CONFORMAL SYMMETRIES AND GENERALIZED RECURRENCES FOR HEAT AND SCHRÖDINGER EQUATIONS IN ONE SPATIAL DIMENSION

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ABSTRACT. A **conformal symmetry** for the heat or time dependent Schrödinger equation $(\partial_t - H)\Psi = 0$ is a linear differential operator L such that $L\Psi$ is a solution of the equation for every solution Ψ. In essence L is a time-dependent constant of the motion. For equations in one spatial dimension it is possible to compute, at least formally, all conformal symmetries, even those which are infinite-order differential operators. An important class of such symmetries coincides with generalized recurrence relations which map the solution space of the time-independent equation $H\phi = E\phi$ to the solution space of the equation $H\theta = (E + \kappa(E))\theta$ for an arbitrary analytic function κ. This symmetry approach provides a significant simplification in the analysis of raising and lowering operators for eigenfunctions of the Schrödinger operator.

1. Introduction.

Let $H = \Delta + V(\mathbf{x})$ be a Hamiltonian operator in n spatial dimensions. A **conformal symmetry** of the equation $(\partial_t - H)\Psi = 0$ is a linear differential operator L in the variables $\mathbf{x}$, t such that $L\Psi$ satisfies the equation whenever Ψ satisfies it. If $[L, \partial_t - H] = 0$, i.e., if L commutes with $\partial_t - H$, it is an **ordinary symmetry** operator. Clearly an ordinary symmetry is a conformal symmetry, but there are many conformal symmetries that are not ordinary. General conformal symmetry differential operators which are of first or second order have proven to be important for deriving special symmetry-adapted solutions of the heat or time-dependent Schrödinger equations, [Blumen and Cole (1974), Olver (1986), Ovsjannikow (1982)], for mapping one of these equations to another, [Kalnins and Miller (1985), (1987a), Niederer (1973), Rosencrans (1976)], and for characterizing solutions of these equations via separation of variables, [Kalnins (1986), Kalnins and Miller (1987b), Miller (1977), Reid (1986), Shapovalov (1976), (1980a), (1981)]. However, there has been very little study of higher (even infinite) order differential conformal symmetry operators or appreciation of their connection to differential recurrence relations for eigenfunctions of the Hamiltonian operator.

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In this paper we consider the very special case of one spatial dimension, where it is possible at least in principle, to determine all conformal symmetries, including those of infinite order. In §2 we relate our problem to classical Lagrangian and Hamiltonian mechanics, and work out the (almost trivial) problem of computing the classical conformal symmetries in one spatial dimension and determining their significance. In §3 we work out the quantum (or linear) analog and completely characterize those conformal symmetries that correspond to generalized recurrences for the heat or Schrödinger operator. In §4 we give very simple derivations of recurrences for two physically important types of potentials: Morse and Pöschl-Teller. We indicate how our method permits determination of the spectra corresponding to these potentials through a simple Lie algebraic procedure which has some advantages over the factorization method. The characterization of the general conformal symmetry for the heat or time-dependent Schrödinger equation is accomplished in §5. Finally in §6, for completeness we review the computation of all potentials which admit first-order conformal differential symmetry operators [Boyer 1975].

2. Classical conformal symmetries. Consider the time-dependent Hamilton-Jacobi equation

$$u_t + \frac{1}{2}u_x^2 + V(x) = 0. \tag{2.1}$$

(Recall the connection between a solution u of (2.1) and Lagrangian mechanics: the total time derivative of u with respect to t is the Lagrangian of the associated mechanical system.) A **(conformal) symmetry** of this equation is a function $L(x,t,u_x,u_t)$ such that

$$\{L, u_t + H\} = R(x,t,u_x,u_t)(u_t + H) \tag{2.2}$$

for some analytic function R where $H(u_x,x) = \frac{1}{2}u_x^2 + V(x)$, i.e., such that $\{L, u_t + H\}$ vanishes whenever the function $u(x,t)$ satisfies (2.1). (We assume that L and R are analytic in u_x, u_t about $u_x = u_t = 0$.) Here $\{.,.\}$ is the Poisson bracket

$$\{f(x,t,u_x,u_t), g(x,t,u_x,u_t)\} = \partial_x f \partial_{u_x} g + \partial_t f \partial_{u_t} g - \partial_x g \partial_{u_x} f - \partial_t g \partial_{u_t} f. \tag{2.3}$$

If $R \equiv 0$, L is an ordinary symmetry of (2.1). For uses of conformal and ordinary symmetries in the study of the time-dependent and -independent Hamilton-Jacobi equation see [Benenti and Francaviglia (1980), Boyer, Kalnins and Miller (1978), Eisenhart (1961), Kalnins and Miller (1987b), Shapovalov (1980b), Woodhouse (1975)]

Since we will be evaluating L only on solutions of (2.1), we can (for solutions) express any even power of u_x as a polynomial in u_t and $V(x)$, and can assume without loss of generality that L takes the form

$$L = a(x,t,u_t)u_x + b(x,t,u_t). \tag{2.4}$$

Substituting this expression into (2.2) we obtain the necessary and sufficient conditions

$$\begin{aligned} a_x &= R, \qquad b_x = -a_t \\ b_t &= (V + u_t)a_x + V'a. \end{aligned} \tag{2.5}$$

