MECHANISMS FOR VARIABLE SEPARATION IN PARTIAL DIFFERENTIAL EQUATIONS AND THEIR RELATIONSHIP TO GROUP THEORY

Willard Miller, Jr.†‡

Abstract. This is a survey of those techniques used to obtain explicit solutions of partial differential equations through the idea of "separation of variables". Frequently, but not always, variable separation is associated with the symmetries of a differential equation. We have a good understanding of variable separation for the scalar equations of mathematical physics. For spinor equations many interesting results are known but there is, as yet, no general theory.

1. Introduction. These lectures constitute a presentation of the basic concepts in the theory of separation of variables for partial differential equations and an exploration of the deep relations between variable separation and the generalized Lie symmetries of these equations. Historically, the theory of variable separation has been developed most intensively and proved most useful for two classes of partial differential equations: first order nonlinear equations and linear equations of all orders. However, the concepts are clearly applicable to general nonlinear equations. We will pay particular attention to variable separation for systems of equations, the part of the field which is of most active current interest.

The primary use of variable separation is for computation of explicit solutions of partial differential equations. The solutions can be calculated by solving ordinary differential equations (the separation equations). Many of the solutions obtained by this method prove so important that these functions are studied and tabulated in their own right: the special functions of mathematical physics. For Hamilton-Jacobi equations variable separation is used to obtain complete integrals which in turn lead to explicit solutions of the associated Hamiltonian system. For linear equations the Fourier method can be used to solve boundary value problems and represent a wide variety of functions as sums or integrals of separated solutions.

Basically, a partial differential equation is (additively) separable in the independent variables $x_1, \dots, x_n$ if the equation admits a nontrivial solution of the form

$$u = \sum_{i=1}^{n} S^{(i)}(x_i).$$

†Supported in part by the National Science Foundation under grant DMS 86–00372
‡School of Mathematics, and Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, MN 55455

Typeset by $\mathcal{AMS}$-TeX

One can also talk about product separation $v = \prod_{i=1}^{n} T^{(i)}(x_i)$ or more complicated types of separation such as $w = \tan[\sum_{i=1}^{n} S^{(i)}(x_i)]$. However a change of dependent variable reduces these other types to additive separation, e.g., $u = \ln v$ or $u = \arctan w$. In §2 we give a precise and useful definition of variable separation.

In §3 we shall review briefly the theory of generalized Lie symmetries of partial differential equations and show that the standard procedure for computing *symmetry adapted* solutions of these equations from a knowledge of the Lie symmetries is an example of variable separation of a particularly simple type.

In §4 we shall apply the theory of variable separation to two particularly simple and physically important problems: orthogonal variable separation for Hamilton-Jacobi and Helmholtz equations. This will lead us to the mathematics of Stäckel form. Then in §5 we will provide an intrinsic characterization of variable separation for these scalar equations in terms of Lie symmetries. In §6 we introduce the basics of variable separation for systems of equations of Dirac type. Finally, in §7 we shall make some general comments about variable separation for systems of equations. The basic problems in this area remain unsolved.

These lectures are concerned with the method of variable separation itself; space does not permit a study of the properties of the separable solutions and the relationship between these properties and Lie symmetries. This relationship is explored in the monograph [1]. Excellent expositions of applications of separable functions to the solution of initial and boundary value problems can be found in [2] and [3].

2. Additive separation. We start with a definition of additive separability for a system of partial differential equations of the form

$$H^{(K)}(x_i, u^{(L)}, u_i^{(L)}, u_{ij}^{(L)}, \dots) = E^{(K)} \tag{2.1}$$

in coordinates $x_1, \dots, x_n$. Here, $u^{(L)}$ are the dependent variables, $u_i^{(L)} = \partial_{x_i} u^{(L)}, u_{ij}^{(L)} = \partial_{x_i}\partial_{x_{ij}} u^{(L)}$, etc., where $1 \le i, j, \dots \le n, 1 \le K, L, \dots \le N$ and the $E^{(K)}$ are parameters. Note that the number of equations, N, equals the number of dependent variables. We assume that each $H^{(K)}$ is a polynomial in a finite number of the variables $u^{(L)}, u_i^{(L)}, u_{ij}^{(L)}, \dots$ with coefficients which are real analytic functions of the variables $x_i, u^{(L)}$, all defined in a common domain $D \times J^N, D \subseteq R^n$ with $(0, \dots, 0) \in D$, and J^N an open set in R^N.

A *solution* of (2.1) is a vector valued function $\mathbf{u} = \mathbf{S}(\mathbf{x}, \mathbf{E})$ where $\mathbf{u} = (u^{(1)}, \dots, u^{(N)})$, defined and analytic for $\mathbf{x} = (x_1, \dots, x_n)$ in a nonzero domain $D' \subseteq D$ and $\mathbf{E}$ in an open interval $R' \subseteq R^N$, such that the substitution of this function into (2.1) renders (2.1) an identity for all $(\mathbf{x}, \mathbf{E}) \in D' \times R'$. An *additive separable solution* is a solution of the form $\mathbf{u} = \sum_{j=1}^{n} \mathbf{S}^{(j)}(x_j, \mathbf{E})$.

Since a separable solution in the coordinates $\mathbf{x}$ satisfies $u_{ij}^{(L)} = 0$ for $i \ne j$, without loss of generality we can set all mixed partial derivatives identically equal to zero and write (2.1) in the form

$$H^{(K)}(x_i, u^{(L)}, u_i^{(L)}, u_{ii}^{(L)}, \dots) = E^{(K)}. \tag{2.2}$$

For convenience we set $u_{i,1}^{(L)} \equiv u_i^{(L)}, u_{i,j+1}^{(L)} = \partial_{x_i} u_{i,j}^{(L)}, j = 1, 2, \dots$ and define m_i to be the largest number j such that $\partial_{u_{i,j}^{(L)}} \mathbf{H} \not\equiv \mathbf{0}$ for some L. To avoid discussion of degenerate cases we require $m_i > 0$ for $i = 1, 2, \ldots, n$. Let D_i denote the total differentiation operators

$$D_i = \partial_{x_i} + \sum_L u^{(L)}_{i,1} \partial_{u^{(L)}} + \sum_{j=1}^{\infty} \sum_L u^{(L)}_{i,j+1} \partial_{u^{(L)}_{i,j}}. \tag{2.3}$$

Our final nondegeneracy assumption is that $\mathbf{u}$ is a separable solution of (2.2) such that the $N \times N$ matrix $(H(i)^K_L) = (H^{(K)}_{u^{(L)}_{i,m_i}})$ is nonsingular for all i, i.e.,

$$\det H(i) \neq 0, \quad i = 1, \ldots, n. \tag{2.4}$$

Then the evident equations $D_j H^{(K)}(\mathbf{x}, \mathbf{u}) = 0$ can be solved uniquely for $u^{(L)}_{i,m_i+1}$:

$$u^{(L)}_{i,m_i+1} = -\sum_{K=1}^{N} \tilde{D}_i H^{(K)} A(i)^L_K, \quad i = 1, \ldots, n, \tag{2.5}$$

where

$$\tilde{D}_i = \partial_{x_i} + \sum_L u^{(L)}_{i,1} \partial_{u^{(L)}} + \sum_{j=1}^{m_i - 1} \sum_L u^{(L)}_{i,j+1} \partial_{u^{(L)}_{i,j}} \tag{2.6}$$

and

$$\sum_{K=1}^{N} H(i)^K_L A(i)^M_K = \delta^M_L, \quad 1 \leq L, M \leq N. \tag{2.7}$$

Here, δ^M_L is the Kronecker delta. Clearly, $\mathbf{u}$ satisfies integrability conditions $D_j \mathbf{u}_{i,m_i+1} = 0, \quad j \neq i$, or

$$\begin{aligned} \tilde{D}_j \tilde{D}_i H^{(L)} &- \sum_{J,K=1}^{N} [\tilde{D}_i H^{(K)} A(i)^J_K \tilde{D}_j H(i)^L_J + \tilde{D}_j H^{(K)} A(j)^J_K \tilde{D}_i H(j)^L_J] \\ &+ \sum_{J,K,P,Q=1}^{N} \tilde{D}_j H^{(P)} A(j)^J_P \tilde{D}_i H^{(K)} A(i)^Q_K H^{(L)}_{u^{(Q)}_{i,m_i}, u^{(J)}_{j,m_j}} = 0. \end{aligned} \tag{2.8}$$

Multiplying both sides of (2.8) by $\det H(i) \det H(j)$ we see that the resulting expression is a polynomial in the derivatives of $\mathbf{u}$. In general, (2.8) is a restriction both on the coefficients of $\mathbf{H}$ and the form of the separable solution $\mathbf{u}$. However, there is an important special case where (2.8) is an identity in the variables $\mathbf{u}, \mathbf{u}_{k,\ell}$. [Indeed, this case will occur if (2.2) admits so many separable solutions that for each $\mathbf{x}^0 \in D$ and each set of real constants $\mathbf{u}^0, \mathbf{u}^0_i, \mathbf{u}^0_{ii}, \ldots, \quad i = 1, 2, \ldots, n$ satisfying $\mathbf{H}(\mathbf{x}^0, \mathbf{u}^0, \mathbf{u}^0_i, \mathbf{u}^0_{ii}, \ldots) = \mathbf{E}$, there is a separable solution $\mathbf{u}(\mathbf{x})$ such that $\mathbf{u}(\mathbf{x}^0) = \mathbf{u}^0, \mathbf{u}_i(\mathbf{x}^0) = \mathbf{u}^0_i, \ldots$.] Then conditions (2.8) reduce to restrictions on the coefficients of $\mathbf{H}$ which are independent of the choice of separable solution. If the polynomial version of (2.8) is an identity we say that $\{x_i\}$ is a *regular separable coordinate system* (for the equation (2.2)).

Suppose $\{x_i\}$ is a regular separable coordinate system and consider the equations

$$\begin{aligned} D_i\mathbf{v} &= \mathbf{v}_{i,1}, \\ D_i\mathbf{v}_{j,1} &= \delta_{i,j}\mathbf{v}_{j,2}, \\ &\vdots \\ D_i\mathbf{v}_{j,m_j-1} &= \delta_{i,j}\mathbf{v}_{j,m_j}, \\ D_i\mathbf{v}_{j,m_j} &= -\delta_{i,j}\tilde{D}_i\mathbf{H}A(i), \quad 1 \le i,j \le n, \end{aligned} \tag{2.9}$$

where $\mathbf{v}$ is an N component vector. The integrability condition for this system of equations is $D_kD_i\mathbf{v}_{j,m_j} = D_iD_k\mathbf{v}_{j,m_j}$, equivalent to (2.8). Since (2.8) is satisfied identically, it follows that for each $\mathbf{x}^0 \in D$ and each set of constants $\mathbf{v}^0_{i,j}, 1 \le i \le n, 0 \le j \le m_i$, such that $\det H(\mathbf{x}^0, \mathbf{v}^0) \ne 0$, there is a unique solution $\mathbf{v}$ of the system (2.9) such that $\mathbf{v}(\mathbf{x}^0) = \mathbf{v}^0, D_i\mathbf{v}(\mathbf{x}^0) = \mathbf{v}^0_{i,1}, D_i\mathbf{v}_{i,j}(\mathbf{x}^0) = \mathbf{v}^0_{i,j+1}$ (see [10, Chapter 1]). Choose $\mathbf{E}$ such that $\mathbf{E} = \mathbf{H}(\mathbf{x}^0, \mathbf{v}^0)$. Then $\mathbf{u} = \mathbf{v}(\mathbf{x})$ is a separable solution of (2.2) such that $\mathbf{u}(\mathbf{x}^0) = \mathbf{v}^0$ and $\mathbf{u}_{i,j}(\mathbf{x}^0) = \mathbf{v}^0_{i,j}, 1 \le i \le n, 1 \le j \le m_i$. Indeed $D_{x_i}\mathbf{H}(\mathbf{x},\mathbf{v}) = \mathbf{0}$, so $\mathbf{H}(\mathbf{x},\mathbf{v}) = \mathbf{E}$.

