Coupling constant metamorphosis and Nth order symmetries in classical and quantum mechanics

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Abstract

We review the fundamentals of coupling constant metamorphosis (CCM) and the Stäckel transform, and apply them to map integrable and superintegrable systems of all orders into other such systems on different manifolds. In general, CCM does not preserve the order of constants of the motion or even take polynomials in the momenta to polynomials in the momenta. We study specializations of these actions which do preserve polynomials and also the structure of the symmetry algebras in both the classical and quantum cases. We give several examples of non-constant curvature 3rd and 4th order superintegrable systems in 2 space dimensions obtained via CCM, with some details on the structure of the symmetry algebras preserved by the transform action.

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1 Introduction

There has been a recent rapid expansion in the number of known classical and quantum superintegrable systems of order 2, [1, 2], and, particularly, of order 3 and higher, [3, 4, 5, 6, 7, 8, 9]. For many of these systems it has been demonstrated that the algebra generated by the fundamental higher order symmetries closes under the Poisson bracket in the classical case, and under the commutator in the quantum case, to form a finite dimensional quadratic or cubic algebra. The representation theory of these algebras and their association with basic properties of the special functions of mathematical physics is of great current interest, [10, 11, 12, 13, 14, 15, 16]. Indeed, the basic properties of Gaussian hypergeometric functions and their various limiting cases, as well as Lamé, Mathieu and Heun functions, and ellipsoidal harmonics all appear as associated with 2nd order superintegrable quantum systems via separation of variables. These functions as well as orthogonal polynomials of a discrete variable, including the general Wilson and Racah polynomials, are bound up with function space models of the irreducible representations of the quadratic algebras associated with 2nd order superintegrable quantum systems. The Painlevé transcendents (not associated with variable separability) appear in the study of 3rd order superintegrable systems. Some examples are known for conformally flat manifolds in n dimensions, [17, 18, 19], but most results are known for 2 and 3 dimensional conformally flat spaces.

There is a disconnect, however, between what is known for 2nd order superintegrable systems and what is known for 3rd and higher order systems. For 2nd order superintegrable systems, classical and quantum, all such systems and all manifolds of dimension 2 on which they occur have been classified and the mechanism of the closure of the quadratic algebra is well understood, [20, 21, 22, 23, 24]. For conformally flat manifolds in 3 dimensions great advances have been made although the classification and structure analysis is not yet complete, [25, 26, 22, 27, 28]. A major tool for obtaining these 2nd order results has been the Stäckel transform [29, 30], a variant of coupling constant metamorphosis [31], which enables a 1-1 invertible transformation between a 2nd order superintegrable system on one manifold and a superintegrable system on another manifold that preserves the symmetry algebra structure. This has given us an elegant method for classification of all 2D superintegrable systems through the important fact that every such system can be shown to be the Stäckel transform of a system on a constant curvature space, [2, 21, 24]. Also it gives important insight into the structure of Koenigs' remarkable potential-free results [32]. Similar results are known for 3D systems but the classification is not yet complete, [26, 27, 28].

For 3rd and higher order superintegrable systems, however, there is no structure and classification theory. Only examples are known, and these are very difficult to obtain. The symmetry algebras can be computed for each example but the mechanism for their closure and structure is not understood. Virtually all known examples are in 2D or 3D Euclidean space. The present paper is a first attempt at refining a tool (CCM/Stäckel transform), that has proved so successful in the classification and structure theory for 2nd order systems, so that it applies to higher order superintegrable systems. There are two basic issues here. The first is that CCM in general doesn't preserve the structure of the symmetry algebras. We have to determine a suitable restriction that does preserve the structure. Secondly, CCM is a classical phenomenon; its extension to the quantum case is not automatic and requires special care. In this paper most of our classical results will be stated for n dimensional systems whereas, for simplicity, the quantum results will be limited mostly to 2 dimensions.

In future papers we will extend the operator CCM to 3 and higher dimensions and employ this tool to attack the structure and classification theory for 3rd and higher order superintegrable systems in all dimensions. An immediate result of the present paper is the explicit display of a large number of higher order superintegrable systems on manifolds not of constant curvature, the existence of which seems not to be widely recognized. We also provide new examples of explicit structure computations for the quadratic algebras of some 3rd and 4th order superintegrable systems on 2D Euclidean space that map to isomorphic systems on nonconstant curvature spaces.

Before proceeding to our results we give some basic definitions that we employ throughout the paper. A classical superintegrable system on an n-dimensional real or complex Riemannian or pseudo-Riemannian manifold is defined by its associated Hamiltonian function $\mathcal{H} = \sum_{ij} g^{ij} p_i p_j + V(\mathbf{x})$ on the phase space of this manifold. Here $g^{ij}(\mathbf{x})$ is the contravariant metric tensor in local coordinates $\mathbf{x}$ and $V(\mathbf{x})$ is a prescribed function that may depend on some parameters. The system is **superintegrable** if it admits $2n-1$ functionally independent generalized symmetries (or constants of the motion) $\mathcal{S}_k, \quad k = 1, \cdots, 2n-1$ with $\mathcal{S}_1 = \mathcal{H}$ where the $\mathcal{S}_k$ are polynomials in the momenta p_j. That is, $\{\mathcal{H}, \mathcal{S}_k\} = 0$ where $\{f, g\} = \sum_{j=1}^{n}(\partial_{x_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{x_j} g)$ is the Poisson bracket for functions $f(\mathbf{x}, \mathbf{p}), g(\mathbf{x}, \mathbf{p})$ on phase space. It is easy to see that $2n-1$ is the maximum possible number of functionally independent symmetries and, locally, such (in general nonpolynomial) symmetries always exist. Most authors, but not us, also demand that the system is **integrable**, i.e., there is a subset of n functionally independent polynomial symmetries, say $\mathcal{S}_1, \cdots, \mathcal{S}_n$, such that $\{\mathcal{S}_j, \mathcal{S}_\ell\} = 0$, $1 \le s, \ell \le n$. If the maximum order of the polynomials corresponding to the generating symmetries is N, we say that the system is Nth order superintegrable.

Superintegrable systems can lay claim to be the most symmetric Hamiltonian systems though many such systems admit no group symmetry; the symmetry is "hidden". Generically, every geometrical trajectory in phase space (but not the time dependence of the trajectory $\mathbf{p}(t), \mathbf{x}(t)$) of the Hamilton equations of motion for the system, is obtained as the common intersection of the (constants of the motion) hypersurfaces $\mathcal{S}_k(\mathbf{p}, \mathbf{x}) = c_k, \quad k = 0, \cdots, 2n-2$. The orbits can be found without solving the equations of motion. Since every known superintegrable system is also integrable, this is better than integrability. A case can be made that the 2nd order superintegrability of the Kepler-Coulomb two-body problem, forcing the existence of conic sections as trajectories, is the reason that Kepler was able to determine the planetary elliptical orbits before the invention of calculus.

There is an analogous definition of superintegrability for quantum systems with Schrödinger operator

$$H = \Delta + V(\mathbf{x}), \quad \Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} g^{ij}) \partial_{x_j},$$

the Laplace-Beltrami operator plus a potential function. Here it is required that there are $2n-1$ functionally independent differential operators, $S_1 = H, S_2, \cdots, S_{2n-1}$ such that and $[H, S_k] \equiv HS_k - S_k H = 0$. Often there is a 1-1 relationship between classical and quantum superintegrable systems associated with a potential and then functional independence refers to the classical system. In those cases where there is no classical analog, however, there is no agreed upon definition of quantum functional independence. A basic motivation for studying these systems is that they can be solved explicitly, often in multiple ways. Typically their symmetry algebras close to form quadratic, cubic, or similar algebras whose representation theory yields spectral information about the quantum system.

In the following sections we review the basic definition and properties of coupling constant metamorphosis (CCM) and the closely related Stäckel transform for classical systems. These concepts apply to any Hamiltonian system with potential, not just superintegrable systems. Then we define specializations of these general concepts that preserve the order of symmetries and also define symmetry algebra isomorphisms. It is these specializations that are needed for the study of superintegrable systems. Then, and most importantly, we find quantum analogs of these classical transforms. At each stage we provide examples, several of them new.

2 Coupling constant metamorphosis

The basic tool that we will employ follows from "coupling constant metamorphosis" (CCM), a general fact about Hamiltonian systems, pointed out in [31]. Let $\mathcal{H}(\mathbf{x},\mathbf{p})+\alpha U(\mathbf{x})$ define a Hamiltonian system in $2n$ dimensional phase space, with canonical coordinates x_j, p_j. Thus the Hamilton-Jacobi equation would take the form $\mathcal{H}(\mathbf{x},\mathbf{p})+\alpha U(\mathbf{x})=E$. Assume that for every value of the parameter α the system admits a constant of the motion $\mathcal{K}(\alpha)$, analytic in α.

