FUNCTIONAL SEPARATION OF VARIABLES FOR LAPLACE EQUATIONS IN TWO DIMENSIONS

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ABSTRACT. We say that a solution Ψ of a partial differential equation in two real variables x_1, x_2 is *functionally separable* in these variables if $\Psi(x_1, x_2) = \phi\,(A(x_1) + B(x_2))$ for single variable functions ϕ, A, B such that $\phi' A' B' \neq 0$. In this paper we classify all possibilities for regular functional separation in local coordinates for equations of the form $\Delta_2 \Psi = f(\Psi, x_1, x_2)$ where Δ_2 is the Laplace-Beltrami operator on a two-dimensional Riemannian or pseudo-Riemannian space. If the dependence of f on x_1, x_2 is nontrivial then separation can occur for conformal Cartesian coordinates on any space. If $f = G(\Psi)$ then for orthogonal coordinates we find that true functional separation, i.e. separation other than additive or multiplicative, occurs precisely for Cartesian coordinates in the Euclidean and pseudo-Euclidean planes. (The sine-Gordon equation provides an example of this separation.) For many of these cases the separated solutions A, B are expressible in terms of elliptic functions. For non-orthogonal coordinates and $f = G(\Psi)$ true functional separation occurs precisely for Cartesian coordinates in the pseudo-Euclidean plane and for a coordinate system on the hyperboloid of one sheet, a pseudo-Riemannian space of constant curvature.

PACS: 02.30.+g, 02.90.+p, 03.65.Fd

1. Introduction. Our aim is to construct explicit closed-form solutions of interesting partial differential equations. The approach we follow is to use a generalization of the classical method of separation of variables to reduce the original pde to a system of ordinary differential equations that we can then attempt to solve. We say that a solution Ψ of a partial differential equation in two real variables x_1, x_2 is *functionally separable* in these variables if $\Psi(x_1, x_2) = \phi\,(A(x_1) + B(x_2))$ for single variable functions ϕ, A, B such that $\phi' A' B' \neq 0$, [1,2,3]. If ϕ' is a constant this corresponds to additive separation of variables; if $\phi(u) = e^u$ it corresponds to multiplicative variable separation. In this paper we classify all possibilities for regular functional separation in local coordinates for equations of the form $\Delta_2 \Psi = f(\Psi, x_1, x_2)$ where Δ_2 is the Laplace-Beltrami operator on a two-dimensional Riemannian or pseudo-Riemannian space. If the dependence of f on x_1, x_2 is nontrivial we show that (depending on the form of f) separation can occur for conformal Cartesian coordinates on any space. If $f = G(\Psi)$ then for orthogonal coordinates we find that true functional separation, i.e. separation other than additive or multiplicative,

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occurs precisely for Cartesian coordinates in the Euclidean and pseudo-Euclidean planes. (The sine-Gordon equation provides an example of this separation.) For many of these cases the separated solutions A, B are expressible in terms of elliptic functions. For non-orthogonal coordinates and $f = G(\Psi)$ true functional separation occurs precisely for Cartesian coordinates in the pseudo-Euclidean plane and for a coordinate system on the hyperboloid of one sheet, a pseudo-Riemannian space of constant curvature.

2. Functional separation for $\Psi_{tt} - \Psi_{xx} = f(\Psi, t, x)$. For our first important example we look for solutions of the equation

$$\Psi_{tt} - \Psi_{xx} = f(\Psi, t, x) \tag{2.1}$$

in the form $\Psi(t,x) = \phi(u), \quad u = A(t) + B(x)$. We will assume that $\phi' \neq 0$ and that all functions are infinitely differentiable. Clearly, functional separation of (2.1) is equivalent to additive separation of the modified equation

$$u_{tt} - u_{xx} + N(u)(u_t^2 - u_x^2) = M(u, t, x) \tag{2.2}$$

$$N(u) = \frac{\phi''}{\phi'}, \quad M(u,t,x) = \frac{f(\phi, t, x)}{\phi'}.$$

Thus, we look for solutions u of (2.2) such that $u_{tx} = 0$.

A set of necessary and sufficient conditions for additive separation of a partial differential equation was worked out by Kalnins and Miller [4,5,6]. We give a brief review of this theory and apply it to our example. Suppose we look for additively separable solutions of the partial differential equation

$$H(x^i, u, u_i, u_{ii}, u_{iii}, \cdots) = 0, \qquad 1 \leq i \leq n \tag{2.3}$$

in the n coordinates x^i. (Here, $u_i = \partial_{x^i} u$, $u_{ii} = \partial_{x^i x^i} u$, $\cdots$, and $u_{ij} \equiv 0$ for $i \neq j$.) We look for solutions of the form:

$$u = \sum_{i=1}^{n} S^{(i)}(x^i)$$

Introducing new notation, let

$$u_{i,1} \equiv u_i, \quad u_{i,j+1} \equiv \partial_{x^i} u_{i,j}, \qquad j = 1, 2, \cdots,$$

let m_i be the largest integer ℓ such that $\partial_{u_{i,\ell}} H = H_{u_{i,\ell}} \not\equiv 0$ and let D_i be the total differentiation operator

$$D_i = \partial_{x^i} + u_{i,1}\partial_u + u_{i,2}\partial_{u_{i,1}} + \cdots + u_{i,m_i+1}\partial_{u_{i,m_i}} + \cdots.$$