Theorem 1. *If $\{x_i\}$ is a regular separable system for the equation $\mathbf{H} = \mathbf{E}$, i.e., if equations (2.8) are satisfied identically, then for every set of $N(m_1 + m_2 + \cdots + m_n + 1)$ constants $\{\mathbf{v}^0, \mathbf{v}^0_{i,j}\}$ with $\mathbf{H}(\mathbf{x}^0, \mathbf{v}^0) = \mathbf{E}$ and $\det H(i)(\mathbf{x}^0, \mathbf{v}^0) \ne 0, 1 \le i \le n$, there is a unique separable solution $\mathbf{u}$ of $\mathbf{H}(\mathbf{x},\mathbf{u}) = \mathbf{E}$ such that $\mathbf{u}(\mathbf{x}^0) = \mathbf{v}^0, \mathbf{u}_{i,j}(\mathbf{x}^0) = \mathbf{v}^0_{i,j}, 1 \le i \le n, 1 \le j \le m_i$.*

If equations (2.8) are not satisfied identically, separable solutions still may exist, but they will depend on fewer than $N(\sum_{i=1}^n m_i + 1)$ parameters. This type of separation is *nonregular*.

An interesting special case of the preceding considerations occurs when $E^{(k)} = E$ in (2.2), i.e., when all of the constants $E^{(k)}$ are equal. (We shall see that this corresponds to systems of equations analogous to the Dirac equations in relativistic quantum theory.) It is easy to see that the following result holds [4,5].

Corollary 1. *If $\{x_i\}$ is a regular separable system for the equation $\mathbf{H} = \mathbf{E}$ with $E^{(k)} = E$, i.e., if equations (2.8) are satisfied identically, then for every set of $N(m_1+m_2+\cdots+m_n+1)$ constants $\{\mathbf{v}^0, \mathbf{v}^0_{i,j}\}$ with $\mathbf{H}(\mathbf{x}^0, \mathbf{v}^0) = \mathbf{E}$ and $\det H(i)(\mathbf{x}^0, \mathbf{v}^0) \ne 0$, $1 \le i \le n$, there is a unique separable solution $\mathbf{u}$ of $\mathbf{H}(\mathbf{x},\mathbf{u}) = \mathbf{E}$ such that $\mathbf{u}(\mathbf{x}^0) = \mathbf{v}$, $\mathbf{u}_{i,j}(\mathbf{x}^0) = \mathbf{v}^0_{i,j}$, $1 \le i \le n$, $1 \le j \le m_i$.*

Another important special case occurs for $N = 1$. Then we have a single partial differential equation

$$H(x_i, u, u_i, u_{ii}, \dots) = E, \tag{2.10}$$

rather than a system. Then conditions (2.8) take the simpler form $(i \ne j)$

$$\begin{aligned} &H_{u_{i,m_i}}H_{u_{j,m_j}}(\tilde{D}_i\tilde{D}_jH) + H_{u_{i,m_i}u_{j,m_j}}(\tilde{D}_iH)(\tilde{D}_jH) \\ &\quad = H_{u_{j,m_j}}(\tilde{D}_iH)(\tilde{D}_jH_{u_{i,m_i}}) + H_{u_{i,m_i}}(\tilde{D}_jH)(\tilde{D}_iH_{u_{j,m_j}}) \end{aligned} \tag{2.11}$$

Corollary 2. *If $\{x_i\}$ is a regular separable system for the equation $H = E$, i.e., if equations (2.11) are satisfied identically, then for every set of $m_1+m_2+\cdots+m_n+1$ constants $\{v^0, v^0_{i,j}\}$ with $H(\mathbf{x}^0, v^0) = E$ and $H_{u_{j,m_j}}(\mathbf{x}^0, v^0) \neq 0$, there is a unique separable solution u of $H(\mathbf{x}, u) = E$ such that $u(\mathbf{x}^0) = v^0$, $u_{i,j}(\mathbf{x}^0) = v^0_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq m_i$.*

We shall postpone the analysis of separation of variables for systems of equations to §6 and restrict our attention here to a single partial differential equation. The following examples exhibit regular and nonregular separation for scalar equations.

Example 1. $H = (x_1 + x_2)(u_{11} + u_{22}) - 2(u_1 + u_2)$. Equations (2.11) are satisfied identically so $\{x_1, x_2\}$ is a regular separable system. The general separable solution depends on five parameters and is given by

$$u = (\alpha x_1^3 + \beta x_1^2 + \gamma x_1 - \frac{1}{2}Ex_1) + (-\alpha x_2^3 + \beta x_2^2 - \gamma x_2 + \delta). \tag{2.12}$$

Example 2. $H = u_{11}^2 + u_1 + u_{22}$. Here we have $u_{111} = -\frac{1}{2}$ (provided $u_{11} \neq 0$) and $u_{222} = 0$ so equations (2.11) are satisfied identically and $\{x_1, x_2\}$, is a regular separable system. The general separable solution depends on five parameters:

$$u = (-\frac{1}{12}x_1^3 + \alpha x_1^2 + \beta x_1) + \left(\frac{1}{2}(E - 4\alpha^2 - \beta)x_2^2 + \gamma x_2 + \delta\right). \tag{2.13}$$

Example 3. $H = x_2 u_{11} + x_1 u_{22} + u_1 + u_2$. Equations (2.11) reduce to the requirement $u_{11} + u_{22} = 0$. The general separable solution depends on four parameters:

$$u = (\alpha x_1^2 + \beta x_1) + (-\alpha x_2^2 + (E - \beta)x_2 + \gamma). \tag{2.14}$$

This is a nonregular separable system.

Example 4. $H = (u_{11} + u_{22})/u$. Equations (2.11) are satisfied identically for $u \neq 0$. The general separable solution depends on five parameters:

$$u = \alpha \exp(x_1\sqrt{E}) + \beta \exp(-x_1\sqrt{E}) + \gamma \exp(x_2\sqrt{E}) + \delta \exp(-x_2\sqrt{E}) \tag{2.15}$$

for $E > 0$, with obvious modifications for $E \leq 0$.

There is a similar theory of additive separation for partial differential equations of the form (2.1) with $E^{(k)} = 0$, i.e., equations not depending on a parameter. For simplicity we consider only the case of a scalar equation; the spinor case is similar. We make the same assumptions on H as before and take the equation in the form (2.10) with $E = 0$:

$$H(x_i, u, u_i, u_{ii}, \ldots) = 0 \tag{2.16}$$

Then a separable solution u of (2.16) must satisfy the integrability conditions (2.11). In case the integrability conditions are identities in the sense that there exist functions $P_{i,j}(x_k, u, u_{k,\ell})$, polynomials in $u_{k,\ell}$ such that

$$\begin{aligned} F_{ij} \equiv\ & H_{u_{i,m_i}} H_{u_{j,m_j}} (\tilde{D}_i \tilde{D}_j H) + H_{u_{i,m_i}, u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H) \\ & - H_{u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H_{u_{i,m_i}}) - H_{u_{i,m_i}} (\tilde{D}_j H)(\tilde{D}_i H_{u_{j,m_j}}) \\ =\ & P_{i,j} H, \qquad i \neq j, \end{aligned} \tag{2.17}$$

we say that $\{x_k\}$ is a *regular separable coordinate* system for the equation $H = 0$.

Theorem 2. *If $\{x_k\}$ is a regular separable system for $H=0$ then for every set of $m_1+m_2+\cdots+m_n+1$ constants $\{v^0, v^0_{i,j}\}$ with $H(x^0,v^0)=0$ and $H_{u_{j,m_j}}(x^0,v^0)\neq 0$, there is a unique separable solution u of $H(x,u)=0$ such that $u(x^0)=v^0$, $u_{i,j}(x^0)=v^0$, $u_{i,j}(x^0)=v^0_{i,j}$, $1\le i\le n$, $1\le j\le m_i$.*

Again we observe that if equations (2.17) are not satisfied identically, separable solutions still may exist but will depend on fewer that $\sum_{i=0}^n m_i+1$ independent parameters. This is *nonregular* separation. Examples 1-4 above for $E=0$ are instances of regular and nonregular separation. Less trivial is

Example 5. $H=(x_2-x_3)u_{11}+(x_3-x_1)u_{22}+(x_1-x_2)u_{33}$. Equations (2.17) are satisfied with $P_{i,j}\not\equiv 0$, so $\{x_k\}$ is a regular separable system for $H=0$, though not for $H=E$. The general separable solution depends on six parameters and is given by

$$u=\frac{1}{6}\alpha(x_1^3+x_2^3+x_3^3)+\frac{1}{2}\beta(x_1^2+x_2^2+x_3^2)+\gamma_1x_1+\gamma_2x_2+\gamma_3x_3+\delta. \tag{2.18}$$

For other definitions of separability see [6].

3. Generalized Lie symmetries.

In this section we list some concepts and formulas from the Lie symmetry theory of partial differential equations that are essential for understanding the significance of variable separation. We limit ourselves here to an enumeration of basic results. For detailed proofs and background material see Olver's recent book[7]. See also [8,9].

In the following all functions are assumed to be locally real analytic. We consider local independent coordinates $x_1,\ldots,x_n$ and a dependent coordinate u. Given an assignment $u=f(x)$ we introduce the notation

$$u^J=\partial^J f(x),\qquad \partial^J=\frac{\partial^{|J|}}{\partial_{x_1}^{j_1}\partial_{x_2}^{j_2}\ldots\partial_{x_n}^{j_n}}, \tag{3.1}$$

to represent the k^{th} order derivatives of f. Here $J=(j_1,\ldots,j_n)$ with each j_i a nonnegative integer and $|J|=j_i+\cdots+j_n=k$. A *generalized vector field* on the $(n+1)$-dimensional space of independent and dependent variables is an expression of the form

$$\hat{Z}=\sum_{i=1}^n \xi_i(x,u^J)\frac{\partial}{\partial x_i}+\varphi(x,u^J)\frac{\partial}{\partial u} \tag{3.2}$$

where ξ_i and φ depend on x,u and finitely many derivates of $u,(|J|\le\ell<\infty)$. Locally this field generates new coordinates $X^*(\alpha),u^*(\alpha)$ obtained by solving

$$\frac{\partial x_i^*(\alpha)}{\partial\alpha}=\xi_i(x^*,u^{*J}),\quad x_i^*(0)=x_i,$$
$$\frac{\partial u^*(\alpha)}{\partial\alpha}=\varphi(x^*,u^{*J}),\quad u^*(0)=u=f(x).$$

If ξ_i and φ depend only on $\mathbf{x}$ and u, then $\hat{Z}$ is the generator of a Lie point transformation. In an obvious way $\hat{Z}$ can be prolonged to a vector field on the jet space with coordinates $(\mathbf{x}, u, u^J)$:

$$Z = \hat{Z} + \sum_{|K|>0} \varphi^K(x, u^J)\frac{\partial}{\partial u^K}, \tag{3.3}$$

where

$$\begin{aligned} \varphi^K &= D^K(\varphi - \sum_{i=1}^{n} u^i \xi_i) + \sum_{i=1}^{n} u^{K,i}\xi_i, \\ K &= (k_1, \dots, k_n), \qquad J = (j_1, \dots, j_n), \\ J, i &= (j_1, \dots, j_{i-1}, j_i + 1, j_{i+1}, \dots, j_n), \\ D^K &= D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n}, \end{aligned} \tag{3.4}$$

And D_i is the *total derivative*

$$D_i = \frac{\partial}{\partial x_i} + \sum_{|K|\geq 0} u^{K,i}\frac{\partial}{\partial u^K}. \tag{3.5}$$

[Although the sums in (3.3) and (3.5) are formally infinite, in practice we will apply these operators only to functions that depend on finitely many derivatives u^K.]

Let $H(\mathbf{x}, u^K) = 0$ be a partial differential equation for u. We say that the generalized vector field $\hat{Z}$ is a *generalized symmetry operator* for this equation provided

$$ZH(\mathbf{x}, u^K) = 0, \tag{3.6}$$

whenever $H(\mathbf{x}, u^K) = 0$ i.e., ZH vanishes for all solutions u of $H = 0$. In the following we will make the technical hypothesis, almost always satisfied by partial differential equations of physical interest, that for each generalized symmetry operator $\hat{Z}$ there are a finite number of functions $\chi_J(\mathbf{x}, u^K)$ such that

$$ZH = \sum_J \chi_J D^J H. \tag{3.7}$$

See [7,page 135], for a discussion of this specialization of (3.6).