The integrability condition for these equations is

$$a_{tt} = -2(V + u_t)a_{xx} - 3V'a_x - V''a. \tag{2.6}$$

To understand the significance of L consider the Hamiltonian system

$$\dot{p} = -H_x, \qquad \dot{x} = H_p \tag{2.7}$$

where $H(p, x) = \frac{1}{2}p^2 + V(x)$. A function $L(x, t, p, -\lambda)$ is a **constant of the motion** for the system (2.7) if

$$\frac{dL}{dt} \equiv L_t + \dot{x}L_x + \dot{p}L_p = 0 \tag{2.8}$$

for solutions of (2.7). Here $\lambda \equiv H(p, x)$ is the **energy**, itself a constant of the motion. Assuming L is of the form $L = a(x, t, -\lambda)p + b(x, t, -\lambda)$ we see that L is a constant of the motion iff a and b satisfy the conditions (2.5), (2.6) with $u_t \equiv -\lambda$.

If L is a constant of the motion with $\partial_t L \neq 0$, then by eliminating p from the equations $L = \mu$, $H = \lambda$ we obtain an implicit equation for the path: $f(x, t) = c$. This is equivalent to the usual solution

$$w \equiv \int \frac{dx}{\sqrt{\lambda - V(x)}} = \sqrt{2}t.$$

Indeed, the general solution of (2.6), with $u_t = -\lambda$, is

$$a = (\lambda - V)^{-\frac{1}{2}}[F(w + \sqrt{2}t) + G(w - \sqrt{2}t)]$$

where F and G are arbitrary functions and $dw(x)/dx = (\lambda - V)^{-\frac{1}{2}}$. It follows that

$$b = -\sqrt{2}F(w + \sqrt{2}t) + \sqrt{2}G(w - \sqrt{2}t) + \kappa$$

where κ is a constant. Thus, since $p = \sqrt{2}(\lambda - V)^{\frac{1}{2}}$, we have

$$L = 2\sqrt{2}G(w - \sqrt{2}t).$$

3. Conformal symmetries and recurrence operators. Now we consider the quantum analog of the classical setting in §2. The basic equation is the heat or time-dependent Schrödinger equation

$$(\partial_t - \partial_{xx} - V(x))\Psi(x,t) = 0. \tag{3.1}$$

Here V and Ψ are complex analytic functions of the complex variables x, t. (No reality or self-adjointness conditions are imposed in our formal treatment. Furthermore, for convenience we adopt a different normalization for (3.1) than for (2.1).) Let H be the operator $H = \partial_{xx} + V(x)$. Formally, we say that the linear operator L, acting on the solution space of (3.1), is a **(conformal) symmetry** if

$$[\partial_t - H, L] \equiv \partial_t L - [H, L] = R(\partial_t - H) \tag{3.2}$$

for some linear operator R. Here $[A, B] = AB - BA$. (We think of L and R as differential operators in t and x, possibly of infinite order.) If $R \equiv 0$ then L is an ordinary symmetry operator for (3.1).

Let

$$U(t) = \exp(tH) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H^n.$$

Then $U(t)^{-1} = U(-t)$ and $\dot{U}(t) = HU(t) = U(t)H$. Setting $L(t) = U(t)X(t)U(t)^{-1}$ and substituting into (3.2), we find

$$\begin{aligned} \partial_t L - [H, L] &= HUXU^{-1} + U\dot{X}U^{-1} \\ &\quad - UXU^{-1}H - HUXU^{-1} + UXU^{-1}H \\ &= U\dot{X}U^{-1} = R(\partial_t - H), \end{aligned} \tag{3.3}$$

so

$$\dot{X}(t) = U^{-1}R(\partial_t - H)U = U^{-1}(t)R(t)U(t)\partial_t \tag{3.4}$$

and

$$X(t) = \left\{ \int_0^t U(s)^{-1} R(s) U(s) ds \right\} \partial_t + L(0). \tag{3.5}$$

This implies (formally) that

$$\begin{aligned} L(t) = &\int_0^t \exp(t-s)H \cdot R(s) \cdot \exp(s-t)H \cdot ds \ (\partial_t - H) \\ &+ \exp(tH) \cdot L(0) \cdot \exp(-tH). \end{aligned} \tag{3.6}$$

Thus $L(t)$ is determined by $R(t)$ and the initial condition $L(t)|_{t=0} = L(0)$. The operator $L(t)$ maps solutions Ψ of (3.1) into solutions:

$$(\partial_t - H)L(t)\Psi = L(\partial_t - H)\Psi + R(\partial_t - H)\Psi = 0, \tag{3.7}$$

see [Levine (1985), Miller (1977)].

Here $L(t)$ can be considered as a **constant of the motion**: Suppose $\Theta(x,t)$, $\Psi(x,t)$ are solutions of (3.1) and

$$(\Theta, \Psi)_t \equiv \int_C \Theta(x,-t)\Psi(x,t)dx \tag{3.8}$$

where C is a contour in the complex x-plane such that all boundary terms vanish in the integration-by-parts formulas to follow. Then

$$\begin{aligned} \frac{d}{dt}(\Theta, L\Psi)_t &= -(\dot{\Theta}, L\Psi) + (\Theta, \dot{L}\Psi) + (\Theta, L\dot{\Psi}) \\ &= -(H\Theta, L\Psi) + (\Theta, [H,L]\Psi) + (\Theta, LH\Psi) \\ &= 0 \end{aligned} \tag{3.9}$$

so $(\Theta, L\Psi)_t$ is a constant.