Then the equation $D_i H(x,u) = 0$ implies

$$u_{i,m_i+1} = -\frac{\tilde{D}_i H}{H_{u_{i,m_i}}}, \qquad i = 1, 2, \cdots, n, \tag{2.4}$$

where

$$\tilde{D}_i = \partial_{x^i} + u_{i,1}\partial_u + u_{i,2}\partial_{u_{i,1}} + \cdots + u_{i,m_i}\partial_{u_{i,m_i-1}}.$$

It follows that u satisfies the integrability conditions $D_j u_{i,m_i+1} = 0, \quad j \neq i$, or

$$\begin{aligned} &H_{u_{i,m_i}} H_{u_{j,m_j}} (\tilde{D}_i \tilde{D}_j H) + H_{u_{i,m_i} u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H) \\ &\quad = H_{u_{j,m_j}} (\tilde{D}_i H)(\tilde{D}_j H_{u_{i,m_i}}) + H_{u_{i,m_i}} (\tilde{D}_j H)(\tilde{D}_i H_{u_{j,m_j}}). \end{aligned} \tag{2.5}$$

Conversely we have the result:

Theorem 1 [4, 5, 6]. *If conditions (2.5) are satisfied identically (for $i \neq j$ and modulo the requirement $H = 0$) in the dependent variables $u, u_{k,\ell}$, then the partial differential equation $H = 0$ admits a $\sum_{i=1}^{n} m_i$ parameter family of separable solutions. Indeed, for every set of $m_1 + m_2 + \cdots + m_n + 1$ constants $\{v^0, v^0_{i,j}\}$ subject to the condition $H(x^0, v^0) = 0$ with $H_{u_{j,m_j}}(x^0, v^0) \neq 0$, there is a unique separable solution u of $H(x, u) = 0$ such that $u(x^0) = v^0$, $u_{i,j}(x^0) = v^0$, $u_{i,j}(x^0) = v^0_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq m_i$.*

If for a particular coordinate system x_i equations (2.5) are satisfied identically modulo $H = 0$ we say that these coordinates permit *regular* separation. If equations (2.5) are not satisfied identically, separable solutions still may exist but will depend on fewer than $\sum_{i=1}^{n} m_i$ independent parameters. This is *nonregular* separation. We will restrict our attention to regular separation.

Now we return to our example (2.2). For ease of calculation we will initially limit ourselves to the case $M = M(u)$, i.e., $f_t = f_x = 0$. Differentiating (2.2) with respect to t we find

$$u_{ttt} = -N' u_t (u_t^2 - u_x^2) - 2N u_t u_{tt} + M' u_t. \tag{2.6}$$

For an additively separable solution we must have $u_{tttx} = 0$. Differentiating (2.6) with respect to x and using (2.2) to eliminate the terms in u_{tt} and u_{xx} we obtain the necessary condition

$$(u_x^3 u_t - u_t^3 u_x)(N'' - 2N'N) + u_x u_t (M'' - 2N'M) = 0.$$

This condition is satisfied identically if and only if the separation conditions

$$1)\ N'' - 2N'N = 0, \qquad 2)\ M'' - 2N'M = 0 \tag{2.7}$$

hold. By Theorem 1, conditions (2.7) are necessary and sufficient for regular separation of equation (2.2). (The corresponding computation for u_{xxx} leads to exactly the same conditions as, by symmetry, it must.) Note that equation 1) admits the first integral $N' = N^2 + c^2$ for c a constant, so we can compute N from an additional integration. Moreover, if N is a solution of 1) then $M = N$ is a solution of 2). Setting $M = NP$ in 2) we thus obtain the first order ordinary differential equation $P' = \xi N^{-2}$ for P, where ξ is a constant.

The solutions are of 4 types:

(1) $c \neq 0$, $N' \neq 0$

$$N = c \tan w, \quad M = k_1 \tan w + k_2 [w \tan w + 1], \quad w = cu + b$$

$$e^{-ia\phi(u)} = \tan w + \sec w, \quad f(\phi) = [\frac{ck_1}{2a} + \frac{ck_2}{2a} \ln(\tan a\phi + \sec a\phi)] \sin 2a\phi - \frac{ck_2}{a} \cos a\phi$$

(2) $c = 0$, $N' \neq 0$

$$N = \frac{-1}{u+b}, \quad M = c_1 (u+b)^2 + \frac{c_2}{u+b}$$

$$\phi(u) = k\ln(u+b), \quad f(\phi) = kc_1 e^{\phi/k} + kc_2 e^{-2\phi/k}$$

(3) $N' \equiv 0,\ N = ic \neq 0$

$$M = c_1 u + c_2, \quad \phi(u) = e^{icu+k}, \quad f(\phi) = c_1\phi\ln\phi + (icc_2 - kc_1)\phi$$

(4) $N' \equiv 0,\ N \equiv 0$

$$M = c_1 u + c_2, \quad \phi(u) = cu, \quad f(\phi) = cc_1\phi + cc_2.$$

In each case we have set an additive constant equal to zero in the expression for $\phi(u)$.