The *characteristic* ϕ of a generalized vector field $\hat{Z}$ is defined by

$$\phi = \varphi - \sum_{i=1}^{n} u^i \xi_i. \tag{3.8}$$

Note that the generalized vector field $\hat{X}(\phi) = \phi\partial_u$ has the prolongation

$$X(\phi) = \phi\partial_u + \sum_{|K|>0} D^K\phi\frac{\partial}{\partial_{u^K}}. \tag{3.9}$$

We call $\hat{X}(\phi)$ the *standard representation* of $\hat{Z}$. It follows from (3.3) and (3.4) that

$$Z = X(\phi) + \sum_{i=1}^{n} \xi_i D_i, \tag{3.10}$$

and from (3.7) that $\hat{Z}$ is a generalized symmetry for the equation $H = 0$ if and only if its standard representation $\hat{X}(\phi) = \phi\partial_u$ is a generalized symmetry. Thus, from the viewpoint of Lie symmetries there is no loss of generality in restricting to operators of the form $\phi\partial_u$.

The *commutator* of two prolongations of standard representation operators is an operator of the same form:

$$[X(\phi_1), X(\phi_2)] \equiv X(\phi_1)X(\phi_2) - X(\phi_2)X(\phi_1) = X(\{\phi_1, \phi_2\}), \tag{3.11}$$

where

$$\begin{gathered} \{\phi_1, \phi_2\} = -\{\phi_2, \phi_1\} \\ \{a_1\phi_1 + a_2\phi_2, \phi_3\} = a_1\{\phi_1, \phi_3\} + a_2\{\phi_2, \phi_3\}, \quad a_i \in R, \\ \{\{\phi_1, \phi_2\}, \phi_3\} + \{\{\phi_2, \phi_3\}, \phi_1\} + \{\{\phi_3, \phi_1\}, \phi_2\} = 0. \end{gathered} \tag{3.13}$$

In terms of this bracket the symmetry condition (3.6)–(3.7) becomes

$$\{\phi, H\} = 0 \text{ whenever } H = 0. \tag{3.14}$$

If ϕ_1 and ϕ_2 depend only on the variables x_i and u^i , then (3.13) reduces to the Poisson bracket:

$$\{\phi_1, \phi_2\} = \sum_{i=1}^{n} (\partial_{x_i}\phi_1\partial_{u^i}\phi_2 - \partial_{x_i}\phi_2\partial_{u^i}\phi_1). \tag{3.15}$$

If ϕ_1 and ϕ_2 are linear in the u^K,

$$\phi_i = \sum_K a_K^{(i)}(\mathbf{x})u^K, \quad i = 1, 2,$$

then the bracket agrees with the usual operator commutator bracket:

$$\begin{gathered} \{\phi_1, \phi_2\} = -[L_1, L_2]u = -(L_1L_2 - L_2L_1)u, \\ L_i = \sum_K a_K^{(i)}(x)D^K, \qquad i = 1, 2, \end{gathered} \tag{3.16}$$

For simplicity we have restricted the above considerations to a scalar partial differential equation with a single dependent variable. However, the extension to spinor equations with many dependent variables is completely straightforward [7].

We are now ready to highlight the simplest point of contact between separation of variables methods and the symmetries of a partial differential equation

$$H(\mathbf{x}, u^K) = 0. \tag{3.17}$$

Let $\hat{Z} \neq 0$ be an infinitesimal point symmetry for (3.17) which is *projectable*, i.e. of the form

$$\hat{Z} = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i} + \varphi(x,u) \frac{\partial}{\partial u}, \tag{3.18}$$

and, in addition, suppose not all ξ_i are zero. Then by Lie's theorem (see [10, pages 34,49, and 50]) we can find new coordinates $s, y_2, \ldots, y_n, v$ such that

$$\mathbf{x} = A(s, \mathbf{y}), \qquad \mathbf{u} = B(s, \mathbf{y}, v), \tag{3.19}$$

and, in the new coordinates

$$\hat{Z} = \frac{\partial}{\partial s}. \tag{3.20}$$

Since $\hat{Z}$ is a point symmetry, it follows from (3.7) (and very mild technical assumptions on H) that H must be of the form

$$H = H_1(s, \mathbf{y}) H_2(\mathbf{y}, v^L), \tag{3.21}$$

where H_2 has nontrivial dependence on the v^L. Thus, each solution $v(s, \mathbf{y})$ of

$$H_2(y, v^L) = 0 \tag{3.22}$$

determines a solution u of $H(\mathbf{x}, u^K) = 0$. The equations $H = 0$ and $H_2 = 0$ are essentially equivalent. However, the second equation depends on s only through derivatives terms in v. Again, under mild technical assumptions on H, we can find solutions $v(\mathbf{y})$ of (3.22), i.e. solutions such that $\partial_s v \equiv 0$. [If $\partial_v H_2 \equiv 0$ then we can find solutions $v(s, \mathbf{y}) = \lambda s + v'(\mathbf{y})$ where λ is a constant.] For $n = 2$, Eq (3.22) reduces to an ordinary differential equation and the solutions of this equation yield *symmetry adapted* solutions of (3.17). Furthermore $(s, \mathbf{y})$ is a separable coordinate system for (3.17), though not necessarily a regular separable system. In general we have reduced a partial differential equation in n variables to one in $n-1$ variables, and this provides an instance of partial separation.

As a simple example, consider the symmetry

$$\hat{Z} = t\partial_x + c\partial_t - \partial_u, \qquad c \neq 0 \tag{3.23}$$

for the Korteweg-de Vries equation

$$\partial_t u - \partial_{xxx} u - u \partial_x u = 0. \tag{3.24}$$

(The symmetry algebra for this equation can be found in many references, e.g., [7]). The requirement $\hat{Z} = \partial_s$ leads to new coordinates (s, y, v) such that

$$t = cs, \quad y = \frac{1}{2} cs^2 + x, \quad v = -s + u.$$

In terms of the new coordinates, (3.24) becomes

$$\partial_s v - c\partial_{yyy} v - cv\partial_y v - 1 = 0. \tag{3.25}$$

The ordinary differential equation obtained by setting $\partial_s v \equiv 0$ is related to the first Painlevé transcendent, [7, page 107].

For a more general approach to symmetry adapted solutions see [9,64].

4. Separability for Hamilton-Jacobi, Helmholtz, and Laplace equations. We now apply the results of §2 to determine the possible regular separable coordinate systems for the Hamilton-Jacobi equation on a pseudo-Riemannian manifold V^n

$$H(x^i, u_i) \equiv \sum_{i,j=1}^{n} g^{ij} u_i u_j = E, \quad g^{ij} = g^{ji}, \tag{4.1}$$

where $u_i = \partial_i u$. In the local coordinates $\{x^i\}$ the metric on V^n is $ds^2 = \sum_{i,j=1}^n g_{ij} dx^i dx^j$ and $\sum_{j=1}^n g_{ij} g^{jk} = \delta_i^k$, $g = \det(g_{ij}) \neq 0$. (Here we change notation again to that of Eisenhart's book [11]. Recall that a solution $u(\mathbf{x}, \lambda_1, \ldots, \lambda_n)$, $\lambda_1 = E$ is a *complete integral* of (4.1) provided $\det(\partial_{x^i \lambda_j} u) \neq 0$, and that the knowledge of a complete integral enables us to solve the associated Hamiltonian system $p_i = -\partial_{x^i} H(\mathbf{x}, \mathbf{p})$, $x^i = \partial_{p_i} H(\mathbf{x}, \mathbf{p})$. Separation of variables is a powerful technique for obtaining explicit complete integrals for many Hamilton-Jacobi equations [12]). Initially we limit ourselves to *orthogonal* coordinates $\{x^i\}$, i.e., coordinates for which $ds^2 = \sum_{i=1}^n H_i^2 (dx^i)^2$, so that $g_{ij} = 0$ if $i \neq j$. Thus (4.1) becomes

$$\sum_{i=1}^{n} H_i^{-2} u_i^2 = E, \tag{4.2}$$

and from the integrability conditions (2.11) we see that $\{x^i\}$ is a regular separable system if and only if

$$\partial_{jk} H_i^{-2} = \partial_j H_i^{-2} \partial_k \ln H_j^{-2} + \partial_k H_i^{-2} \partial_j \ln H_k^{-2}, \quad j \neq k. \tag{4.3}$$

These are the standard Levi-Civita separability conditions and are well known to be equivalent to the requirement that the metric coefficients be in Stäckel form with respect to the coordinates $\{x^i\}$ (see [11,13,14]). That is, there exists an $n \times n$ matrix $s_{ji}(x^j)$, whose j^{th} row depends only on x^j, such that $S = \det(s_{ji}) \neq 0$, (a *Stäckel matrix*), and

$$H_j^{-2} = \frac{s^{j1}}{S}, \tag{4.4}$$

where s^{j1} is the $(j, 1)$ minor of (s_{ij}).

We will work out the proof of this fact, indeed we will generalize it, and study the theory of Stäckel matrices in §6. Here we only note that if the H_i^{-2} are in Stäckel form and u satisfies the set of ordinary differential equations (the separation equations)

$$u_i^2 + \sum_{j=1}^{n} \lambda_j s_{ij}(x^i) = 0, \quad i = 1, \ldots, n, \tag{4.5}$$

where $\lambda_1 = -E$, $\lambda_2, \ldots, \lambda_n$ are parameters and $u_i = u_i(x^i)$, then u is a separable solution of (4.2).

For the Hamilton-Jacobi equation with potential

$$\sum_{i=1}^{n} H_i^{-2} u_i^2 + V(x) = E, \tag{4.6}$$

the results are similar. The integrability conditions reduce to (4.3) and

$$\partial_{ik} V - \partial_k \ln H_j^{-2} \partial_j V - \partial_j \ln H_k^{-2} \partial_k V = 0, \quad j \neq k. \tag{4.7}$$

As shown in reference[15], this last condition means precisely that the potential function can be expressed in the form

$$V = \sum_{i=1}^{n} f^{(i)}(x^i) H_i^{-2}. \tag{4.8}$$

We will prove a generalization of this result in §6.

A regular orthogonal separable system for the Hamilton-Jacobi equation (4.2) with $E = 0$ is characterized by the integrability conditions

$$\partial_{jk} H_i^{-2} - \partial_j H_i^{-2} \partial_k \ln H_j^{-2} - \partial_k H_i^{-2} \partial_j \ln H_k^{-2} = \rho_{jk}(x) H_i^{-2} H_j^2 H_k^2, \qquad j \neq k, \tag{4.9}$$

for some functions ρ_{jk}. These equations are equivalent to

$$\partial_{jk} \ln K_i^{-2} + \partial_j \ln K_i^{-2} \partial_k \ln K_i^{-2} - \partial_j \ln K_i^{-2} \partial_k \ln K_j^{-2} - \partial_j \ln K_k^{-2} \partial_k \ln K_i^{-2} = 0, \tag{4.10}$$

for $K_i^{-2} = H_i^{-2}/H_1^{-2}$. Furthermore, as shown in reference [16], the equations are equivalent to the requirement that $H_i^{-2} = Q(\mathbf{x}) H_i^{-2}$, where the metric $ds^2 = \sum_{j=1}^{n} H_j^2 (dx^j)^2$ is in Stäckel form. The separation equations have the appearance (4.5) with $E = 0$.

Next we study the problem of (multiplicative) separation of variables for the Helmholtz (or Schrödinger) equation

$$(\Delta + V(\mathbf{x}))\Psi(\mathbf{x}) = E\Psi(\mathbf{x}), \tag{4.11}$$

on the pseudo-Riemannian manifold V^n. Here

$$\Delta = \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j) \tag{4.12}$$

is the Laplace-Beltrami operator on V^n, defined independent of local coordinates [11]. To convert this product separation problem, $\Psi = \Pi_{i=1}^{n} \Psi^{(i)}(x^i)$, to the standard additive separation form, we introduce the new dependent variable $u = \ln \Psi$. Further, we restrict ourselves to orthogonal separable systems

$$ds^2 = \sum_{i=1}^{n} H_i^2 (dx^i)^2.$$

Then (4.11) becomes

$$H \equiv \sum_{i=1}^{n} [H_i^{-2}(u_{ii} + u_i^2) + s_i u_i] + V = E, \tag{4.13}$$

where

$$s_i = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g} H_i^{-2}), \quad \sqrt{g} = H_1 H_2 \ldots H_n. \tag{4.14}$$

The integrability conditions (2.11) for regular separation lead to (4.3), (Stäckel form), upon comparison of the coefficients of u_i^2. Comparison of the coefficients of u_{ii} in (2.11) yields the *Robertson condition*[13]

$$\partial_{ij} \ln(\sqrt{g} H_i^{-2}) = 0, \qquad i \neq j. \tag{4.15}$$

Comparison of the constant terms in (2.11) yields the conditions (4.7) on the potential $V(\mathbf{x})$, i.e., the potential must be expressable in the form (4.8) to permit separation. There are no additional consequences of the integrability conditions. The separation equations take the form

$$u_{ii} + u_i^2 + g_i(x^i) u_i + f_i(x^i) + \sum_{j=1}^{n} \lambda_j s_{ij}(x^i) = 0, \tag{4.16}$$

where $\lambda_1 = -E$.

