)$ vanish on the ellipsoidal coordinate hypersurfaces $x^i = u_r$.

How are the eigenvalues λ_i, $i = 2, \cdots, n$ to be computed? By direct substitution into the separation equations. If this is done then the eigenvalues can be determined from the formula

(2.12)
$$\sum_{i=1}^{N} \lambda_i U_{m+i-N} + \sum_{i=0}^{k}\sum_{j=0}^{p}\left[N^2 + N[1 - \frac{i}{2} - \frac{3j}{2}] + \frac{1}{4}(i+j-1)(3j-2i)\right] \times$$
$$V_{k-i}E_{p-j}U_{m+i+j-N-1} = 0$$

where

$$U_p = \frac{1}{p!}\left[\frac{\partial}{\partial\rho}\right]^p \Pi_{i=1}^{M}(\rho - u_i)\,|_{\rho=0} = \frac{1}{p!}\sum_{i_1,\cdots,i_p\neq} u_{i_1}u_{i_2}\cdots u_{i_p}(-1)^p,$$

$$V_j = \frac{1}{j!}\left[\frac{\partial}{\partial\rho}\right]^j \Pi_{j=1}^{k}(\rho - e_{\alpha_j})\,|_{\rho=0} = \frac{1}{j!}\sum_{\alpha_{i_1},\cdots,\alpha_{i_j}\neq} e_{\alpha_{i_1}}e_{\alpha_{i_2}}\cdots e_{\alpha_{i_j}}(-1)^j,$$

$$E_q = \frac{1}{q!}\left[\frac{\partial}{\partial\rho}\right]^q \Pi_{r=1}^{M-k}(\rho - e_{A_r})\,|_{\rho=0} = \frac{1}{q!}\sum_{A_1,\cdots,A_q\neq} e_{A_1}e_{A_2}\cdots e_{A_q}(-1)^q$$

and $\{e_A\} = \{e_i\} - \{e_{\alpha_j}\}$.

The polynomial eigenfunctions obtained in this way form a complete set. Indeed if we consider the cartesian coordinates $z^i = rs_i$, $i = 1, \cdots, n+1$, then for given integer σ solutions of (2.10) can be obtained by looking for solutions of Laplace's equation $\Delta\Phi = 0$ where $\Phi = P_\sigma(z^1, \cdots, z^{n+1})$ is a homogeneous polynomial of degree σ in the variables z^i. In particular the space of all polynomials in the variables z^i can be decomposed into subspaces of polynomials of which a typical element is

$$(\Pi_{j=1}^{k} z^{\alpha_j})P_M\left((z^1)^2, \cdots, (z^{n+1})^2\right).$$

If we choose the new variables $u_i = (z^i)^2$ then the polynomial $P(u_1, \cdots, u_{n+1})$ satisfies the equation

$$\left[\sum_{i=1}^{n+1}(4u_i\frac{\partial^2}{\partial u_i^2} + 2\frac{\partial}{\partial u_i}) + \sum_j 4\frac{\partial}{\partial u_{\alpha_j}}\right] P_M(u_1, \cdots, u_{n+1}) = 0. \qquad (2.13)$$

There are $C_M^{n+M} = (n+M)!/n!M!$ possible homogeneous polynomials of degree $P_M(u_1, \cdots, u_{n+1})$ and the equation (2.13) imposes at most C_{M-1}^{n+M-1} conditions on them. Therefore, the number $N(n, M)$ of solutions is such that $N(n, M) \geq C_M^{n+M} - C_{M-1}^{n+M-1} = C_M^{n+M-1}$. In fact the equality sign holds true. Indeed it is not difficult to see that (2.13) cannot admit solutions of the form $(u_1 + \cdots + u_{n+1})^k p(u_1, \cdots, u_{n+1})$ for p a polynomial when $k = 1, 2, \cdots$. The dimension of the space of all such polynomials is C_{M-1}^{n+M-1}. Therefore $C_M^{n+M} - C_{M-1}^{n+M-1} \geq N(n, M)$. Hence equality holds. This is exactly the number of independent solutions of the Niven equations.

