SEPARATION OF VARIABLES AND THE XXZ GAUDIN MAGNET

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ABSTRACT. In this work we generalise previous results connecting (rational) Gaudin magnet models and classical separation of variables. It is shown that the connection persists for the case of linear r-matrix algebra which corresponds to the trigonometric 4×4 r-matrix (of the XXZ type). We comment also on the corresponding problem for the elliptic (XYZ) r-matrix. A prescription for obtaining integrable systems associated with multiple poles of an L-operator is given.

1. Introduction. Separation of variables for the Hamilton-Jacobi and Schrödinger equations have long been known as methods for explicit solution of these equations in appropriate circumstances. The technical requirements for this method of solution have quite fully developed in recent years (see [1–8]). In particular the relationship between the separable systems and the Gaudin magnet [4,9] integrable systems models has been established via r-matrix algebra methods, where the r-matrix corresponds to the rational or so called XXX case, [4–8]. This relationship works very clearly with separable coordinate systems on spaces of constant curvature. The question we answer here is how these notions can be extended to include the so-called trigonometric r-matrix algebra in the XXZ case. To do this let us

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recall the fundamental ideas of the r-matrix formalism (see [10,11] and references in there). For a classical mechanical system the basic (linear) r-matrix algebra is

$$\{L(u)\otimes I, I\otimes L(v)\} = [r(u-v),\ L(u)\otimes I + I\otimes L(v)]\,, \tag{1.1}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket and $[\cdot,\cdot]$ the matrix commutator bracket. The operator $L(u)$ is taken to be the 2×2 matrix

$$L(u)_{11} = -L(u)_{22} = A(u), \qquad L(u)_{12} = B(u), \qquad L(u)_{21} = C(u),$$

and $r(u)$ is a suitable 4×4 matrix of scalars solving the classical Yang-Baxter equation [10,11]; u being arbitrary constant is called the spectral parameter. In the case of the XXZ r-matrix algebra the non zero elements of r can be taken to be

$$r(u)_{11} = r(u)_{44} = \coth(u), \qquad r(u)_{23} = r(u)_{32} = \frac{1}{\sinh(u)}. \tag{1.2}$$

In component form, the r-matrix algebra relations are

$$\begin{aligned}
&\{A(u),A(v)\} = \{B(u),B(v)\} = \{C(u),C(v)\} = 0\,,\\
&\{A(u),B(v)\} = \frac{1}{\sinh(u-v)}\left(\cosh(u-v)B(v) - B(u)\right),\\
&\{A(u),C(v)\} = \frac{1}{\sinh(u-v)}\left(-\cosh(u-v)C(v) + C(u)\right),\\
&\{B(u),C(v)\} = \frac{-2}{\sinh(u-v)}\left(A(u) - A(v)\right).
\end{aligned} \tag{1.3}$$

If we now make the ansatz $A(u) = \coth(u)S_3$, $B(u) = (1/\sinh(u))S_+$ and $C(u) = (1/\sinh(u))S_-$ these relations imply

$$\{S_3, S_\pm\} = \pm S_\pm\,, \qquad \{S_+, S_-\} = 2S_3\,. \tag{1.4}$$

To relate this observation to the separation of variables methods, we form the $L(u)$ operator with elements

$$\begin{aligned}
&B(u) = \sum_{\alpha=1}^{n} \frac{1}{\sinh(u-e_\alpha)}\, S_{+\alpha}\,, \qquad C(u) = \sum_{\alpha=1}^{n} \frac{1}{\sinh(u-e_\alpha)}\, S_{-\alpha}\,,\\
&A(u) = \sum_{\alpha=1}^{n} \coth(u-e_\alpha)\, S_{3\alpha}\,,
\end{aligned} \tag{1.5}$$

where

$$\{S_{3\alpha}, S_{\pm\beta}\} = \pm\delta_{\alpha\beta}S_{\pm\alpha}\,, \qquad \{S_{+\alpha}, S_{-\beta}\} = 2\delta_{\alpha\beta}S_{3\alpha}\,. \tag{1.6}$$

The r-matrix algebra relations, (1.1) or (1.3), imply

$$\{\det L(u), \det L(v)\} = 0\,, \tag{1.7}$$

i.e., that $\det L(u)$ is a generating function of the constants of the motion. In particular we have