We will only consider the action of $L(t)$ on the solution space of (3.1). Then each term ∂_x^2 in the expansion

$$L(t) = \sum_{n,m=0}^{\infty} \ell(x,t)_{n,m}\partial_x^n\partial_t^m$$

can be replaced successively by $\partial_t - V(x)$, if at each stage the terms in the expansion are reordered so that the derivative terms act directly on the solution space of (3.1). Thus $L(t)$ can be placed in the **canonical form**

$$L(t) = a(x,t,\lambda)\partial_x + b(x,t,\lambda). \tag{3.10}$$

where $\lambda \equiv \partial_t$ and a, b are analytic in (x,t,λ) in a neighborhood of $(x_0, 0, 0)$. Here we consider

$$a(x,t,\lambda) = \sum_{m=0}^{\infty} a_m(x,t)\partial_t^m$$

with a similar interpretation for b. The action of $L(t)$ on constant energy solutions (E is a constant)

$$\Psi(x,t) = e^{Et}\phi(x), \qquad H\phi = E\phi \tag{3.11}$$

can easily be made rigorous:

$$L\Psi = e^{Et}\{a(x,t,E)\partial_x + b(x,t,E)\}\phi(x). \tag{3.12}$$

Furthermore, if the solution Ψ of (3.1) can be represented in the form

$$\Psi(x,t) = \int_{C'} e^{Et}\phi(x,E)\, dE \tag{3.13}$$

where C' is an appropriate contour in the complex E-plane we can interpret L as

$$L(t)\Psi = \int_{C'} e^{Et}\{a(x,t,E)\partial_x + b(x,t,E)\}\phi(x,E)\, dE. \tag{3.14}$$

Now let us determine the conditions on a and b so that $L(t)$, (3.10), is a symmetry of (3.1). Substituting (3.10) into (3.2) and equating coefficients of ∂_{xx}, ∂_x and the constant terms on both sides of the resulting expression we find the conditions

$$\begin{aligned} &i)\ R = 2a_x \\ &ii)\ a_t - a_{xx} - 2b_x = 0 \\ &iii)\ b_t - b_{xx} + aV_{xx} = R(\lambda - V(x)). \end{aligned}$$

Substituting i) into iii) we obtain the conditions

$$\begin{aligned} b_x &= \frac{1}{2}a_t - \frac{1}{2}a_{xx} \\ b_t &= \frac{1}{2}a_{tx} - \frac{1}{2}a_{xxx} + 2a_x\lambda - 2a_xV - aV_x. \end{aligned} \tag{3.15}$$

The integrability condition for these equations is

$$a_{tt} - 2a_{xxt} + a_{xxxx} + 4a_{xx}(V - \lambda) + 6a_xV_x + 2aV_{xx} = 0. \tag{3.16}$$

Theorem 1. *Condition (3.16) is necessary and sufficient for $L(t)$, expression (3.10), to be a symmetry.*

Proof. If a satisfies (3.16) then b is determined uniquely by (3.15) (up to a trivial added term which may only depend on λ).

It is not difficult to find all solutions of (3.16) which are of the form

$$a = \exp(t\kappa(\lambda))f(x, \lambda). \tag{3.17}$$

In this case (3.16) reduces to a 4th-order ordinary differential equation:

$$f_{xxxx} + (4V - 4\lambda - 2\kappa)f_{xx} + 6V_xf_x + (2V_{xx} + \kappa^2)f = 0. \tag{3.18}$$

Corollary 1. *Suppose $L(t) = \exp(t\kappa(\lambda))[f(x,\lambda)\partial_x + g(x,\lambda)]$ is a conformal symmetry of (3.1). Then the operator $L'(t) = \exp(t\kappa'(\lambda'))[f'(x,\lambda')\partial_x + g'(x,\lambda')]$ where*

$$\begin{aligned} \kappa' &= -\kappa, & \lambda' &= \lambda + \kappa \\ f'(x,\lambda') &= f(x, \lambda+\kappa), & g'(x,\lambda') &= -g(x,\lambda+\kappa) - f_x(x,\lambda+\kappa) \end{aligned}$$

is also a conformal symmetry.

Proof. This follows from the invariance of (3.18) under the transformation $\kappa' = -\kappa$, $\lambda' = \lambda + \kappa$.

This result shows that each symmetry of the form (3.17) has an "adjoint symmetry". See [Boyer and Miller 1974] for an application of a similar idea for symmetries of the Schrödinger equation in two spatial dimensions.

