Superintegrability on two dimensional complex Euclidean space

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Abstract

In this work we examine the basis functions for those quantum mechanical systems on two dimensional complex Euclidean space, with non zero potential, that admit separation in at least two coordinate systems. We present all of these cases from a unified point of view. In particular, all of the polynomial special functions that arise via variable separation have their essential features expressed in terms of their zeros.

1 Introduction

In a previous work [1] we have examined the basis functions for those classical and quantum mechanical systems in real two dimensional Euclidean space which admit a solution via separation of variables in more than one coordinate system. (We recall that the notion of a *superintegrable system* relates to a potential for which the solution via separation of variables is possible in more than one coordinate system, [2, 3, 4]. The interest in these systems results from the fact that they have extra integrals of the motion that can be used to break the degeneracy of the energy eigenstates.) The methodical search for such potentials was initiated by Smorodinsky, Winternitz et. al. in two and three dimensions, [5, 6, 7]. (For some related work, see [8, 9, 10, 11].) In this paper we look at the same problem for the case of complex Euclidean space. In so doing we show that there are a number of additional potentials possible other than those of the real case alone. These additional potentials are complexifications of potentials for the Klein Gordan equation on real flat space with indefinite metric.

Our basic problem is to solve Schrödinger's equation

$$H\Psi = -\frac{1}{2}\Delta\Psi + V(x,y)\Psi = -\frac{1}{2}(\partial_x^2 + \partial_y^2)\Psi + V(x,y)\Psi = E\Psi \qquad (1)$$

via an ansatz of the form $\Psi = \psi_1(u_1)\psi_2(u_2)$ in some suitable coordinate system $x = f(u_1, u_2), y = g(u_1, u_2)$. We are interested in potentials $V(x, y)$ for which this is possible for two or more distinct coordinate systems. For the case of real Euclidean space there are exactly four such potentials and a detailed analysis of their finite solutions or bound states has been given previously, e.g., [1, 8, 10]. We summarize the results of previous work that we need in order to give a readable presentation. In addition to the four potentials possible for real Euclidean space we find in the complex case three more potentials. For one of the potentials there is the possibility of finite polynomial solutions. We give the details of the eigenfunction solutions and the structure of the quadratic algebra associated with each system. The quadratic algebra yields essential information for the expansion of the basis functions of one separable system in terms of the basis functions for a second system.

There are seven possible potentials. The four potentials already treated in the real Euclidean case are.

$$V_1(x,y) = \frac{1}{2}\omega^2(x^2+y^2) + \frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}, \tag{2}$$

$$V_2(x,y) = \frac{1}{2}\omega^2(4x^2+y^2) + \frac{k_2^2 - \frac{1}{4}}{y^2}, \tag{3}$$

$$V_3(x,y) = \frac{-\alpha}{\sqrt{x^2+y^2}} + \frac{k_1^2 - \frac{1}{4}}{\sqrt{x^2+y^2}[\sqrt{x^2+y^2}+x]} + \frac{k_2^2 - \frac{1}{4}}{\sqrt{x^2+y^2}[\sqrt{x^2+y^2}-x]} \tag{4}$$

$$V_4(x,y) = \frac{-\alpha}{\sqrt{x^2+y^2}} + B_1\frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}} + B_2\frac{\sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{x^2+y^2}}. \tag{5}$$

These potentials are thoroughly treated in our previous article. We now turn our attention to the *new* potentials that occur in the complex case. The possible orthogonal separable coordinate systems in complex Euclidean space, together with their defining symmetry operators for the zero potential case [13], are listed in Table 1 and the remaining potentials are listed in Table 2. Here,

$$M = y\partial_x - x\partial_y, \quad P_1 = \partial_x, \quad P_2 = \partial_y.$$

Table 1.

Separable Coordinate Systems on Two-Dimensional Complex Euclidean Space

Coordinate System	Integrals of Motion	Coordinates
I. Polar $r > 0, \theta \in [0, \pi]$	$I = M^2$	$x = r\cos\theta$ $y = r\sin\theta$
II. Cartesian, $x, y \in R$	$I = P_1^2$	x, y
III. Light Cone $x, y \in R$	$I = (P_1 + iP_2)^2$	$\hat{x} = x - iy$ $\hat{y} = x + iy$
IV. Elliptic $\alpha > 0, \beta \in [0, 2\pi]$	$I = M^2 - a^2P_2^2$, $a \neq 0$	$x = a\cosh\alpha\cos\beta$ $y = a\sinh\alpha\sin\beta$
V. Parabolic $\xi, \eta > 0$	$I = \{M, P_2\}$	$x = \frac{1}{2}(\xi^2 - \eta^2)$ $y = \xi\eta$
VI. Hyperbolic	$I = M^2 + (P_1 + iP_2)^2$	$x = \frac{(u^2+u^2v^2+v^2)}{2vu}$ $y = \frac{i(u^2-u^2v^2+v^2)}{2vu}$
VII. Semi-Hyperbolic	$I = \{M, P_1 + iP_2\}$ $+(P_1 - iP_2)^2$	$x = -\frac{1}{4}(w-z)^2 + \frac{1}{2}(w+z)$ $iy = -\frac{1}{4}(w-z)^2 - \frac{1}{2}(w+z)$