(1.8)

$$-\det L(u) = A^2(u) + B(u)C(u) = \sum_{\alpha=1}^{n} \left[\frac{C_\alpha}{\sinh^2(u-e_\alpha)} + H_\alpha \coth(u-e_\alpha) \right] + H_0^2,$$

where $C_\alpha = (S_{3\alpha})^2 + S_{+\alpha}S_{-\alpha}$ are the Casimir elements of the algebra generated by elements $S_{+\alpha},\ S_{-\alpha}$ and $S_{3\alpha}$. Furthermore $H_0 = \sum_\alpha S_{3\alpha}$ and

$$H_\alpha = \sum_{\beta\neq\alpha} \left(2S_{3\alpha}S_{3\beta}\coth(e_\alpha - e_\beta) + \frac{1}{\sinh(e_\alpha - e_\beta)}(S_{+\alpha}S_{-\beta} + S_{+\beta}S_{-\alpha}) \right). \tag{1.9}$$

With the following realization of the algebra in terms of the canonical coordinates x_α and p_α, $\{p_\alpha, x_\beta\} = -\delta_{\alpha\beta}$:

$$S_{+\alpha} = \frac{i}{2}x_\alpha^2\,, \qquad S_{-\alpha} = \frac{i}{2}p_\alpha^2\,, \qquad S_{3\alpha} = -\frac{1}{2}x_\alpha p_\alpha\,, \tag{1.10}$$

the constants (1.9) have the form

$$H_\alpha = \frac{1}{4}\sum_{\beta\neq\alpha} \frac{-1}{\sinh(e_\alpha - e_\beta)} \left(x_\alpha^2 p_\beta^2 + x_\beta^2 p_\alpha^2 - 2x_\alpha x_\beta p_\alpha p_\beta \cosh(e_\alpha - e_\beta) \right), \tag{1.12}$$

and $H_0 = -\frac{1}{2}\sum_\alpha x_\alpha p_\alpha$. Notice that all $C_\alpha = 0$ in such a representation.

2. Variable Separation for the XXZ Magnet. Proceeding as in [4,7,8], we choose separable coordinates such that $B(u) = 0$, i.e., $u = u_j,\ j = 1, \dots, n-1$. This implies $\sum_\alpha x_\alpha^2/\sinh(u - e_\alpha) = 0$ for $u = u_1, \dots, u_{n-1}$, which in turn implies that we choose coordinates according to

$$x_\alpha^2 = e^{u_n} \frac{\Pi_{j=1}^{n-1}\sinh(u_j - e_\alpha)}{\Pi_{\beta\neq\alpha}\sinh(e_\beta - e_\alpha)}\,, \tag{2.1}$$

motivated by the general formula

$$\sum_{\alpha=1}^{n} \frac{x_\alpha^2}{\sinh(u - e_\alpha)} = e^{u_n}\frac{\Pi_{j=1}^{n-1}\sinh(u - u_j)}{\Pi_{\beta=1}^{n}\sinh(u - e_\beta)}\,. \tag{2.2}$$

For each u_j we can define the canonically conjugate coordinate v_j as follows:

$$v_j = A(u_j) = -\frac{1}{2}\sum_{\alpha=1}^{n}\coth(u_j - e_\alpha)\,x_\alpha p_\alpha\,, \quad 1 \le j \le n-1\,, \qquad v_n = H_0\,. \tag{2.3}$$

The coordinates $u_i,\ v_j\ (i,j = 1, \dots, n)$ satisfy the canonical bracket relations

$$\{u_i, u_j\} = 0\,, \quad \{v_i, u_j\} = \delta_{ij}\,, \quad \{v_i, v_j\} = 0\,. \tag{2.4}$$

The changing of variables x_α, p_α for the new variables u_i, v_i is the procedure of separation of variables. The matrix elements of the L-operator can be expressed in terms of these variables according to the formulas

$$\begin{aligned} A(u) =& iB(u)\left[2\cosh\left(u+\Sigma_{j=1}^{n-1}u_j-\Sigma_{\alpha=1}^{n}e_\alpha\right)\ v_n \right.\\ &\left. +\sum_{j=1}^{n-1}\frac{-2v_j}{\sinh(u-u_j)}\frac{\Pi_{\alpha=1}^{n}\sinh(u_j-e_\alpha)}{\Pi_{k\neq j}\sinh(u_j-u_k)}\right]e^{-u_n}. \end{aligned} \tag{2.5}$$

The entry $C(u)$ can be computed by using the formula

$$\begin{aligned} p_\alpha =& x_\alpha e^{-u_n}\left[2\cosh\left(\Sigma_{j=1}^{n-1}u_j-\Sigma_{\gamma\neq\alpha}e_\gamma\right)\ v_n \right.\\ &\left. +\sum_{j=1}^{n-1}\frac{-2v_j}{\sinh(e_\alpha-u_j)}\frac{\Pi_{\alpha=1}^{n}\sinh(u_j-e_\alpha)}{\Pi_{k\neq j}\sinh(u_j-u_k)}\right]. \end{aligned} \tag{2.6}$$

This gives the relation between the coordinates x_α, p_α and u_i, v_i where $v_n = H_0$.

