Notes on the Solution of Homework 2 of Math 5467- Spring 2004

The following notes are written by the grader of the course. They are not complete solutions of the problems.

- **Problem 1**: Let $\tilde{f}(t)$ be the approximation of $f(t)$ by partial sums. Note that the given function is even, so all coefficients of sine terms are zero.

$$\tilde{f}(t) = 8\cos(t) + \frac{8}{9}\cos(3t) + \frac{8}{25}\cos(5t) + \frac{8}{49}\cos(7t)$$

  The periodic extension of this function on the real axis is continuous. The general formula for the coefficients of the expansion is $a_n = 8/(n^2)$ when n is odd, $a_n = 0$ when n is even. The convergence is faster of order $O(1/n^2)$. Also, note that we do not observe any Gibbs phenomenon. Another observation is that $\tilde{f}(t)$ is differentiable for all t since it is a finite sum of cosine terms. On the other hand, $f(t)$ is NOT differentiable at $t = 0$, there we see a corner which is smoothed out in the approximation.

- **Problem 2:** $f(t) = -t$ is an odd function, and its periodic extension is discontinuous at the boundaries. So, all coefficients of the cosine terms and the constant coefficient are zero. The coefficients of sine terms

$$b_n = 2\frac{\cos(n\pi)}{n}$$

  The convergence is slower than the previous problem, of order $O(1/n)$. Also, you should observe the Gibbs phenomenon. For any function $f(t)$ with discontinuity, we can not have uniform convergence of the Fourier series.

- **Problm 3:**

$$f_2(t) = \pi^2 - 3t^2 = \sum_{n=1}^{\infty} \frac{-12\cos(n\pi)}{n^2}\cos(nt)$$

  1. let $t = \pi$ in $f_2(t)$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

  2. let $t = 0$ in $f_2(t)$.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

  .

  3. Use Parseval's equality applied to the coefficients of Fourier series of the second function (note that $b_n = 0$ and $a_0 = 0$).

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(t)]^2 = \sum_{n=1}^{\infty} |a_n|^2$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

- **Problem 4:** Let us evaluate the Fourier transform of $\Pi(2t-3)$:

$$G_1(\lambda) = \int_{-\infty}^{\infty} \Pi(2t-3)e^{-j\lambda t}dt = \int_{-\infty}^{\infty} \Pi(u)e^{-j\lambda(u+3)/2}du/2$$

$$G_1(\lambda) = \frac{1}{2}e^{-j3\lambda/2}\widehat{\Pi}(\frac{\lambda}{2})$$

  where $\widehat{\Pi}(\lambda) = 2\sin(\lambda/2)/\lambda$. Similarly, you can derive the rest. I highly recommend that you derive it from the basic equation, and it may involve a simple substituion. Note that if you expand the function in time domain, it will be compressed in the frequency domain (Heisenberg's uncertainty principle).

- **Problem 5**: Here λ denotes the radian frequency, other books use ω.

  1. We need to use time-reversal property. $\mathcal{F}(f(-t)) = \hat{f}(-\lambda)$. In more details,
  $$\int_{\infty}^{-\infty} f(-t)e^{-2j\pi\lambda t}dt = -\int_{\infty}^{-\infty} f(u)e^{-2j\pi(-\lambda)u}du = \int_{-\infty}^{\infty} f(u)e^{-2j\pi(-\lambda)u}du$$
  where $u = -t$. It is trivial to notice by using this substitution, $dt = -du$, and the upper limit of the integral will be $-\infty$, lower limit is ∞. So, when interchange the upper and lower limit, the sign will cancel as shown above.
  2. We need to use the scaling property. $\mathcal{F}(2f(2t)) = \hat{f}(\lambda/2)$. If you shrink the domain of the function in the time domain, its domain is expanded in the frequency domain. This is known as *uncertainty principle*.
  3. $\mathcal{F}(e^{4it}f(t)) = \tilde{f}(\lambda - 4)$.
  4. $\mathcal{F}(f'(t)) = j\lambda\tilde{f}(\lambda)$.
  5. $\mathcal{F}((f * f)(t)) = (\tilde{f}(\lambda))^2$
  6. $\mathcal{F}((f(t).f(t)) = (1/2\pi)\tilde{f}(\lambda) * \tilde{f}(\lambda)$.

  You should observe the symmetry property of Fourier transform: If $\mathcal{F}(f(t)) = F(\lambda)$, then $\mathcal{F}(F(t)) = 2\pi f(-\lambda)$. You can use this property to derive part (6) from part (5).

- **Problem 6**: This is easy. Use the geometric progression formula.

- **Problem 7**: Again, this a problem on Parseval's equality.
$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iat}e^{-int}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(a-n)t}dt = \frac{(-1)^n \sin(a\pi)}{\pi(a-n)}$$
$$\sum_{-\infty}^{\infty} |C_n|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(t)|^2 dt = 1$$
  From both equations you can derive the required summation formula.

- **Problem 8**:
  We first show that $\{h_0, \cdots, h_{n-1}\}$ is an ON set in $L^2[0,1]$.