Table 2. Complex Superintegrable Systems

Potential $V(\vec{x})$	Coordinate System
$V_5 = \frac{B}{2}(x - iy)$	Cartesian Light Cone Semi-Hyperbolic
$V_6 = \frac{\alpha}{2\sqrt{x+iy}}$	Parabolic Semi-Hyperbolic Light Cone
$V_7 = \frac{1}{2}\left[\frac{\alpha(x^2+y^2)}{(x+iy)^4} + \frac{\beta}{(x+iy)^2} + \gamma(x^2+y^2)\right]$	Polar Hyperbolic

2 Potential V_5

The Schrödinger equation for potential V_5 admits separable solutions in three distinct coordinate systems. In cartesian coordinates this equation has the form

$$-\frac{1}{2}(\partial_x^2 + \partial_y^2)\Psi + \frac{1}{2}B(x - iy)\Psi = E\Psi. \tag{6}$$

It admits separable solutions $\Psi = \psi(x)\varphi(y)$ which satisfy

$$\partial_x^2\psi(x) + Bx\psi(x) = (2E + \lambda)\psi(x), \qquad \partial_y^2\varphi(y) - iyB\varphi(y) = (2E - \lambda)\varphi(y).$$

The solutions obtained in this way are products of Airy functions [14], typically

$$\psi(x) = Ai\Big((-B)^{1/3}x - \frac{2E+\lambda}{B}\Big), \quad \varphi(y) = Ai\Big((iB)^{1/3}y - \frac{2E-\lambda}{B}\Big).$$

The operator K, whose eigenvalue is the separation constant λ, is defined by

$$K\Psi = (-\partial_x^2 + \partial_y^2)\Psi + B(x+iy)\Psi.$$

In light cone coordinates Schrödinger's equation has the form

$$-4\partial_{\hat{x}}\partial_{\hat{y}}\Psi + B\hat{y}\Psi = 2E\Psi \tag{7}$$

which admits a solution of the form $\Psi = h(\hat{x})k(\hat{y})$ where

$$\partial_{\hat{x}}h(\hat{x}) = \rho h(\hat{x}), \qquad -4\rho\partial_{\hat{y}}k(\hat{y}) + (B\hat{y} - 2E)k(\hat{y}) = 0.$$

These equations have the solution

$$\Psi = \exp(\rho\hat{x} + \frac{E}{2\rho}\hat{y} - \frac{B}{8\rho}\hat{y}^2).$$

The solutions are clearly eigenfunctions of the operator $N = \partial_x + i\partial_y = 2\partial_{\hat{x}}$.

Finally, if we write the Schrödinger equation in semi-hyperbolic coordinates it becomes

$$(\partial_w^2 - \partial_z^2)\Psi + B(w^2 - z^2)\Psi = 2E(w-z)\Psi. \tag{8}$$

The solutions of this equation can be obtained via the separation of variables ansatz $\Psi = W(w)Z(z)$. The separation equations are

$$\partial_x^2 X + Bx^2 X = (2Ex + \lambda)X,$$

where $X = W, Z$ when $x = w, z$, respectively. The solutions of these equations are typically

$$X(x) = D_\nu\Big(\sqrt{2}(-B)^{\frac{1}{4}}(x - \frac{E}{B})\Big),$$

where $\nu = (-\frac{1}{2}\sqrt{-B})(-\lambda - (\frac{E}{B})^2) - \frac{1}{2}$, and $D_\nu(z)$ is a parabolic cylinder function [14]. The operator, whose eigenvalue is the separation parameter, is

$$L = \frac{1}{z-w}(z\partial_w^2 - w\partial_z^2) + Bzw.$$

The operators K, L, N and H which define these bases satisfy the following quadratic algebra relations [15, 1] :

$$[N,K] = -\frac{1}{2}B, \quad [N,L] = -N^2 - B, \quad [K,L] = \frac{1}{2}N^3 + \{N,K\} + \frac{1}{4}BN, \tag{9}$$

$$4H^2 + N^4 + 4\{N^2, K\} - 4BL - 8K - BN^2 = 0.$$

Here, $[A,B] = AB - BA$, $\{A,B\} = AB + BA$. There are no solutions which are the analogue of bound states in this case.