It follows that every orthogonal coordinate system permitting product separation of the Helmholtz equation (4.11) corresponds to a Stäckel form; hence it permits additive separation of the Hamilton-Jacobi equation (4.1). Eisenhart has shown, [11], that the additional Robertson condition for product separation is equivalent to the requirement $R_{ij} = 0$ for $i \neq j$, where R_{ij} is the Ricci tensor of V^n expressed in the Stäckel coordinates $\{x^i\}$. It follows that the Robertson condition is automatically satisfied in Euclidean space, a space of constant curvature or any Einstein space.

More generally we can introduce the notion of R-separation for the Helmholtz equation (4.13) in orthogonal coordinates $\{x^i\}$. Here, R-separable solutions take the form $\Psi = \exp(R(\mathbf{x}))\Pi_{i=1}^n \Psi^{(i)}(x^i) = e^R \Theta$, where $R(\mathbf{x})$ is a fixed function, independent of parameters. If $R \equiv 0$ we have *separation*, and if $R(x) = \sum_{i=1}^n R^{(i)}(x^i)$ we have *trivial* R-separation. Otherwise the R-separation is *nontrivial*. Writing $u = \ln \Theta = \ln \Psi - R$, we have the following generalization of (4.13):

$$H \equiv \sum_{i=1}^{n} \left[H_i^{-2}(u_{ii} + u_i^2) + (2H_i^{-2} \partial_i R + s_i) u_i + H_i^{-2}(\partial_{ii} R + (\partial_i R)^2) + s_i \partial_i R \right] + V = E. \tag{4.17}$$

Comparing the coefficients of u_i^2 in the integrability conditions (2.11) we again find the metric $ds^2 = \sum_{i=1}^n H_i^2 (dx^i)^2$ must be in Stäckel form. Comparison of the coefficients of u_{ii} yields

$$\partial_{ij}[2R + \ln(\sqrt{g} H_i^{-2})] = 0, \quad i \neq j. \tag{4.18}$$

Finally, comparison of the constant terms in (1.6) and use of (4.18) leads to requirement (4.7) for the *modified potential*,

$$\tilde{V} = V - \frac{1}{2}\sum_{i=1}^{n} H_i^{-2}(\partial_i \ell_i + \frac{1}{2}\ell_i^2), \tag{4.19}$$

where

$$\ell_i = \partial_i \ln(\sqrt{g}H_i^{-2}) = \partial_i \ln\frac{\sqrt{g}}{S}. \tag{4.20}$$

We see that whenever $\tilde{V}$ satisfies (4.7), hence (4.8), equation (4.11) permits orthogonal R-separation with

$$R = -\frac{1}{2}\ln\frac{g}{S} + \sum_{i=1}^{n} L^{(i)}(x^i), \tag{4.21}$$

where the functions $L^{(i)}$ are arbitrary. Thus through appropriate choice of V, every additively separable coordinate system $\{x^i\}$ for the zero-potential Hamilton-Jacobi equation can occur as a multiplicatively separable system for the Helmholtz equation. Details are given in reference [17].

The question arises whether nontrivial R-separation occurs for $V = 0$. From (4.15), (4.18) and Eisenhart's formulation of Robertson's condition as $R_{ij} = 0$, $i \neq j$, we see that only trivial orthogonal R-separation can occur in an Einstein space. However, as shown in reference [17], nontrivial R-separation can occur for $V = 0$, even in conformally flat spaces. An example is

$$\begin{aligned} ds^2 &= (x+y+z)[(x-y)(x-z)dx^2 + (y-z)(y-x)dy^2 + (z-x)(z-y)dz^2], \\ e^R &= (x+y+z)^{-\frac{1}{4}}. \end{aligned} \tag{4.22}$$

Orthogonal R-separation for the Laplace equation on V^n, $\Delta\Psi(\mathbf{x}) = 0$, can be treated by analogy to the foregoing [18,19], but will not be presented here.

Also, because of space and time limitations we shall not treat here the general case of (possibly) nonorthogonal separation for the Hamilton-Jacobi, Helmholtz, heat and Laplace equations. The details of the general case can be found in [18,20-25].

For explicit computations of the possible separable coordinate systems for an equation on a fixed manifold (e.g., Euclidean space or a space of constant curvature), see [1,15,26-30].

5. Intrinsic characterization of variable separation for scalar equations.
For the Hamilton-Jacobi, Helmholtz, and Laplace equations in V^n, introduced in the previous section, ($R-$)separable coordinate systems can always be characterized intrinsically, i.e., in a coordinate-free manner. Consider first the problem of

additive orthogonal separation for the Hamilton-Jacobi equation (4.1). Let H be the quadratic form

$$H = \sum_{i,j=1}^{n} g^{ij} u_i u_j, \quad u_i = \partial_{y_i} u, \tag{5.1}$$

where $(g^{ij}(y))$ is the metric expressed in terms of the general coordinates $\{y^k\}$ and u is the dependent variable. Expression (5.1) is invariant under a change of independent coordinates. Now assume that (4.1) is separable in the orthogonal coordinates $\{x^i\}$. From the separation equations (4.5), we are led to the quadratic forms $(\sum_{j=1}^{n} A^{\ell j} s_{jk}(x^j) = \delta^{\ell}_{k})$

$$A^{\ell} = \sum_{j=1}^{n} A^{\ell j} u_j^2 \quad \ell = 1, 2, \ldots, n, \tag{5.2}$$

which can easily be shown to have the following properties:

(1) $A^1 = H$.
(2) The n element set (A^{ℓ}) is linearly independent (as a set of quadratic forms).
(3) The differentials of the separable coordinates, $\omega^j = dx^j$, constitute a simultaneous eigenbasis for the (A^{ℓ}). [Here, ρ is a root of a quadratic form $A = (a^{ij})$ with respect to the metric g^{ij} if $\det(a^{ij} - \rho g^{ij}) = 0$, and $\omega = \sum_{\ell} \lambda_{\ell} dx^{\ell}$ is an eigenform corresponding to ρ if $\omega \neq 0$ and $\sum_{j=1}^{n}(a^{ij} - \rho g^{ij})\lambda_j = 0$.]
(4) $A^{\ell}(\mathbf{x}, u_x) = -\lambda_{\ell}$ for each additively separable solution $\mathbf{u} = \sum_{i=1}^{n} u^{(i)}(x^i, \lambda)$ of (4.1).

The following converse of these statements holds [24,31]:

Theorem 3. *Let (A^{ℓ}), $A^1 = H$, be a linearly independent set of n second order symmetric quadratic forms such that*

(1) $\{A^{\ell}, A^m\} = 0$, $1 \leq \ell, m \leq n$,
(2) *The (A^{ℓ}) have a common eigenbasis $\{\omega^{(j)}\}$.*

Then there is a separable coordinate system $\{x^j\}$ for the Hamilton-Jacobi equation $H(\mathbf{y}, v_y) = E$ on V^n such that $\omega^{(j)} = f^{(j)}(x)dx^j$ for some functions $f^{(j)}$. The separable solutions u are determined by $A^{\ell}(\mathbf{x}, u_{\mathbf{x}}) = -\lambda_{\ell}$.

The main point of this theorem is that, under the required hypotheses the eigenforms ω^{ℓ} of the quadratic forms a^{ij} are normalizable, i.e., that up to multiplication by a nonzero function, ω^{ℓ} is the differential of a coordinate. This fact, which is proved through use of the commutation relations $\{A^{\ell}, A^m\} = 0$ to verify appropriate integrability conditions, permits us to compute the coordinates directly from a knowledge of the symmetry operators.

As an example of the use of Theorem 3 we consider the Hamilton-Jacobi equation for two dimensional Minkowski space. In cartesian coordinates this equation is

$$H \equiv u_x^2 - u_t^2 = E.$$

The vector space of all symmetries of the form $L = a(x,t)u_x + b(x,t)u_y$ (Killing vectors) is easily shown to be closed under the bracket $\{\cdot,\cdot\}$; hence the symmetries form a Lie algebra. Furthermore we have $\{H, L\} \equiv 0$ for each linear symmetry. The Lie algebra is three dimensional over the field of real scalars, with basis

$$\begin{aligned} L_1 &= u_x, & L_2 &= u_t, & L_3 &= tu_x + xu_t, \\ \{L_1, L_2\} &= 0, & \{L_3, L_1\} &= L_2, & \{L_3, L_2\} &= L_1. \end{aligned} \tag{5.3}$$

It can be shown that every symmetry which is quadratic in the first derivatives of u (second order Killing tensor) is a polynomial in the linear symmetries L_i. (This is true for all spaces of constant curvature, e.g., [24].) Thus all candidates for variable separation can be built from the basis symmetries (5.3).

Consider, for example, the quadratic symmetry $A^2 = 2L_3L_1$. With respect to cartesian coordinates, the corresponding symmetric quadratic forms are

$$A^2 \sim \begin{pmatrix} 2t & x \\ x & 0 \end{pmatrix}, \quad A^1 = H \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly, A^2 has roots $\rho = t \pm \sqrt{t^2 - x^2}$ (assuming $t > |x|$) with a basis of eigenforms $\omega_1 = (t + \sqrt{t^2 - x^2})dx - xdt$, $\omega_2 = (t - \sqrt{t^2 - x^2})dx - xdt$. By Theorem 3, A^2 does define a regular separable coordinate system $\{\xi^1, \xi^2\}$ for (5.2) and there exist functions f_i such that $d\xi^i = f_i\omega_i$, $i = 1,2$. We find $f_1 = [\xi^2((\xi^2)^2 - (\xi^1)^2)]^{-1}$, $f_2 = -[\xi^1((\xi^2)^2 - \xi^1)^2)]^{-1}$,

$$t = \frac{1}{2}((\xi^1)^2 + (\xi^2)^2), \qquad x = \xi^1\xi^2. \tag{5.4}$$

On the other hand the symmetry

$$A = 2L_3(L_1 - L_2) \tag{5.5}$$

has two equal roots and only one eigenform. Thus A cannot determine a separable coordinate system.

For manifolds of dimension $n \geq 3$ there is a second way that a system of $n-1$ commuting symmetries may fail to determine separable coordinates: although each quadratic symmetry determines a basis of eigenforms, there is no basis of eigenforms for all symmetries simultaneously. See [24] for a simple example. Reference [31] considers *partial* separation for Hamilton-Jacobi equations.

These results extend to R-separation of the Helmholtz equation on V^n,

$$\Delta\Psi(\mathbf{x}) = E\Psi(\mathbf{x}), \tag{5.6}$$

where in local coordinates

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \partial_i(\sqrt{g}g^{ij}\partial_j). \tag{5.7}$$

Here a linear differential operator A on V^n is a *symmetry operator* for Δ if

$$[\Delta, A] \equiv \Delta A - A\Delta = 0. \tag{5.8}$$

[This agrees with the bracket (3.16).] Note that uniquely associated with every second order symmetry operator

$$A = \sum_{\substack{i=1\\j=1}}^{n} a^{ij}\partial_{ij} + \sum_{i=1}^{n} b^i \partial_i + c$$

in local coordinates $\{y^\ell\}$, is the second order Killing tensor

$$A = \sum_{i,j=1}^{n} a^{ij} u_i u_j .$$

Indeed, $[\Delta, A] = 0$ implies $\{H, A\} = 0$, though the converse is false.