The equation for the eigenvalue curve $\Gamma:\ \det(L(u)-\lambda I)=0$, has the form

$$\lambda^2 - A(u)^2 - B(u)C(u) = 0\,.$$

If we put $u=u_j,\ j=1,\ldots,n-1$ into this equation then $\lambda=\pm v_j$. Thus variables u_j and v_j $(j=1,\ldots,n-1)$ lie on the curve Γ:

$$v_j^2-\sum_{\alpha=1}^{n}H_\alpha\coth(u_j-e_\alpha)-H_0^2\equiv v_j^2+\det L(u_j)=0\,. \tag{2.7}$$

Equations (2.7) are the separation equations for the degrees of freedom connected with the values of the integrals H_α. (Note that $\sum_{\alpha=1}^{n}H_\alpha=0$.)

For illustrative reasons it is more transparent to use the variables $A_i=e^{e_i}$ and $U_i=e^{u_i}$. Then many of the expressions given have algebraic form. For example, the nearest object we have to a Hamiltonian in the case of XXZ r-matrix algebra is $H=\sum_{i=1}^{n}A_i^2H_i$ which has the form

$$\begin{aligned} H =& \sum_{i=1}^{n-1}e^{-\sum_{j\neq i}u_j}\left(e^{-2u_i}v_i^2+H_0^2\right)\frac{\Pi_{\alpha=1}^{n}\sinh(u_i-e_\alpha)}{\Pi_{j\neq i}\sinh(u_i-u_j)}\\ =& (4\Pi_{j=1}^{n}A_j)^{-1}(-1)^n\sum_{i=1}^{n-1}\left(p_{U_i}^2+H_0^2\right)\frac{\Pi_{k=1}^{n}(A_k^2-U_i^2)}{U_i^2\Pi_{j\neq i}(U_i^2-U_j^2)}\,. \end{aligned} \tag{2.8}$$

Note that

$$e^{u_n}=\frac{U_1\cdots U_{n-1}}{A_1\cdots A_n}\sum_{\alpha=1}^{n}A_\alpha x_\alpha^2\,.$$

If we adopt the standard procedure and write $p_{U_i}=\partial W/\partial U_i$ then the equation $H=E$ admits separation of variables via the usual ansatz $W=\sum_{i=1}^{n-1}W_i(U_i)$. The separation equations can be written in the alternate form

$$\left[\Pi_{k=1}^{n}(U_j^2-A_k^2)\right]\left(\frac{\partial W_j}{\partial U_j}\right)^2=H_0U_j^{2n-2}+H_0(-1)^n\left(\Pi_kA_k^2\right)U_j^{-2}+\sum_{k=1}^{n-1}\lambda_kU_j^{2k-2},$$

where the λ_k are related to the H_j via

$$P_{2n-4}(u) \equiv 2u^{-2}\left[\Pi_{k=1}^{n}(u^2 - A_k^2)\right]\sum_{i=1}^{n}\frac{H_i A_i^2}{u^2 - A_i^2},$$

$$\lambda_k = H_0 \sum_{\alpha_1<\ldots<\alpha_{n-k}} (-1)^{n-k} A_{\alpha_1}^2 \cdots A_{\alpha_{n-k}}^2 + \frac{1}{(2k-2)!}P_{2n-4}^{(2k-2)}(0),$$

$$\lambda_{n-1} = E - H_0 \sum_{k=1}^{n} A_k^2 .$$

From what has been developed so far we see that separation of variables goes through for XXZ r-matrix algebras constructed in this way. In the previous article [8] for the case of spaces of constant curvature we essentially have the rational r-matrix algebra and it is possible to formulate using well defined limiting procedures the cases of integrable systems for which some of the e_i parameters are equal. What was also established previously was the construction of integrable systems given on the algebra with commutation relations

$$\{(Z_j^J)_\ell, (Z_k^{J'})_m\} = -\delta_{JJ'}(Z_{j+k-N_J}^J)_s\epsilon_{\ell ms}, \tag{2.9}$$

where $\ell, m, s = 1, 2, 3,\ 0 < j < N_J,\ 0 < k < N_{J'},\ 0 < J \leq p$ and $\epsilon_{\ell ms}$ is the usual totally antisymmetric tensor, and the vector Z_j^J has the form