For each of these cases the separated solutions $u = A(t) + B(x)$ can be obtained by solving ordinary differential equations. Indeed the mechanism for separation is generalized differential Stäckel form [6, 7], a nontrivial extension of classical Stäckel form [8,9]. In case (2) the basic separation equations are

$$(A')^2 = u_t^2 = \alpha A^2 + \beta A + \gamma + 2c_1 A^3 \tag{2.8a}$$

$$(B')^2 = u_x^2 = \alpha B^2 - \beta B + \gamma - 2c_1 B^3 + c_2 \tag{2.8b}$$

with two additional separation equations

$$A'' = u_{tt} = \alpha A + \beta/2 \quad + 3c_1 A^2 \tag{2.8c}$$

$$B'' = u_{xx} = \alpha B - \beta/2 \quad - 3c_1 B^2 \tag{2.8d}$$

obtained by differentiating equation (2.8a) with respect to t and (2.8b) with respect to x and canceling the common factors $2A'$, $2B'$. (Here for simplicity we have set $b = 0$.) The constants $\alpha, \beta, \gamma, c_1$ and c_2 are separation parameters. We observe that

$$(2.2) = (2.8c) - (2.8d) - \frac{1}{u}(2.8a) + \frac{1}{u}(2.8b)$$

so a solution $u = A(t) + B(x)$ of (2.8a), (2.8b) necessarily satisfies (2.2). Note that for general values of $\alpha, \beta, \gamma, c_1, c_2$ the problem of finding functionally separable solutions has been reduced to quadratures and inversion, i.e., to the problem of evaluating integrals of the form

$$t = \int_\delta^A \frac{dA}{\sqrt{\alpha A^2 + \beta A + \gamma + 2c_1 A^3}}$$

and solving for A as a function of t. It is well known for these integrals that A, B are elliptic functions [10, Chapter VII, §5]. (The possible explicit solutions for the wave equation in Cartesian coordinates ($f \equiv 0$) are given in [2, page 126]). Furthermore, for fixed f the separated solution depends on 4 independent parameters, in agreement with the prediction of Theorem 1. We see that the *Stäckel matrix* associated with this separation is

$$\begin{pmatrix} A^2 & A & 1 & 2A^3 & 0 \\ B^2 & -B & 1 & -2B^3 & 1 \\ A & \frac{1}{2} & 0 & 3A^2 & 0 \\ B & -\frac{1}{2} & 0 & -3B^2 & 0 \end{pmatrix}. \tag{2.9}$$

(We make a brief digression to recall the meaning of Stäckel matrices and Stäckel form [8,9]. We consider a classical example: orthogonal separation for the Hamilton-Jacobi equation for a free particle on an n dimensional Riemannian space

$$\sum_{i=1}^{n} H_i^{-2} u_i^2 = E.$$

Here the metric, expressed in terms of the local orthogonal coordinates $\{x^i\}$, is

$$ds^2 = \sum_{i=1}^{n} H_i^2 (dx^i)^2.$$

Assume additive separation so that $\partial_j u_i = \partial_j \partial_i u = 0$ for $i \neq j$. We determine conditions such that any solution of the *separation equations*

$$u_i^2 + \sum_{j=1}^{n} s_{ij}(x^i)\lambda^j = 0, \qquad i = 1, \cdots, n \quad \lambda^1 = -E$$

is automatically a separable solution of the Hamilton-Jacobi equation. Here $\partial_k s_{ij}(x^i) = 0$ for $k \neq i$ and $\det(s_{ij}) \neq 0$ so that the separation equations are ordinary differential equations. The constants λ^j are called *separation parameters.* We say that $S = (s_{ij})$ is a *Stäckel matrix* in the sense that the ith row of S depends only on the variable x^i. Now it is not difficult to show that the Hamilton-Jacobi equation can be recovered from the separation equations if and only if the metric coefficients can be expressed in the form $H_i^{-2} = (S^{-1})^{1i}$. If this is the case the metric coefficients are said to be in *Stäckel form* with respect to these coordinates. Similar constructions work for second order linear differential equations, such as the Schrödinger equation. For generalized differential Stäckel form the construction of the original partial differential equation from the (nonsingular) Stäckel matrix is analogous except that now more than one row may depend on the variable x^i and the elements of the matrix may be functions of the dependent variables such as u_i, in addition to the explicit functions of the independent variables x^i which is allowed for classical Stäckel matrices.)