Theorem 1 *Coupling constant metamorphosis. The Hamiltonian $\mathcal{H}'=(\mathcal{H}-E)/U$ admits the constant of the motion $\mathcal{K}'=\mathcal{K}(-\mathcal{H}')$, where now E is a parameter.*

PROOF: Note that if F, G are functions on phase space of the form $G(\mathbf{x},\mathbf{p})$, $F=F(a)=F(a,\mathbf{x},\mathbf{p})$ where $a=\alpha(\mathbf{x},\mathbf{p})$ then

$$\{F,G\}=\{F(a),G\}|_{a=\alpha(\mathbf{x},\mathbf{p})}+\partial_a F(a)|_{a=\alpha(\mathbf{x},\mathbf{p})}\{\alpha,G\}.$$

By assumption, $\{\mathcal{K}(\alpha),\mathcal{H}\}=-\alpha\{\mathcal{K}(\alpha),U\}$ for any value of the parameter α. Thus

$$\{\mathcal{K}(\alpha),\mathcal{H}'\}=\frac{\{U,\mathcal{K}(\alpha)\}}{U}(\mathcal{H}'+\alpha).$$

Now

$$\{\mathcal{K}(-\mathcal{H}'),\mathcal{H}'\}=\left[\partial_\alpha\mathcal{K}(\alpha)\{\mathcal{H}',\mathcal{H}'\}+\frac{\{U,\mathcal{K}(\alpha)\}}{U}(\mathcal{H}'+\alpha)\right]_{\alpha=-\mathcal{H}'}=0.$$

Q.E.D

Corollary 1 *Let $\mathcal{K}_1(\alpha),\mathcal{K}_2(\alpha)$ be constants of the motion for the system $\mathcal{H}(\mathbf{x},\mathbf{p})+\alpha U(\mathbf{x})$. Then $\{\mathcal{K}_1,\mathcal{K}_2\}(\alpha)\equiv\{\mathcal{K}_1(\alpha),\mathcal{K}_2(\alpha)\}$ is also a constant of the motion and*

$$\{\mathcal{K}_1(-\mathcal{H}'),\mathcal{K}_2(-\mathcal{H}')\}=\{\mathcal{K}_1,\mathcal{K}_2\}(-\mathcal{H}').$$

Clearly CCM takes integrable systems to integrable systems and superintegrable systems to superintegrable systems. We are concerned with the case where

$$\mathcal{H}=\sum_{i,j=1}^{n}g^{ij}p_ip_j+V(\mathbf{x})+\alpha U(\mathbf{x})\equiv\mathcal{H}_0+V+\alpha U \qquad (1)$$

is a classical Hamiltonian system on an n-dimensional pseudo-Riemannian manifold and are interested only in those constants of the motion $\mathcal{K}$ that are polynomial in the momenta. As we shall see, in the case of 2nd order constants of the motion there is special structure. The 2nd order constants of the motion are typically at most linear in α, so they transform to 2nd order symmetries again. In this case CCM agrees with the Stäckel transform that we shall take up in the next section. However, in general the order of constants of the motion is not preserved by coupling constant metamorphosis.

Example 1 *The system*

$$\mathcal{H} = p_1^2 + p_2^2 + b_1\sqrt{x_1} + b_2 x_2$$

admits the 2nd order constant of the motion $\mathcal{K}^{(2)} = p_2^2 + b_2 x_2$ *and the 3rd order constant of the motion* $\mathcal{K}^{(3)} = p_1^3 + \frac{3}{2}b_1\sqrt{x_1}p_1 - \frac{3b_1^2}{4b_2}p_2$, *([4] and references contained therein). If we choose* $\alpha U = \alpha\sqrt{x_1}$ *then the transform of* $\mathcal{K}^{(3)}$ *will be 5th order. If we choose* $\alpha U = \alpha x_2$ *then the transform of* $\mathcal{K}^{(3)}$ *will be rational, but nonpolynomial. Thus to obtain useful structure results from this general transform, and to obtain results that have the possibility of carrying over to the quantum case, we need to restrict the generality of the transform action.*

3 The Jacobi transform

Here we study a specialization of coupling constant metamorphosis to the case where $V = 0$. The special version of the transform we study takes Nth order constants of the motion for Hamiltonian systems to Nth order constants of the motion. An Nth order constant of the motion $\mathcal{K}(\mathbf{x}, \mathbf{p})$ for the system

$$\mathcal{H} = \sum_{i,j=1}^{n} g^{ij} p_i p_j + U(\mathbf{x}) = \mathcal{H}_0 + U \tag{2}$$

is a function on the phase space such that $\{\mathcal{K}, \mathcal{H}\} = 0$ where

$$\mathcal{K} = \mathcal{K}_N + \mathcal{K}_{N-2} + \mathcal{K}_{N-4} + \cdots + \mathcal{K}_0, \quad n \text{ even},$$

$$\mathcal{K} = \mathcal{K}_N + \mathcal{K}_{N-2} + \mathcal{K}_{N-4} + \cdots + \mathcal{K}_1, \quad n \text{ odd}.$$

Here, $\mathcal{K}_N \neq 0$ and $\mathcal{K}_j$ is homogeneous in $\mathbf{p}$ of order j. This implies the conditions

$$\{\mathcal{K}_N, \mathcal{H}_0\} = 0, \tag{3}$$

$$\{\mathcal{K}_{N-2k}, U\} + \{\mathcal{K}_{N-2k-2}, \mathcal{H}_0\} = 0, \quad k = 0, 1, \cdots, [N/2] - 1, \tag{4}$$

and, for N odd,

$$\{\mathcal{K}_1, U\} = 0. \tag{5}$$

The case $N = 1$ is very special. Then $\mathcal{K} = \mathcal{K}_1$ and the conditions are

$$\{\mathcal{K}, \mathcal{H}_0\} = 0, \quad \{\mathcal{K}, U\} = 0,$$

so $\mathcal{K}$ is a Killing vector and U is invariant under the local group action generated by the Killing vector.

For $N = 2$, $\mathcal{K} = \mathcal{K}_2 + \mathcal{K}_0$ and the conditions are

$$\{\mathcal{K}_2, \mathcal{H}_0\} = 0, \quad \{\mathcal{K}_2, U\} + \{\mathcal{K}_0, \mathcal{H}_0\} = 0, \tag{6}$$

so $\mathcal{K}_2$ is a 2nd order Killing tensor and U satisfies (linear) Bertrand-Darboux integrability conditions.

For $N = 3$, $\mathcal{K} = \mathcal{K}_3 + \mathcal{K}_1$ and the conditions are

$$\{\mathcal{K}_3, \mathcal{H}_0\} = 0, \quad \{\mathcal{K}_3, U\} + \{\mathcal{K}_1, \mathcal{H}_0\} = 0, \quad \{\mathcal{K}_1, U\} = 0$$

so $\mathcal{K}_3$ is a 3rd order Killing tensor. The integrability conditions for the last 2 equations lead to nonlinear PDEs for U.

Theorem 2 *Suppose the system (2) admits an Nth order constant of the motion $\mathcal{K}$ where $N \geq 1$. Then*

$$\hat{\mathcal{K}} = \sum_{j=0}^{[N/2]} \left(-\frac{\mathcal{H}_0 - E}{U}\right)^j \mathcal{K}_{N-2j}$$

is an Nth order constant of the motion for the system $(\mathcal{H}_0 - E)/U$.

PROOF: It follows from the general conditions (3), (4), (5). That

$$\mathcal{K}(\alpha) = \sum_{j=0}^{[N/2]} \alpha^j \mathcal{K}_{N-2j}$$

is a constant of the motion for the system $\mathcal{H}(\alpha) = \mathcal{H}_0 + \alpha U$. Then from Theorem 1, we have that $\mathcal{K}(-\frac{\mathcal{H}_0 - E}{U})$ is an Nth order constant of the motion for the system $(\mathcal{H}_0 - E)/U$. Q.E.D.

Note that if we set $E = 0$ then $\hat{\mathcal{K}}$ becomes an Nth order Killing tensor for the free system $\mathcal{H}_0/U$.

Corollary 2 *Suppose the system $\mathcal{H}_0 + U$ is Nth order superintegrable. Then the free system $\mathcal{H}_0/U$ is also Nth order superintegrable.*

We will call $\hat{\mathcal{K}}$ a *Jacobi transform* of $\mathcal{K}$, in recognition of its close relationship to the Jacobi metric, [33] page 172, and to distinguish it from the Stäckel transform and more general coupling constant metamorphosis. Note that the Jacobi transform for general parameter E is invertible.

Corollary 2 tells us that each of the 3rd order superintegrable systems found by Gravel in 2D Euclidean space, [4], yields superintegrable systems on conformally flat manifolds, usually not of constant curvature.