$$Z_j^J = \begin{pmatrix} \frac{1}{4}\sum_i (p_i^J p_{j+1-i}^J + x_i^J x_{j+1-i}^J) \\ \frac{i}{4}\sum_i (p_i^J p_{j+1-i}^J - x_i^J x_{j+1-i}^J) \\ \frac{i}{2}\sum_i p_i^J x_{j+1-i}^J \end{pmatrix} \tag{2.10}$$

in the coordinate representation. Indeed, if we adopt the limiting procedure

$$\begin{aligned} A_j^J &\to A_1^J + {}^J\epsilon_{j-1}^1, \quad j = 1, \ldots, N_J, \quad J = 1, \ldots, p, \\ p_j^J &\to \sqrt{a_j^J}\left(p_1^J + \sum_{i=2}^{N_J} {}^J\epsilon_{j+1-i}^{i-1}\, p_i^J\right), \\ x_j^J &\to \sqrt{a_j^J}\left(x_1^J + \sum_{i=2}^{N_J} {}^J\epsilon_{j+1-i}^{i-1}\, x_i^J\right), \end{aligned} \tag{2.11}$$

where

$${}^J\epsilon_{j+1-i}^{i-1} = \Pi_{\ell=2}^{i}({}^J\epsilon_{j-1}^1 - {}^J\epsilon_{\ell-2}^1), \qquad a_j^J = \frac{1}{\Pi_{k\neq j}({}^J\epsilon_{j-1}^1 - {}^J\epsilon_{k-1}^1)},$$

and $N_1 + \ldots + N_p = n + 1$, then the Hamiltonian H has the form

$$H = (\Pi_{j=1}^{n+1} A_j)^{-1} \sum_{i=1}^{n} \left(p_{U_i}^2 + H_0^2\right) \frac{\Pi_{k=1}^{p}(A_k^2 - U_i^2)^{N_k}}{U_i^2 \Pi_{j\neq i}(U_i^2 - U_j^2)} \tag{2.12}$$

with obvious separation equations. (We require ${}^J\epsilon_0^i = 0$ and take the limit as the ${}^J\epsilon_h^1 \to 0$ for $h = 1, \dots, N_J - 1$, see [8].) The generating function for the constants can be derived by applying these procedures to $\det L(u)$. We will, however, adopt a different and more general strategy. If we leave the matrix elements of $L(u)$ in the form (1.5) and subject the resulting expression for $\det L(U)$

$$-\frac{1}{2}\det L(U) = \frac{1}{2}\left(\sum_{\alpha=1}^{n} S_{3\alpha}\right)^2 + 2U^2\left[\sum_{i=1}^{2}\left(\sum_{\alpha=1}^{n}\frac{A_\alpha S_{i\alpha}}{U^2 - A_\alpha^2}\right)^2\right.$$
$$\left. + \sum_{\alpha=1}^{n}\frac{S_{3\alpha}}{U^2 - A_\alpha^2}\left(-\sum_{\alpha=1}^{n} S_{3\alpha} + U^2\sum_{\alpha=1}^{n}\frac{S_{3\alpha}}{U^2 - A_\alpha^2}\right)\right], \tag{2.13}$$

where $U = e^u$, and $S_{\pm\alpha} = S_{1\alpha} \pm iS_{2\alpha}$, to the transformations

$$A_j^J \to A_1^J + {}^J\epsilon_{j-1}^1, \qquad j = 1, \dots, N_J, \quad J = 1, \dots, p,$$
$${}^J S_1 \delta_{k0} + \sum_{j=2}^{N_J} ({}^J\epsilon_{j-1}^1)^k ({}^J S_j) = Z_{N_J - k}^J, \quad k = 0, \dots, N_J,$$

then we arrive at a general expression for the generating function $\det L(U)$. The constants of the motion are obtained by the usual means of expanding the expression following from (2.13) in partial fractions and reading off the independant components. In the case of degenerate roots the expression can be readily modified. Accordingly we have

$$-\frac{1}{2}\det L(U) = \frac{1}{2}\left(\sum_{J=1}^{p}(Z_{N_J}^J)_3\right)^2$$
$$+ 2U^2\left[\sum_{i=1}^{2}\left(\sum_{J=1}^{p}\sum_{j=0}^{N_J-1}\frac{1}{j!}\left(\left(\frac{\partial}{\partial A_j}\right)^j\frac{A_j}{U^2 - A_j^2}\right)(Z_{N_J-j}^J)_i\right)^2\right.$$
$$+ \sum_{J=1}^{p}\sum_{j=0}^{N_J-1}\frac{1}{j!}\left(\left(\frac{\partial}{\partial A_j}\right)^j\frac{1}{U^2 - A_j^2}\right)(Z_{N_J-j}^J)_3 \times \tag{2.14}$$
$$\left.\times \sum_{J=1}^{p}\left(-(Z_{N_J}^J)_3 + U^2\sum_{j=0}^{N_J-1}\frac{1}{j!}\left(\left(\frac{\partial}{\partial A_j}\right)^j\frac{1}{U^2 - A_j^2}\right)(Z_{N_J-j}^J)_3\right)\right].$$