In fact for each choice of integers $p_1, p_2, \cdots, p_n$ such that$\sum_{i=1}^n p_i = M$ there are solutions u_{ir}, $i = 1, \cdots, n$; $r = 1, \cdots, p_r$ for which

$$e_1 < u_{11} < \cdots < u_{1p_1} < e_2 < u_{21} < \cdots < u_{2p_2} < e_3 < \cdots < u_{np_n} < e_{n+1}. \tag{2.14}$$

There are exactly C_M^{n+M-1} such solutions. This follows from a straightforward generalisation of a theorem due to Stieltjes [5]. Indeed, consider the function

$$\Phi = \Pi_{i=1}^n \left[\Pi_{j=1}^{p_j}(\Pi_{q=1}^{n+1}|u_{ij} - e_q|^{\kappa_q})\right] \Pi_{i\neq j}(\Pi_{k\neq p}|u_{ik} - u_{jp}|)$$

where the κ_q are positive and half integral and the variables u_{ij} are in the ranges given by the inequalities (2.14). This product is zero when some of the u_{ij} are on the boundary of the domain (2.14). When the variables are unequal to each other and also to e_i, $i = 1, \cdots, n+1$ then Φ is positive and is a continuous bounded function of the variables. Hence there is a set of interior values for which Φ attains its upper bound. This set satisfies the critical point conditions

$$\frac{\partial}{\partial u_{ij}} \log \Phi = 0, \quad \forall i, j.$$

These restrictions are just the generalised Niven conditions,

$$\sum_{q=1}^{n+1} \frac{\kappa_q}{(u_{ij} - e_q)} + \sum_{i\neq j}\sum_{k\neq p} \frac{1}{(u_{ik} - u_{jp})} = 0.$$

If the variables are complex then finite solutions can be constructed in this way but the completeness property is no longer valid.

For a graph of the type (2.7) analogous calculations can be made. Let us look for solutions of the form $\Psi = \psi\ \varphi$, where $\psi = \Pi_{i=1}^n \psi_i(x^i)$, and $\varphi = \Pi_{j=1}^{n+1}\varphi_j(S_{p_j})$. Each of the φ_j functions is chosen to satisfy the individual Laplace equation (2.3) for $n = p_j$. The Laplace equation

$$\Delta\Psi = -\sigma(\sigma + n - 1)\Psi$$

then becomes

$$\sum_{i=1}^n \left[\frac{\partial^2}{\partial {s_i}^2} + \frac{p_i}{s_i}\frac{\partial}{\partial s_i} - \frac{n_i(n_i + p_i - 1)}{(s_i)^2}\right]\psi = 0 \tag{2.15}$$

where $s_i = {}_e s_i$, $i = 1, \cdots, n+1$.

If we try a solution of the form

$$\psi = (\Pi_{j=1}^p {}_e s_{\alpha_j})\Pi_{i=1}^M(\sum_{p=1}^{n+1} \frac{{}_e s_p^2}{u_i - e_p}) \left(\Pi_{i=1}^{n+1}({}_e s_i^{n_i})\right)$$

then the generalised form of the Niven equations is

$$\sum_{j=1}^p \frac{2}{(u_i - e_{\alpha_j})} + \sum_{q=1}^{n+1} \frac{(2n_q + p_q + 1)}{(u_i - e_q)} + \sum_{r\neq i} \frac{4}{(u_i - u_r)} = 0 \tag{2.16}$$

where $\sigma = p + 2M + \sum_{i=1}^{n+1} n_i$ and ${}_e s_{\alpha_j}, j = 1, \cdots, k$ correspond to boxes in the diagram for which $p_{\alpha_j} = 0$ and consequently $0 \le p \le k$. These equations provide a complete solution for the problem of separation of variables on the n-sphere. In analogy with the proof for ellipsoidal coordinates described by the graph (2.2), the space of all polynomial solutions of (2.15) is spanned by polynomials of the form

$$\Pi_{j=1}^{p}({}_e z^{\alpha_j})(\Pi_{i=1}^{n+1}(z^i)^{n_i})P_M\left((z^1)^2, \cdots, (z^{n+1})^2\right).$$

With variables $u_i = (z^i)^2$ as before, the polynomials $P_M(u_1, \cdots, u_{n+1})$ satisfy

$$\left[\sum_{i=1}^{n+1}\left(4u_i\frac{\partial^2}{\partial u_i^2} + 2(2n_i + p_i + 1)\frac{\partial}{\partial u_i}\right) + \sum_j 4\frac{\partial}{\partial u_{\alpha_j}}\right] P_M(u_1, \cdots, u_{n+1}) = 0.$$

We can argue as before that there are only C_M^{n+M-1} possible solutions and this is exactly the number of solutions of the Niven equations.