3 Potential V_6

For potential V_6 the Schrödinger equation admits separable solutions in three coordinate systems.

In parabolic coordinates the Schrödinger equation has the form

$$(\partial_\xi^2 + \partial_\eta^2)\Psi - \alpha\sqrt{2}(\xi - i\eta)\Psi = 2E(\xi^2 + \eta^2)\Psi. \tag{10}$$

The separation equations are

$$\partial_\xi^2 X - \alpha\sqrt{2}\xi X = (2E\xi^2 + \lambda)X, \quad \partial_\eta^2 Y + \alpha\sqrt{2}i\eta Y = (2E\eta^2 - \lambda)Y,$$

where $\Psi = X(\xi)Y(\eta)$. Solutions to these equations are typically

$$X(\xi) = D_\mu\Big((8E)^{\frac{1}{4}}(\xi + \frac{\alpha}{\sqrt{2E}})\Big), \quad Y(\eta) = D_\kappa\Big((8E)^{\frac{1}{4}}(\xi + \frac{i\alpha}{\sqrt{2E}})\Big),$$

where

$$\mu = \frac{1}{\sqrt{8E}}(-\lambda + \frac{\alpha^2}{4E}) - \frac{1}{2}, \quad \kappa = \frac{1}{\sqrt{8E}}(\lambda - \frac{\alpha^2}{4E}) - \frac{1}{2},$$

and $D_\mu(x)$ is a parabolic cylinder function [14]. The operator whose eigenvalue is λ is

$$L = \frac{1}{\xi^2 + \eta^2}(\xi^2\partial_\xi^2 - \eta^2\partial_\eta^2) + \frac{i\sqrt{2}\alpha}{\xi + i\eta}.$$

In light cone coordinates the Schrödinger equation has the form

$$-4\partial_{\hat{x}}\partial_{\hat{y}}\Psi + \frac{\alpha}{\sqrt{\hat{y}}}\Psi = 2E\Psi, \tag{11}$$

which admits a solution $\Psi = p(\hat{x})q(\hat{y})$ where

$$\partial_{\hat{x}}p(\hat{x}) = \rho p(\hat{x}), \quad -4\rho\partial_{\hat{y}}q(\hat{y}) + (\frac{\alpha}{\sqrt{\hat{y}}} - 2E)q(\hat{y}) = 0.$$

These equations have the solution

$$\Psi = \exp(\rho\hat{x} + \frac{E}{2\rho}\hat{y} - \frac{\alpha}{2\rho}\sqrt{\hat{y}}).$$

The solutions are clearly eigenfunctions of the operator $N = \partial_x + i\partial_y = 2\partial_{\hat{x}}$.

For semi-hyperbolic coordinates the Schrödinger equation has the form

$$(\partial_w^2 - \partial_z^2)\Psi - i\alpha\sqrt{2}\Psi = 2E(w-z). \tag{12}$$

The separation equations are

$$\partial_w^2 R(w) + (-2Ew + \rho - \frac{i\alpha}{\sqrt{2}})R(w) = 0, \qquad \partial_z^2 S(z) + (-2Ez + \rho + \frac{i\alpha}{\sqrt{2}})S(z) = 0,$$

where $\Psi = R(w)S(z)$. These equations have Airy function solutions

$$Ai\Big((2E)^{\frac{1}{3}}w - \frac{\rho - i\frac{\alpha}{\sqrt{2}}}{2E}\Big), \qquad Ai\Big((2E)^{\frac{1}{3}}w - \frac{\rho + i\frac{\alpha}{\sqrt{2}}}{2E}\Big).$$

These solutions are eigenfunctions of the operator

$$M = \frac{1}{w-z}(z\partial_w^2 - w\partial_z^2 - \frac{i\alpha}{\sqrt{2}}(w+z)).$$

The basis elements L, M, T and H satisfy the quadratic algebra relations [15, 1]

$$[L, N] = H + N^2, \quad [M, N] = -\frac{1}{2}H, \quad [M, L] = \frac{1}{4}N^3 + \{N, M\} + \frac{1}{8}HN, \tag{13}$$

$$HL + 2HM + \frac{1}{8}N^2H + \{N^2, M\} + \frac{1}{8}N^4 - \frac{1}{4}\alpha^2 = 0.$$

4 Potential V_7

The Schrödinger equation for potential V_7 admits separable solutions in two coordinate systems.