To solve equation (3.18) we make use of the following fact [Whittaker 1929]: Let $u(x)$ and $v(x)$ be solutions of the differential equations $u'' - p(x)u = 0$, $v'' - q(x)v = 0$. Then $y(x) = u(x)v(x)$ satisfies

$$\begin{aligned} (p-q)y'''' - (p'-q')y''' - 2(p^2-q^2)y'' + (-pp' + qq' + 5p'q - 5pq')y' \\ + (p'^2 - q'^2 - (p-q)(p''+q'') + (p-q)^3)y = 0. \end{aligned} \tag{3.19}$$

Now consider the equations

$$\begin{aligned} &i)\ u'' + Vu = (\lambda + \kappa)u \\ &ii)\ v'' + Vv = \lambda v, \end{aligned} \tag{3.20}$$

i.e., $p = \lambda + \kappa - V$, $q = \lambda - V$. Then (3.19) reduces to (3.18) with $f = uv$.

Theorem 2. *Every symmetry operator* $L = a\partial_x + b$ *such that* $a = \exp(t\kappa(\lambda))f(x,\lambda)$, $b = \exp(t\kappa(\lambda))g(x,\lambda)$ *with* $\kappa \neq 0$ *is of the form*

$$(3.21)\qquad \begin{aligned} f(x,\lambda) &= \sum_{i,j=1}^{2} c_{ij}(\lambda)u_i(x,\lambda)v_j(x,\lambda) \\ g(x,\lambda) &= -\sum_{i,j=1}^{2} c_{ij}(\lambda)u_i(x,\lambda)v_{jx}(x,\lambda) \end{aligned}$$

where $\{u_1, u_2\}$ *is a basis of solutions for (3.20i) and* $\{v_1, v_2\}$ *is a basis of solutions for (3.20ii).*

We ignore the trivial constant $c(\lambda)$ which can always be added to b. Note that the operators L characterized by Theorem 2 are recursion operators for the solutions ϕ of $H\phi = E\phi$. Indeed setting $\Psi = e^{tE}\phi$ we have $L\Psi = e^{t(E+\kappa(E))}\theta$ where $H\theta = (E+\kappa(E))\theta$. Furthermore, it is always possible to map a particular solution $\phi_1(x,E)$ of $H\phi = E\phi$ to a particular solution $\theta_1(x, E+\kappa(E))$ of $H\theta = (E+\kappa(E))\theta$. Indeed, let $\{\phi_1, \phi_2\}$ be a basis for the solution space of $H\phi = E\phi$ with ϕ_2 normalized so that the Wronskian

$$\phi_2(x,E)\phi_1'(x,E) - \phi_2'(x,E)\phi_1(x,E) \equiv 1.$$

Then

$$L = \exp(t\kappa(\lambda))\theta_1(x, \lambda+\kappa(\lambda))\left(\phi_2(x,\lambda)\partial_x - \phi_2'(x,\lambda)\right)$$

is a conformal symmetry operator such that $L\Psi_1 = e^{t(E+\kappa(E))}\theta_1$, $L\Psi_2 = 0$.

The case $\kappa(\lambda) \equiv 0$ needs special treatment. Then (3.18) can be written as

$$\frac{d}{dx}\{f_{xxx} - 4f_x\lambda + 4f_xV + 2fV_x\} = 0$$

so

$$(3.22)\qquad f_{xxx} - 4f_x\lambda + 4f_xV + 2fV_x = c(\lambda)$$

where c is independent of x. It follows that the symmetry operator $L = a\partial_x + b$ must have the form

$$(3.23)\qquad a = f(x,\lambda), \qquad b = -\frac{1}{2}c(\lambda)t - \frac{1}{2}f_x(x,\lambda).$$

This operator will be independent of t provided $c(\lambda) \equiv 0$. We now determine the general solution of (3.22) in this case. Here we make use of the fact that if $u(x)$ and $v(x)$ are solutions of the same differential equation $w'' - p(x)w = 0$ then $y(x) = u(x)v(x)$ satisfies

$$(3.24)\qquad y''' - 4py' - 2p'y = 0.$$

If $p = -V + \lambda$ then (3.24) reduces to (3.22) with $c(\lambda) \equiv 0$.

Theorem 3. *Every symmetry operator $L = a\partial_x + b$ such that $a_t = b_t = 0$ is of the form $a = f(x, \lambda)$, $b = -\frac{1}{2}f_x(x, \lambda)$ with*

$$f(x,\lambda) = c_1(\lambda)v_1^2(x,\lambda) + c_2(\lambda)v_1(x,\lambda)v_2(x,\lambda) + c_3(\lambda)v_2^2(x,\lambda) \tag{3.25}$$

where $\{v_1, v_2\}$ is a basis of solutions for (3.20ii).

The operators L characterized by Theorem 3 are symmetry operators for the solutions ϕ of $H\phi = E\phi$: setting $\Psi = e^{tE}\phi$ we have $L\Psi = e^{tE}\theta$ where $H\theta = E\theta$.

Symmetries of the form described by Theorems 2 and 3 can be composed to yield a product of the same form.