Case (1) is similar. If we write $u = A(t) + B(x)$ in place of $cu + b$ we find the basic separation equations

$$(A')^2 = u_t^2 = \alpha e^{2iA} + \beta + \gamma e^{-2iA} + \frac{ck_1}{2a} + \frac{ck_2}{2a}A \tag{2.10a}$$

$$(B')^2 = u_x^2 = -\alpha e^{-2iB} + \beta - \gamma e^{2iB} - \frac{ck_2}{2a}B \tag{2.10b}$$

with two additional separation equations

$$A'' = u_{tt} = i\alpha e^{2iA} - i\gamma e^{-2iA} + \frac{ck_2}{4a} \tag{2.10c}$$

$$B'' = u_{xx} = i\alpha e^{-2iB} - i\gamma e^{2iB} - \frac{ck_2}{4a} \tag{2.10d}$$

obtained by differentiating equation (2.10a) with respect to t and (2.10b) with respect to x and canceling the common factors $2A'$, $2B'$. Cases (3) and (4) are even simpler. For general values of $\alpha, \beta, \gamma, c_1, c_2$ the problem of finding functionally separable solutions has been reduced to quadratures and inversion, i.e., to the problem of evaluating integrals of the form

$$t = \int_\delta^A \frac{dA}{\sqrt{\alpha e^{2iA} + \beta + \gamma e^{-2iA} + kA}}$$

and solving for A as a function of t. For many special cases A, B can be expressed iin terms of elliptic functions, but for general values of the parameters with $\alpha\beta k \neq 0$ these functions have not been tabulated. (Note that for $k_2 = 0$ the type (1) equation (2.1) is the sine-Gordon equation [2, 11, 12 (§5.2), 13] and equations (2.10ab) can be integrated explicitly.)

It is easy to modify the derivation of the separation conditions for (2.2) in the case where not both f_t, f_x vanish. The separation conditions (2.7) then become

$$(2.11) \quad 1)\ N'' - 2N'N = 0, \quad 2)\ M_{uu} - 2N'M = 0, \quad 3)\ M_{ux} = M_{ut} = M_{tx} = 0$$

If $N' \neq 0$ then conditions 2) and 3) imply that $f_t = f_x = 0$, which is contrary to our assumption. It follows easily that only solutions of types (3) and (4) listed above can occur in this case. The solutions given for these types are still valid except that now $c_2 = g(t) + h(x)$ where g and h are arbitrary functions.

Next we consider an example with very different behavior, equation (2.1) expressed in the nonorthogonal Cartesian coordinates s, y where $t = s + y$, $x = s - y$. Then the equation becomes

$$\Psi_{sy} = F(\Psi, s, y)$$

and we seek solutions of the form $\Psi(s, y) = \phi(u)$, $u = A(s) + B(y)$. This is equivalent to additive separation of the modified equation

$$(2.12) \qquad u_s u_y = M(u, s, y), \quad M(u, s, y) = \frac{F(\phi, s, y)}{\phi''}.$$

(Here, we are assuming that $F \neq 0$. If $F = 0$ then the only regular separable solution is $\phi(u) = au + b$.) We initially limit ourselves to the case $M = M(u)$, i.e., $F_s = F_y = 0$. Differentiating (2.12) we find

$$(2.13) \qquad u_{ss} = M' \frac{u_s}{u_y}, \quad u_{yy} = M' \frac{u_y}{u_s}.$$

Then the requirement $u_{ssy} = 0$ together with (2.12) and (2.13) implies the necessary and sufficient condition for regular separation:

$$(2.14) \qquad M''M - (M')^2 = 0.$$

The solution is $M(u) = \alpha e^{ku}$. The possible functions $\phi(u)$ are the solutions of the ordinary differential equation

$$\phi'' = \alpha^{-1} e^{-ku} F(\phi).$$

The separation equations are

$$-\alpha e^{kA} + cA' = 0, \quad B' - ce^{kB} = 0$$

and can be solved explicitly. Here c is the separation parameter. For fixed M the separable solutions of (2.11) have 2 independent parameters, as predicted by Theorem 1. (See [1, page 692] for the explicit solution of the Sine-Gordon equation $\Psi_{sy} = \sin\Psi$ by this method.)

If F depends on the variables s, y the necessary and sufficient conditions for separation are

(2.15)
$$M_{uu}M^2 - M_u^2 M + M_{sy}M - M_s M_y = 0, \ M_{us}M - M_s M_u = 0, \ M_{uy}M - M_u M_y = 0.$$

These conditions imply that $M = P(u)Z(s, y)$ where

$$(\ln P)'' = aP^{-1}, \quad (\ln Z)_{sy} = -aZ$$

and a is a constant. Thus $F(\phi, s, y) = G(\phi)Z(s, y)$ and $\phi(u)$ is obtained by solving the ordinary differential equation $\phi'' = G(\phi)/P(u)$. For $a = 0$ a change of variables of the form $S(s), Y(y)$ reduces the problem to (2.14). For $a \neq 0$ the equation for Z is the Liouville equation with general solution $Z = -\frac{2}{a}S'(s)Y'(y)/\left(S(s) + Y(y)\right)^2$ for arbitrary functions S, Y, [14, 15 (page 60)]. With an obvious change of variables, preserving the separation, we have $Z(s, y) = -\frac{2}{a}/(s+y)^2$. Furthermore the equation for P has the general solution $P(u) = \frac{a}{2k^2}\cosh^2 k(u + b)$ and the envelope solution $P(u) = -\frac{a}{2}u^2$. For the general solution we have $u = \pm(2k)^{-1}\ln|(\alpha y - \beta)(\gamma s + \delta)/(-\gamma y + \delta)(\alpha s + \beta)| - b$. For the envelope solution we find $u = (\alpha\gamma - \beta\delta)(s + y)/(\gamma s + \delta)(-\gamma y + \delta)$. Here, $\alpha\delta - \beta\gamma \neq 0$. In each case for fixed M there are two independent parameters in the expression for u.