The operators that describe the separation can be obtained from those given for generic coordinates as follows. If we choose the generic coordinates on the sphere of dimension $N = \sum_{i=1}^{n+1} p_i + n$ then the operators are obtained by taking the first $p_1 + 1$ of the e_i's equal then the next $p_2 + 1$ of the e_i's equal and so on in the expressions given above for the operators that describe the generic coordinates.

3. Separation of Variables on E_n. The results we have developed for the sphere S_n can readily be adapted to Euclidean n space. Specifically the extensions of the work on S_n applies to finding solutions of Laplace's equation $\Delta\Psi = 0$ in E_n. If we choose cartesian coordinates defined by z^i, $i = 1, \cdots, n$ then ellipsoidal coordinates y^i are given by

$$(z^j)^2 = c^2\frac{\Pi_{i=1}^{n}(y^i - e_j)}{\Pi_{j\neq i}(e_i - e_j)}, \quad j = 1, \cdots, n \tag{3.1}$$

where $e_1 < y^1 < e_2 < y^2 < \cdots < e_n < y^n$. This coordinate system is denoted by the box symbol

$$[[e_1|e_2|\cdots|e_n]].$$

The infinitesmal distance is given by

$$ds^2 = -\frac{1}{4}c^2\sum_{i=1}^{n}\frac{\Pi_{j\neq i}(y^i - y^j)}{\Pi_{k=1}^{n}(y^i - e_k)}(dy^i)^2.$$

Laplace's equation $\Delta\Psi = 0$ for in E_n, expressed in terms of the y^i, has the form

$$-\frac{4}{c^2}\sum_{i=1}^{n}\frac{4}{\Pi_{j\neq i}(y^i - y^j)}\sqrt{Q_i}\frac{\partial}{\partial y^i}[\sqrt{Q_i}\frac{\partial\Psi}{\partial y^i}] = 0 \tag{3.2}$$

where $Q_j = \Pi_{i=1}^{n}(y^j - e_i)$. The separation equations are

$$4\sqrt{Q_i}\frac{\partial}{\partial y^i}[\sqrt{Q_i}\frac{\partial\Psi_i}{\partial y^i}] + \sum_{j=1}^{n-1}\lambda_j(y^i)^{n-1-j}\Psi_i = 0 \tag{3.3}$$

where $\Psi = \Pi_{i=1}^{n}\Psi_i(y^i)$. The separation constants λ_j are eigenvalues of commuting second order symmetric operators in the enveloping algebra of the Euclidean group $E(n)$, generated by the Lie derivatives

$$I_{ij} = z^i\frac{\partial}{\partial z^j} - z^j\frac{\partial}{\partial z^i} \quad i,j=1,\cdots,n;\ i>j, \quad \text{and} \quad P_j = \frac{\partial}{\partial z^j}, \quad j=1,\cdots,n.$$

Specifically these operators are

$$I_k^n = \sum_{i>j} S_k^{ij} I_{ij}^2 + c^2\sum_{i=1}^{n} S_k^i P_i^2 \tag{3.4}$$

where S_k^{ij} are defined as in the case of S_n, and $S_k^i = \frac{1}{k!}\sum_{i_1,\cdots,i_k\neq} e_{i_1}\cdots e_{i_k}$. The separable solutions satisfy the eigenvalue equations

$$I_k^n\Psi = \lambda_j\Psi. \tag{3.5}$$

The analogue of the coordinates given by (2.7) for S_n is represented diagramatically by a graph of the form

$$\begin{array}{cccccc} [[\ e_1 & |\ e_2 & |\ e_3\ | & \cdots & |\ e_n &]] \\ \downarrow & \downarrow & \downarrow & & \downarrow & \\ S_{p_1} & S_{p_2} & S_{p_3} & \cdots & S_{p_n} & \end{array} \tag{3.6}$$

where P_{p_i} denotes coordinates on a sphere of dimension p_i and $p_i = 0,1,2,\cdots$. The arrows indicate that the coordinates attached to each e_i come from S_{p_i}. The graph, given a suitable choice of coordinates, would typically look like