Corollary 2. *Let $u_1(x,\lambda)$, $v_1(x,\lambda)$, $\kappa_1(\lambda)$, resp. $u_2(x,\lambda)$, $v_2(x,\lambda)$, $\kappa_2(\lambda)$, satisfy equations (3.20). Then the operators*

$$L_i = e^{t\kappa_i(\lambda)}(u_i v_i \partial_x - u_i(\partial_x v_i)), \quad i = 1, 2$$

are symmetries of (3.1) and $L_1 L_2$ is a symmetry of the same form:

$$L_1 L_2 = e^{t[\kappa_1(\kappa_2(\lambda)+\lambda)+\kappa_2(\lambda)]}(uv\partial_x - u(\partial_x v)),$$

where

$$\begin{aligned} v(x,\lambda) &= v_2(x,\lambda), \\ u(x,\lambda) &= c(\lambda)u_1(x, \kappa_2(\lambda) + \lambda) \end{aligned}$$

and

$$c(\lambda) = v_1(x, \kappa_2(\lambda) + \lambda)\partial_x u_2(x,\lambda) - u_2(x,\lambda)\partial_x v_1(x, \kappa_2(\lambda) + \lambda),$$

independent of x.

Although Theorems 2 and 3 exhibit clearly the structure of these generalized recurrences, other methods for actually computing the recurrences may be simpler. It may not be necessary to obtain solutions of the second-order equations (3.20) to determine a generalized recurrence. For example, let us compute the possible solutions of (3.18) of the form $f(x,\lambda) = \exp(\mu(\lambda)x)$. We find

$$2V_{xx} + 6\mu V_x + 4\mu^2 V = 4\lambda\mu^2 - (\mu^2 - \kappa)^2.$$

Since the potential $V(x)$ cannot depend on λ, it follows that (for nonconstant V) μ is independent of λ and

$$\begin{aligned} V(x) &= c_1 e^{-\mu x} + c_2 e^{-2\mu x} + c_3 \\ \kappa(\lambda) &= \mu^2 \pm 2\mu\sqrt{\lambda - c_3}. \end{aligned} \tag{3.26}$$

The symmetry operator is

$$L = \exp(t\kappa(\lambda))[e^{\mu x}\partial_x + (\frac{\kappa}{2\mu} - \frac{\mu}{2})e^{\mu x} - \kappa^{-1}\mu c_1].$$

A second (trivial) example will help to fix some of the ideas. The equation

$$(\partial_t - \partial_{xx})\Psi = 0$$

admits the symmetry $L = \partial_x$. Since L is independent of t it maps solutions of

$$H\theta \equiv \theta_{xx} = E\theta$$

to solutions. A basis of solutions for (3.27) is $v_1(x) = e^{\sqrt{E}x}$, $v_2(x) = e^{-\sqrt{E}x}$. Using Theorem 3 we can define a symmetry operator $\tilde{L} = f\partial_x - \frac{1}{2}f_x$ with $f = v_1 v_2$. Clearly $\tilde{L} = L$.

The next instructive example was, in essence, given in [Whittaker 1929] but without the heat equation setting presented here. Consider the equation

$$(\partial_t - \partial_{xx} - e^{2x})\Psi = 0$$

with $H = \partial_{xx} + e^{2x}$. The equation

$$H\theta(x) = E\theta(x)$$

has a basis of solutions consisting of the Bessel and Neumann functions, $\theta(x) = J_\nu(e^x)$, $Y_\nu(e^x)$, $E = \nu^2$, [Erdélyi 1953 (p. 30)]. It is easy to see that the operator $L = \exp\{(2\nu+1)t\}(-e^{-x}\partial_x + \nu e^{-x})$ is a symmetry of (3.28) which induces a recurrence mapping solutions θ of $H\theta = \nu^2\theta$ to solutions ϕ of $H\phi = (\nu+1)^2\phi$. Indeed this follows from the well known recurrence relations

$$e^{-x}(-\frac{d}{dx} + \nu)C_\nu(e^x) = C_{\nu+1}(e^x), \qquad C_\nu = J_\nu, Y_\nu,$$

see [Erdélyi 1953 (p. 11), Watson 1966]. (Note: These relations arise naturally by considering the symmetries of the Helmholtz equation in 2-dimensional Euclidean space and are somewhat less natural in the context of (3.28). Furthermore, L has a branch point at $E = 0$ and cannot be expanded in a power series in ∂_t as in interpretation (3.10). However, interpretations (3.12) and (3.13) are still meaningful and Theorem 2 remains valid.) Applying Theorem 2 with $V = e^{2x}$, $\lambda = \nu^2$, $\kappa = 2\nu + 1$ we choose $f = -\frac{\pi}{2}J_{\nu+1}(e^x)Y_\nu(e^x) + \frac{\pi}{2}Y_{\nu+1}(e^x)J_\nu(e^x)$. Then $a = -e^{-x}e^{t(2\nu+1)}$ in view of the well known identity [Erdélyi 1953 (p. 79)]. It follows from (3.21) that $b = \nu e^{-x}e^{t(2\nu+1)}$ in agreement with our expression for L. We see that Wronskian determinant and similar identities for solutions of second order ordinary differential equations are closely related to Theorems 2 and 3.

In [Whittaker 1929] there is derived a recurrence formula for Mathieu functions which can be interpreted very simply in terms of the equation

$$(\partial_t - \partial_{xx} - 16q\cos 2x)\Psi(x,t) = 0$$

and Theorem 2.

