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Math 5467. Midterm Exam II (Solutions)

April 19, 2004

(There are a total of 100 points on this exam, with an additional 25 points for the takehome problem. Notes are permitted.)

Problem 1 *(20 points) Suppose the half band filter*

$$P(\omega) = 1 + \sum_{n \text{ odd}} \mathbf{p}(n)e^{-in\omega}$$

satisfies $P(0) = 2$. From these facts alone, deduce that $P(\pi) = 0$, i.e., the filter is low pass.

Solution*:*

$$P(0) = 2 = 1 + \sum_{n \text{ odd}} \mathbf{p}(n)(1)^n \Longrightarrow \sum_{n \text{ odd}} \mathbf{p}(n) = 1.$$

$$P(\pi) = 1 + \sum_{n \text{ odd}} \mathbf{p}(n)(-1)^n = 1 - \sum_{n \text{ odd}} \mathbf{p}(n) = 1 - 1 = 0.$$

Problem 2 *(30 points) Let $\phi(t)$ be a bounded continuous scaling function with support in the bounded interval $[0, a)$ for a multiresolution analysis. Here $a = \sup\{x : \phi(x) \neq 0\}$. The function is normalized by $\int \phi(t)\, dt = 1$. The integer translates of $\phi(t)$ form a basis (not necessarily orthonormal) for the subspace $V_0 \subset L^2(R)$. The dilation equation satisfied by $\phi(t)$ is*

$$\phi(t) = \frac{1}{2}\phi(2t) + \phi(2t-1) + \frac{1}{2}\phi(2t-2).$$

1. *(15 points) From these facts determine whether the basis functions $\{\phi(t-k)\}$ are orthonormal. Justify your answer.*

   Solution*: A necessary condition for the integer translates of the scaling function to form an ON set is that the filter coefficients $\mathbf{c}[k]$ be doubleshift orthogonal. In particular N must be odd. However, in this case $N = 2$ so the translates can't be ON. Note also that the filter coefficients are $(2^{-3/2}, 2^{-1/2}, 2^{-3/2})$, clearly not doubleshift orthogonal.*

2. *(15 points) Use the dilation equation to show that $\phi(0) = 0$. Show also that $\phi(2) = 0$.*

   Solution*: Set $t = 0$ in the dilation equation. Then*

   $$\phi(0) = \frac{1}{2}\phi(0) + \phi(-1) + \frac{1}{2}\phi(-2).$$

   Since the support of $\phi(t)$ is contained in $[0, a)$ with $a \geq 0$ we must have $\phi(-1) = \phi(-2) = 0$. Thus $\phi(0) = \frac{1}{2}\phi(0)$, or $\phi(0) = 0$. Now set $t = 1$ in the dilation equation. Then

   $$\phi(1) = \frac{1}{2}\phi(2) + \phi(1) + \frac{1}{2}\phi(0).$$

   But $\phi(0) = 0$ and $\phi(1)$ cancels out of the equation, so $\phi(2) = 0$.

3. *EXTRA CREDIT. Can you show that $a \leq 2$? HINT: If $a > 2$ then $2a - 2 = a + (a-2) > a$ and also $2a, 2a - 1 > a$.*

   Solution*: Suppose $a > 2$. Then $2a - 2 > a$, $2a > a$ and $2a - 1 > a$. Since $a = \sup\{x : \phi(x) \neq 0\}$, we can find a number b such that $2 < b <$*

a, $\phi(b) \neq 0$ and also $2b-2 > a$, $2b > a$ and $2b-1 > a$. But from the dilation equation

$$\phi(b) = \frac{1}{2}\phi(2b) + \phi(2b-1) + \frac{1}{2}\phi(2b-2).$$

Since all terms on the right-hand side of this equation are 0 we must have $\phi(b) = 0$. This is a contradiction! Hence $a \leq 2$.