Corollary 3 *The Jacobi transform satisfies the properties*

$$\widehat{\{\mathcal{K},\mathcal{L}\}} = \{\hat{\mathcal{K}},\hat{\mathcal{L}}\},\ \ \widehat{\mathcal{K}\mathcal{L}} = \hat{\mathcal{K}}\hat{\mathcal{L}},$$

and, if $\mathcal{K}$, $\mathcal{L}$ *are of the same order,* $a\widehat{\mathcal{K}+b}\mathcal{L} = a\hat{\mathcal{K}} + b\hat{\mathcal{L}}$. *Thus it defines a homomorphism from the graded symmetry algebra of the system* $\mathcal{H}_0 + U$ *to the graded symmetry algebra of the system* $(\mathcal{H}_0 - E)/U$.

Example 2 *Consider the system of Example 1:* $\mathcal{H} = p_1^2 + p_2^2 + b_1\sqrt{x_1} + b_2x_2$, *and let* $U = b_1\sqrt{x_1} + b_2x_2 + b_3$ *for some fixed* b_1, b_2, b_3 *with* $b_1b_2 \neq 0$. *The new Hamiltonian is*

$$\hat{\mathcal{H}} = \frac{p_1^2 + p_2^2 - E}{b_1\sqrt{x_1} + b_2x_2 + b_3}.$$

and the Jacobi transforms of $\mathcal{K}^{(2)}, \mathcal{K}^{(3)}$ *are*

$$\hat{\mathcal{K}}^{(2)} = p_2^2 - b_2x_2\left(\frac{p_1^2 + p_2^2 - E}{b_1\sqrt{x_1} + b_2x_2 + b_3}\right),$$

$$\hat{\mathcal{K}}^{(3)} = p_1^3 - (\frac{3}{2}b_1\sqrt{x_1}p_1 - \frac{3b_1^2}{4b_2}p_2)\left(\frac{p_1^2 + p_2^2 - E}{b_1\sqrt{x_1} + b_2x_2 + b_3}\right).$$

4 The Stäckel transform

Using the same notation as in the previous section, and a particular nonzero potential $U = V(\mathbf{x}, \mathbf{b}_0)$ we define the Stäckel transform for a system $\mathcal{H} = \mathcal{H}_0 + V(\mathbf{x}, \mathbf{b})$, [29]. The transform of $\mathcal{K} = \mathcal{K}_1$ is $\tilde{\mathcal{K}} = \mathcal{K}_1$ The transform of $\mathcal{K} = \mathcal{K}_2 + \mathcal{K}_0$ is $\tilde{\mathcal{K}} = \mathcal{K} - \frac{\mathcal{K}_0^U}{U}\mathcal{H}$. (Here $\mathcal{K}_j$ is a homogeneous polynomial in $\mathbf{p}$ of order $2j$, and $\mathcal{K}_j^U$ is the restriction of $\mathcal{K}_j$ to the potential $V = U$.) The transform maps 1st and 2nd order constants of the motion for $\mathcal{H}$ to constants of the motion for the system $\mathcal{H}/U$. Thus the system $\mathcal{H}$ is 2nd order superintegrable iff the system $\mathcal{H}/U$ is 2nd order superintegrable. For completeness we review briefly the direct proofs of the basic theoretic facts.

Theorem 3 *Let $\mathcal{K}$ be a 2nd order constant of the motion for the system $\mathcal{H}$ and U be a particular instance of the potential V. Then $\tilde{\mathcal{K}}$ is a 2nd order constant of the motion for the system $\mathcal{H}/U$.*

PROOF:

$$\{\tilde{\mathcal{K}}, \frac{\mathcal{H}}{U}\} = \{\mathcal{K} - \frac{\mathcal{K}_0^U}{U}\mathcal{H}, \frac{\mathcal{H}}{U}\} = -\frac{\mathcal{H}}{U^2}\left(\{\mathcal{K}_2, U\} + \{\mathcal{K}_0^U, \mathcal{H}_0\}\right) = 0$$

Q.E.D.

Corollary 4 *Let $\mathcal{K}, \mathcal{L}$ be 2nd order constants of the motion for the system $\mathcal{H}$ and let $\tilde{\mathcal{K}}$, $\tilde{\mathcal{L}}$ be their respective Stäckel transforms determined by the potential U. If $\{\mathcal{K}, \mathcal{L}\} = 0$ then $\{\tilde{\mathcal{K}}, \tilde{\mathcal{L}}\} = 0$.*

PROOF: Suppose $\{\mathcal{K}, \mathcal{L}\} = 0$. We have

$$\{\mathcal{K}, \mathcal{L}\} = \{\mathcal{K}_2, \mathcal{L}_2\} + (\{\mathcal{K}_2, \mathcal{L}_0\} + \{\mathcal{K}_0, \mathcal{L}_2\}) = 0,$$

where the first term on the right is of order 3 and the second term is of order 1. Thus

$$\{\mathcal{K}_2, \mathcal{L}_2\} = \{\mathcal{K}_2, \mathcal{L}_0\} + \{\mathcal{K}_0, \mathcal{L}_2\} = 0.$$

Then, a straightforward computation yields

$$\{\tilde{\mathcal{K}}, \tilde{\mathcal{L}}\} = \{\mathcal{K}, \mathcal{L}\} - \frac{\mathcal{H}}{U}\left(\{\mathcal{K}_2, \mathcal{L}_0^U\} + \{\mathcal{K}_0^U, \mathcal{L}_2\}\right) = 0.$$

Q.E.D.

Corollary 5 *Let $\{\mathcal{K}, \mathcal{L}\} = 0$ be as in Corollary 4 and assume that one instance of the potential V is the constant 1, i.e., $V(\mathbf{x}, \mathbf{b}_0) = 1$. Then if $\{\tilde{\mathcal{K}}, \tilde{\mathcal{L}}\} = 0$ we must have $\{\mathcal{K}, \mathcal{L}\} = 0$.*

PROOF: Suppose $\{\tilde{\mathcal{K}}, \tilde{\mathcal{L}}\} = 0$. Then the order 3 and order 1 terms on the left hand side of this expression must vanish separately:

$$\{\mathcal{K}_2, \mathcal{L}_2\} - \frac{\mathcal{H}_0}{U}\left(\{\mathcal{K}_2, \mathcal{L}_0^U\} + \{\mathcal{K}_0^U, \mathcal{L}_2\}\right) = 0, \tag{7}$$

$$\{\mathcal{K}_2, \mathcal{L}_0\} + \{\mathcal{K}_0, \mathcal{L}_1\} - \frac{V}{U}\left(\{\mathcal{K}_2, \mathcal{L}_0^U\} + \{\mathcal{K}_0^U, \mathcal{L}_2\}\right) = 0, \tag{8}$$

Identity (7) says that

$$\{\mathcal{K}_2, \mathcal{L}_2\} = \mathcal{H}_0 \mathcal{X} \tag{9}$$

where $\mathcal{X} = \frac{1}{U}\left(\{\mathcal{K}_2, \mathcal{L}_0^U\} + \{\mathcal{K}_0^U, \mathcal{L}_2\}\right)$. Since $\mathcal{K}_2, \mathcal{L}_2$ are 2nd order Killing tensors of $\mathcal{H}_0$, it follows easily from the Jacobi relation for the Poisson bracket that $\mathcal{X}$ is a Killing vector. From identity (8) we obtain the result

$$\{\mathcal{K}, \mathcal{L}\} + \mathcal{H}\mathcal{X} = \{\tilde{\mathcal{K}}, \tilde{\mathcal{L}}\} = 0. \tag{10}$$

Taking the Poisson bracket of the left hand side of this last identity with $\mathcal{H}$ we see that $\mathcal{X}$ is a first order constant of the motion for system $\mathcal{H}$. From (8) we have

$$\mathcal{X} = \frac{1}{U}\left(\{\mathcal{K}_2, \mathcal{L}_0^U\} + \{\mathcal{K}_0^U, \mathcal{L}_2\}\right) = \frac{1}{V}\left(\{\mathcal{K}_2, \mathcal{L}_0^V\} + \{\mathcal{K}_0^V, \mathcal{L}_2\}\right)$$

for any nonzero choice of potential V. Choosing $V = 1$ we find

$$\mathcal{X} = \{\mathcal{K}_2, \mathcal{L}_0^1\} + \{\mathcal{K}_0^1, \mathcal{L}_2\}. \tag{11}$$

From relation (6) with $V = 1$ we have $\{\mathcal{K}_2, 1\} + \{\mathcal{K}_0^1, \mathcal{H}_0\} = 0$ so $\{\mathcal{K}_0^1, \mathcal{H}_0\} = 0$. Since the metric is nondegenerate, this implies that $\mathcal{K}_0^1 = c_1$, a constant. Similarly, $\mathcal{L}_0^1 = c_2$ is constant. Thus (11) implies $\mathcal{X} = 0$, which together with (10) implies $\{\mathcal{K}, \mathcal{L}\} = 0$. Q.E.D.