From this expression constants of the motion can be deduced just as before. The separation of variables proceeds as usual in the case of the choice of coordinates as given in [8]. The expressions for the coordinates corresponding to multiple roots with signature $N_1, N_2, ..., N_p$ can be obtained from the generic case by the limiting procedures already outlined. In rational form the generic coordinates are

$$x_i^2 = [\Pi_{j=1}^n (A_j^2 - U_i^2)][\Pi_{s=1}^{n-1} U_s]\frac{\Pi_{k\neq i} A_k}{\Pi_{\ell\neq i}(A_\ell^2 - A_i^2)}. \tag{2.15}$$

For the case of signature $N_1, N_2, ..., N_p$ the coordinates are given by the relations

$$\sum_{i=1}^{j} x_i^J x_{j+1-i}^J = [\Pi_{k=1}^n U_k] \left\{ \sum_{r=1}^{j-2} (N_J - r - 3)[\Pi_{q=0}^{r-2}(N_J + q - 2)] \times \right.$$
$$\left. \times \frac{[\Pi_{L\neq J}(A_1^L)^{N_L}]}{(2^{N_J})A_1^J r!} \left(\frac{\partial}{\partial A_1^J}\right)^r \frac{\Pi_{i=1}^n((A_1^J)^2 - U_i^2)}{A_1^J \Pi_{L\neq J}(2A_1^L)^{N_L}} \right\}.$$

This gives a complete description of the separation of variables procedure for the signature $N_1, N_2, ..., N_p$ case. We illustrate these ideas with two examples.

A. The case of signature 2,1 and dimension 3. In this case the generating function assumes the form

$$\det L(u) = \frac{1}{\sinh^2(u-e_1)} \left\{ \frac{1}{2}(Z_1^1.Z_1^1 + Z_2^1.Z_2^1) - \frac{1}{\sinh(e_1-e_3)} \big((Z_1^1)_2(Z_1^2)_2 \right.$$
$$\left. +(Z_1^1)_1(Z_1^2)_1\big) + \cosh(e_1-e_3)(Z_1^1)_3(Z_1^2)_3 \right\}$$
$$+ \frac{Z_1^1.Z_1^1\coth(u-e_1)}{2\sinh^4(u-e_1)} \left\{ \frac{1}{\sinh(e_1-e_3)} \big((Z_2^1)_1(Z_1^2)_1 + (Z_2^1)_2(Z_1^2)_2\big) \right.$$
$$+ \cosh(e_1-e_3)(Z_2^1)_3(Z_1^2)_3 - \frac{1}{\sinh^2(e_1-e_3)} \big((Z_1^1)_3(Z_1^2)_3$$
$$\left. + \cosh(e_1-e_3)\big((Z_1^1)_2(Z_1^2)_2 + (Z_1^1)_1(Z_1^2)_1\big)\big) \right\}$$
$$- \coth(u-e_3) \left\{ \frac{1}{\sinh(e_1-e_3)} \big(- \big((Z_1^2)_2(Z_2^1)_2 + (Z_1^2)_1(Z_2^1)_1\big) \right.$$
$$+ \cosh(e_1-e_3)(Z_1^2)_3(Z_2^1)_3\big) + \frac{1}{\sinh^2(e_1-e_3)} \big((Z_1^2)_3(Z_1^1)_3$$
$$\left. + \cosh(e_1-e_3)\,\big((Z_1^2)_1(Z_2^1)_1 + (Z_1^2)_2(Z_2^1)_2\big)\big) \right\}.$$