Theorem 4. *Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system $\{x^j\}$ for the Helmholtz equation $\Delta\Psi = E\Psi$ are that there exist n second order differential operators $A^1 = \Delta, A^2, \ldots, A^n$ such that*

(1) $[A^i, A^j] = 0, \quad 1 \le i, j \le n.$
(2) *The associated set of Killing tensors (A^i) is linearly independent.*
(3) *There is a basis $\{\omega^{(j)} : 1 \le j \le n\}$ of simultaneous eigenforms for the A.*

If these conditions are satisfied then there exist functions $f^j(\mathbf{x})$ such that $\omega^{(j)} = f^j dx^j$, $1 \le j \le n$. Theorem 4 follows from Theorem 3 through exploitation of the commutation relations (1). Indeed, these relations can be used to show that the separation conditions for $\tilde{V}$ are valid.

Corollary 3. *Suppose the second order differential operators $A^1 = \Delta$, $A^2, \ldots, A^n$ satisfy conditions 1-3 of Theorem 4 and in addition that they are self-adjoint form with respect to the measure $dV = \sqrt{g}dy^1 \ldots dy^n$:*

$$A^\ell = \frac{1}{\sqrt{g}} \sum_{\substack{i=1\\j=1}}^{n} \partial_y i(\sqrt{g} a^{ij}_{(\ell)} \partial_{y^j}) + c_\ell(\mathbf{y}), \quad \ell = 1, 2, \ldots, n, \tag{5.9}$$

(a form which is independent of the choice of local coordinates $\{y^j\}$). Then the R-separable solutions $\Psi = e^R \prod_{i=1}^n \Psi^{(i)}(x^i)$ of $\Delta\Psi = E\Psi$ are characterized as the eigenfunctions of the A^ℓ:

$$A^\ell \Psi = -\lambda_\ell \Psi, \quad \ell = 1, 2, \ldots, n, \tag{5.10}$$

where $\lambda_1 = -E$ and $\lambda_2, \ldots, \lambda_n$ are separation constants.

This result is implied by theorems stated without published proof by Shapovalov, e.g. [18,33,34]. The results were obtained independently and proved in [17].

For the Hamilton-Jacobi equation

$$H(x, \partial_i u) = 0 \tag{5.11}$$

i.e., $E = 0$, there is an analogous characterization of additive separation by conformal symmetries. A function $L(x, \partial_i u)$ is said to be a *conformal symmetry* provided there is a function $q(x, \partial_i u)$ such that

$$\{L, H\} = qH. \tag{5.12}$$

Theorem 5 [16,31],**.** *Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system $\{x_j\}$ for the Hamilton-Jacobi equation (5.11) are that there exist $n-1$ symmetric quadratic functions $B^k = \sum_{i,j=1}^{n} b^{ij}_{(k)}(\mathbf{x})u_i u_j$, $2 \le k \le n$, such that*

(1) *Each B^k is a conformal symmetry.*
(2) *$\{B^i, B^j\} = 0$, $2 \le i, j \le n$.*
(3) *The set (H, B^k) is linearly independent (as n quadratic forms).*
(4) *There is a basis $\{\omega^{(j)} : 1 \le j \le n\}$ of simultaneous eigenforms for the $\{B^k\}$.*

If conditions **1-4** *are satisfied then there exist functions $f^j(\mathbf{x})$ such that $\omega^{(j)} = f^j dx^j$, $1 \le j \le n$.*

Again, these results extend to R-separation of the Laplace equation on V^n:

$$\Delta\Psi(\mathbf{x}) = 0, \tag{5.13}$$

see [18,19,34] .

The preceding theorems characterize orthogonal R-separation for Hamilton-Jacobi, Helmholtz, and Laplace equations in terms of symmetries. There are similar results for nonorthogonal separation of these equations, though the results are more complicated to state and prove: see [20,25,31,34].

We can reach several important conclusions concerning variable separation and R-separation for Hamilton-Jacobi, Helmholtz, and Laplace (or wave) equations. First, one must recognize the intrinsic geometric nature of R-separation. The apparently technical conditions for R-separation are equivalent to the existence of an n-dimensional family of commuting symmetry operators which can be simultaneously diagonalized. In spaces, such as those of constant curvature, for which all symmetry operators can be constructed from the Lie symmetry algebra, all R-separation questions become problems in algebra [1]. These problems have been solved explicitly for the Hamilton-Jacobi and Helmholtz equations on n-dimensional spheres and n-dimensional Euclidean space, [26,27,35].

Second, comparing Theorems 3 and 4, it is obvious that R-separation, not ordinary separation, for the Helmholtz equation is the natural analogy of additive separation for the Hamilton-Jacobi equation. Finally, we note the close relationship between variable separation and quantization theory. Corresponding to a separable system $\{x^j\}$ for the Hamilton-Jacobi equation we have an involutive family $\{A^\ell\}$ of quadratic constants of the motion. The Helmholtz equation R-separates in these same coordinates if and only if second order operators $\{A^\ell\}$ can be found (with the pure second order terms in A^ℓ agreeing with those of A^ℓ) such that the A^ℓ pairwise commute.

There are similar results relating symmetry and R-separation for heat and time-dependent Schrödinger equations, [1,23,33,29,34].

For early references to variable separation see [14,36,37].

6. Spinor equations.

In this section our aim will be to study multiplicative separation for spinor equations that take the form of the Dirac equation, i.e.,

$$H\psi(x) = E\psi(x) \tag{6.1}$$

where H is an $N \times N$ first order matrix differential operator in the variables $x^1, \ldots, x^n$, ψ is an N-component spinor and E is an eigenvalue. However, it is instructive to begin by studying additive separation for the system

$$\sum_{i=1}^{n} H^i \partial_i \varphi(x) = D \tag{6.2}$$

where $\partial_i = \partial_{x^i}$,

$$D = \begin{pmatrix} E^{(1)} & & 0 \\ & \ddots & \\ 0 & & E^{(N)} \end{pmatrix},$$

φ is an $N \times N$ matrix, and the $N \times N$ matrices $H^i(x)$ are nonsingular. Applying Theorem 1, we see that necessary and sufficient conditions that the coordinates $\{x^i\}$ are a regular separable system for (6.2) are

$$\partial_{jk} H^i = \partial_j H^k (H^k)^{-1} \partial_k H^i + \partial_k H^j (H^j)^{-1} \partial_j H^i, \quad j \neq k. \tag{6.3}$$

(Note that for $N = 1$ equations (6.3) agree with the Levi-Civita conditions (4.3) where $H^i \equiv H_i^{-2}$.)

We will determine the significance of these conditions. Suppose $S_{ij}(x)$, $1 \leq i$, $j \leq n$ are $N \times N$ matrices such that $\partial_k S_{ij}(x) = 0$ for $k \neq i$ and $\det S = \det(S_{ij}) \neq 0$ where S is a (bordered) $Nn \times Nn$ matrix. Let $A = S^{-1} = (A^{ij})$ where each A^{ij} is an $N \times N$ matrix. Then

$$\sum_{i=1}^{n} A^{ki} S_{ij} = \sum_{i=1}^{n} S_{ji} A^{ik} = \delta_j^k, \quad 1 \leq j, k \leq n. \tag{6.4}$$

Now consider the separated equations

$$\varphi_i = \sum_{j=1}^{n} S_{ij}(x^i) D^{(j)}, i = 1, \ldots, n \tag{6.5}$$

where each $D^{(j)}$ is an $N \times N$ diagonal matrix with independent parameters $d_I^{(j)}$, $1 \leq I \leq N$, as diagonal elements. Suppose $D^{(1)} = D$ and set $A^{1i} = H^i$. Then it follows from (6.4) that

$$\sum_{i=1}^{n} H^i \varphi_i = D, \tag{6.6}$$

and, under the assumption that each H^i is nonsingular, equation (6.6) has a family of separated solutions depending on $N(n + 1)$ constants. Hence $\{x^i\}$ is a regular separable system and the matrices H^i must satisfy conditions (6.3).

To prove the converse, we first suppose that $(S_{ij}(x^i))$ is a nonsingular bordered $nN \times nN$ matrix such that conditions (6.5) hold with $A^{1\ell} \equiv H^\ell$ nonsingular for $1 \leq \ell \leq n$. It follows from the second equation (6.4) that

$$\sum_{i=1}^{n} S_{ji} \partial_\ell A^{ik} = 0, \quad j \neq \ell.$$

Hence

$$\partial_\ell A^{ik} = A^{i\ell} f(\ell, k, x) \tag{6.7}$$

where f is an $N \times N$ matrix, independent of i. Setting $i = 1$ we find

$$f(\ell, k, x) = (H^\ell)^{-1} \partial_\ell H^k.$$

Hence,

$$\partial_\ell A^{ik} = A^{i\ell} (H^\ell)^{-1} \partial_\ell H^k. \tag{6.8}$$

As is easily checked, the integrability conditions $\partial_j(\partial_\ell A^{ik}) = \partial_\ell(\partial_j A^{ik})$, $j \neq \ell$, for this system of equations are precisely (6.3). Conversely, if conditions (6.3) hold for nonsingular H^k, we can find solutions (A^{ij}) of (6.8) such that $A^{1k} = H^k$ and (locally) $\det A = \det(A^{ik}) \neq 0$. Then, differentiating the second equation (6.4) with respect to x^ℓ and making use of (6.8), we find

$$\sum_{i=1}^n \partial_\ell S_{ji} A^{ik} = 0, \qquad j \neq \ell,$$

so that $\partial_\ell S_{ji}(x) = 0$ for $j \neq \ell$. This proves that the separation equations (6.5) are the mechanism for regular variable separation in equations (6.2) and that the conditions (6.3) imply that the matrices H^i are in Stäckel form: $H^i = (S^{-1})^{1i}$ where $S_{ij} = S_{ij}(x^i)$. (The scalar case $N = 1$ corresponds to the classical Stäckel form [14].)

Theorem 6. *Suppose $\{H^{(i)}\}$ satisfies equations (6.3), i.e., it is in Stäckel form. Let $\{C_{ij}(x^i)\}$ be a particular Stäckel form matrix:*

$$\sum_{i=1}^n H^{(i)} C_{ij}(x^i) = \delta_j^1 I, \quad 1 \leq j \leq n,$$
$$\det(C_{ij}) \neq 0.$$

Then the most general Stäckel form matrix $\{D_{ij}(x^i)\}$ associated with $\{H^{(i)}\}$ can be expressed as $D_{ij}(x^i) = \sum_{\ell=1} C_{i\ell}(x^i) F_j^\ell$ where $\{F_j^\ell\}$ is a constant nonsingular (bordered) $Nn \times Nn$ matrix such That $F_j^1 = \delta_j^1 I$

If we require regular separation of the system

$$\sum_{i=1}^n H^i \partial_i \varphi(x) + V(x) = D \tag{6.9}$$

where assumptions (6.2) hold and $V(x)$ is an $N \times N$ matrix, the separation conditions are (6.3) again (which imply that the H^i are in Stäckel form) and

$$\partial_{jk} V = \partial_j H^k (H^k)^{-1} \partial_k V + \partial_k H^j (H^j)^{-1} \partial_j V, \qquad j \neq k. \tag{6.10}$$

Condition (6.10) simply says that

$$V = \sum_{i=1}^{n} H^i S_i(x^i) \tag{6.11}$$

for some $N \times N$ matrices S_i with $\partial_j S_i = 0$, $i \neq j$. Indeed it is straightforward to verify (using (6.3)) that this expression satisfies (6.10). On the other hand, suppose that $V'(x)$ is an invertible matrix such that the matrices $H^{'i} = (V')^{-1}H^i$ are also in Stäckel form. Then both the $\{H^{'i}\}$ and the $\{H^i\}$ satisfy (6.3). A direct computation shows that the necessary and sufficient condition that V' maps the Stäckel form H^i to another Stäckel form is that it satisfies (6.10). Thus if V' satisfies (6.10) and is invertible then $(V')^{-1}H^i = A^{'1i}$ where $\sum_i A^{'ji}C'_{i\ell}(xi) = \delta^j_\ell$. Thus $\sum_i (V')^{-1}H^i C'_{i1}(x^i) = 1$ or $V' = \sum_i H^i C'_{i1}(x^i)$ which is of the form (6.11). In particular the identity matrix I satisfies (6.10) and is nonsingular so there exist matrices $C_i(x^i)$ such that $I = \sum_i H^i C_i(x^i)$. Now let V be any solution of (6.10). Clearly we can always find a constant μ such that $V(x) + \mu I \equiv V'(x)$ is locally nonsingular. Since V' is also a solution of (6.10) we can find matrices $C'_{i1}(x^i)$ such that

$$V'(x) = \sum_i H^i C'_{i1}(x^i) = V(x) + \mu \sum_i H^i C_i(x^i).$$

Setting, $S_i(x^i) = C'_{i1}(x^i) - \mu C_i(x^i)$ we obtain (6.11).