$$\begin{aligned} z^{P_1+1} &= ({}_e z^1)({}_1 s_1) \\ z^{P_1+2} &= ({}_e z^1)({}_1 s_2) \\ &\cdots \\ z^{P_1+p_1+1} &= ({}_e z^1)({}_1 s_{p_1+1}) \\ &\cdots \\ &\cdots \\ z^{P_n+1} &= ({}_e z^n)({}_n s_1) \\ z^{P_n+2} &= ({}_e z^n)({}_n s_2) \\ &\cdots \\ &\cdots \\ z^{P_n+p_n+1} &= ({}_e z^n)({}_n s_{p_{n+1}+1}) \end{aligned} \tag{3.7}$$

where $P_1 = 0$, $P_i = \sum_{j=1}^{i-1}(p_j+1)$ and $\sum_{m=1}^{p_k+1}({}_k s_m)^2 = 1$. By applying these rules to graphs of the form (3.6) we recover a class of separable coordinates of Laplace's equation. If we seek solutions $\Psi(z^i)$ of Laplace's equation of the form

$$\Psi(z^i) = \Pi_{j=1}^{k} z^{\alpha_j}\Pi_{i=1}^{M}\left(\sum_{p=1}^{n}\frac{(z^p)^2}{u_i - e_p} - 1\right) \tag{3.8}$$

where $0 \le k \le n$, $1 \le \alpha_j \le n$, and $\alpha_j \ne \alpha_k$ if $j \ne k$, we will be successful provided the u_j satisfy the Niven equations (2.16).

In terms of ellipsoidal coordinates, the solutions $\Psi_i(y^i)$ can be written in the form

$$\Psi_i(y^i) = \sqrt{\Pi_{j=1}^k (y^i - e_{\alpha_j})}\Pi_{r=1}^M (y^i - u_r).$$

The eigenvalues λ_i, $i = 2, \cdots, n-1$ can be computed in exactly the same way as for the sphere (2.2), i.e., via the formulas (2.12). Let us now choose a graph and look for solutions of the form $\Psi = \psi\ \varphi$ where $\psi = \Pi_{i=1}^n \psi_i(y^i)$ and $\varphi = \Pi_{j=1}^n \varphi(S_{p_j})$. Each of the φ_j functions is chosen to satisfy the individual Laplace equation (2.3) on S_{p_j}. Laplace's equation in Euclidean space then becomes

$$\Delta\psi = \sum_{i=1}^n \left[\frac{\partial^2}{\partial z^i} + \frac{p_i}{z^i}\frac{\partial}{\partial z^i} - \frac{n_i(n_i + p_i - 1)}{(z^i)^2}\right]\psi = 0 \tag{3.9}$$

where $z^i = {}_e z^i$, $i = 1, \cdots, n$.

If we try a solution of the form

$$\psi = \Pi_{j=1}^p z^{\alpha_j}\Pi_{i=1}^M \left(\sum_{p=1}^n \frac{(z^p)^2}{u_i - e_p} - 1\right)\Pi_{k=1}^n z^{n_k}, \tag{3.10}$$

then the generalised Niven equations have the form

$$N[e_1, \cdots, e_n; e_{\alpha_1}, \cdots, e_{\alpha_k}; u_1, \cdots, u_M] = 0.$$

These equations provide a solution for coordinate systems of the type described by the box diagram (3.6). The corresponding operators that describe the separation can be obtained by taking the first $p_1 + 1$ of the e_i's equal, then the next $p_2 + 1$ e_i's equal, and so on.

Ellipsoidal coordinates are not the only ones that are basic to the construction of separable systems in E_n. Parabolic coordinates are also basic [7], [8]. The question that we ask is can we extend the construction of Niven and if so what does it mean? In fact, it *is* possible to generate polynomial solutions in parabolic coordinates. To see this we recall that parabolic coordinates y^i can be taken as

$$\begin{aligned} z^1 &= \frac{c}{2}(y^1 + \cdots + y^n - e_1 - \cdots - e_{n-1}) \\ (z^j)^2 &= -c^2\frac{\Pi_{i=1}^n (y^i - e_{j-1}))}{\Pi_{j\ne i+1}(e_i - e_{j-1})}, \quad j = 2, \cdots, n \end{aligned} \tag{3.11}$$

where $y^1 < e_1 < y^2 < e_2 < \cdots < e_{n-1} < y^n$ and the z^j are cartesian coordinates in E_n. This coordinate system is denoted by the box symbol

$$\{e_1|e_2|e_3|\cdots|e_{n-1}\}. \tag{3.12}$$

The infinitesmal distance is given by

$$ds^2 = -\frac{c^2}{4}\sum_{i=1}^n \frac{\Pi_{j\ne i}(y^i - y^j)}{\Pi_{k=1}^{n-1}(y^i - e_k)}(dy^i)^2.$$