3. Functional separation for pseudo-Riemannian spaces. To generalize the first example of §2 we search for solutions of the equation

$$\Delta_2\phi(u) = f(\phi(u), x_1, x_2) \tag{3.1}$$

such that $u = A(x_1) + B(x_2)$ where x_1, x_2 is an orthogonal coordinate system in a pseudo-Riemannian space, i.e., with signature $(+1, -1)$. Here Δ_2 is the Laplace-Beltrami operator [16]

$$\begin{aligned}\Delta_2 =& \frac{1}{H_1H_2}\left[\partial_1\left(\frac{H_2}{H_1}\partial_1\right) + \partial_2\left(\frac{H_1}{H_2}\partial_2\right)\right] \\ =& \frac{1}{H_1^2}\partial_{11} + \frac{1}{H_2^2}\partial_{22} + h_1\partial_1 + h_2\partial_2\end{aligned} \tag{3.2}$$

where

$$h_1 = \frac{1}{H_1H_2}\partial_1\left(\frac{H_2}{H_1}\right), \quad h_2 = \frac{1}{H_1H_2}\partial_2\left(\frac{H_1}{H_2}\right) \tag{3.3}$$

and the metric for the pseudo-Riemannian space is given (in terms of the local coordinates x_1, x_2) by

$$ds^2 = H_1^2(x_1,x_2)\ dx_1^2 + H_2^2(x_1,x_2)\ dx_2^2.$$

It follows that we must determine additively separable solutions for the equation

(3.4)
$$\frac{1}{H_1^2}u_{11} + \frac{1}{H_2^2}u_{22} + h_1u_1 + h_2u_2 + N(u)(\frac{u_1^2}{H_1^2} + \frac{u_2^2}{H_2^2}) = M(u,x_1,x_2)$$
$$N(u) = \frac{\phi''}{\phi'}, \quad M(u,x_1,x_2) = \frac{f(\phi,x_1,x_2)}{\phi'}.$$

According to Theorem 1 we must compute u_{111} and u_{222} in terms of lower order derivatives and then use (3.4) and the requirement $u_{1112} = 0$ to obtain a polynomial identity in u, u_1, u_2, u_{22}. This computation is tedious but straightforward.

Initially, we require that $N \neq 0$. Equating the coefficients of the term u_2u_{22} we find the condition $(H_1^{-2} - H_2^{-2})\partial_1(\ln \frac{H_1^2}{H_2^2})N = 0$. Due to the signature requirement on the metric the first factor cannot vanish. Thus $\partial_1(\ln \frac{H_1^2}{H_2^2}) = 0$, which implies $h_1 = 0$. By symmetry it follows that $h_2 = 0$ also. Thus $H_1^2/H_2^2 = k$ is a constant, negative because of the signature requirement. By suitable rescaling of the coordinates we can obtain $H_1^2 = -H_2^2 = \rho(x_1,x_2)$.

Now, multiplying both sides of (3.4) by $\rho(x_1,x_2)$ and making the identifications $x_1 = t$, $x_2 = x$, $\tilde{M}(u,x_1,x_2) = \rho(x_1,x_2)f(\phi,x_1,x_2)/\phi'$ we obtain equation (2.2) again. Thus the analysis of this example in §2 completely resolves our more general problem in the case $N \neq 0$. Indeed, if $N'f \neq 0$ then $\rho f = G(\phi)$, the coordinates are conformal to Cartesian coordinates in the pseudo-Euclidean plane and the solutions are of types (1) and (2) given in §2. If $f = 0$ the coordinates are again conformally Cartesian and the possibilities for N and ϕ correspond to types (1) and (2) given in §2.

If $N' = 0$ but $Nf \neq 0$ then either $\rho f = G(\phi)$ and we have the type (3) solutions of §2, or $\rho f = G(\phi)[g(x_1) + h(x_2) + \tilde{\lambda}]$ where $g + h$ is nonconstant. For this latter possibility we must have $c_1 = 0$ in the type (3) solutions; these solutions correspond to multiplicative separation of the Schrödinger equation $\Delta_2\Psi - V\Psi = \lambda\Psi$ where $V = [g(x_1) + h(x_2)]/\rho(x_1,x_2)$ is a separable potential. The general theory for such solutions is discussed in [17, Chapter 1], for example.

If $N' = f = 0$ but $N \neq 0$ then we must have $c_1 = c_2 = 0$ in the type (3) solutions. These solutions correspond to multiplicative separation of the Laplace equation $\Delta_2\Psi = 0$.