The appropriate separation equations for this case are

$$\varphi_i = \sum_{j=1}^{n} S_{ij}(x^i) D^{(j)} + S_i(x^i), \qquad i = 1, \ldots, n.$$

It follows easily that the separated solutions φ satisfy the "eigenvalue" equations

$$\sum_{i=1}^{n} A^{ki}\varphi_i + V^k = D^{(k)}, \qquad 1 \leq k \leq n. \tag{6.12}$$

where $V^k = \sum_{i=1}^{n} A^{ki}S_i$. However the expressions on the left-hand side of (6.12) have no apparent connection with symmetries of (6.9).

We *can* obtain a relation between symmetry and separation for systems of linear equations. To motivate this connection let us consider the system (6.9) again where the diagonal matrix $D = EI$, a multiple of the identity matrix I. We write $\varphi(x) = \sum_{i=1}^{n} \varphi^{(i)}(x^i)$. Then we can obtain (multiplicative) separable solutions of the linear equation

$$\sum_{i=1}^{n} H^i \partial_i \psi + V\psi = E\psi$$

for the N-component spinor ψ if $\psi = \Pi_{i=1}^{n} \Theta^{(i)}(x^i)\xi$ where $\Theta^{(i)} = \exp \varphi^{(i)}$ and ξ is a constant N-spinor. This characterization of ψ will hold provided $\Theta^{(i)}\Theta^{(j)} = \Theta^{(j)}\Theta^{(i)}$ for $1 \leq i,\, j \leq n$.

With these motivational items out of the way, we proceed to an examination of multiplicative separation for equations of Dirac type:

$$\mathbf{H}\psi = E\psi, \qquad E \text{ constant} \tag{6.13}$$

where

$$\mathbf{H} = \sum_{i=1}^{n} H^i(x)\partial_i + V(x),$$

H^i, V are $N \times N$ matrices and ψ is an N-component spinor. We require that the H^i, $1 \le i \le n$, are nonsingular matrices. We define a *factorizable system* for (6.13) as a set of equations

$$\partial_i\psi = (\sum_{j=1}^{n} C_{ij}(x)\lambda^j - C_i(x))\psi \qquad i = 1, \ldots, n. \tag{6.14}$$

where the $C_{ij}(x)$, $C_i(x)$ are $N \times N$ matrices such that $\det(C_{ij}) \neq 0$, the λ^j are independent parameters with $\lambda^1 = E$, and for every initial point x^0 and N-spinor ξ there is a solution $\psi(x)$ of (6.14) which also satisfies (6.13) and has the property $\psi(x^0) = \xi$. The integrability conditions for (6.14): $\partial_i(\partial_j\psi) = \partial_j(\partial_i\psi)$, $i \neq j$, imply

$$\begin{aligned} &a) \quad C_{jk}C_{i\ell} + C_{j\ell}C_{ik} = C_{ik}C_{j\ell} + C_{i\ell}C_{jk}, \\ &b) \quad \partial_j C_{ik} - C_{ik}C_j - C_iC_{jk} = \partial_i C_{jk} - C_{jk}C_i - C_jC_{ik}, \\ &c) \quad \partial_i C_j + C_iC_j = \partial_j C_i + C_jC_i. \end{aligned} \tag{6.15}$$

Let the $nN \times nN$ bordered matrix $A = (A^{ij})$ be the inverse of (C_{jk}) :

$$\sum_{j=1}^{n} A^{ij}C_{jk} = \sum_{j=1}^{n} C_{kj}A^{ji} = \delta^i_k I. \tag{6.16}$$

It follows that the solutions ψ of (6.14) satisfy the eigenvalue equations

$$A^k\psi \equiv \sum_{i=1}^{n} A^{ki}(x)\partial_i\psi + B^k(x)\psi = \lambda^k\psi, \quad 1 \le k \le n.$$

Clearly,

$$\begin{aligned} A^{1i}(x) = H^i(x), B^1(x) = V(x), \\ B^i(x) = \sum_{j=1}^{n} A^{ij}(x)C_j(x). \end{aligned} \tag{6.17}$$

Theorem 7. *The integrability conditions (6.15) for a factorizable system are satisfied if and only if there exists $N \times N$ matrices $A^{ki}(x)$, $B^k(x)$, $1 \le k, i \le n$ such that*

(1) *The operators*

$$\mathbf{A}^k = \sum_{i=1}^{n} A^{ki}\partial_i + B^k, \qquad k = 1, \ldots, n$$

commute, i.e.,

$$\mathbf{A}^k \mathbf{A}^\ell = \mathbf{A}^\ell \mathbf{A}^k.$$

(2) $\mathbf{H} = \mathbf{A}^1$

(3) $\sum_{i=1}^{n} A^{ki} C_{ij} = I\delta_j^k$

where I is the $N \times N$ identity matrix.

Proof. Condition (1) is equivalent to the requirements

$$\begin{aligned}
&i) && A^{ji}A^{k\ell} + A^{j\ell}A^{ki} = A^{ki}A^{j\ell} + A^{k\ell}A^{ji},\\
&ii) && \sum_i (A^{ji}\partial_i A^{k\ell} - A^{ki}\partial_i A^{j\ell}) = A^{k\ell}B^j - A^{j\ell}B^k\\
& && \qquad + B^k A^{j\ell} - B^j A^{k\ell},\\
&iii) && \sum_i (A^{ji}\partial_i B^k - A^{ki}\partial_i B^j) = B^k B^j - B^j B^k.
\end{aligned} \tag{6.18}$$

Let us assume that conditions (6.15) hold and define

$$B_{i\ell;hm} = C_{ih}C_{\ell m} + C_{im}C_{\ell h} = C_{\ell h}C_{im} + C_{\ell m}C_{ih}$$

(by (6.15a)). It follows that

$$\sum_{i,\ell}(A^{ki}A^{j\ell} + A^{k\ell}A^{ji} - A^{j\ell}A^{ki} - A^{ji}A^{k\ell})B_{i\ell;hm} = 0, \tag{6.19}$$

for all k, j, h, m. But

$$\sum_{i,\ell} A^{k\ell}A^{ji}B_{i\ell;hm} = (\delta_m^k \delta_h^j + \delta_h^k \delta_m^j)I,$$

so $(B_{i\ell,hm})$ is invertable. It follows that (6.18i) holds. A straightforward computation shows that (6.18iii) is a direct consequence of (6.15c), (6.16) and (6.17). Multiplying (6.15b) on the right by $A^{k\ell}$, summing on k and using the identity $\sum_k \partial_j C_{ik} A^{k\ell} = -\sum_k C_{ik}\partial_j A^{k\ell}$, we obtain

$$\begin{aligned}
&\sum_k (C_{ik}\partial_j A^{k\ell} + C_{ik}C_j A^{k\ell}) + C_i \delta_j^\ell\\
&\qquad = \sum_k (C_{jk}\partial_i A^{k\ell} + C_{jk}C_i A^{k\ell}) + C_j \delta_i^\ell.
\end{aligned}$$

Multiplying this equation on the left by

$$A^{tj}A^{si} - A^{sj}A^{ti},$$

summing on j, i and using (6.18i) we obtain (6.18ii). The proof of the converse is similar.

This result implies that a factorizable system is equivalent to the existence of a family of $n-1$ commuting matrix symmetry operators $\mathbf{A}^k$ of $\mathbf{H}\psi = E\psi$ such that $\det(A^{ki}) \neq 0$. Note that conditions (6.15) and (6.18) take the same form in all coordinate systems. Moreover, by solving each of the ordinary differential equations successively in a given coordinate system we can "factor" a solution as a product of n $N \times N$ matrices. For example, if $n = 2$ a solution might take the form

$$\psi(x, y) = \Theta^{(1)}(x, y, y^0)\Theta^{(2)}(x, x^0, y^0)\xi$$

where $\Theta^{(1)}(x^0, y^0, y^0) = \Theta^{(2)}(x^0, x^0, y^0) = I$.

Factorizable systems also retain their form under changes of frame $\psi = R(x)\psi'$ where $R(x)$ is a given nonsingular $N \times N$ matrix (independent of the parameters λ^j). The symmetry operators $\mathbf{A}^k$ acting on ψ induce symmetry operators

$$\begin{aligned} \mathbf{A}'^k = R^{-1}\mathbf{A}^k R = \sum_i (R^{-1}A^{ki}R\partial_i + \\ R^{-1}A^{ki}\partial_i R) + R^{-1}B^k R \\ = \sum_i A'^{ki}\partial i + B'^k \end{aligned} \tag{6.20}$$

acting on ψ'. By choosing R appropriately it is always possible to transform the factorizable system in a form where $B'^k \equiv C'_k \equiv 0$. Indeed, the condition that $B'^k \equiv 0$ corresponds to the equations

$$\partial_i R = -C_i R. \tag{6.21}$$

The integrability conditions for these equations are identical to (6.15c). Thus a solution R exists and is determined uniquely (up to multiplication on the right by a constant nonsingular matrix). It follows from (6.17) that if $B'^k \equiv 0$ then $C'_k \equiv 0$.

We say that a factorizable system is *separable* if (by a change of frame $\psi = R\psi'$ if necessary) the equations (6.14) take the form

$$\partial_i\psi' = (\sum_{j=1}^n C_{ij}(x^i)\lambda^j - C_i(x^i))\psi' \qquad i = 1, \ldots, n. \tag{6.14$'$}$$

in a *particular* coordinate system $\{x^1, \ldots, x^n\}$, i.e., $\partial_\ell C_{ij} = \partial_\ell C_i = 0$ if $\ell \neq i$.

Let $C_i(x^i, \lambda) = \sum_{j=1}^n C_{ij}(x^i)\lambda^j - C_i(x^i)$ and let ξ be a constant N-spinor.

Theorem 8. *Suppose* $(6.14)'$ *is a separable system for (6.13) in the coordinates* $\{x^\ell\}$ *and let* $\mathbf{x}_0$ *be a fixed vector. Then (6.1) admits solutions of the form*

$$\psi(\mathbf{x}) = R(\mathbf{x})\Theta^{(1)}(x^1)\dots\Theta^{(n)}(x^n)\xi \tag{6.22}$$

where the Θ^ℓ *are* $N\times N$ *matrices such that* $\Theta^{(i)}(x^i)\Theta^{(j)}(x^j) = \Theta^{(j)}(x^j)\Theta^{(i)}(x^i)$ *for all* i,j *and* $\Theta^{(i)}(x_0^i) = I$.

Proof. Here R is the frame change that leads to $(6.14)'$. Consider the equation

$$\begin{aligned}\partial_\ell\Theta^{(\ell)}(x^\ell) &= C_\ell(x^\ell,\lambda)\Theta^{(\ell)}(x^\ell),\\ \Theta^{(\ell)}(x_0^\ell) &= I.\end{aligned}$$

This equation with initial condition has a unique solution for each ℓ. It follows from eqns. (6.15) that $C_i(x^i,\lambda)C_j(x^j,\lambda) = C_j(x^j,\lambda)C_i(x^i,\lambda)$ for $i\neq j$. We claim that

$$\Theta^{(i)^{-1}}(x^i)C_j\Theta^{(i)}(x^i) = C_j$$

for $i\neq j$. Indeed this is clearly true for $x^i = x_0^i$ and

$$\begin{aligned}\partial_i(\Theta^{(i)-1}C_j\Theta^{(i)}) &= -\Theta^{(i)-1}C_i\Theta^{(i)}\Theta^{(i)-1}C_j\Theta^{(i)}\\ &\quad + \Theta^{(i)-1}C_jC_i\Theta^{(i)}\\ &= \Theta^{(i)-1}(C_jC_i - C_iC_j)\Theta^{(i)} = 0\end{aligned}$$

by (6.15c).