The constants of the motion can be deduced from the coefficents of independent functions of u. In the coordinate representation these constants have the form

$$H_1 = x_1^2p_2^2 + p_1^2x_2^2 - 2x_1x_2p_1p_2 + \frac{1}{\sinh(e_1-e_3)} \left(x_1^2p_3^2 + x_3^2p_1^2 - 2x_1x_3p_1p_3\right),$$
$$H_2 = \frac{2}{\sinh(e_1-e_3)} \left(x_1x_2p_3^2 + p_1p_2x_3^2 - (x_1x_3p_2p_3 + p_1p_3x_2x_3)\cosh(e_1-e_3)\right)$$
$$- \frac{1}{\sinh^2(e_1-e_3)} (x_1p_3 + p_1x_3)^2,$$

where we have used the notation $x_1 = x_1^1$, $x_2 = x_2^1$ and $x_3 = x_1^2$, with similar relations for the p_i's. The coordinates are given by the formulas

$$x_1^2 = -\frac{\sinh(u_1-e_1)\sinh(u_2-e_1)}{\sinh(e_1-e_3)},$$
$$2x_1x_2 = -\frac{\sinh(u_1-e_1)\sinh(u_2-e_1)\cosh(e_1-e_3)}{\sinh^2(e_1-e_3)} - \frac{1}{\sinh(e_1-e_3)} \times$$
$$\times \left(\sinh(u_1-e_1)\cosh(u_2-e_1) + \sinh(u_1-e_3)\cosh(u_2-e_3)\right),$$
$$x_3^2 = \frac{\sinh(u_1-e_3)\sinh(u_2-e_3)}{\sinh^2(e_1-e_3)}.$$

B. The Case of Signature 3 and Dimension 3. The generating function has the form

$$
\begin{aligned}
L(U) = {} & \frac{1}{(U^2-A_1^2)^6} 32A_1^8 Z_1.Z_2 \\
& + \frac{1}{(U^2-A_1^2)^5} 16A_1^2 \left[5Z_1.Z_1 + (Z_1)_2(Z_2)_2 + 2A_1\left((Z_1)_3(Z_2)_3 + (Z_1)_1(Z_2)_1\right)\right] \\
& + \frac{1}{(U^2-A_1^2)^4} 2A_1^4 \left[33Z_1.Z_1 + 36Z_1.Z_2 + 8A_1^2 Z_1.Z_3 + 4A_1^2 Z_2.Z_3\right] \\
& + \frac{1}{(U^2-A_1^2)^3} 2A_1^2 \left[26A_1 Z_1.Z_2 + 8A_1^2 Z_2.Z_2\right. \\
& \left. +14A_1^2 Z_1.Z_3 + Z_1.Z_1 + 4A_1^3 Z_3 Z_2 + (Z_1)_3^2\right] \\
& + \frac{1}{(U^2-A_1^2)^2} 2\left[A_1^4 Z_3.Z_3 + 6A_1^3 Z_2.Z_3\right. \\
& + A_1^2\left(5\left((Z_2)_2^2 + (Z_2)_1^2\right) + 6(Z_3)_2(Z_1)_1 + 4(Z_2)_3^2\right) \\
& \left. +A_1\left(6(Z_2)_2(Z_1)_2 + (Z_2)_1(Z_1)_1 + 4(Z_2)_3(Z_1)_3 + (Z_1)^2\right)\right] \\
& + \frac{1}{U^2-A_1^2} 2\left[A_1^2 Z_3.Z_3 + 2A_1 Z_2.Z_3 + (Z_2)_2^2 + (Z_2)_1^2 + (Z_1)_3(Z_3)_3\right].
\end{aligned}
$$

The coordinates are given by

$$
\begin{aligned}
x_1^2 &= \frac{1}{4}U_1U_2 - \frac{A_1^2}{4}\left(\frac{U_1}{U_2}+\frac{U_2}{U_1}\right) + \frac{A_1^4}{4U_1U_2}, \\
2x_1x_2 &= -\frac{3}{8A_1}U_1U_2 - \frac{A_1}{8}\left(\frac{U_1}{U_2}+\frac{U_2}{U_1}\right) + \frac{5A_1^3}{8U_1U_2}, \\
2x_1x_3 + x_2^2 &= \frac{3U_1U_2}{8A_1^2} + \frac{1}{8}\left(\frac{U_1}{U_2}+\frac{U_2}{U_1}\right) + \frac{3A_1^2}{8U_1U_2}.
\end{aligned}
$$

3. The XYZ Magnet. These methods can be extended to the case of elliptic or XYZ r-matrix algebras. The only difference is that in this case a solution of the problem via separation of variables is not yet known[1] but the coalescing of indices goes through just as before. Indeed, the operator $L(u)$ can be taken just as in (1.1). The non zero elements of the r-matrix in this case are

$$
\begin{aligned}
& r(u)_{11} = r(u)_{44} = \frac{\mathrm{cn}(u)}{\mathrm{sn}(u)}, \qquad r(u)_{14} = r(u)_{41} = \frac{1-\mathrm{dn}(u)}{2\mathrm{sn}(u)}, \\
& r(u)_{23} = r(u)_{32} = \frac{1+\mathrm{dn}(u)}{2\mathrm{sn}(u)}.
\end{aligned}
\tag{3.1}
$$