Now suppose that $N = 0$. Then by equating the coefficients of the term u_{22} in the separation conditions we obtain the requirement

(3.5)
$$\partial_{12} \ln \left(\frac{H_1^2}{H_2^2}\right) = 0.$$

This means that by replacing our coordinates with suitably renormalized coordinates $X_1(x_1), X_2(x_2)$, if necessary, (this preserves functional separation) we can obtain the metric coefficients in the form $H_1^2(x_1,x_2) = -H_1^2(x_1,x_2) = \rho(x_1,x_2)$, with respect to the new coordinates x_1, x_2. Now, multiplying both sides of (3.4) by the factor $\rho(x_1, x_2)$ and noticing that $h_1 = h_2 = 0$, we obtain (2.2) with $t = x_1, x = x_2$, $N = 0$ and $\tilde{M}(u, t, x) = \rho(t, x) f(\phi, t, x)/\phi'$. If $M \neq 0$ it follows from (2.11) and the type (4) solutions of §2 that either $\rho f = G(\phi)$, $(M' \neq 0)$, or $\rho f = g(t) + h(x) + \tilde{\lambda}$ where $g + h$ is variable. These latter solutions correspond to additive separation of the equation $\Delta_2 \Psi - V\Psi = \lambda$ where $V = [g(x_1) + h(x_2)]/\rho(x_1, x_2)$ is a separable potential. The general theory for such solutions in n variables is treated in [6, 7].

If $N = M = 0$ we obtain the special case of the type (4) solutions of §2 with $f = 0$. These solutions correspond to additive separation of the Laplace equation $\Delta_2 \Psi = 0$.

Theorem 2. *The equation*

$$\Delta_2 \phi(u) = f(\phi(u), x_1, x_2)$$

where Δ_2 is the Laplace-Beltrami operator on a pseudo-Riemannian space with signature $(+1, -1)$ admits (regular) solutions such that $u = A(x_1) + B(x_2)$ in a local orthogonal coordinate system x_1, x_2 for some function ϕ with $\phi' \neq 0$ under the following circumstances. (Here, $N(u) = \phi''/\phi'$, $N' = N^2 + c^2$ for c a constant, $M(u) = f(\phi)/\phi'$ and ds^2 is the metric on the pseudo-Riemannian space.) In all cases the coordinates are conformally Cartesian: $ds^2 = \rho(x_1, x_2)(dx_1^2 - dx_2^2)$.

(1) $c \neq 0,\ N' \neq 0$

$$N = c \tan w, \quad M = k_1 \tan w + k_2[w \tan w + 1], \quad w = cu + b$$

$$e^{-ia\phi(u)} = \tan w + \sec w$$

$$\rho(x_1, x_2) f(\phi) = [\frac{ck_1}{2a} + \frac{ck_2}{2a} \ln(\tan a\phi + \sec a\phi)] \sin 2a\phi - \frac{ck_2}{a} \cos a\phi$$

(2) $c = 0,\ N' \neq 0$

$$N = \frac{-1}{u+b}, \quad M = c_1(u+b)^2 + \frac{c_2}{u+b}$$

$$\phi(u) = k \ln(u+b), \quad \rho(x_1, x_2) f(\phi) = kc_1 e^{\phi/k} + kc_2 e^{-2\phi/k}$$

(3) $N = ic \neq 0,\ M' \neq 0$

$$M = c_1 u + c_2, \quad \phi(u) = e^{icu+k}, \quad \rho(x_1, x_2) f(\phi) = c_1 \phi \ln \phi + (icc_2 - kc_1)\phi$$

(4) $N = ic \neq 0,\ M' = 0$

$$M = c_2, \quad \phi(u) = e^{icu+k}, \quad \rho(x_1, x_2) f(\phi) = icc_2 \phi[g(x_1) + h(x_2) + \lambda/icc_2]$$

Stäckel form coordinates providing multiplicative separation of the Schrödinger equation $\Delta_2 \Psi - V\Psi = \lambda\Psi$ on the pseudo-Riemannian space where $V = icc_2[g(x_1) + h(x_2)]/\rho(x_1, x_2)$ is a separable potential.

(5) $N = 0,\ M' \neq 0$

$$M = c_1 u + c_2, \quad \phi(u) = cu, \quad \rho(x_1, x_2) f(\phi) = cc_1 \phi + cc_2$$

(6) $N = M' = 0$

$$M = c_2, \quad \phi(u) = cu, \quad \rho(x_1, x_2) f(\phi) = cc_2[g(x_1) + h(x_2) + \lambda/cc_2]$$

(Here, $g + h$ is nonconstant.) Stäckel form coordinates providing additive separation of the equation $\Delta_2\Psi - V\Psi = \lambda$ on the pseudo-Riemannian space where $V = cc_2[g(x_1) + h(x_2)]/\rho(x_1, x_2)$ is a separable potential.

The type (4), (6) multiplicative and additive separations have been studied in detail for several manifolds [18, 19, 20, 21]. It is known that for pseudo-Euclidean space separation is possible in 10 coordinate systems whereas for the single sheeted hyperboloid it is possible in 9 coordinate systems.

Next we search for solutions of the equation

$$\Delta_2 \phi(u) = f(\phi(u), x_1, x_2) \tag{3.6}$$

such that $u = A(x_1) + B(x_2)$ where x_1, x_2 is a *nonorthogonal* coordinate system in a pseudo-Riemannian space. Here Δ_2 is the Laplace-Beltrami operator

$$\Delta_2 = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{2} \partial_i \left(g^{ij} \sqrt{g} \partial_j \right), \quad g = \det(g_{ij}), \quad \sum_j g^{ij} g_{jk} = \delta^i_k,$$

where the metric for the pseudo-Riemannian space is given (in terms of the local coordinates x_1, x_2) by

$$ds^2 = g_{11}(x_1, x_2)\ dx_1^2 + 2g_{12}(x_1, x_2)\ dx_1 dx_2 + g_{22}(x_1, x_2)\ dx_2^2, \quad g_{12} = g_{21}.$$

The nonorthogonality requirement is that $g^{12} \neq 0$.