It follows that $\Theta^{(j)-1}\Theta^{(i)}\Theta^{(j)} \equiv \Theta^{(i)}$ for all x^i, x^j near x_0^i, x_0^j. Indeed this is true for $x^j = x_0^j$ and

$$\begin{aligned}\partial_j(\Theta^{(j)-1}\Theta^{(i)}\Theta^{(j)}) &= -\Theta^{(j)-1}C_j\Theta^{(i)}\Theta^j\\ &\quad + \Theta^{(j)-1}\Theta^{(i)}C_j\Theta^j\\ &= \Theta^{(j)-1}(\Theta^{(i)}C_j - C_j\Theta^{(i)})\Theta^{(j)}\\ &= 0.\end{aligned}$$

Setting $\psi' = \Theta^{(1)}\dots\Theta^{(n)}\xi$ we see that ψ' satisfies $(6.14)'$ with $\psi'(\mathbf{x}_0) = \xi$. $\square$

The "Dirac" equation corresponding to $(6.14)'$ is of the form $\sum H^i\partial_i\psi' + V\psi' = E\psi'$ with $E = \lambda^1$, $H^i = A^{1i}$, $V = \sum_{j=1}^n H^jC_j(x^j)$. It follows immediately that H^i is in Stäckel form, i.e., it satisfies equations (6.4) and that V is a Stäckel multiplier, i.e., it satisfies equations (6.10). (On the other hand, a set of Stäckel form matrices H^i and a Stäckel multiplier V in a given coordinate system do not necessarily correspond to a multiplicative separable system.)

Theorem 9. *Suppose* $\{x^i\}$ *is an* $(R-)$*separable coordinate system for the equation*

$$(\sum_i H^i(x)\partial_i + (V+\mu W))\psi = E\psi$$

where μ is a parameter, $H^i, V, W, \{x^i\}$ are independent of μ and W is nonsingular. Then $\{x^i\}$ is also an $(R-)$separable coordinate system for the equation

$$(\sum_i W^{-1}H^i\partial_i + W^{-1}V)\psi' = E\psi'.$$

This result generalizes a result of [38] in the case $N = 1$.

It is easy to find examples of factorizable systems for which there is no associated separable or R-separable coordinate system. The author is not aware of a general (coordinate independent) criterion for determining when a factorizable system is separable. Moreover, a characterization of the separable coordinates in terms of properties of the commuting symmetry operators $\mathbf{A}^k$ in Theorem 7 is still open.

In the case of the Dirac equation itself somewhat more is known, but many problems remain. Corresponding to arbitrary local coordinates $\{x^1, x^2, x^3, x^4\}$ on a four-manifold with contravariant metric tensor (g^{ij}) of signature $(1, 1, 1, -1)$ the Dirac equation takes the form

$$(\sum_{i=1}^{4} \gamma^i(\partial_i - \Gamma_i) + m)\psi = 0. \tag{6.23}$$

Here γ^i, Γ_i are 4×4 matrices, $m \neq 0$ is the rest mass of the electron and ψ is a 4-spinor. The γ^i are determined up to similarity by the requirement

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij}I, \quad 1 \le i, j \le 4. \tag{6.24}$$

Expressions for the Christoffel matrices $\Gamma_i(x)$ can be found in [39-41], for example. (In Minkowski space with cartesian coordinates $\{x, y, z, t\}$ and diagonal metric $(1, 1, 1, -1)$ we have $\Gamma_i \equiv 0$.) For a fixed orthogonal coordinate system $\{g^{ij}\}$ is diagonal and $g^{ii} \neq 0$. It follows that each of the matrices $\gamma^i(x)$ is nonsingular so that the general theory developed in this section applies. All of the examples of variable separation for the Dirac equation known to the author are separable systems in the sense of $(6.14)'$. Most of the known examples occur for Minkowski space and correspond to symmetry operations associated with the evident Poincaré symmetries of (6.23) and or to Casimir operators associated with subalgebras of the Poincaré group. A Poincaré symmetry takes the form $\mathbf{A} = Id + B$ where $d(x)$ is a (scalar) first order partial differential operator, I is the identity matrix and $B(x)$ is a 4×4 matrix. For such a symmetry, standard local Lie theory can be used to show that a coordinate system $\{y^i\}$ exists such that $\mathbf{A} = I\partial_{y^1} + B$. For independent commuting Poincaré symmetry operators $\mathbf{A}^1, \ldots, \mathbf{A}^\ell$, $1 \le \ell \le 3$, we can construct a coordinate system $\{y^i\}$ such that $\mathbf{A}^i = I\partial_{y^i} + B^i$. See [42-86] for examples. New interest in variable separation for the Dirac equation was ignited when Chandrasekhar [49] found a separable system for this equation in a Kerr metric gravitational background, i.e., in the geometry of a spinning black hole [50]. In the limit that the Kerr metric degenerates to a Minkowski space metric, Chandrasekhar's separable system led to a new Minkowski separable system corresponding to oblate spheroidal coordinates in an unexpected frame. Moreover, one of the associated symmetry operators $\mathbf{A}$ was not associated with the Poincaré symmetries. In [44] it was pointed out that $\mathbf{A}$ is associated with a Killing-Yano tensor and in [51] that the appropriate frame for separation is defined by the eigenvectors of this tensor. Also in [51] it was shown that the tensor $A^{ij} = \frac{1}{2}(A^iA^j + A^jA^i)$ where $\mathbf{A} = \sum_i A^i\partial_i + B$ can be expressed as $A^{ij} = a^{ij}I$ and the tensor a^{ij} determines the separable coordinates in a manner analogous to Theorem 4. However, these are observations, not proofs, and the problem of intrinsic characterization of separable systems for the Dirac equation remains open even for the case of Minkowski space. The Minkowski case should be settled very soon because Kamran et. al. [52] have now classified (under the action of the Poincaré group) all triplets of commuting symmetry operators for the Dirac equation. They find that at least two of the symmetries always correspond to Poincaré symmetries. Thus separable coordinate systems will always have solutions such that dependence on at least two of the variables is exponential. (In particular the general separable coordinates of ellipsoidal or paraboloidal type for the Hamilton-Jacobi equation in Minkowski space cannot separate the Dirac equation.) The availability of this classification should enable us to characterize the separable systems in an invariant manner.

As an alternate procedure to that of Kamran et. al. I propose an approach to separation of the Dirac equation in the spirit of Eisenhart. Given a Dirac equation with potential in a four-dimensional space time, not necessarily Minkowski, we can always write it in the form

$$\sum_{i=1}^{4} \gamma^i\partial_i\psi + V\psi = -m\psi$$

where the true potential and the Christoffel matrices have been incorporated into V. We assume that a fixed frame has been chosen, as have the orthonormal coordinates $\{x^i\}$:

$$ds^2 = \sum_{i=1}^{4} h_i^2(dx^i)^2.$$

Here

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2h_i^{-2}\delta^{ij}I, \qquad \mathbf{A}^1 = \sum_{i=1}^{4} \gamma^i\partial_i + V.$$

Since we chose a fixed frame, the γ^i and V are functions of h^i. Our problem is to determine a possible frame change R , ($\psi = R\Phi$), such that the Dirac equation for Φ is separable in the coodinates x^i. The conditions for separation are:

(1) factorization conditions

$$[\mathbf{A}^i, \mathbf{A}^j] = 0, \quad 1 \le i, j \le 4$$

(2) Stäckel form conditions

$$\partial_k A^{i\ell} + [\partial_k RR^{-1}, A^{i\ell}] = A^{ik}(\gamma^k)^{-1}(\partial_k\gamma^\ell + [\gamma^\ell, \partial_k RR^{-1}])$$

(3) Stäckel multiplier conditions

$$\begin{aligned}\partial_\ell B^i + [B^i, \partial_\ell RR^{-1}] = & A^{i\ell}(H^\ell)^{-1}\{\partial_\ell V + [V, \partial_\ell RR^{-1}] + \sum_s H^s \partial_{\ell s} RR^{-1} \\ & - \sum_s \partial_\ell RR^{-1} H^s \partial_s RR^{-1}\} \\ & + \sum_s (\partial_\ell RR^{-1} A^{is} \partial_s RR^{-1} - A^{is}\partial_{s\ell} RR^{-1})\end{aligned}$$

(4) conditions that the metric belong to the desired spacetime. In particular, the metric should be of signature $(+,+,+,-)$ and for Minkowski space the Riemann curvature tensor should vanish

$$R_{ijk\ell} = 0.$$

When two of the symmetry operators $\mathbf{A}^i$ can be taken in Poincaré form it appears feasable to solve these equations explicitly.

7. Comments and problems.

Among the unsolved problems in this subject is that of finding all solutions of the separation equations (2.8), or (more simply) of the scalar separation equations (2.11). In other words the most general mechanism for variable separation is still not determined. In all cases known to the author this mechanism is a variant of Stäckel form. To understand some of the variations consider Example 1 again:

$$(x_1 + x_2)(\partial_{11} u + \partial_{22} u) - 2(\partial_1 u + \partial_2 u) = E.$$

This equation admits a five-parameter separable solution in the coordinates x_1, x_2:

$$\begin{aligned} u = & (\alpha x_1^3 + \beta x_1^2 + \gamma x_1 - \frac{1}{2} E x_1) \\ & + (-\alpha x_2^3 + \beta_2^2 - \gamma x_2 + \delta). \end{aligned}$$

The mechanism of separation is puzzling until one realizes that the appropriate separation equations are

$$\begin{array}{lcccr} \partial_1 u & +E/2 & -\gamma & -2\beta x_1 & -3\alpha x_1^2 = 0, \\ \partial_{11} u & & & -2\beta & -6\alpha x_1 = 0, \\ \partial_2 u & & +\gamma & -2\beta x_2 & +3\alpha x_2^2 = 0, \\ \partial_{22} u & & & -2\beta & +6\alpha x_2 = 0. \end{array}$$

The associated "Stäckel matrix" responsible for the separation is

$$\begin{bmatrix} \frac{1}{2} & -1 & -2x_1 & -3x_1^2 \\ 0 & 0 & -2 & -6x_1 \\ 0 & 1 & -2x_2 & 3x_2^2 \\ 0 & 0 & -2 & 6x_2 \end{bmatrix}.$$

This is not a true Stäckel matrix since more than one row depends on a given variable x_i. Moreover, the second and fourth rows are the derivatives of the first and third rows, respectively. It is a nontrivial example of a differential-Stäckel matrix, [53].

In [53] it was shown that *all* additive separable systems for n^{th} order linear differential equations are associated with differential Stäckel matrices. However, there is no apparent connection between separation and symmetry for the equations.

In [54] differential Stäckel matrices were further generalized by allowing each row of the matrices to depend not only on a single coordinate x^i but also on a finite number of derivatives of the dependent variable u with respect to x^i. These matrices correspond to additive separation in classes of nonlinear equations.

Other interesting unsolved problems concern multiplicative separation for spinor equations of the form

$$H\psi = D\psi$$

where H is an $N \times N$ first order matrix differential operator in the variables $x^1, \ldots, x^n$, ψ is an N-component spinor and D is an $N \times N$ diagonal constant matrix. In particular, some or all of the diagonal elements of D may be zero. Maxwell's equations in an arbitrary space-time (and more generally, equations for massless particles with spin) are of this form with $D = 0$. Here even the definition of separation is not yet clear, nor is the relationship between symmetry and separation. For some partial results see [49,55]. We mention some of the results known for the Kerr metric:

$$\begin{aligned} ds^2 =& (1 - \frac{2Mr}{\rho^2})dt^2 - (\frac{\rho^2}{\Delta})dr^2 - \rho^2 d\theta^2 \\ &+ (\frac{2aMr\sin^2\theta}{\rho^2})dtd\phi - [(r^2+a^2) + \frac{2a^2Mr\sin^2\theta}{\rho^2}]\sin^2\theta d\phi^2, \end{aligned}$$

where

$$\Delta = r^2 + a^2 - 2Mr, \qquad \rho^2 = r^2 + a^2\cos^2\theta.$$

M is mass of black hole.
a is angular momentum per unit mass of black hole.

When $a = 0$ the Kerr metric reduces to the Schwarzschild metric.

When $M = 0$ this is the Minkowski (flat) space metric in oblate spheroidal coordinates.

The following equations are known to "separate" in these coordinates:

(1) The Hamilton-Jacobi equation. This follows from the standard theory of variable separation.
(2) The Dirac equation (spin $\frac{1}{2}$ particles with mass). See [49,56]. This can be understood from the theory of the preceding section.
(3) Neutrino equation (spin $\frac{1}{2}$ particles with zero mass). See [49]. This is essentially the $m = 0$ case of the Dirac equation.