An alternate way of proving Corollary 5 is to demonstrate that there is an "inverse" Stäckel transform that takes the system $\mathcal{H}/U$ to $\mathcal{H}$ via the special potential $1/U$. The outcome of applying the initial transform to a 2nd order constant of the motion $\mathcal{K}$ of $\mathcal{H}$ and then transforming back is $\mathcal{K} - \mathcal{K}_0^1\mathcal{H}$, where $\mathcal{K}_0^1$ is a constant. If each 2nd order symmetry $\mathcal{K}$ is normalized by the requirement $\mathcal{K}_0^1 = 0$, (by adding a suitable constant) then this action is the identity operator.

These results show that the Stäckel transform takes 2nd order superintegrable systems to 2nd order superintegrable systems, preserves variable separability, and is invertible. As stated in this generality for second order symmetries, the Stäckel transform is not a special case of coupling constant metamorphosis, although the two transforms are closely related. However in the situation where the potential functions $V(\mathbf{x}, \mathbf{b})$ form a finite dimensional vector space, which is usual in the study of 2nd order superintegrability, then the transforms coincide. In this case, by redefining parameters if necessary, we can assume V is linear in $\mathbf{b}$.

Now we will investigate extensions of the Stäckel transform to higher order constants of the motion, under the assumption that $V(\mathbf{x}, \mathbf{b})$ is linear in $\mathbf{b} = (b_0, b_1, \cdots b_M)$, U is of the form $U(\mathbf{x}) = V(\mathbf{x}, \mathbf{b}^0)$ and the potentials $V(\mathbf{x}, \mathbf{b})$ span a space of dimension $M + 1$. In particular,

$$V(\mathbf{x}, \mathbf{b}) = b_0 + \sum_{i=1}^{M} U^{(i)}(\mathbf{x}) b_i \tag{12}$$

where the set of functions $\{1, U^{(1)}(\mathbf{x}), \cdots, U^{(M)}(\mathbf{x})\}$ is linearly independent. In the study of 2nd order superintegrability, typically the 2nd order constants of the motion are linear in the $\mathbf{b}$ and the algebra generated by these symmetries via products and commutators has the property that a constant of the motion of order N depends polynomially on the parameters with order $\leq [N/2]$. Thus we consider only those higher order constants of the motion of order N of the form

$$\mathcal{K} = \sum_{j=0}^{[N/2]} \mathcal{K}_{N-2j}(\mathbf{p}, \mathbf{b}) \tag{13}$$

where $\mathcal{K}_{N-2j}(a\mathbf{p}, \mathbf{b}) = a^{N-2j}\mathcal{K}_{N-2j}(\mathbf{p}, \mathbf{b})$ and $\mathcal{K}_{N-2j}(\mathbf{p}, a\mathbf{b}) = a^j\mathcal{K}_{N-2j}(\mathbf{p}, \mathbf{b})$ for any parameter a. Let $\mathcal{K}(\mathbf{b})$ be such an Nth order constant of the motion. Then

$$\mathcal{K}(\alpha) \equiv \mathcal{K}(\mathbf{p}, \mathbf{b} + \alpha\mathbf{b}^{(0)}) \tag{14}$$

is an Nth order constant of the motion for the system with Hamiltonian $\mathcal{H}_0 + V(\mathbf{x}, \mathbf{b}) + \alpha U(\mathbf{x})$. Applying Theorem 1 we have

Theorem 4 *Let $\mathcal{K}$ be an Nth order constant of the motion for the system $\mathcal{H}_0 + V(\mathbf{x}, \mathbf{b})$ where V is of the form (12) and $\mathcal{K}$ is of the form (13). Let $\mathcal{K}(\alpha)$ be defined by (14). Then*

$$\tilde{\mathcal{K}} = \mathcal{K}\left(-\frac{\mathcal{H}_0 + V(\mathbf{x}, \mathbf{b})}{U(\mathbf{x})}\right) = \sum_{j=0}^{[N/2]} \tilde{\mathcal{K}}_{N-2j}(\mathbf{p}, \mathbf{b})$$

is an Nth order constant of the motion for the system $(\mathcal{H}_0 + V(\mathbf{x}, \mathbf{b}))/U(\mathbf{x})$, where

$$\tilde{\mathcal{K}}_{N-2j}(a\mathbf{p}, \mathbf{b}) = a^{N-2j}\tilde{\mathcal{K}}_{N-2j}(\mathbf{p}, \mathbf{b}), \quad \tilde{\mathcal{K}}_{N-2j}(\mathbf{p}, a\mathbf{b}) = a^j\tilde{\mathcal{K}}_{N-2j}(\mathbf{p}, \mathbf{b}) \tag{15}$$

for any parameter a.

Example 3 *This example of a 4th order superintegrable system is taken from [9] and corresponds to the choice $k = 2$ for the potential $V = Ar^2 + B/r^2\cos^2(kt) + C/r^2\sin^2(kt)$ for suitable A, B, C, as written in polar coordinates. The structure relations and transform are new. Let*

$$\mathcal{H} = p_1^2 + p_2^2 + a(x_1^2 + x_2^2) + b\frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} + c\frac{(x_1^2 + x_2^2)}{x_1^2 x_2^2}$$

There are two basic constants of the motion, one of 2nd order,

$$\mathcal{K}_2 = (x_1p_2 - x_2p_1)^2 + 4b\frac{x_1^2 x_2^2}{(x_1^2 - x_2^2)^2} + c\frac{(x_1^4 + x_2^4)}{x_1^2 x_2^2}$$

and one of 4th order,

$$\mathcal{K}_4 = (p_1^2 - p_2^2)^2 + [2ax_1^2 + 2b\frac{(x_1^2+x_2^2)}{(x_1^2-x_2^2)^2} - 2c\frac{(x_1^2-x_2^2)}{x_1^2x_2^2}]p_1^2$$

$$+[-4ax_1x_2 + 8b\frac{x_1x_2}{(x_1^2-x_2^2)^2}]p_1p_2 + [2ax_2^2 + 2b\frac{(x_1^2+x_2^2)}{(x_1^2-x_2^2)^2} + 2c\frac{(x_1^2-x_2^2)}{x_1^2x_2^2}]p_2^2$$

$$+a^2(x_1^2-x_2^2)^2 + \frac{b^2}{(x_1^2-x_2^2)^2} + c^2\frac{(x_1^2-x_2^2)^2}{x_1^4x_2^4} + 8ab\frac{x_1^2x_2^2}{(x_1^2-x_2^2)} + 2\frac{bc}{x_1^2x_2^2}$$

These constants of the motion generate a closed Poisson algebra. Let $\mathcal{R} = \{\mathcal{K}_2, \mathcal{K}_4\}$*.The relations are*

$$\{\mathcal{K}_2, \mathcal{R}\} = 32(\mathcal{H}^2 - 2\mathcal{K}_4)\mathcal{K}_2 - 64(b+2c)\mathcal{K}_4 + 64(b-c)\mathcal{H}^2 - 128ab\mathcal{K}_1 - 128ab(b+2c),$$

$$\{\mathcal{K}_4, \mathcal{R}\} = 32\mathcal{K}_4(\mathcal{K}_4 - \mathcal{H}^2) + 128a\mathcal{K}_2\mathcal{H}^2 - 384a^2\mathcal{K}_2^2 + 128ab\mathcal{K}_4 - 64(b+4c)a\mathcal{H}^2$$
$$+256a^2(2c-b)\mathcal{K}_2 + 128a^2(b^2+40c^2+20bc)$$

There is a Casimir constraint

$$\mathcal{R}^2 = 64\mathcal{K}_2\mathcal{K}_4(\mathcal{H}^2 - \mathcal{K}_4) - 64b\mathcal{H}^4 + 128(b-c)\mathcal{K}_4\mathcal{H}^2 - 64(b+2c)\mathcal{K}_4^2 - 128a\mathcal{K}_2^2\mathcal{H}^2 +$$
$$256a^2\mathcal{K}_2^3 - 256ab\mathcal{K}_2\mathcal{K}_4 + 128a(b+4c)\mathcal{H}^2\mathcal{K}_2 + 256a^2(b-c)\mathcal{K}_2^2 - 256ab(b+2c)\mathcal{K}_4$$
$$+256a(7bc+b^2-2c^2)\mathcal{H}^2 - 256a^2(b^2+4c^2+20bc)\mathcal{K}_2 - 256a^2(2c+b)(b^2+16bc-4c^2).$$