It follows that we must determine (additively) regular separable solutions for the equation

(3.7)
$$g^{11} u_{11} + g^{22} u_{22} + h_1 u_1 + h_2 u_2 + N(u)(g^{11} u_1^2 + 2g^{12} u_1 u_2 + g^{22} u_2^2) = M(u, x_1, x_2),$$
$$N(u) = \frac{\phi''}{\phi'}, \quad M(u, x_1, x_2) = \frac{f(\phi, x_1, x_2)}{\phi'},$$

where here

$$h_j = \frac{1}{\sqrt{g}} \sum_{i=1}^{2} \partial_i \left(g^{ij} \sqrt{g} \right), \quad j = 1, 2.$$

The case $N = 0$, i.e. $\phi(u) = u$, is degenerate in this problem so we require $N \neq 0$. The analysis is similar to the proof of Theorem 2. According to Theorem 1 we must compute u_{111} and u_{222} in terms of lower order derivatives and then use (3.4) and the requirement $u_{1112} = 0$ to obtain a polynomial identity in u, u_1, u_2, u_{22}. This computation is tedious but straightforward. First we assume $g^{11} g^{22} \neq 0$. Then equating coefficients of u_{22}^2 in the identity we find $N = 0$, a contradiction. If we assume $g^{11} \neq 0, g^{22} = 0$ and equate coefficients of $u_1 u_2 u_{22}$ we find again that

$N = 0$, another contradiction. By symmetry, $g^{22} \neq 0, g^{11} = 0$ is also impossible. Thus the only remaining possibility is $g^{11} = g^{22} = 0$. Setting $g^{12} = \rho^{-1}(x_1, x_2)$ we can write (3.7) in the simple form

$$u_1 u_2 = \frac{f(\phi, x_1, x_2)}{\phi''}\rho(x_1, x_2). \tag{3.8}$$

This is just equation (2.12) with the identifications $x_1 = s, x_2 = y, \tilde{M}(u, s, y) = \rho(s, y) f(\phi, s, y)/\phi''$. It follows from the analysis of (2.12) that there are two types of solutions. For the first type we can assume $\rho f = G(\phi)$ and the solutions are given following (2.14). (Here the coordinates are conformal to Cartesian coordinates.) For the second type we can assume $\rho f = G(\phi)(s + y)^{-2}$ and the solutions are given following (2.15). (Here the coordinates are conformal to coordinates on the one-sheeted hyperboloid, a space of constant curvature [18, 19, 20]. Explicitly, if $dS^2 = ds\, dy/(s+y)^2$ then setting $z_1 = (sy+1)/2(s+y), z_2 = (s-y)/2(s+y), z_3 = (sy-1)/2(s+y)$ we find $z_1^2 + z_2^2 - z_3^2 = 1/4$ and $dS^2 = -dz_1^2 - dz_2^2 + dz_3^2$.)

Theorem 3. *The equation*

$$\Delta_2 \phi(u) = f(\phi(u), x_1, x_2)$$

admits (regular) solutions such that $u = A(x_1) + B(x_2)$ and $\phi'' \neq 0$ in a local nonorthogonal coordinate system x_1, x_2 on a pseudo-Riemannian space in precisely two cases:

1. *The coordinates are conformal to Cartesian coordinates in the pseudo-Euclidean plane: $ds^2 = \rho(x_1, x_2) dx_1 dx_2$, $\rho f = G(\phi)$.*
2. *The coordinates $ds^2 = \rho(x_1, x_2)\frac{dx_1 dx_2}{(x_1+x_2)^2}$, $\rho f = G(\phi)/(x_1 + x_2)^2$ are conformal to coordinates $d\tilde{s}^2 = \frac{dx_1 dx_2}{(x_1+x_2)^2}$ on the hyperboloid of one sheet.*

4. Functional separation for Riemannian spaces. Now we seek solutions of the equation

$$\Delta_2 \phi(u) = f(\phi(u), x_1, x_2) \tag{4.1}$$

such that $u = A(x_1) + B(x_2)$ where x_1, x_2 is an orthogonal coordinate system in a Riemannian space, i.e., with signature $(+1, +1)$. Again Δ_2 is the Laplace-Beltrami operator (3.2) It follows that we must determine additively separable solutions for the equation (3.4). The computations here are almost identical with those used in the proof of Theorem 2. The only difference is that here we require that H_1^2 and H_2^2 are positive, whereas in §3 $H_1^2 H_2^2$ was negative.

The result is as follows.