(4) Maxwell's equations (spin 1 particles with zero mass). See [49,57,58]. The separation in this case is not well understood.
(5) Rarita-Schwinger equations (spin $\frac{3}{2}$ particles with zero mass). See [59]. Poorly understood.
(6) Gravitational perturbation equations (spin 2 particles, zero mass). See [58,60]. Poorly understood.

One reasonably successful approach to obtaining explicit solutions of mass zero, spin s equations is the method of Hertz potentials. To explain this we adopt the Newman-Penrose spinor formalism in Minkowski space, [61,62].In this formalism the covariant zero-mass field equation for a free spin-s field is

$$\nabla^{AX'}\phi_{AB\cdots K} = 0,$$

where $\phi_{AB\cdots K}$ is a totally symmetric spinor with $2s$ indices.

We define the spinor d'Alembertian operator (a scalar) as

$$\Box \equiv \nabla_{AX'}\nabla^{AX'}.$$

A *Hertz potential* is a totally symmetric $2s$-spinor $P_{AB\cdots K}$ satisfying

$$\Box \bar{P}^{X'N'\cdots W'} = 2\nabla^{A(X'}G_A^{N'\cdots W')}$$

where $G_{AN'\cdots W'}$ is an arbitrary gauge spinor with one unprimed and $2s-1$ symmetrized primed indices. The basic result is [60]:

Theorem (Kegeles and Cohen). *The spinor*

$$\begin{aligned}\phi_{AB\cdots K} =& \nabla_{AM'}\nabla_{BN'}\cdots\nabla_{KW'}\bar{P}^{M'N'\cdots W'} \\ & - \nabla_{(BN'}\ldots\nabla_{KW'}G_{A)}^{N'\cdots W'}\end{aligned}$$

is a solution of the zero-mass field equation

$$\nabla^{AX'}\phi_{AB\cdots K} = 0.$$

(See also [63].)

This result has been extended to curved spacetimes for $s = \frac{1}{2}, 1, \frac{3}{2}, 2$. For the Kerr metric, G can be chosen such that the equation for the Hertz potential has only one nontrivial component and variables separate in a simple fashion. This separable Hertz potential then yields a "separable" solution of the field equations. (However, the field equation solution is not strictly separable according to the definition provided in this paper.) This method is not general; it works only for algebraically special spacetimes.

References

1. W. Miller, Jr., *Symmetry and Separation of Variables*, Addison-Wesley, Reading, Massachusetts, 1977.

2. H. Bateman, *Partial Differential Equations of Mathematical Physics, (1st ed., 1932)*, Cambridge Univ. Press, London and New York, 1969.
3. P. Morse and H. Feshbach, *Methods of Theoretical Physics, Part I*, McGraw-Hill, New York, 1953.
4. E. G. Kalnins and W. Miller, *Modern Developments in Analytical Mechanics*, Acta Academiae Scientiarum Taurinensis, Torino, 1983.
5. W. Miller, *Lecture Notes in Physics*, Springer-Verlag, New York, 1983.
6. T.W. Koornwinder, *Proceedings of the Scheveningen Conferences of Differential Equations. Lecture Notes in Mathematics*, Springer-Verlag, 1980.
7. P.J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics **107, Springer-Verlag, New York** (1986).
8. G.W. Bluman and J.D. Cole, *Applied Mathematical Series # 13*, Springer-Verlag, 1974.
9. L.V. Ovsjannikow, *Group Properties of Differential Equations*, Academic Press, 1982.
10. L. P. Eisenhart, *Continuous Groups of Transformations*, Dover Reprint, Dover, Delaware, 1961.
11. L.P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 2^{nd} printing, 1949.
12. V.I. Arnold, *Graduate Texts in Mathematics, 60*, Springer-Verlag, New York, 1978.
13. L.P. Eisenhart, *Separable systems of Stäckel*, Ann. Math. **35** (1934), 284–305.
14. P. Stäckel, *Über die integration der Hamilton-Jacobischen differentialgeichung mittels separation der variabeln*, Habilitationschrift, Halle (1891).
15. E.G. Kalnins and W. Miller, Jr., *R-separation of variables for the four-dimensional flat Laplace and Hamilton-Jacobi equations*, Trans. A.M.S. **244** (1978), 241–261.
16. E.G. Kalnins and W. Miller, Jr., *Conformal Killing tensors and variable separation for the Hamilton-Jacobi equation*, SIAM J. Math. Anal **14** (1983), 126-137.
17. E.G. Kalnins and W. Miller, Jr., *The theory of orthogonal R-separation for Helmholtz equations*, Advances in Mathematics **51** (1984), 91–106.
18. V.N. Shapovalov, Differential Equations **16** (1980), 1864.
19. E.G. Kalnins and W. Miller, Jr., *Intrinsic characterization of orthogonal R-separation for Laplace equations*, J. Phys. A: Math. Gen **15** (1982), 2699–2709.
20. E.G. Kalnins and W. Miller, Jr., *The general theory of R-separation for Helmholtz equations*, J. Math. Phys.
21. S. Benenti and M. Francaviglia, *The theory of separability of the Hamilton-Jacobi equation and its application to general relativity*, General Relativity and gravitation, Vol. 1, A. Held, ed., Plenum, New York, 1980.
22. S. Benenti, *Separability structures on Riemannian manifolds*, Proceedings of the Conference on Differential Geometric Methods in Mathematical Physics, Lecture Notes in Mathematics, Vol. 836, Springer-Verlag, Berlin, 1980.
23. E.G. Kalnins and W. Miller, Jr., *R-separation of variables for the time-dependent Hamilton -Jacobi and Schrödinger equations*, J. Math. Phys **28** (1987), 1005–1015.
24. E.G. Kalnins and W. Miller, Jr., *Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations*, SIAM J. Math. Anal **11** (1980), 1011-1026.
25. E.G. Kalnins and W. Miller, Jr., *Killing tensors and nonorthogonal variable separation for the Hamilton-Jacobi equation*, SIAM J. Math. Anal **12** (1981), 617–638.
26. E.G. Kalnins and W. Miller, Jr., *Separation of variables on n-dimensional Riemannian manifolds I. The n-sphere and Euclidean n-space*, J. Math. Phys. **27** (1986), 1721–1736.
27. E.G. Kalnins, *Separation of Variables for Riemannian Spaces of Constant Curvature*, Pitman, Monographs and Surveys in Pure and Applied Mathematics 28, Longman, Essex, England, 1986.
28. M. Bôcher, *Die Reihentwickelungen der Potentialtheorie*, B.G. Teubner, Leipzig, 1894.
29. G.J. Reid, *R-separation for heat and Schrödinger equations I.*, SIAM J. Math. Anal. **17** (1986), 646.
30. C.P. Boyer, E.G. Kalnins and P. Winternitz, *Separation of variables for the Hamilton-Jacobi equation on complex projective spaces*, SIAM J. Math. Anal **16** (1985), 93–109.
31. V.N. Shapovalov, *Stäckel spaces*, Siberian Math. J. **20** (1980), 790-800.
32. N.M.J. Woodhouse, *Killing tensors and separation of the Hamilton-Jacobi equation*, Commun. Math. Phys **44** (1975), 9–38.

33. V.N. Shapovalov, *Separation of variables in the nonstationary Schrödinger equation*, Sov. Phys. J. **17** (1976), 1718.
34. V.N. Shapovalov, *Separation of variables in second-order linear differential equations*, Differential Equations **10** (1981), 1212.
35. P. Winternitz, I. Lukac and Ya A. Smorodinskii, *Quantum numbers of the little group of the Poincaré group*, Soviet J. Nucl. Phys **7** (1968), 139–145.
36. C. Neumann, *De problemate quodan mechanico, quod ad priman integralium ultraellipticorum classen revocatur*, J. Reine Angew. Math **56** (1859), 46-63.
37. E. Jacobi, *Vorlesungen über Dynamik*, Supplement band, Berlin, 1884.
38. C.P. Boyer, E.G. Kalnins and W. Miller, Jr., *Stäckel-equivalent integrable Hamiltonian systems*, SIAM J. Math. Anal **17** (1986), 778–797.
39. D.R. Brill and J.A. Wheeler, *Interaction of neutrinos and gravitational fields*, Revs. Mod. Phys. **29** (1957), 465–479.
40. V. Fock, *Geometrisierung der Diracschen theorie des elektrons*, Zeitschrift für Physik **57** (1936), 261–277.
41. V. Bargmann, *Bemerkungen zur allgemein-relativistischen fassung der quantentheorie*, Stzber. Preuss, Akad. Wiss., Physick-Math. **28** (1932), 346–354.
42. R.G. McLenaghan and Ph. Spindel, *Quantum numbers for Dirac spinor fields in a curved space time*, Phys. Rev. D **20** (1979), 409–413.
43. N. Kamran and R.G. McLenaghan, *Symmetry operators for neutrino and Dirac fields on curved spacetime*, Phys. Rev. **D 30** (1984), 357–362.
44. B. Carter and R.G. McLenaghan, *Generalized total angular momentum operator for the Dirac equation in curved space-time*, Phys. Rev. **D 19** (1979), 1093–1097.
45. A.H. Cook, *On separable solutions of Dirac's equation for the electron*, Proc. R. Soc. Lond. A **383** (1982), 247–278.
46. V.G. Bagrov, I.L. Buckbinder and D.M. Gittman, J. Phys. A **9** (1976), 1955–1965.
47. V.G. Bagrov, D.M. Gittman, A.V. Shapovalov and V.N. Shapovalov, Izv. viyssh. ucheb. Zared., Fiz **no.6** (1977), 105–114; 7, 46–51.
48. V.G. Bagrov, D.M. Gittman, V.N. Zhadorzhnyi, N.B. Sukhomlin and V.N. Shapovalov, Izv. vyssh. ucheb. zaved, Fiz **no.2** (1978), 13–18.
49. S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press, 1984.
50. R.P. Kerr, *Gravitational field of spinning mass as an example of algebraically special metrics*, Phys. Rev. Lett **11** (1963), 237–239.
51. E.G. Kalnins, G.C. Williams and W. Miller, *Matrix operator symmetries of the Dirac equation and separation of variables*, J. Math. Phys. **27** (1986), 1893-1900.
52. N. Kamran, M. Légaré, R.G. McLenaghan and P. Winternitz, *The classification of complete sets of operators commuting with the Dirac operator in Minkowski space-time*, CRM-1483 (1987).
53. E.G. Kalnins & W. Miller, *Differential-Stäckel matrices*, J. Math. Phys. **26** (1985), 1560-1565.
54. E.G. Kalnins and W. Miller, *Generalized Stäckel matrices*, J. Math. Phys. **26** (1985), 2168-2173.
55. E.G. Kalnins and W. Miller, *Electromagnetic waves in Kerr geometry*, Proc. R. Soc. Lond. **A 408** (1986), 23-30.
56. R. Güven, *The solution of Dirac's equation in a class of type D backgrounds*, Proc. R. Soc. Lond A **356** (1977), 465–470.
57. S. Teukolsky, *Rotating black holes: separable wave equations for gravitational and electromagnetic perturbation*, Phys. Rev. Lett **29** (1972), 1114–1118.
58. N. Kamran and R.G. McLenaghan, *Separation of variables for higher spin, zero rest-mass field equations on type D vacuum backgrounds with cosmological constant*, Gravitation and Geometry, a Festschift in honor of Ivor Robinson, W. Rindler and A. Trautman, eds., Bibliopolis Publishers, 1986.
59. N. Kamran, *Separation of variables for the Rarita-Schwinger equation on all type D vacuum backgrounds*, J. Math. Phys **26** (1985), 1740–1742.
60. L.S. Kegeles and J.M. Cohen, *Constructive procedure for perturbations of spacetimes*, Phys. Rev. D. **19** (1979), 1641–1664.

61. E.T. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, J. Math. Phys **3** (1962), 566–573.
62. R. Penrose and W. Rindler, *Vol. 1, Cambridge Monographs on Mathematical Physics*, Cambridge University Press, 1984.
63. G.F. Torres Del Castillo, *Killing spinors and massless spinor fields*, Proc. R. Soc. Lond. **A** **400** (1985), 119–126.
64. P.J. Olver and P. Rosenau, *Group-invariant solutions of differential equations*, SIAM J. Appl. Math **47** (1987), 263–278.