  Proof: We have $\int_0^1 h_k(t)h_\ell(t)dt = 0$ if $k \neq \ell$ because $h_k(t)h_\ell(t) = 0$ for all $t \in (0,1)$. However,
$$\int_0^1 h_k^2(t)dt = n\int_{\frac{k}{n}}^{\frac{k+1}{n}} dt = 1.$$
  Now we show that, for $f(t)$ continuous, $f_n(t) \to f(t)$ pointwise (even uniformly) in t as $n \to \infty$.

  Proof: We have
$$f_n(t) = \sum_{k=0}^{n-1} n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(\tau)d\tau\phi(nt-k).$$
  Thus,
$$f_n(t) = n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(\tau)d\tau \quad \text{for } \frac{k}{n} \leq t < \frac{k+1}{n}.$$
  By the mean value theorem of calculus, there is a point $t_{n,k}$ in the interval $(\frac{k}{n}, \frac{k+1}{n})$ such that the integral expression on the right is equal to $f(t_{n,k})$. Thus
$$f_n(t) = f(t_{n,k}) \quad \text{for } \frac{k}{n} \leq t < \frac{k+1}{n}.$$

Since f is continuous on the closed bounded set $[0, 1]$, it is uniformly continuous on this set. Thus for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $|f(t) - f(t')| < \epsilon$ whenever $|t - t'| < \delta(\epsilon)$. Now given $t \in [0, 1)$ and $\epsilon > 0$, choose $n > \frac{1}{\delta(\epsilon)}$. Then

$$|f(t) - f_n(t)| = |f(t) - f(t_{n,k})| < \epsilon, \text{ because } |t - t_{n.k}| < \frac{1}{n} < \delta(\epsilon)$$

so $f_n(t) \to f(t)$, uniformly in t as $n \to \infty$.

A solution of the graphing problem using MATLAB. The following MATLAB code will give a simultaneous plot of the two functions.

```
t= 0:1/(2^7):1-1/(2^7);
f4= zeros(1,2^7);
N=4;
for Integer =0:N-1
for number = (2^7)*Integer/N + 1 : (2^7)*Integer/N + (2^7)/N;
f4(number)=1- (Integer*(Integer+1)/(N)^2)-(1/(3*N^2));
end
end
f8= zeros(1,2^7);
N=8;
for Integer =0:N-1
for number = (2^7)*Integer/N + 1 : (2^7)*Integer/N + (2^7)/N;
f8(number)=1- (Integer*(Integer+1)/(N)^2)-(1/(3*N^2));
end
end
plot(t,f4,t,f8,t,f16,t , t.)
title('Plot of f(t)=t and Haar approximations for N=4,8')
```

If we needed to use many different values of N, it would be easier to define a function $haar(N)$ to compute $f_N(t)$ for us, save it in the file haar.m, and then write a simple code to plot $haar(N)$ for different values of N. (You would need to open MATLAB in the directory containing haar.m, or make sure that this directory is in the startup path of MATLAB. For example, the file haar.m could take the form

```
function x = haar(N)
% This function computes the Haar approximation f_N(t)
% to f(t)=t at 128 equally spaced points on the
% unit interval, for any positive integer N.
x= zeros(1,2^7);
for Integer =0:N-1
for number = (2^7)*Integer/N + 1 : (2^7)*Integer/N + (2^7)/N;
x(number)=1- (Integer*(Integer+1)/(N)^2)-(1/(3*N^2));
end;
end;
```

Then the main program would be very short (even if we add the graphs for N=16 and N=32):

```
t= 0:1/(2^7):1-1/(2^7);
plot(t,haar(4),t,haar(8),t,haar(16),t,haar(32),t , t.)
title('Plot of f(t)=t and Haar approximations for N=4,8,16,32')
```

A more flexible (but less accurate) approach, useful for treating several different funtions $f(t)$ and values of N, would be easier to define a function $Haar(f, N)$ to compute $f_N(t)$ for us (using the MATLAB function $quad(f, a, b)$, Simpson's rule for evaluation of $\int_a^b f(t)dt$), save the definition in the file Haar.m, and then write a simple code to plot $Haar(f, N)$ for different choices of f, N. (You would need to open MATLAB in the directory containing Haar.m, or make sure that this directory is in the startup path of MATLAB. For example, the file Haar.m could take the form

```
function x = Haar(f,N)
% This function computes the Haar approximation f_N(t)
% to the function f(t) at 128 equally spaced points
%on the unit interval
x= zeros(1,2^7);
for Integer =0:N-1
for number = (2^7)*Integer/N + 1 : (2^7)*Integer/N + (2^7)/N;
x(number)=N*quad(f,Integer/N,(Integer+1)/N);
end;
end;
```

Then the main program would be very short:

```
t= 0:1/(2^7):1-1/(2^7);
plot (t,Haar('t.',4),t,Haar('t.',8),t,Haar('t.',16),t,t.)
title('Plot of f(t)=t and Haar approximations for N=4,8,16')
```

or

```
plot (t, Haar('sin(2*pi*t)',8),t, Haar('sin(2*pi*t)',16),t,sin(2*pi*t))
title('Plot of f(t)=sin(2*pi*t) and Haar approximations for N=8,16')
```