Theorem 4. *The equation*

$$\Delta_2 \phi(u) = f(\phi(u), x_1, x_2)$$

where Δ_2 is the Laplace-Beltrami operator on a Riemannian space admits (regular) solutions such that $u = A(x_1) + B(x_2)$ in a local orthogonal coordinate system

x_1, x_2 for some function ϕ with $\phi' \neq 0$ under the following circumstances. (Here, $N(u) = \phi''/\phi'$, $N' = N^2 + c^2$ for c a constant, $M(u) = f(\phi, x_1, x_2)/\phi'$ and ds^2 is the metric on the Riemannian space.) In all cases the coordinates are conformally Cartesian: $ds^2 = \rho(x_1, x_2)(dx_1^2 + dx_2^2)$.

(1) $c \neq 0,\ N' \neq 0$

$$N = c \tan w, \quad M = k_1 \tan w + k_2[w \tan w + 1], \quad w = cu + b$$

$$e^{-ia\phi(u)} = \tan w + \sec w$$

$$\rho(x_1, x_2) f(\phi) = [\frac{ck_1}{2a} + \frac{ck_2}{2a} \ln(\tan a\phi + \sec a\phi)] \sin 2a\phi - \frac{ck_2}{a} \cos a\phi$$

(2) $c = 0,\ N' \neq 0$

$$N = \frac{-1}{u+b}, \quad M = c_1(u+b)^2 + \frac{c_2}{u+b}$$

$$\phi(u) = k \ln(u+b), \quad \rho(x_1, x_2) f(\phi) = kc_1 e^{\phi/k} + kc_2 e^{-2\phi/k}$$

(3) $N = ic \neq 0,\ M' \neq 0$

$$M = c_1 u + c_2, \quad \phi(u) = e^{icu+k}, \quad \rho(x_1, x_2) f(\phi) = c_1 \phi \ln \phi + (icc_2 - kc_1)\phi$$

(4) $N = ic \neq 0,\ M' = 0$

$$M = c_2, \quad \phi(u) = e^{icu+k}, \quad \rho(x_1, x_2) f(\phi) = icc_2\phi[g(x_1) + h(x_2) + \lambda/icc_2]$$

Stäckel form coordinates providing multiplicative separation of the Schrödinger equation $\Delta_2\Psi - V\Psi = \lambda\Psi$ on the Riemannian space where

$$V = icc_2[g(x_1) + h(x_2)]/\rho(x_1, x_2)$$

is a separable potential.

(5) $N = 0,\ M' \neq 0$

$$M = c_1 u + c_2, \quad \phi(u) = cu, \quad \rho(x_1, x_2) f(\phi) = cc_1\phi + cc_2$$

(6) $N = M' = 0$

$$M = c_2, \quad \phi(u) = cu, \quad \rho(x_1, x_2) f(\phi) = cc_2[g(x_1) + h(x_2) + \lambda/cc_2]$$

(Here, $g + h$ is nonconstant.) Stäckel form coordinates providing additive separation of the equation $\Delta_2\Psi - V\Psi = \lambda$ on the Riemannian space where $V = cc_2[g(x_1) + h(x_2)]/\rho(x_1, x_2)$ is a separable potential.

In case (1) the basic separation equations are

$$\begin{aligned}(A')^2 = u_1^2 =& \alpha e^{2iA} + \beta + \gamma e^{-2iA} + \frac{ck_1}{2a} + \frac{ck_2}{2a} A\\ -(B')^2 = -u_2^2 =& -\alpha e^{-2iB} + \beta - \gamma e^{2iB} \quad - \frac{ck_2}{2a} B\end{aligned}$$

with two additional separation equations

$$\begin{aligned}A'' = u_{11} =& i\alpha e^{2iA} \quad - i\gamma e^{-2iA} \quad + \frac{ck_2}{4a}\\ -B'' = -u_{22} =& i\alpha e^{-2iB} \quad - i\gamma e^{2iB} \quad - \frac{ck_2}{4a}.\end{aligned}$$

In case (3) the basic separation equations are

$$\begin{aligned}(A')^2 = u_1^2 =& \alpha A^2 + \beta A + \gamma + 2c_1 A^3\\ -(B')^2 = -u_2^2 =& \alpha B^2 - \beta B + \gamma - 2c_1 B^3 + c_2\end{aligned}$$

with the two additional separation equations

$$\begin{aligned}A'' = u_{11} =& \alpha A + \beta/2 \quad + 3c_1 A^2\\ -B'' = -u_{22} =& \alpha B - \beta/2 \quad - 3c_1 B^2.\end{aligned}$$

As an example of the possible functionally separable explicit solutions of the Laplace equation $u_{x_1 x_1} + u_{x_2 x_2} = 0$ via this method see [2, page 117].

The type (4), (6) multiplicative and additive separations have been studied in detail for constant curvature manifolds [18, 19, 20, 21]. It is known that for Euclidean space separation is possible in 4 coordinate systems whereas for the double sheeted hyperboloid it is possible in 9 coordinate systems and for the sphere it is possible in 2 coordinate systems.

From the proof of Theorem 3 we see that functional separation in nonorthogonal variables occurs only if $g^{11} = g^{22} = 0$. This is not possible for coordinates in a Riemannian space.

Theorem 5. *The equation*

$$\Delta_2 \phi(u) = f(\phi(u), x_1, x_2)$$

in a local nonorthogonal coordinate system x_1, x_2 on a Riemannian space admits no regular separable solutions $u = A(x_1) + B(x_2)$ with $\phi'' \neq 0$.

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