Then the Stäckel transformed system

$$\tilde{\mathcal{H}} = \frac{p_1^2 + p_2^2 + a(x_1^2+x_2^2) + b\frac{(x_1^2+x_2^2)}{(x_1^2-x_2^2)^2} + c\frac{(x_1^2+x_2^2)}{x_1^2x_2^2} + d}{(x_1^2+x_2^2) + B\frac{(x_1^2+x_2^2)}{(x_1^2-x_2^2)^2} + C\frac{(x_1^2+x_2^2)}{x_1^2x_2^2} + D}$$

is also superintegrable with 4th and 2nd order generating constants of the motion. Note that for $B = C = 0$, $D = 4$, *the transformed system is defined on a Darboux space of type 3, whereas if* $B = D = 0$, $C = 1$ *the transformed system is defined on a Darboux space of type 2, [2].*

5 2D quantum symmetries

Here we begin the study of quantum symmetries. The quantization is much simpler in the 2D case than for dimensions greater than 2, and for 1st and 2nd order symmetries, so we begin with these special cases to gain insight. Here the metric, expressed in Cartesian-like coordinates, is $ds^2 = \lambda(\mathbf{x})(dx_1^2 + dx_2^2)$, and the Hamiltonian system $\mathcal{H} = (p_1^2 + p_2^2)/\lambda(\mathbf{x}) + V(\mathbf{x})$ is replaced by the Hamiltonian (Schrödinger) operator with potential

$$H = \frac{1}{\lambda(\mathbf{x})}(\partial_{11} + \partial_{22}) + V(\mathbf{x}). \qquad (16)$$

5.1 2nd order operator symmetries

A 2nd order symmetry of the Hamiltonian system $\mathcal{K} = \sum_{k,j=1}^{2} a^{kj}(\mathbf{x})p_k p_j + W(\mathbf{x})$, with $a^{kj} = a^{jk}$, corresponds to the operator

$$K = \frac{1}{\lambda(\mathbf{x})} \sum_{k,j=1}^{2} \partial_k \left(\lambda(\mathbf{x}) a^{kj}(\mathbf{x}) \partial_j\right) + W(\mathbf{x}), \quad a^{kj} = a^{jk}.$$

These operators are formally self-adjoint with respect to the bilinear product

$$< f, g >_\lambda = \int f(\mathbf{x}) g(\mathbf{x}) \lambda(\mathbf{x}) dx_1 dx_2$$

on the manifold, i.e.,

$$< f, Hg >_\lambda = < Hf, g >_\lambda, \quad < f, Kg >_\lambda = < Kf, g >_\lambda$$

for all local C^∞ functions f, g with compact support on the manifold, where the domain of integration is $\mathcal{C}^2$ or $\mathcal{R}^2$. If the functions defining a differential operator are singular on a 1-dimensional or 0-dimensional set, we we restrict the support of f, g to be bounded away from this set. We define the formal adjoint T^* of a linear operator T on the space C_0^∞ by

$$< T^* f, g >_\lambda = < f, Tg >_\lambda \tag{17}$$

for all $f, g \in C_0^\infty$. The operators H, K are formally self-adjoint: $H^* = H, K^* = K$.

If the Schrödinger equation admits a multiplicative separable solution in particular coordinates x_1, x_2 then the Schrödinger operator can be written as

$$H = \frac{1}{X^{(1)}(x_1) + X^{(2)}(x_2)} \left(\partial_{11} + \partial_{22} + V^{(1)}(x_1) + V^{(2)}(x_2)\right) \tag{18}$$

where the 2nd order symmetry responsible for the separation is

$$K = \frac{1}{X^{(1)}(x_1) + X^{(2)}(x_2)} \big(X^{(2)}(x_2)\partial_{11} - X^{(1)}(x_1)\partial_{22} \tag{19}$$
$$+ X^{(2)}(x_2)V^{(1)}(x_1) - X^{(1)}(x_1)V^{(2)}(x_2)\big).$$

Thus the metric is $\lambda(\mathbf{x}) = X^{(1)}(x_1) + X^{(2)}(x_2)$ and the potential is $V(\mathbf{x}) = (V^{(1)}(x_1) + V^{(2)}(x_2))/(X^{(1)}(x_1) + X^{(2)}(x_2))$.

A 1st order symmetry of the Hamiltonian system $\mathcal{L} = \sum_{k=1}^{2} a^k(\mathbf{x}) p_k$ corresponds to the operator

$$L = \sum_{k=1}^{2} \left(a^k(\mathbf{x}) \partial_k + \frac{\partial_k(\lambda(\mathbf{x}) a^k(\mathbf{x}))}{2\lambda(\mathbf{x})} \right).$$

It is easy to show that L is formally skew-adjoint, i.e., $L^* = -L$.

The following results that relate the operator commutator $[A, B] = AB - BA$ and the Poisson bracket are straightforward to verify.

Lemma 1

$$\{\mathcal{H}, \mathcal{K}\} = 0 \iff [H, K] = 0.$$

This result is not generally true for higher dimensional manifolds.

Lemma 2

$$\{\mathcal{H}, \mathcal{L}\} = 0 \iff [H, L] = 0.$$

The classical Stäckel transform for these systems can easily be extended to the operator case. Suppose V is a parametrized potential and let U be a special instance of that potential. Let $K = \frac{1}{\lambda}\sum \partial_i(\lambda a^{ij}\partial_j) + W = K_2 + K_0$, where $K_0 = W$, be a 2nd order formally self-adjoint symmetry operator of H and K_0^U be the restriction of K_0 to $V = U$. Then

$$\tilde{K} = K - K_0^U U^{-1} H$$

is the corresponding formally self-adjoint symmetry operator of $\tilde{H} = U^{-1}H$, with respect to the metric $d\tilde{s}^2 = U\lambda(dx_1^2 + dx_2^2)$. Here the order of operators in a product is important and a function represents the operation of multiplying on the left by that function.

Theorem 5 *1.*

$$[\tilde{H}, \tilde{K}] = 0 \iff [H, K] = 0.$$

2.

$$\tilde{K} = \sum_{ij} \frac{1}{U\lambda}\partial_i \left((a^{ij} - \delta^{ij}\frac{W_U}{U\lambda})U\lambda \right) \partial_j + \left(W - \frac{W_U V}{U} \right).$$

PROOF:

1. This is a straightforward verification, using the identities

$$[H_0, K_2] = 0, \quad [H_0 + V, K_2 + K_0] = 0, \quad [H_0 + U, K_2 + K_0^U] = 0, \tag{20}$$

$$[A, BC] = B[A, C] + [A, B]C, \quad [A, U^{-1}] = -U^{-1}[A, U]U^{-1}$$

for linear operators A, B, C and nonzero function U.

2. This follows from the fact that $\partial_i K_0^U \equiv \partial_i W^U = \lambda \sum_j a^{ij}\partial_j U$.

Q.E.D.

Note that the second part of the theorem shows that $\tilde{K}$ is indeed formally self-adjoint on the manifold with metric $U\lambda(dx_1^2+dx_2^2)$. Another way to see this is to use the formal definition of adjoint. With respect to the inner product on the space with weight function $U\lambda$ we have $<f,g>_{U\lambda}=<f,Ug>_\lambda=<Uf,g>_\lambda$. Thus

$$<\tilde{K}f,g>_{U\lambda}=<(UK-UK_0^U U^{-1}H)f,g>_\lambda=<f,(KU-HK_0^U)g>_\lambda$$

$$=<f,(U^{-1}KU-U^{-1}HK_0^U)g>_{U\lambda}\ .$$

This shows that $\tilde{K}^*=U^{-1}KU-U^{-1}HK_0^U=\tilde{K}$, where the final equality follows directly from identities (20).

Corollary 6 *If $K^{(1)},K^{(2)}$ are 2nd order symmetry operators for H, then*

$$[\tilde{K}^{(1)},\tilde{K}^{(2)}]=0 \iff [K^{(1)},K^{(2)}]=0.$$

Since one can always add a constant to a potential, it follows that $1/U$ defines an inverse Stäckel transform of $\tilde{H}$ to H.