We now make the ansatz

$$
\begin{aligned}
& A(u) = \frac{\mathrm{cn}(u)}{\mathrm{sn}(u)} S_3, \quad B(u) = \frac{1}{2\mathrm{sn}(u)}\left[(1+\mathrm{dn}(u))S_- + (1-\mathrm{dn}(u))S_+\right], \\
& C(u) = \frac{1}{2\mathrm{sn}(u)}\left[(1+\mathrm{dn}(u))S_+ + (1-\mathrm{dn}(u))S_-\right].
\end{aligned}
\tag{3.2}
$$

[1]See [12] where the variable separation has been done for the periodic classical XYZ-chain from which the system in question can be obtained through the limit.

Here the $S_\pm$, S_3 obey the same commutation relations as (1.4). We choose the $L(u)$ operator to be

(3.3)

$$A(u) = \sum_{\alpha=1}^{n} \frac{\mathrm{cn}(u-e_\alpha)}{\mathrm{sn}(u-e_\alpha)} S_{3\alpha}\,,$$

$$B(u) = \sum_{\alpha=1}^{n} \frac{1}{2\mathrm{sn}(u-e_\alpha)} \left[(1+\mathrm{dn}(u-e_\alpha))S_{-\alpha} + (1-\mathrm{dn}(u-e_\alpha))S_{+\alpha}\right]\,,$$

$$C(u) = \sum_{\alpha=1}^{n} \frac{1}{2\mathrm{sn}(u-e_\alpha)} \left[(1+\mathrm{dn}(u-e_\alpha))S_{+\alpha} + (1-\mathrm{dn}(u-e_\alpha))S_{-\alpha}\right]\,.$$

The determinant of $L(u)$ is once again a generator of the constants of the motion. It has the form

$$\det L(u) = \sum_{\alpha=1}^{n} H_\alpha E(u-e_\alpha+iK') + H_0 \tag{3.4}$$

where

$$H_\alpha = 2k^2 \sum_{\beta\neq\alpha} \frac{1}{\mathrm{sn}(e_\alpha-e_\beta)} \left[S_{1\alpha}S_{1\beta} + \mathrm{dn}(e_\alpha-e_\beta)S_{2\alpha}S_{2\beta} + \mathrm{cn}(e_\alpha-e_\beta)S_{3\alpha}S_{3\beta}\right]\,,$$

$$H_0 = 2k^2 {\sum_{\alpha,\beta}}' \frac{E(e_\beta-e_\alpha)}{\mathrm{sn}(e_\beta-e_\alpha)}\left[S_{1\alpha}S_{1\beta} + \mathrm{dn}(e_\alpha-e_\beta)S_{2\alpha}S_{2\beta} + \mathrm{cn}(e_\alpha-e_\beta)S_{3\alpha}S_{3\beta}\right]$$

(3.5)

$$-{\sum_{\alpha,\beta}}' \left[k^2\mathrm{cn}(e_\alpha-e_\beta)S_{2\alpha}S_{2\beta} + \mathrm{dn}(e_\alpha-e_\beta)S_{3\alpha}S_{3\beta}\right] - \sum_{\alpha=1}^{n}\left[k^2 S_{2\alpha}^2 + S_{3\alpha}^2\right].$$

Here $E(z) = \int^z \mathrm{dn}^2(u)du$ is Jacobi's epsilon function. The same is now true as for the case of XXZ r-matrix algebras: if we subject the e_α's and the $S_{i\beta}$'s to the transformations given by (2.11), then we arrive at the generating function for the constants of motion for a root structure having the signature $N_1, N_2, ..., N_p$. The expression for this function is

$$\det L(u) = \sum_{k=1}^{3} \left(\sum_{r=0}^{N_J-1} \left(\frac{\partial}{\partial^J e_1}\right)^r f_k(u-{}^J e_1)(Z^J_{N_J-r})_k \right)^2, \tag{3.6}$$

where $f_1(z) = 1/\mathrm{sn}(z)$, $f_2(z) = \mathrm{dn}(z)/\mathrm{sn}(z)$, and $f_3(z) = \mathrm{cn}(z)/\mathrm{sn}(z)$.