In polar coordinates the equation has the form

$$(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 - \frac{\alpha}{r^2}e^{-4i\theta} - \frac{\beta}{r^2}e^{-2i\theta} - \gamma^2 r^2)\Psi = -2E\Psi. \tag{14}$$

The θ separation equation has the form

$$(\partial_\theta^2 - \alpha e^{-4i\theta} - \beta e^{-2i\theta} - \lambda)\Lambda(\theta) = 0,$$

which has solutions of the form

$$\Lambda(\theta)=e^{\frac{\alpha z}{2}}z^{-\frac{1}{2}-\frac{\beta}{4\alpha}-p}L_p^{2p+\frac{\beta}{2\alpha}-1}(-\alpha z),$$

where $z=e^{-2i\theta}$, $\lambda=-[2p+1+\frac{\beta}{2\alpha}]^2$ and $L_p^q(x)$ is a Laguerre polynomial with p an integer [14]. The corresponding separation equation in r is

$$(\partial_r^2+\frac{1}{r}\partial_r+\frac{\lambda}{r^2}-\gamma^2r^2+2E)R(r)=0,$$

which has solutions

$$R(r)=e^{-\frac{1}{2}\gamma r^2}r^{\frac{1}{2}+\frac{\beta}{4\alpha}+p}L_m^{\frac{1}{2}+\frac{\beta}{4\alpha}+p}(\gamma r^2)$$

for which $E=\gamma(2(m+p+1)+\frac{\beta}{2\alpha})$. The operator which corresponds to the eigenvalue λ is

$$M=\partial_\theta^2-\alpha e^{-4i\theta}-\beta e^{-2i\theta}.$$

For hyperbolic coordinates the Schrödinger equation has the form

$$[\frac{1}{u^2-v^2}(v\partial_v(v\partial_v)-u\partial_u(u\partial_u))+\frac{\beta}{u^2v^2}-\alpha^2\frac{u^2+v^2}{u^4v^4}+\gamma^2(u^2+v^2)]\Psi=2E\Psi. \tag{15}$$

The separation equations are

$$[t\partial_t(t\partial_t)-\frac{\alpha^2}{t^4}+\frac{\beta}{t^2}-\gamma^2t^4+2Et^2+\mu]T(t)=0,$$

where $\Psi=U(u)V(v)$ and $T=U,V$ when $t=u,v$, respectively. The operator which characterises this separation is

$$L=\frac{1}{u^2-v^2}(v^2u\partial_u(u\partial_u)-u^2v\partial_v(v\partial_v))+\alpha^2(\frac{1}{u^2v^2}+\frac{1}{u^4}+\frac{1}{v^4})-\beta(\frac{1}{u^2}+\frac{1}{v^2})-\gamma^2u^2v^2.$$

To obtain polynomial eigenfunctions we can look for solutions of the form [16]

$$\Psi=\exp\Big(\frac{\alpha}{2}[\frac{1}{u^2}+\frac{1}{v^2}]-\frac{1}{2}\gamma(u^2+v^2)\Big)\Pi_{i=1}^n(\frac{1}{u^2}-\theta_i)(\frac{1}{v^2}-\theta_i)(uv)^{1+n+\frac{\beta}{2\alpha}},$$

where the zeros satisfy

$$2\sum_{j\neq i}\frac{\theta_i^2}{\theta_i-\theta_j}-(\frac{\beta}{2\alpha}+2n+2)\theta_i+\gamma+\alpha^2\theta_i^2=0,\quad i=1,\cdots,n.$$

The energy eigenvalues and the separation constant are

$$E = \gamma(2(n+1) + \frac{\beta}{2\alpha}), \quad \lambda = -[2n+1+\frac{\beta}{2\alpha}]^2 - 2\alpha\gamma + 4\gamma\sum_{\kappa=1}^{n}\frac{1}{\theta_\kappa}.$$

The relations of the corresponding quadratic algebra are

$$[R,L] = -8L^2 - 24M^2 - 16\{L,M\} - 8\beta H - 32\alpha\gamma, \tag{16}$$

$$[R,M] = 16M^2 - 8M - 8L + 8\beta H + 8\{L,M\}, \tag{17}$$

$$\begin{aligned} R^2 \;=\; & 16M^2 + \frac{16}{3}(L^2M + ML^2 + LML) - \frac{176}{3}L^2 + \frac{80}{3}M^2 \qquad (18) \\ & -16\alpha H^2 + 16\beta(HL + HM) - 16\{L,M\} + (64\gamma\alpha - \frac{256}{3})M \\ & -\frac{256}{3}L + \frac{128}{3}\beta H + \frac{64}{3}\gamma\alpha + 16\gamma\beta^2, \end{aligned}$$

where $R = [L,M]$.

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