4. Examples of recurrences in quantum theory. The recurrences described by Theorem 2 have the property that they map the solution space of $H\phi = E\phi$ into the solution space of $H\theta = (E + \kappa(E))\theta$ for any E. However, in many physical systems modelled by these equations, particularly the Schrödinger equation, one imposes boundary conditions and square integrability on the solution space and obtains a discrete spectrum $\{E_n, n = 0, 1, \cdots\}$, [Landau and Lifshitz 1958] or a continuous spectrum (or both). Typically the eigenspace corresponding to E_n is one-dimensional with single basis function ϕ_n. In this case one is interested in differential recurrence operators L_n^+, L_n^- such that $L_n^{\pm}\phi_n = c_{n\pm1}\phi_{n\pm1}$ for constants $c_{n\pm1}$, [Infeld and Hull 1951, Levine 1988]. As is implied by the comments following Theorem 2, one can always find *special* conformal symmetry operators L which have these properties when retricted to the spectrum $\{E_n\}$. (A *general* conformal symmetry of the type given in Theorem 2 will not have these properties since it won't necessarily map eigenfunctions to eigenfunctions, i.e., it won't necessarily preserve integrability or a particular set of boundary conditions.) Such a special recurrence operator has an expression in terms of Theorem 2 which is of the same complexity as the eigenfunctions $\{\phi_n\}$. However, it is sometimes the case that we can directly find elementary solutions of (3.18) with the appropriate properties. This approach to obtaining these recurrences is much simpler than earlier approaches known to the authors and has the advantage that it demonstrates explicitly that the recurrence operators are restrictions of conformal symmetry operators for the time-dependent equation. Furthermore, this approach can sometimes be used to obtain a direct algebraic derivation of the spectrum, similar to the factorization method [Inoui 1948, Infeld and Hull 1951, Miller 1968]. In distinction to the factorization method, however, the raising and lowering operators are never artificial constructs but are always interpretable as symmetries of the underlying time-dependent Schrödinger equation.

Here we consider two instructive examples of recurrences for the discrete spectra of Schrödinger operators. We will continue to use the formal analytic equation (3.1). See the cited references for the reality and boundary conditions, and explicit eigenfunctions for each example.

Example 1. The Morse potential.

Here,

$$V(x) = D[2\exp(-\mu x) - \exp(-2\mu x)] \tag{4.1}$$

where D, μ are positive parameters. As follows from (3.26), the Schrödinger equation admits the generalized symmetries

$$L^{\pm} = e^{t\kappa^{\pm}}\left[e^{\mu x}\partial_x + (\frac{\kappa^{\pm}}{2\mu} - \frac{\mu}{2})e^{\mu x} - \frac{2\mu D}{\kappa^{\pm}}\right].$$

where $\kappa^{\pm}(\lambda) = \mu^2 \pm 2\mu\sqrt{\lambda}$. Since $\kappa^+(\lambda) + \kappa^-(\lambda + \kappa^+(\lambda)) = 0$ for $\sqrt{\lambda} + \mu \geq 0$ and $\kappa^-(\lambda) + \kappa^+(\lambda + \kappa^-(\lambda)) = 0$ for $\sqrt{\lambda} - \mu \geq 0$, we can easily verify that

$$\begin{aligned} L^+L^- &\sim D - \frac{4D^2}{(\mu - 2\sqrt{\lambda})^2}, \\ L^-L^+ &\sim D - \frac{4D^2}{(\mu + 2\sqrt{\lambda})^2} \end{aligned} \tag{4.2}$$

where equality is meant in the sense that the two sides agree when applied to a solution of (3.1). Thus we have the commutation relations

$$\begin{aligned} [L^+, L^-] &\sim \frac{-32D^2\mu\sqrt{\lambda}}{(\mu^2 - 4\lambda)^2} \\ [\lambda, L^+] &\sim (\mu^2 + 2\mu\sqrt{\lambda})L^+ \\ [\lambda, L^-] &\sim (\mu^2 - 2\mu\sqrt{\lambda})L^-, \end{aligned} \tag{4.3}$$

an analog of the commutation relations for the Lie algebra $s\ell(2)$. Even though L^+, L^-, λ don't generate a finite dimensional Lie algebra, one can easily mimic the (weight vector) approach to the representation theory of $s\ell(2)$ to determine the irreducible representations of the associative algebra generated by these three operators. Note the "Casimir operator" C acting on the solution space of (3.1):

$$C = L^+L^- + \frac{4D^2}{(\mu - 2\sqrt{\lambda})^2} \sim L^-L^+ + \frac{4D^2}{(\mu + 2\sqrt{\lambda})^2} \sim D.$$

To see how close is the analogy to the representation theory of $s\ell(2)$ we look for a "lowest weight vector" Ψ_0 for λ, i.e., a nonzero solution of the equations

$$\begin{aligned} (\lambda - H)\Psi_0 &= 0 \\ \lambda\Psi_0 &= E_0\Psi_0 \\ L^-\Psi_0 &= 0. \end{aligned} \tag{4.4}$$

Evaluating $C\Psi_0 = D\Psi_0$ we find $4D^2/(\mu - 2\sqrt{E_0})^2 = D$ or

$$E_0 = \mu^2(\frac{\sqrt{D}}{\mu} - \frac{1}{2})^2,$$

assuming $\mu - 2\sqrt{E_0} \geq 0$. Recursively applying L^+ to get $\Psi_n = (L^+)^n\Psi_0$ with eigenvalues E_n satisfying the recurrence $E_{n+1} = E_n + \kappa^+(E_n) = (\mu + \sqrt{E_n})^2$, we find the spectrum [Morse 1929, Ter Haar 1946, Levine 1988]

$$E_n = \mu^2[\frac{\sqrt{D}}{\mu} - (n + \frac{1}{2})]^2, \qquad n = 0, 1, 2, \cdots .$$

This algebraic approach can be made rigorous and shown to generate the discrete and continuous spectrum associated with the Morse potential.