As an example, the generating function corresponding to signature 2,1 is

$$\det L(u) = H_1 E(u-e_1+iK') + H_2 E(u-e_3+iK') + H_3 + \frac{1}{\mathrm{sn}^4(u-e_1)}H_4$$
$$+ \frac{\mathrm{cn}(u-e_1)\mathrm{dn}(u-e_1)}{\mathrm{sn}^3(u-e_1)}H_5 + \frac{1}{\mathrm{sn}^2(u-e_1)}H_6 + \frac{1}{\mathrm{sn}^2(u-e_3)}H_7,$$

where

$$
\begin{aligned}
H_1 &= \frac{2}{k^2\mathrm{sn}(e_1-e_3)}\Big[(Z_1^2)_1(Z_2^1)_1 - k^2\mathrm{cn}(e_1-e_3)(Z_1^2)_3(Z_2^1)_3 \\
&\quad - \frac{k^2\mathrm{cn}(e_1-e_3)\mathrm{dn}(e_1-e_3)}{\mathrm{sn}(e_1-e_3)}(Z_1^2)_1(Z_1^1)_1 + \frac{\mathrm{dn}(e_1-e_3)}{\mathrm{sn}(e_1-e_3)}(Z_1^2)_3(Z_1^1)_3\Big]\,, \\
H_2 &= -H_1\,, \\
H_3 &= \frac{E(e_1-e_3)}{k^2\mathrm{sn}(e_1-e_3)}H_1 + k^2\left((Z_1^1)_1^2 - (Z_2^1)_2^2 - (Z_1^2)_2^2\right) - (Z_1^2)_2^2 - (Z_1^2)_3^2 \\
&\quad - 2k^2\mathrm{sn}(e_1-e_3)(Z_1^2)_1(Z_1^1)_1 + 2\mathrm{dn}(e_1-e_3)(Z_1^2)_3(Z_2^1)_3\,, \\
H_4 &= Z_1^1.Z_1^1\,, \qquad H_5 = 2Z_1^1.Z_2^1\,, \\
H_6 &= Z_2^1.Z_2^1 - (Z_1^1)_1^2 - (Z_1^1)_2^2 - k^2(Z_1^1)_3^2 + \frac{2}{\mathrm{sn}(e_1-e_3)}(Z_1^2)_1(Z_1^1)_1 \\
&\quad - 2\frac{\mathrm{cn}(e_1-e_3)}{\mathrm{sn}(e_1-e_3)}(Z_1^2)_3(Z_2^1)_3\,, \\
H_7 &= Z_1^2.Z_1^2\,.
\end{aligned}
$$

We note that the ideas developed here also work in the case of separation of variables for spaces of constant Riemannian curvature, as developed in previous articles [6–8]. Indeed, in that case the rational r-matrix algebra is as before and the non zero elements of the r-matrix are

$$r(u)_{11} = r(u)_{44} = r(u)_{23} = r(u)_{32} = 1. \tag{3.7}$$

The generating function of the constants of the motion for signature $N_1, ..., N_p$ is then

$$\det L(u) = \sum_{k=1}^{3}\left(\sum_{J=1}^{p}\sum_{j=0}^{N_J-1}\frac{(Z_{j+1}^J)_k}{(u-{}^Je_1)^{N_J-j}} + \epsilon_k\right)^2 . \tag{3.8}$$

This is the generalisation of the generating function for separable coordinates on spaces of constant curvature of dimension $n = \sum_{j=1}^{p} N_J + 1$. Indeed, if we use the form (3.8) and if $\epsilon_k = 0$ for $k = 1,2,3$ then we have the generating function on the sphere for generic ellipsoidal coordinates, and if $\epsilon_1 = -1/4,\ \epsilon_2 = 1/4,\ \epsilon_3 = 0$ then we have the generating function of ellipsoidal coordinates in n-dimensional Euclidean space. As an example consider the system with signature 2,1. The generating function is then

$$
\begin{aligned}
\det L(u) &= \frac{\frac{(Z_1^2).(Z_2^1)}{e_1-e_3} - \frac{(Z_1^1).(Z_2^1)}{(e_1-e_3)^2}}{u-e_1} + \frac{\frac{(Z_1^1).(Z_2^1)}{2} - \frac{(Z_1^2).(Z_1^1)}{e_1-e_3}}{(u-e_1)^2} \\
&\quad + \frac{(Z_1^1).(Z_1^2)}{(u-e_1)^3} + \frac{(Z_1^1).(Z_1^1)}{(u-e_1)^4} - \frac{\frac{(Z_1^2).(Z_2^1)}{e_1-e_3} - \frac{(Z_1^1).(Z_2^1)}{(e_1-e_3)^2}}{u-e_3} + \frac{(Z_1^2).(Z_1^2)}{(u-e_3)^2}\,.
\end{aligned}
$$

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