5.2 3rd order operator symmetries

A classical 3rd order symmetry takes the form $\mathcal{K}=\mathcal{K}_3+\mathcal{K}_1$ where

$$\mathcal{K}_3=\sum_{k,j,i=1}^{2}a^{kji}(\mathbf{x})p_kp_jp_i,\quad \mathcal{K}_1=\sum_{\ell=1}^{2}b^\ell(\mathbf{x})p_\ell.$$

The conditions $\{\mathcal{K}_3,\mathcal{H}_0\}=0$ are

$$\begin{array}{rcl} 2a_i^{iii} &=& -3\left((\ln\lambda)_i a^{iii}+(\ln\lambda)_j a^{jii}\right),\ i\neq j \qquad (21)\\ 3a_i^{jii}+a_j^{iii} &=& -3\left((\ln\lambda)_i a^{iij}+(\ln\lambda)_j a^{ijj}\right),\quad i\neq j\\ 2\left(a_1^{122}+a_2^{112}\right) &=& -(\ln\lambda)_1 a^{122}-(\ln\lambda)_1 a^{111}-(\ln\lambda)_2 a^{222}-(\ln\lambda)_2 a^{112}, \end{array}$$

which are just the requirements that the a^{kji} be the components of a 3rd order Killing tensor. The conditions $\{\mathcal{K}_3,V\}+\{\mathcal{K}_1,\mathcal{H}_0\}=0$ are

$$b_2^1+b_1^2=3\sum_{s=1}^{2}\lambda a^{s21}V_s, \qquad (22)$$

$$b_j^j=\frac{3}{2}\sum_{s=1}^{2}\lambda a^{sjj}V_s-\frac{1}{2}\sum_{s=1}^{2}(\ln\lambda)_s b^s,\quad j=1,2,$$

and the condition $\{\mathcal{K}_1, V\} = 0$ is

$$\sum_{s=1}^{2} b^s V_s = 0. \tag{23}$$

Now let's consider a 3rd order operator symmetry K that is skew-adjoint. The detailed conditions $[K, H] = 0$ are complicated. However, we will restrict ourselves to systems with potentials that simultaneously admit a 3rd order classical symmetry and the related 3rd order quantum symmetry. A characteristic feature of such systems, and one that we will exploit, is that if U is such a potential then so is αU for all scalars α. If $\mathcal{K}$ is the classical symmetry then we can write the operator symmetry in the form $K = K'_3 + K'_1$ where K'_3, K'_1 are skew-adjoint of respective orders 3 and 1,

$$K'_3 = \sum_{k,j,i=1}^{2} \left(a^{kji}\partial_{kji} + \frac{3}{2\lambda}(a^{kji}\lambda)_i \partial_{kj} + \frac{1}{2\lambda}(a^{kji}\lambda)_{kj}\partial_i \right),$$

$$K'_1 = \sum_{i=1}^{2} \left(B^i\partial_i + \frac{1}{2\lambda}(B^i\lambda)_i \right),$$

and the terms a^{kji} satisfy (21).

Now replace V by αU. Then the symmetry condition is $[K(\alpha), H_0+\alpha U] = 0$ for all α where $K = K'_3 + K'_1(\alpha)$. We assume that $K(\alpha)$ is analytic in α about $\alpha = 0$. Then, K'_3 is independent of α and the dependence of K'_1 on α is at most 1st order. Thus we can write $B^i(\alpha) = c^i + \alpha b^i$ or $K'_1(\alpha) = K''_1 + \alpha K_1$. The symmetry condition can be written as

$$\begin{aligned} 0 &= [K'_3 + K''_1 + \alpha K_1, H_0 + \alpha U] \\ &= [K'_3 + K''_1, H_0] + \alpha([K'_3 + K''_1, U] + [K_1, H_0]) + \alpha^2[K_1, U], \end{aligned}$$

for all α. Setting $K_3 = K'_3 + K''_1$ we have the identities

$$[K_3, H_0] = 0, \quad [K_3, U] + [K_1, H_0] = 0, \quad [K_1, U] = 0. \tag{24}$$

Note that the 2nd order terms in the 2nd identity are precisely the classical conditions (22). The 3rd identity is precisely the classical condition (23). The operator K''_1 determines the transition from the classical constant of the motion to the operator symmetry. Now define

$$\tilde{K} = K_3 - K_1 U^{-1}(H_0 + b),$$

where the operator order is important and b is a constant.. A straightforward computation using identities, (24) yields $[\tilde{K}, U^{-1}(H_0 + b)] = 0$, so $\tilde{K}$ is a 3rd symmetry operator for the Hamiltonian $U^{-1}(H_0 + b)$.

Theorem 6 *Let $H(\alpha) = H_0 + \alpha U$, let $K(\alpha)$ be a 3rd order skew-adjoint symmetry of H, analytic in α about $\alpha = 0$. Then there are 1st and 3rd order skew-adjoint operators K_1, K_3 such that $K(\alpha) = K_3 + \alpha K_1$ and identities (24) are satisfied. The operator $\tilde{K} = K_3 - K_1^U U^{-1}(H_0 + b)$ is a 3rd order symmetry for the system $\tilde{H} = U^{-1}(H_0 + b)$.*

Corollary 7 *$\tilde{K}^* = -\tilde{K}$ so $\tilde{K}$ is a 3rd order formally skew-adjoint symmetry of $\tilde{H}$.*

PROOF: This is a consequence of $K^* = -K, \tilde{H}^* = \tilde{H}$ and relations (24). Q.E.D.

Note: The preceding argument has to be modified in the special case that the system admits a 1st order α-independent symmetry L: $[L, H(\alpha)] = 0$. Then $K_1'(\alpha)$ need not be at most 1st order as a polynomial in α. Indeed we can add a term $f(\alpha)L$ to K_1' without changing the commutation relations. However, the conclusion (24) remains correct.

Example 4 *(The 9-1 anisotropic oscillator) Let $H(\alpha) = \partial_{11} + \partial_{22} + \alpha(9x_1^2 + x_2^2)$. This is a superintegrable system with generating 2nd and 3rd order symmetries*

$$L = \partial_{22} + \alpha x_2^2, \quad K = \{x_1\partial_2 - x_2\partial_1, \partial_{22}\} + \frac{\alpha}{3}(\{x_2^3, \partial_1\} - 9\{x_1x_2^2, \partial_2\}),$$

where $\{S_1, S_2\} \equiv S_1S_2 + S_2S_1$. Let $U = (9x_1^2 + x_2^2) + c$. It follows that the system

$$\tilde{H} = \frac{1}{(9x_1^2 + x_2^2) + c}(\partial_{11} + \partial_{22} + b)$$

is superintegrable with one 2nd and one 3rd order symmetry.

5.3 4th order operator symmetries

Next we consider the case of a 4th order constant of the motion

$$\mathcal{K} = \sum_{\ell,k,j,i=1}^{2} a^{\ell kji}(\mathbf{x})p_\ell p_k p_j p_i + \sum_{m,q=1}^{2} b^{mq}(\mathbf{x})p_m p_q + W(\mathbf{x}) = \mathcal{K}_4 + \mathcal{K}_2 + \mathcal{K}_0, \quad (25)$$

This must satisfy the conditions

$$\begin{aligned} a_i^{iiii} &= -2\sum_{s=1}^{2} a^{siii}(\ln\lambda)_s \qquad (26)\\ 4a_i^{jiii}+a_j^{iiii} &= -6\sum_{s=1}^{2} a^{siij}(\ln\lambda)_s, \quad i\neq j\\ 3a_i^{jjii}+2a_j^{iiij} &= -\sum_{s=1}^{2} a^{siii}(\ln\lambda)_s - 3\sum_{s=1}^{2} a^{sijj}(\ln\lambda)_s, \quad i\neq j \qquad (27)\end{aligned}$$

$$2b_i^{ij}+b_j^{ii} = 6\lambda\sum_{s=1}^{2} a^{sjii}V_s - \sum_{s=1}^{2} b^{sj}(\ln\lambda)_s, \quad i\neq j \qquad (28)$$

$$b_i^{ii} = 2\lambda\sum_{s=1}^{2} a^{siii}V_s - \sum_{s=1}^{2} b^{sj}(\ln\lambda)_s,$$

and

$$\lambda\sum_{s=1}^{2} b^{si}V_s = W_i. \qquad (29)$$

Note that the $a^{\ell kji}$ is a 4th order Killing tensor.

If K is a 4th order symmetry operator, there exist functions $a^{\ell kji}, \tilde b^{ij}, \tilde W$ such that K has the unique self-adjoint form

$$K = \sum_{\ell,k,j,i=1}^{2} \frac{1}{\lambda}\partial_{ij}\left(a^{\ell kji}\lambda\partial_{k\ell}\right) + \sum_{i,j=1}^{2}\frac{1}{\lambda}\partial_i\left(\tilde b^{ij}\lambda\partial_j\right) + \tilde W = K_4' + K_2' + K_0', \qquad (30)$$

where the functions $\tilde b^{ij}(x_1,x_2), \tilde W(x_1,x_2)$ contain the parameter dependence. Equating coefficients of the 5th derivative terms in the operator condition $[K,H]=0$ we obtain exactly the Killing tensor conditions (26).