Example 2. Pöschl-Teller potentials.

This is a class of potentials of the form

$$V(x) = \frac{A'}{\cos^2 \frac{x}{2}} + \frac{B'}{\sin^2 \frac{x}{2}} = \frac{A}{1 + \cos x} + \frac{B}{1 - \cos x}. \tag{4.5}$$

The most general potential admitting a symmetry such that $f(x,t) = \sin x$ is (4.5), to within an additive constant. For such a potential we have the generalized symmetries

$$L^{\pm} = e^{t\kappa^{\pm}}[\sin x\partial_x - (\frac{\kappa^{\pm} + 1}{2})\cos x + \frac{B - A}{\kappa^{\pm}}]$$

where $\kappa^{\pm}(\lambda) = -1 \pm 2\sqrt{-\lambda}$. Since $\kappa^+(\lambda) + \kappa^-(\lambda + \kappa^+(\lambda)) = 0$ for $\sqrt{-\lambda} - 1 \geq 0$ and $\kappa^-(\lambda) + \kappa^+(\lambda + \kappa^-(\lambda)) = 0$ for $\sqrt{-\lambda} + 1 \geq 0$, we can easily verify that

$$\begin{aligned} L^+L^- &\sim -A - B + \lambda - \sqrt{-\lambda} - \frac{(B - A)^2}{(1 + 2\sqrt{-\lambda})^2} \\ L^-L^+ &\sim -A - B + \lambda + \sqrt{-\lambda} - \frac{(B - A)^2}{(-1 + 2\sqrt{-\lambda})^2}. \end{aligned} \tag{4.6}$$

Thus we have the commutation relations

$$
\begin{aligned}
[L^+, L^-] &\sim -2\sqrt{-\lambda} + \frac{8(B-A)^2\sqrt{-\lambda}}{(1+4\lambda)^2}, \\
[\lambda, L^+] &\sim (-1+2\sqrt{-\lambda})L^+ \\
[\lambda, L^-] &\sim (-1-2\sqrt{-\lambda})L^-,
\end{aligned}
\tag{4.7}
$$

another analog of the commutation relations for the Lie algebra $s\ell(2)$. We have the "Casimir operator" C acting on the solution space of (3.1):

$$C = L^+L^- + \sqrt{-\lambda} - \lambda + \frac{(B-A)^2}{(1+2\sqrt{-\lambda})^2} \sim L^-L^+ - \sqrt{-\lambda} - \lambda + \frac{(B-A)^2}{(1-2\sqrt{-\lambda})^2} \sim -A-B.$$

To see how close is the analogy to the representation theory of $s\ell(2)$ we look for a "highest weight vector" Ψ_0 for λ, i.e., a nonzero solution of the equations

$$
\begin{aligned}
(\lambda - H)\Psi_0 &= 0 \\
\lambda\Psi_0 &= E_0\Psi_0 \\
L^+\Psi_0 &= 0.
\end{aligned}
\tag{4.8}
$$

Thus, from $C\Psi_0 = -(A+B)\Psi_0$ we have

$$A + B = E_0 + \sqrt{-E_0} - \frac{(B-A)^2}{(-1+2\sqrt{-E_0})^2}.$$

Setting $A = (1-4\alpha^2)/8$, $B = (1-4\beta^2)/8$ and $\sqrt{-E_0} = (1+u)/2$, we find the solutions $u = \pm(\alpha+\beta)$, $\pm(\alpha-\beta)$. Since A and B depend on α^2, β^2, without loss of generality we can assume $\sqrt{-E_0} = (\alpha+\beta+1)/2$. Recursively applying L^- to get $\Psi_n = (L^-)^n\Psi_0$ with eigenvalues E_n satisfying the recurrence $E_{n+1} = E_n + \kappa^-(E_n) = E_n - 1 - 2\sqrt{-E_n}$, we find the spectrum [Pöschl and Teller 1933, Alhassid et al. 1983]

$$E_n = -(n + \frac{\alpha+\beta+1}{2})^2, \qquad n = 0, 1, 2, \cdots.$$

Again this algebraic approach can be made rigorous and shown to generate the discrete spectra associated with the Pöschl-Teller potentials.
NOTE: The two preceding examples correspond to type F and type E factorizations, respectively, [Infeld and Hull 1951], and interpret these factorizations as symmetries. Indeed the solutions f of (3.18) with f independent of λ correspond to all the type B,D,E and F factorizations and special cases of type A and C factorizations. The general type A and C factorizations are not associated with the time-dependent Schrödinger equation.

