Homework Problem Set #3

Math 5467

February 26, 2004

Exercise 1 *Let*

$$\hat{f}(\lambda) = \begin{cases} (1-\lambda^2) & \text{for } |\lambda| \le 1 \\ 0 & \text{for } |\lambda| > 1 \end{cases}$$

1. *Show that*
$$f(t) = \frac{2}{\pi}\left[\frac{\sin t - t\cos t}{t^3}\right].$$

2. *Choose* $\Omega = 1$ *and use the Shannon sampling theorem to write* $f(t)$ *as a series.*

3. *Graph the sum of the first 30 terms in the series.*

4. *Compare the preceding graph with the graph of* $f(t)$.

Exercise 2 *Suppose the only nonzero components of the input vector* $\mathbf{x}$ *and the impulse response vector* $\mathbf{h}$ *are* $\mathbf{x}(0) = 1$, $\mathbf{x}(1) = 2$, *and* $\mathbf{h}(0) = \frac{1}{4}$, $\mathbf{h}(1) = \frac{1}{2}$ *and* $\mathbf{h}(2) = \frac{1}{4}$. *Compute the outputs* $\mathbf{y}(n) = \mathbf{x} * \mathbf{h}(n)$. *Verify in the frequency domain that* $Y(\omega) = H(\omega)X(\omega)$.

Exercise 3

- *Write down the infinite matrix* $(\downarrow 3)$ *that executes the downsampling:* $(\downarrow 3)\mathbf{x}(n) = \mathbf{x}(3n)$.

- *Write down the infinite matrix $(\uparrow 3)$ that executes the upsampling:*

$$(\uparrow 3)\mathbf{y}(n) = \begin{cases} \mathbf{y}(\frac{n}{3}) & \text{if 3 divides } n \\ 0 & \text{otherwise} \end{cases}.$$

- *Multiply the matrices $(\uparrow 3)(\downarrow 3)$ and $(\downarrow 3)(\uparrow 3)$. Describe the output each of these product operations.*

Exercise 4 *(Filtering with the FFT) Let*

$$f(t) = e^{-\frac{t^2}{10}} (\sin t + 2\cos 2t + 0.5 \sin t \sin 60t).$$

Discretize f by setting $y_k = f(2k\pi/256), k = 1, \cdots, 256$. Use MATLAB's FFT to compute $\hat{y}_k$ for $0 \le k \le 255$. (Note that $y_{n-k} = \overline{y_k}$. Thus the low frequency coefficients are $\hat{y}_0, \cdots, \hat{y}_m$ and $\hat{y}_{256-m}, \cdots, \hat{y}_{256}$ for some small integer m. Filter out the high-frequency terms by setting $\hat{y}_k = 0$ for $m \le k \le 255 - m$ with $m = 6$. Then apply the inverse FFT to these filtered $\hat{y}_k$ to compute the filtered y_k. Plot the results and compare with the original unfiltered signal. Experiment with several different values of m.

Exercise 5 *(Compression with the FFT) Consider the signal $f(t)$ as given in the previous problem. Let $tol = 1.0$ In the previous problem compress the transformed signal by setting $\hat{y}_k = 0$ whenever $|\hat{y}_k| < tol$. Apply the inverse FFT to the compressed transfirmed signal to get a compressed signal y_k. Plot the results and compare with the original uncompressed signal. Experiment with several different values of tol. Keep track of the percentage of Fourier coefficients that have been filtered out.*

The following MATLAB routines should be useful:

You can discretize the interval $[0, 2\pi]$ and read in the signal as a vector by using the commands

```
t=linspace(0,2*pi,2^8);
f=exp(-t.^2/10).*(sin(t)+2*cos(2*t)+0.5*sin(t).*sin(60*t));
```

If $hatf$ is the FFT of f, i.e., $hatf = fft(f)$, you can filter out high frequency components from $hatf$ with a command such as

```
filterhatf=[ hatf(1: m) zeros(1, 2^8-2*m) hatf(2^8-m+1:2^8)]
```

```
function fc=compress( f, r)
% Input the vector f and ratio r: 0<= r <=1.
% The output is the vector fc in which the smallest
% 100r% of the terms f_k, in absolute value, are set
% equal to zero.
if (r<0) | (r>1)
   error ('r should be between 0 and 1')
end;
N=length(f); Nr=floor(N*r);
ff=sort(abs(w));
tol=abs(ww(Nr+1);
fc=(abs(w)>=tol).*w;
```

```
function L2error =fftcomp(t,f,r)
% Input: time vector t, signal vector f, compression rate r, (between
% 0 and 1)
Output: graph of f, graph of the compression of f, and the relative L2
error
if (r<0) | (r>1)
   error ('r should be between 0 and 1')
end;
hatf=fft(f);
hatfc=compress(hatf,r);
fc=ifft(hatfc);
plot(t,f,t,fc)
L2error=norm(f-fc,2)/norm(f)
```