The remaining conditions on K intertwine $\lambda, a^{\ell kji}, \tilde b^{ji}$, $\tilde W$ and V, and are complicated. Rather than solve them directly, we use the fact that the system with potential αU must be solvable for all α, and require that the symmetry $K(\alpha)$ is analytic in α about $\alpha=0$. The following argument for the form of K is correct, up to addition of operators $f(\alpha)L_2$ or $g(\alpha)$ where L_2 is a 2nd order self-adjoint α-independent symmetry operator. Modulo this remark, $K_1'(\alpha)$ must be at most a 1st order polynomial in α and $K_0'(\alpha)$ must be at most quadratic. We can make the unique decomposition

$$\tilde b^{ji}(\mathbf{x}) = c^{ji}(\mathbf{x}) + \alpha b^{ji}(\mathbf{x})$$

$$\tilde{W} = U^{(0)}(\mathbf{x}) + \alpha U^{(1)}(\mathbf{x}) + \alpha^2 W(\mathbf{x}).$$

Substituting into $[K(\alpha), H_0 + \alpha U] = 0$ and equating the 3rd derivative terms that are linear in α, we get exactly conditions (28) Equating the coefficients of the 0th derivative terms that are quadratic in α we get exactly conditions (29).

Now we write $K_2'(\alpha) = A_2 + \alpha B_2,$, $K_0'(\alpha) = A_0 + \alpha B_0 + \alpha^2 C_0$. It follows that

$$[K_4, H_0] = 0, \quad [K_4, U] + [K_2, H_0] = 0, \quad [K_2, U] + [K_0, H_0] = 0, \tag{31}$$

where

$$K = K_4 + K_2 + K_0, \quad K_4 = K_4' + A_2 + A_0, \quad K_2 = B_2 + B_0, \quad K_0 = C_0.$$

Now define

$$\tilde{K} = K_4 - K_2 U^{-1}(H_0 + b) + K_0 \left(U^{-1}(H_0 + b)\right)^2,$$

where the operator order is important. A straightforward computation using identities, (31) yields $[\tilde{K}, U^{-1}(H_0 + b)] = 0$. Thus $\tilde{K}$ is a 4th order symmetry operator for the Hamiltonian $U^{-1}(H_0 + b)$.

Theorem 7 *Let $H(\alpha) = H_0 + \alpha U$, let $K(\alpha)$ be a 4th order self-adjoint symmetry of $H(\alpha)$, analytic at $\alpha = 0$. Then there are 0th, 2nd and 4th order self-adjoint operators K_0, K_2, K_4 such that $K(\alpha) = K_4 + \alpha K_2 + \alpha^2 K_0$ and identities (31) are satisfied. The operator $\tilde{K} = K_4 - K_2 U^{-1}(H_0 + b) + K_0(U^{-1}(H_0 + b))^2$ is a 4th order symmetry for the system $\tilde{H} = U^{-1}(H_0 + b)$.*

Corollary 8 *$\tilde{K}^* = \tilde{K}$ so $\tilde{K}$ is a 4th order formally self-adjoint symmetry of $\tilde{H}$.*

Example 5 *This is an extension of Example 3 to the quantum case, [9]. Let*

$$H = \partial_{11} + \partial_{22} + a(x_1^2 + x_2^2) + b\frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} + c\frac{(x_1^2 + x_2^2)}{x_1^2 x_2^2}$$

There are two basic self-adjoint symmetry operators, one of 2nd order,

$$K_2 = (x_1\partial_2 - x_2\partial_1)^2 + 4b\frac{x_1^2 x_2^2}{(x_1^2 - x - 2^2)^2} + c\frac{(x_1^4 + x_2^4)}{x_1^2 x_2^2}$$

and one of 4th order,

$$K_4 = (\partial_{11} - \partial_{22})^2 + [2ax_1^2 + 2b\frac{(x_1^2 + x_2^2)}{(x_1^2 - x_2^2)^2} - 2c\frac{(x_1^2 - x_2^2)}{x_1^2 x_2^2}]\partial_{11}$$

$$+[-4ax_1x_2+8b\frac{x_1x_2}{(x_1^2-x_2^2)^2}]\partial_{12}+[2ax_2^2+2b\frac{(x_1^2+x_2^2)}{(x_1^2-x_2^2)^2}+2c\frac{(x_1^2-x_2^2)}{x_1^2x_2^2}]\partial_{22}$$

$$+(2ax_1-\frac{4c}{x_1^3})\partial_1+(2ax_2-\frac{4c}{x_2^3})\partial_2+a^2(x_1^2-x_2^2)^2+\frac{b^2}{(x_1^2-x_2^2)^2}$$

$$+c^2\frac{(x_1^2-x_2^2)^2}{x_1^4x_2^4}+8ab\frac{x_1^2x_2^2}{(x_1^2-x_2^2)}+2\frac{bc}{x_1^2x_2^2}+6c(\frac{1}{x_1^4}+\frac{1}{x_2^4}).$$

These operators generate a closed symmetry algebra. Let $R=[K_2,K_4]$*.The relations are*

$$[K_2,R]=32H^2K_2-32\{K_4,K_2\}-64(b+2c+4)K_4+64(b-c+2)H^2$$

$$-128a(b+1)K_1-128a(b^2+2bc+4b+6c+4),$$

$$[K_4,R]=32K_4^2-32H^2K_4+128aK_2H^2-384a^2K_2^2+128a(b+1)K_4-64a(b+4c+6)H^2$$

$$+256a^2(2c-b+14)K_2+128a^2(b^2+4c^2+20bc+18b+8c-8)$$

There is also the Casimir operator

$$R^2=32H^2\{C_1,C_2\}-\frac{32}{3}\{C_1,C_2,C_2\}-(64\beta+128\gamma+\frac{2816}{3})C_2^2+(128(\beta-\gamma)+\frac{2816}{3})H^2C_2$$

$$-(192+64\beta)H^4-128\alpha H^2C_1^2-128\alpha(\beta+1)\{C_1,C_2\}+\frac{128}{3}\alpha(12\gamma+3\beta+50)H^2C_1+256\alpha C_1^3$$

$$-\frac{256}{3}\alpha(44\beta+44+18\gamma+3\beta^2+6\beta\gamma)C_2-256\alpha^2(2\gamma-\beta46)C_1^2+\frac{256}{3}\alpha(42+22\gamma+40\beta$$

$$+3\beta^2-6\gamma^2+21\beta\gamma)H^2-\frac{256}{3}\alpha^2(152\gamma-88+182\beta+3\beta^2+12\gamma^2+60\beta\gamma)C_1$$

$$+\frac{256}{3}(280\gamma^2-80\beta^2+24\gamma^3+320\gamma+48\beta-4\beta\gamma-84\beta\gamma^3-54\gamma\beta^2-3\beta^2+28).$$

Then the Stäckel transformed system

$$\tilde{H}=\frac{1}{(x_1^2+x_2^2)+B\frac{(x_1^2+x_2^2)}{(x_1^2-x_2^2)^2}+C\frac{(x_1^2+x_2^2)}{x_1^2y_2^2}+D}(\partial_{11}+\partial_{22}+$$

$$\left. a(x_1^2+x_2^2)+b\frac{(x_1^2+x_2^2)}{(x_1^2-x_2^2)^2}+c\frac{(x_1^2+x_2^2)}{x_1^2x_2^2}+d\right)$$

is also superintegrable with 4th and 2nd order generating self-adjoint symmetries. (Actually, we have here proved only that the transformed system is superintegrable for $a=b=c=0$*, but the more general case will follow from Theorem 8.)*

5.4 Nth order operator symmetries

A possible structure of the Nth order operator case is now clear, though it is far from clear whether the structure includes all cases. Suppose the system $H(\alpha) = H_0 + V + \alpha U \equiv H + \alpha U$ admits a truly Nth order symmetry operator $K(\alpha)$ analytic in α about $\alpha = 0$, where $N \geq 2$ and K is self-adjoint for even N, skew-adjoint for odd N. Then we can write

$$K(\alpha) = K'_N + \sum_{j=1}^{[n/2]} K'_{N-2j}(\alpha)$$

where each K'_{N-2j} is self-adjoint or skew-adjoint, depending on the parity of N. The symmetry condition is

$$[K'_N + \sum_{j=1}^{[N/2]} K'_{N-2j}(\alpha), H + \alpha U] = 0, \tag{32}$$

where the $K'_{N-2j}(\alpha)$ are analytic in α. Suppose, modulo terms of the form $f^{(i)}(\alpha)L_{N-2j}$ where L_{n-2j} is an α-independent symmetry of $H + \alpha U$ for $j > 0$, we have

$$K'_{N-2j} = \sum_{i=0}^{j} A^{(i)}_{N-2j}\alpha^i, \quad j = 0, 1, \cdots, [N/2]$$

where the $A^{(i)}_{N-2j}$ are independent of α. Setting $K_{N-2j} = \sum_{h=0}^{[(N-2j)/2]} A^{(j)}_{N-2j-2h}$ we have $K = \sum_{j=0}^{[N/2]} K_{N-2j}$ and the symmetry condition (32) becomes

$$[\sum_{j=0}^{[N/2]} \alpha^j K_{N-2j}, H + \alpha U] = 0, \tag{33}$$

or

$$[K_{N-2j}, U] + [K_{N-2j-2}, H] = 0, \quad j = 0, 1, \cdots, [N/2], \tag{34}$$

where we define $K_{N-2j} \equiv 0$ for $j > [N/2]$ and $j < 0$.