5. General symmetries. Now we look for more general symmetry operators than those characterized by Theorems 2 and 3. Consider the equations

$$
\begin{aligned}
&i)\ u_{xx}(x,t) + (V(x) - \lambda)u(x,t) = u_t(x,t) \\
&ii)\ v_{xx}(x) + (V(x) - \lambda)v(x) = 0.
\end{aligned}
\tag{5.1}
$$

It is straightforward to verify that if $y(x,t) = u(x,t)v(x)$ where u and v satisfy (5.1) then y satisfies the equation

$$y_{tt} - 2y_{xxt} + y_{xxxx} - 4y_{xx}\lambda + 4y_{xx}V + 6y_xV_x + 2yV_{xx} = 0, \tag{5.2}$$

identical to (3.16).

Theorem 4. *Let $w(x,t)$ and $v(x)$ be solutions of the equations*

$$\begin{aligned} &i)\ w_{xx} + V(x)w = w_t \\ &ii)\ v_{xx} + V(x)v = \lambda v. \end{aligned} \tag{5.3}$$

Then $L = a\partial_x + b = wve^{-\lambda t}\partial_x - wv_x e^{-\lambda t}$ is a symmetry operator for (2.1).

Of course, by taking sums or integrals of the "separated symmetries" (5.3) we can obtain still more general symmetries of (2.1).

The **initial value problem** for symmetries of (2.1) is the problem of determining a symmetry $L(t) = a(x,t,\lambda)\partial_x + b(x,t,\lambda)$ such that $L(0) = A(x,\lambda)\partial_x + B(x,\lambda)$ where A and B are given functions. In principle, Theorem 4 reduces the initial value problem for symmetries to the Cauchy problem for (3.1). (Here our trearment is formal. A precise, rigorous treatment would require the imposition of reality conditions and other restrictions on the solution space of (3.1). Two of the most important cases are:

(1) The heat equation. Here x is real and $t \geq 0$.
(2) The Schrödinger equation. Here $x = iy$ and $t = is$ where y and s are real.

We shall not enter into these details.)

Let $w_\ell(x,t,\lambda)$, $\ell = 1,2$, be solutions of (5.3i) and $\{v_1(x,\lambda), v_2(x,\lambda)\}$ be a basis of solutions of (5.3ii). Then by Theorem 4,

$$L(t) = e^{-\lambda t}\{(w_1v_1 + w_2v_2)\partial_x - (w_1v_{1x} + w_2v_{2x})\}$$

is a symmetry of (3.1) with initial value

$$\begin{aligned} w_1(x,0,\lambda)v_1(x,\lambda) + w_2(x,0,\lambda)v_2(x,\lambda) &= A(x,\lambda) \\ w_1(x,0,\lambda)v_{1x}(x,\lambda) + w_2(x,0,\lambda)v_{2x}(x,\lambda) &= -B(x,\lambda). \end{aligned}$$

Since $\{v_1, v_2\}$ is a basis the Wronskian $v_1v_{2x} - v_2v_{1x}$ is nonzero, so $w_1(x,0,\lambda)$ and $w_2(x,0,\lambda)$ can be uniquely determined by A and B. It follows that unique solution of the Cauchy problem for (3.1) leads to unique solution of the initial value problem for symmetries.

6. First order symmetries. Most of the examples presented thus far are symmetry operators L of infinite order in ∂_t. It is of interest to determine the potentials $V(x)$ such that (2.1) admits finite order differential operators L. The answer for symmetries first order in ∂_x and ∂_t is well known [Boyer 1975] but Theorem 1 yields a particularly simple derivation. L is at most first order in ∂_t iff $\partial_\lambda a = 0$. It follows from (3.16) that $a_{xx} = 0$. Thus $a = f_1(t)x + f_2(t)$ and

$$\dddot{f}_1 x + \dddot{f}_2 + 2f_1(3V_x + xV_{xx}) + 2f_2V_{xx} = 0. \tag{6.1}$$

(1) ($f_1 \neq 0$): Then there exist constants α, β, γ such that $(x+\alpha)V_{xx} + 3V_x + \beta x + \gamma = 0$. Replacing $x+\alpha$ by x and relabling γ we can write this expression as

$$\frac{d}{dx}(xV_x) + 2V_x + \beta x + \gamma = 0$$

with solution

$$V = \frac{\beta}{8}x^2 + \frac{\gamma}{3}x + \frac{\delta}{2} + \frac{\rho}{x^2}.$$

Substituting back into (6.1) we find the following possibilities for V (up to a constant multiple):

$$x^2, \quad x, \quad 1, \quad \frac{1}{x^2} + \delta + \beta x^2. \tag{6.2}$$

here we have replaced $x + \alpha$ by x if necessary.

(2) $(f_1 \equiv 0, f_2 \neq 0)$: There exists a constant α such that $V_{xx} = \alpha$. Thus $V = \frac{1}{2}\alpha x^2 + \beta x + \gamma$ and we obtain the normal forms for V: x^2, x, 1.

It is straightforward to show that each of the potentials (6.2) actually admits first order symmetry operators [Boyer (1975)].

Theorem 5. *The potentials such that equation (2.1) admits a nontrivial first order symmetry operator are those given by (6.2), up to a scalar multiple and translation in x.*

A much more difficult problem is : What are all potentials which admit symmetry operators of finite order in ∂_t?

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