Problem 3 *(35 points) Let $\phi(t)$ and $w(t)$ be the Haar scaling and wavelet functions. Let V_j and W_j be the spaces generated by $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$ and $w_{j,k}(t) = 2^{j/2}w(2^j t - k)$, $k = 0, \pm 1, \cdots$, respectively. Let $f(t) \in L^2(R)$ be defined given by*

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

1. *(15 points) Verify that the orthogonal projection $f_1(t)$ of $f(t)$ on the subspace V_1 of functions constant on intervals of half-integer length is*

$$f_1(t) = \begin{cases} 1/4 & 0 \leq t < 1/2 \\ 3/4 & 1/2 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution: *By definition*

$$f_1(t) = \sum_k a_{1k}\phi_{1k}(t), \qquad a_{1k} = (f, \phi_{1k}).$$

Since only ϕ_{10} and ϕ_{11} have support whose intersection with the support of f is nonzero, only a_{10}, a_{11} can be nonzero. We have

$$a_{10} = (f, \phi_{10}) = \sqrt{2}\int_0^{\frac{1}{2}} t\ dt = \frac{\sqrt{2}}{8},$$

$$a_{11} = (f, \phi_{11}) = \sqrt{2}\int_{\frac{1}{2}}^1 t\ dt = \frac{3\sqrt{2}}{8}.$$

Thus

$$f_1(t) = a_{10}\phi_{10}(t) + a_{11}\phi_{11}(t) = \frac{\sqrt{2}}{8}\sqrt{2}\phi(2t) + \frac{3\sqrt{2}}{8}\sqrt{2}\phi(2t-1) = \begin{cases} 1/4 & 0 \leq t < 1/2 \\ 3/4 & 1/2 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

2. *(10 points) Express f_1 in terms of the $\phi_{1,k}$ basis for V_1.*

Solution*:*

$$f_1(t) = \frac{\sqrt{2}}{8}\phi_{10}(t) + \frac{3\sqrt{2}}{8}\phi_{11}(t).$$

3. *(10 points) Decompose f_1 into its component parts in W_0, and V_0. In other words, find the Haar wavelet decomposition for f_1.*

Solution:

$$\phi_{00} = \frac{1}{\sqrt{2}}\phi_{10} + \frac{1}{\sqrt{2}}\phi_{11}, \quad w_{00} = \frac{1}{\sqrt{2}}\phi_{10} - \frac{1}{\sqrt{2}}\phi_{11},$$

so

$$\phi_{10} = \frac{1}{\sqrt{2}}\phi_{00} + \frac{1}{\sqrt{2}}w_{00}, \quad \phi_{11} = \frac{1}{\sqrt{2}}\phi_{00} - \frac{1}{\sqrt{2}}w_{00}.$$

Thus

$$f_1(t) = \frac{\sqrt{2}}{8}\phi_{10}(t) + \frac{3\sqrt{2}}{8}\phi_{11}(t) = \frac{1}{2}\phi_{00}(t) - \frac{1}{4}w_{00}(t) = f_0(t) + w_0(t),$$

where

$$f_0(t) = \begin{cases} 1/2 & 0 \le t < 1 \\ 0 & \text{otherwise,} \end{cases} \qquad w_0(t) = \begin{cases} -1/4 & 0 \le t < 1/2 \\ 1/4 & 1/2 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4 *(15 points) The reverse Daubechies 4-tap filter is related to Daubechies $D_4 = db2$ by reversing the order of the filter coefficients in the z-transform. Thus the transform of the reverse filter is*

$$C(z) = \frac{1}{4\sqrt{2}}\left((1-\sqrt{3}) + (3-\sqrt{3})z^{-1} + (3+\sqrt{3})z^{-2} + (1+\sqrt{3})z^{-3}\right).$$

It satisfies the properties $C(1) = \sqrt{2}$, $C(-1) = 0$ (low pass) and $|C(z)|^2 + |C(-z)|^2 = 2$ (double-shift orthogonality). Is it true that the reverse filter also has a root of degree 2 at $z = -1$, i.e., that $C'(-1) = 0$? Justify your answer.

Solution 1*:*

$$C'(z) = \frac{1}{4\sqrt{2}}\left(-(3-\sqrt{3})z^{-2} - 2(3+\sqrt{3})z^{-3} - 3(1+\sqrt{3})z^{-4}\right),$$

so

$$C'(-1) = \frac{1}{4\sqrt{2}}\left(-(3-\sqrt{3}) + 2(3+\sqrt{3}) - 3(1+\sqrt{3})\right) = 0.$$

Solution 2*: We can factor $C(z)$ to get*

$$C(z) = \frac{z^{-3}}{4\sqrt{2}}(1+z)^2\left((1-\sqrt{3})z + (1+\sqrt{3})\right).$$