Laplace's equation in E_n, expressed in parabolic coordinates, has the form (3.2) with $Q_j = \Pi_{i=1}^{n-1}(y^j - e_i)$. Using the expression

$$\Omega(u) = \sum_{j=2}^{n} \frac{(z^j)^2}{u - e_j} + 2cz^1 - c^2 u = c^2 \frac{\Pi_{j=1}^{n}(u - y^j)}{\Pi_{i=1}^{n-1}(u - e_i)},$$

we seek solutions $\Psi(z^\ell)$ of the form

$$\Psi(z^\ell) = \Pi_{j=1}^{k} z^{\alpha_j} \Pi_{i=1}^{M} \Omega(u_i) \tag{3.13}$$

where $2 \le k \le n-1$, $2 \le \alpha_j \le n-1$, $\alpha_j \ne \alpha_k$. The u_i must satisfy the corresponding Niven equations:

$$N[e_1, \cdots, e_{n-1}; e_{\alpha_1}, \cdots, e_{\alpha_k}; u_1, \cdots, u_M] = 0.$$

It is clearly possible to generalise this result to the case of coordinate systems corresponding to the diagram

$$\begin{array}{ccccccc} \{ & e_1 & | \; e_2 & | \; e_3 \; | & \cdots & | \; e_{n-1} & \}. \\ & \downarrow & \downarrow & \downarrow & & \downarrow & \\ & S_{p_1} & S_{p_2} & S_{p_3} & \cdots & S_{p_{n-1}} & \end{array} \tag{3.14}$$

4. Finite Solutions in Cyclidic Coordinates. In the previous section we have seen how the Niven ansatz has been able to give polynomial solutions of Laplace's equation $\Delta\Psi = 0$. The question we answer in this section is whether analogous solutions exist for the case of cyclidic coordinates. Cyclidic coordinates are the generic orthogonal coordinates for which Laplace's equation admits a solution via the R separation ansatz [11]. After reviewing the basic properties of cyclidic coordinates we answer this question in the affirmative.

A natural way for the realisation of cyclidic coordinates is in terms of projective coordinates w^j, $j = 1, \cdots, n+2$, given in terms of cartesian coordinates z^k by

$$\begin{aligned} w^k &= 2z^k \rho^2, \quad k = 1, \cdots, n \\ w^{n+1} &= i\rho^2\Big(\sum_{k=1}^{n}(z^k)^2 + 1\Big) \\ w^{n+2} &= \rho^2\Big(\sum_{k=1}^{n}(z^k)^2 - 1\Big), \end{aligned} \tag{4.1}$$

that satisfy $\sum_{j=1}^{n+2}(w^j)^2 = 0$.

The surface defined by

$$\sum_{j=1}^{n+2} \frac{(w^j)^2}{(y - e_j)} = 0 \tag{4.2}$$

where $e_1 < e_2 < e_3 < \cdots < e_{n+2}$, is a cyclide in n dimensions. Coordinates $y = y^k$, $k = 1, \cdots, n$ such that $e_1 < y^1 < e_2 < y^2 < \cdots < y^n < e_{n+1}$ are just the orthogonal cyclidic coordinates in n dimensions. This coordinate system is known to provide an R separation of variables for Laplace's equation. Indeed any function $\Psi(z^1,\cdots,z^n)$ can be expressed in terms of the projective coordinates by observing that

$$z^k = \frac{-w^k}{iw^{n+1}+w^{n+2}}, \qquad k=1,\cdots,n.$$

If $\Psi(z^1,\cdots,z^n)$ is a solution of Laplace's equation then

$$\sum_{k=1}^{n} \frac{\partial^2\Psi}{\partial(w^k)^2} = 0 \quad \text{and} \quad \frac{\partial^2\Psi}{\partial(w^{n+1})^2} + \frac{\partial^2\Psi}{\partial(w^{n+2})^2} = 0.$$