Now define

$$\tilde{K} = \sum_{h=0}^{[N/2]} (-1)^h K_{N-2h} \left(U^{-1}(H + b)\right)^h,$$

where b is a constant. From relations (34) we have

$$[\tilde{K}, U^{-1}(H + b)] = \sum_{h=0}^{[N/2]} (-1)^h [K_{N-2h}, U^{-1}(H + b)] \left(U^{-1}(H + b)\right)^h = \tag{35}$$

$$U^{-1}\sum_{h=0}^{[N/2]}\left([K_{N-2h},U](-1)^{h+1}(U^{-1}(H+b))^{h+1}+[K_{N-2h},H](-1)^h(U^{-1}(H+b))^h\right)$$

$$=U^{-1}\sum_{h=1}^{[N/2]}(-1)^h\left([K_{N-2h+2},U]+[K^U_{N-2h},H]\right)(U^{-1}(H+b))^h=0.$$

Thus $[\tilde{K},\tilde{H}]=0$.

Theorem 8 *Let $H(\alpha)=H_0+V+\alpha U\equiv H+\alpha U$ and $N\geq 2$. Let $K(\alpha)$ be a nonzero Nth order operator symmetry of $H(\alpha)$ analytic at $\alpha=0$, self-adjoint for even N and skew-adjoint for odd N. Suppose further there are operators K_{N-2j} such that $K(\alpha)=\sum_{j=0}^{[N/2]}K_{N-2j}\alpha^j$ and identities (33), (34) are satisfied. Then the operator $\tilde{K}=\sum_{h=0}^{[N/2]}(-1)^hK_{N-2h}(U^{-1}(H+b))^h$ is an Nth order symmetry for the system $\tilde{H}=U^{-1}(H+b)$.*

Corollary 9 *Let $K(\alpha)$ $L(\alpha)$ be Nth and Mth order operator symmetries, respectively, of $H(\alpha)$, each satisfying the conditions of Theorem 8. Then*

$$[\tilde{L},\tilde{K}]=\widetilde{[L,K]},\quad \tilde{L}\tilde{K}=\widetilde{LK}.$$

Example 6 *(The 9-1 anisotropic oscillator) This is a generalization of Example 4 to a full Stäckel transform. Let $H(0)=\partial_{11}+\partial_{22}+a(9x_1^2+x_2^2)$ and L be as in Example 4 with α replaced by $a+\alpha$, and $U=(9x_1^2+x_2^2)+c$. It follows that the system*

$$\tilde{H}=\frac{1}{(9x_1^2+x_2^2)+c}\left(\partial_{11}+\partial_{22}+a(9x^2+y^2)+b)\right)$$

is superintegrable with one 2nd and one 3rd order symmetry.

Note that Theorem 8 does not require that the quantum system go to a classical system, only that a scalable potential term can be split off. Thus it applies to “hybrid” quantum systems that have a classical part.

Example 7 *(The hybrid 9-1 anisotropic oscillator) Let $H(0)=\partial_{11}+\partial_{22}+a(9x_1^2+x_2^2)-2/x_2^2$. This is a superintegrable system with generating 2nd and 3rd order symmetries,*

$$L=\partial_{22}+ax_2^2,\quad K=\{x_1\partial_2-x_2\partial_1,\partial_{22}\}+\{\frac{a}{3}x_2^3+\frac{1}{x_2},\partial_1\}-\{3x_1(ax_2^2+\frac{1}{x_2^2}),\partial_2\}.$$

Note that this system does not have a classical limit. [Using a different normalization that makes clear the classical limit, Gravel writes this Hamiltonian *as* $H(0) = -(\hbar^2/2)(\partial_{11}+\partial_{22}) + a(9x_1^2+x_2^2) + \hbar^2/x_2^2$*.] Let* $U = (9x_1^2+x_2^2)+c$*. It follows that the system*

$$\tilde{H} = \frac{1}{(9x_1^2+x_2^2)+c}\left(\partial_{11}+\partial_{22}+a(9x_1^2+x_2^2)-\frac{2}{x_2^2}+b\right)$$

is superintegrable with one 2nd and one 3rd order symmetry.

Example 8 *(A translated hybrid 9-1 anisotropic oscillator) This is a slight modification of Example 7. Let* $H(0) = \partial_{11}+\partial_{22}+a(9x_1^2+x_2^2)+cx_1 - 2/x_2^2$*. This is a superintegrable system with generating 2nd and 3rd order symmetries, and no classical limit. Let* $U = x_1$*. It follows that the system*

$$\tilde{H} = \frac{1}{x_1}\left(\partial_{11}+\partial_{22}+a(9x_1^2+x_2^2)+cx_1-\frac{2}{x_2^2}+b\right)$$

is superintegrable with one 2nd and one 3rd order symmetry. This space is a Darboux space of type 1, [2].

Example 9 *Let* $H(0) = \partial_{11}+\partial_{22}+a/x_1^2 - 2/x_2^2$*. This is a superintegrable system with two linearly independent 2nd and three linearly independent 3rd order symmetries, [4]. This system does not have a classical limit. [Using a different normalization that makes clear the classical limit, Gravel writes this Hamiltonian as* $H(0) = -(\hbar^2/2)(\partial_{11}+\partial_{22})+a/x_1^2+\hbar^2/x_2^2$*.] Let* $U = 1/x_1^2+c$*. It follows that the system*

$$\tilde{H} = \frac{x_1^2}{1+cx_1^2}\left(\partial_{11}+\partial_{22}+\frac{a}{x_1^2}-\frac{2}{x_2^2}+b\right)$$

is superintegrable with with two linearly independent 2nd and three linearly independent 3rd order symmetries. In the case $c = 0$ *this is a superintegrable system on a space of nonzero constant curvature. Indeed, for* x_1, x_2 *real, it is the upper half space metric of non-Euclidean geometry.*

In the operator case where $V = 0$ in Theorem 8 there is always a corresponding classical system. Indeed, equations (33) and (34) clarify the close relationship between symmetries of quantum systems with potentials invariant under scaling and classical constants of the motion. If we set $\alpha = 1/\hbar^2$, $V = 0$ in (33) we can rewrite this expression as

$$[\sum_{j=0}^{[N/2]} \hbar^{N-2j} K_{N-2j}, \hbar^2 H_0 + U] = 0.$$

Further if we write the differential terms in the operators K_{N-2j} as

$$K_{N-2j} = \sum_{i_s=1,2} a^{i_1\cdots i_{N-2j}} \partial_{i_1} \cdots \partial_{i_{N-2j}} + \text{ lower order terms,}$$

we can associate these operators with the phase space functions

$$\mathcal{K}_{N-2j}(\mathbf{x}, \mathbf{p}) = \sum_{i_s=1,2} a^{i_1\cdots i_{N-2j}} p_{i_1} \cdots p_{i_{N-2j}}.$$

Then by equating coefficients of the highest order derivative terms in equations (34) we obtain the Poisson bracket relations

$$\{\mathcal{K}_{N-2j}, U\} + \{\mathcal{K}_{N-2j-2}, \mathcal{H}_0\} = 0, \quad j = 0, 1, \cdots, [N/2], \tag{36}$$

so that $\mathcal{K} = \sum_{j=0}^{[N/2]} \mathcal{K}_{N-2j}$ is an Nth order constant of the motion for the system with Hamiltonian $\mathcal{H} = \mathcal{H}_0 + U$.

6 Conclusions and outlook

We have found specializations of classical CCM that preserve the order of symmetries and determine symmetry algebra homomorphisms, and for 2D manifolds we have extended them to the quantum case. Generally speaking, these transforms apply to systems with a nonconstant potential that admits scaling in at least one parameter. They do not apply to quantum systems with no classical counterpart in which the potential is fixed. This tool makes it clear that superintegrable systems occur for a wide variety of manifolds, not just on constant curvature spaces. For 2nd order superintegrable systems the Stäckel transform has been used effectively in 2D to show that all such systems are transforms of systems on constant curvature spaces, and this has lead to an elegant classification of all such systems. It is our aim to develop CCM to investigate the possibility of a similar classification for 3rd and higher order superintegrable systems.

For simplicity, we have restricted our quantum constructions to 2D manifolds though some partial results hold in n dimensions. There appears to be no insurmountable barrier to extending these results to 3D and higher conformally flat manifolds, but the details have not yet been worked out. Clearly gauge transformations are required and the gauge will be a function of the scalar curvature of the manifold.

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