Therefore if the general projective coordinates w^j, $j=1,\cdots,n+2$ are related to the coordinates y^k, $k=1,\cdots,n$ in such a way that $\sum_{j=1}^{n+2}(w^j)^2=0$ then the solution of Laplace's equation is also a solution of $\sum_{j=1}^{n+2}\frac{\partial^2\Psi}{\partial(w^j)^2}=0$. The question we now address is can finite type solutions be found in these coordinates? To consider this possibility, take coordinates

$$(w^j)^2 = y^{n+2}\frac{\Pi_{\ell=1}^{n+1}(y^\ell - e_j)}{\Pi_{j\neq\ell}(e_\ell - e_j)}, \; j=1,\cdots,n+2 \tag{4.3}$$

that satisfy (4.2) for $y=y^\ell$ and $y^{n+2}=\sum_{j=1}^{n+2}(w^j)^2$. When $y^{n+2}=0$ then (4.3) and (4.2) can be solved to give

$$(w^j)^2 = -\left(\sum_{k=1}^{n+2} e_k(w^k)^2\right)\frac{\Pi_{i=1}^{n+1}(y^i - e_j)}{\Pi_{j\neq i}(e_i - e_j)}, \; j=1,\cdots,n+2.$$

These expressions can be obtained by the limiting process $y^{n+2}\to 0$ in such a way that $y^{n+2}y^{n+1}\to -(\sum_{k=1}^{n+2}e_k(w^k)^2)$. If we look for solutions of the form

$$\Psi(w^\ell) = (\Pi_{j=1}^{k} w^{\alpha}j)\Pi_{i=1}^{M}\left(\sum_{p=1}^{n+2}\frac{(w^p)^2}{u_i - e_p}\right) \tag{4.4}$$

then the corresponding Niven equations

$$N[e_1,\cdots,e_{n+2};e_{\alpha_1},\cdots,e_{\alpha_k};u_1,\cdots,u_M]=0$$

hold. These solutions are homogeneous polynomials of degree $2M+k=\sigma$. If, indeed, we take the limit specified above, then the limiting solutions of Laplace's equation are

$$\Psi(w^\ell) = \left(\sum_{k=1}^{n+2} e_k(w^k)^2\right)^{\sigma/2}\Pi_{j=1}^{k} w^{\alpha_j}\Pi_{i=1}^{M}\left(\sum_{p=1}^{n+2}\frac{(w^p)^2}{u_i - e_p}\right). \tag{4.5}$$

In order to obtain the explicit form of these solutions we can, without loss of generality, let the parameter $e_{n+2}\to\infty$. The resulting coordinates have the form

$$(w^j)^2 = (w^{n+2})^2\frac{\Pi_{i=1}^{n+1}(y^i - e_j)}{\Pi_{j\neq i}(e_i - e_j)}, \quad j=1,\cdots,n+1,$$

and the corresponding solutions of Laplace's equation are

$$\Psi(z^\ell) = (v^{n+2})^\sigma \Pi_{j=1}^k v^{\alpha_j} \Pi_{i=1}^M \left(\sum_{p=1}^{n+1} \frac{(v^p)^2}{u_i - e_p} \right)$$

where $v^j = w^j/\rho^2$, $j = 1, \cdots, n+2$.

The general solutions of finite type can be obtained from the expression (4.4) by using the known limiting procedures outlined in [10]. The various possible coordinate types then can be be characterised by the elementary divisors of the two quadratic forms. One example of these types is given in the following section.

5. Separation of Variables in Complex Spaces. The results developed for E_n and S_n can be extended to the case of complexified spaces of constant curvature. We illustrate this for the case of $S_{n\mathbf{C}}$. Basically, what occurs in this case is the possibility that some of the parameters e_i are equal. The procedure for dealing with all these possibilities has been given in [10]. Here, we give an example of how this works in the simplest case in which two of the parameters e_i are equal. Much of what we have done before transfers itself directly to the complex case from the real case. For a metric of the form (2.2) with $e_1 = e_2$, a suitable choice of coordinates $\{x^i\}$ on $S_{n\mathbf{C}}$ is

$$\begin{aligned} (s_1 - is_2)^2 &= \frac{\Pi_{i=1}^n (x^i - e_1)}{\Pi_{j\neq 1}(e_j - e_1)} \\ (s_1^2 + s_2^2) &= \frac{\partial}{\partial e_1} \frac{\Pi_{i=1}^n (x^i - e_1)}{\Pi_{j\neq 1}(e_j - e_1)} \qquad (5.1) \\ s_k^2 &= (\frac{\Pi_{i=1}^n (x^i - e_k)}{(e_1 - e_k)^2 \Pi_{j\neq 1,k}(e_j - e_k)}, \qquad k = 3, 4, \cdots, n+1. \end{aligned}$$

These coordinates correspond to a graph of the form

$$[e_1^2 | e_3^1 | \cdots | e_{n-1}^1] \qquad (5.2)$$

in the notation of [10].

The separation equations are the same as in (2.4) but with $R_i = (x^i - e_1)^2 \Pi_{j=3}^{n+1}(x^i - e_j)$. Solutions can be sought in the form

$$\Psi(s_i) = \Pi_{j=1}^k s_{\alpha_j} \Pi_{i=1}^M \left(\frac{(s_1 - is_2)^2}{(u_i - e_1)^2} + \frac{s_1^2 + s_2^2}{(u_i - e_1)} + \sum_{j=3}^{n+1} \frac{s_j^2}{(u_i - e_j)} \right) \qquad (5.3)$$

where $0 \leq k \leq n-1$, $3 \leq \alpha_j \leq n+1$, $\alpha_j \neq \alpha_k$. It follows from the equality

$$\frac{(s_1 - is_2)^2}{(u - e_1)^2} + \frac{s_1^2 + s_2^2}{(u - e_1)} + \sum_{j=3}^{n+1} \frac{s_j^2}{(u - e_j)} = \frac{\Pi_{i=1}^n (u - x^i)}{(u - e_1)^2 \Pi_{j=3}^{n+1}(u - e_j)}$$

that the analogue of the Niven equations is

$$\sum_{j=3}^{n+1} \frac{\frac{1}{4}}{(u_i - e_j)} + \frac{\frac{1}{2}}{(u_i - e_i)} + \sum_{j=1}^{k} \frac{\frac{1}{2}}{(u_i - e_{\alpha_j})} + \sum_{r\neq i} \frac{1}{(u_i - u_r)} = 0.$$

As stated above this process can be repeated and applied to any combination of coordinate systems for which some of the e_i's are equal. We should also mention here that similar results apply to coordinate systems in $E_{n\mathbf{C}}$.

References

1. I.V. Komarov and V.B. Kuznetsov, *Quantum Euler-Manakov top on the 3-sphere S_3*, J. Phys. A (To appear).
2. P. Olevski, *The separation of variables in the equation $\Delta u + \lambda u = 0$ for space of constant curvature in two and three dimensions*, Math. Sb. **27** (1950), 379.
3. W.D. Niven, *VI. On ellipsoidal harmonics*, Phil.Trans. **CLXXXII** (1891), 231.
4. E.W. Hobson, *The theory of spherical and ellipsoidal harmonics*, Chelsea, New York, 1965.
5. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, England, 1973.
6. M. Gaudin, *La fonction d′onde de Bethe*, Masson, Paris, 1983.
7. E.G. Kalnins and W.Miller Jr., *Separation of variables on n-dimensional Riemannian manifolds.1.The n-sphere S_n and Euclidean n-space R_n*, J. Math. Phys. **27** (1986), 1721.
8. E.G. Kalnins, *Separation of Variables for Riemannian Spaces of Constant Curvature*, Pitman Monographs and Surveys in Pure and Applied Mathematics 28, Longman Scientific and Technical, Essex, England, 1986.
9. N.J.Vilenkin, *Special Functions and the Theory of Group Representations*, Translations of Mathematical Monographs Vol22 American Math.Society, Providence, Rhode Island, 1968.
10. E.G. Kalnins, W. Miller Jr. and G.J. Reid, *Separation of variables for complex Riemannian spaces of constant curvature 1. Orthogonal separable coordinates for coordinates S_{nC} and E_{nC}*, Proc. Roy. Soc. Lond. **A394** (1984), 183.
11. M. Bôcher, *Die Reihentwickelungen der Potentialtheorie*, B.G. Teubner, Leipzig, 1894.