Teukolsky-Starobinsky identities for arbitrary spin

by

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Abstract

The Teukolsky-Starobinsky identities are proven for arbitrary spin s. A pair of covariant equations are given which admit solutions in terms of Teukolsky functions for general s . The method of proof is shown to extend to the general class of spacetimes considered by Torres del Castillo.

Introduction

Gravitational and electromagnetic perturbations in Kerr geometry are known to be intimately connected to Teukolsky functions [1]. This came about because of investigations by Teukolsky who showed that in the Newman-Penrose formalism [2] separable solutions were possible for certain Maxwell and Weyl scalars in Kerr geometry. The resulting separable solutions are known as Teukolsky functions. In addition to the problem of gravitational and electromagnetic perturbations these functions reappear when the neutrino [3] and Rarita-Schwinger [4] equations are solved in a background of Kerr geometry. These functions satisfy what are known as the Teukolsky-Starobinsky identities. In this work we prove these identities for any spin s. This result is established relatively easily. One of the difficulties with the Kerr metric is that for $s > 2$ these functions do not appear to come from any covariant equation. We rectify this situation by introducing covariant equations which admit Teukolsky functions for general s as their solutions. No claim is made that these equations have physical significance. Finally we note that the method of proof applies to the more general class of spacetimes studied by Torres del Castillo [5] who proved these results for $s \leq 2$.

The Teukolsky-Starobinsky Identities

We consistently use in this article the spinor notation of Penrose and Rindler [6] and the null tetrad formalism of Chandrasekhar [7]. Specifically we restrict ourselves to the Kinnersley null tetrad of vectors with components

$$\begin{aligned} l^a &= \frac{1}{\sqrt{2}\,\Delta}(r^2+a^2, \Delta, 0, a) \\ n^a &= \frac{1}{\sqrt{2}\,\tilde{\rho}\tilde{\rho}^*}(r^2+a^2, -\Delta, 0, a) \\ m^a &= \frac{1}{\sqrt{2}\,\tilde{\rho}}(ia\sin\theta, 0, 1, i\csc\theta) \\ \overline{m}^a &= \frac{1}{\sqrt{2}\,\tilde{\rho}^*}(-ia\sin\theta, 0, 1, -i\csc\theta) \end{aligned} \tag{1}$$

where

$$\Delta = r^2 - 2Mr + a^2 \quad , \quad \rho^2 = r^2 + a^2\cos^2\theta \quad \text{and} \quad \tilde{\rho} = r + ia\cos\theta \tag{2}$$

The Kerr solution has the line element

$$\begin{aligned} ds^2 = (1 - \frac{2Mr}{\tilde{\rho}\tilde{\rho}^*})\,dt^2 - \frac{\tilde{\rho}\tilde{\rho}^*}{\Delta}\,dr^2 - \tilde{\rho}\tilde{\rho}^*\,d\theta^2 - \big[\,(r^2+a^2) + \frac{2a^2Mr\sin^2\theta}{\tilde{\rho}\tilde{\rho}^*}\,\big]\sin^2\theta\,d\phi^2 \\ + \frac{2aMr\sin^2\theta}{\tilde{\rho}\tilde{\rho}^*}\,dt\,d\phi \end{aligned} \tag{3}$$

The differential operators $\mathcal{D}_n$, $\mathcal{D}_n^\dagger$, $\mathcal{L}_n$ and $\mathcal{L}_n^\dagger$ are defined as

$$\begin{aligned} \mathcal{D}_n &= \partial_r + \frac{iK}{\Delta} + 2n\frac{(r-M)}{\Delta} \\ \mathcal{D}_n^\dagger &= \partial_r - \frac{iK}{\Delta} + 2n\frac{(r-M)}{\Delta} \\ \mathcal{L}_n &= \partial_\theta + Q + n\cot\theta \\ \mathcal{L}_n^\dagger &= \partial_\theta - Q + n\cot\theta \end{aligned} \tag{4}$$

where

$$K = (r^2+a^2)\sigma + am \quad \text{and} \quad Q = a\sigma\sin\theta + m\csc\theta \tag{5}$$

Teukolsky functions P_{+s} and P_{-s} in the variable r satisfy

$$\begin{aligned} (\Delta\mathcal{D}_{1-s}\mathcal{D}_0^\dagger - 2(2|s|-1)i\sigma r)P_{+s} &= \lambda P_{+s} \\ (\Delta\mathcal{D}_{1-s}^\dagger\mathcal{D}_0 + 2(2|s|-1)i\sigma r)P_{-s} &= \lambda P_{-s} \end{aligned} \tag{6}$$

where $s=\frac{1}{2},1,\ldots$. The first result proven is

THEOREM 1

If $s=\frac{1}{2},1,\ldots$ then

$$\Delta^s\mathcal{D}_0^{2s}\big[\,\Delta\mathcal{D}_{1-s}^\dagger\mathcal{D}_0+2(2s-1)i\sigma r\,\big]=\big[\,\Delta\mathcal{D}_{1-s}\mathcal{D}_0^\dagger-2(2s-1)i\sigma r\,\big]\Delta^s\mathcal{D}_0^{2s} \tag{7}$$

PROOF

By induction on s. Noting that for $s=\frac{1}{2}$

$$\Delta^{\frac{1}{2}}\mathcal{D}_0(\Delta\mathcal{D}_{\frac{1}{2}}^\dagger\mathcal{D}_0)=(\Delta\mathcal{D}_{\frac{1}{2}}\mathcal{D}_0^\dagger)\Delta^{\frac{1}{2}}\mathcal{D}_0 \tag{8}$$

If we now assume the result is true for a given s then

$$\begin{aligned}&\Delta^{s+\frac{1}{2}}\mathcal{D}_0^{2(s+\frac{1}{2})}\big[\,\Delta\mathcal{D}_{1-(s+\frac{1}{2})}^\dagger\mathcal{D}_0+2(\,2(s+\frac{1}{2})-1)i\sigma r\,\big]\\ =&\Delta^{s+\frac{1}{2}}\mathcal{D}_0^{2s+1}\big[\,\Delta(\mathcal{D}_{1-s}^\dagger-\frac{r-M}{\Delta})\mathcal{D}_0+4si\sigma r\,\big]\\ =&\Delta^{\frac{1}{2}}\mathcal{D}_{-s}\big[\,\Delta^s\mathcal{D}_0^{2s}(\Delta\mathcal{D}_{1-s}^\dagger\mathcal{D}_0+2(2s-1)i\sigma r)+\Delta^s\mathcal{D}_0^{2s}(2i\sigma r-(r-M)\mathcal{D}_0)\,\big]\end{aligned} \tag{9}$$

and

$$\begin{aligned}&\big[\,\Delta\mathcal{D}_{1-(s+\frac{1}{2})}\mathcal{D}_0^\dagger-2(\,2(s+\frac{1}{2})-1)i\sigma r\,\big]\Delta^{s+\frac{1}{2}}\mathcal{D}_0^{2(s+\frac{1}{2})}\\ =&\Delta^{\frac{1}{2}}(\Delta\mathcal{D}_{1-s}\mathcal{D}_{\frac{1}{2}}^\dagger-4si\sigma r)\mathcal{D}_{-s}\Delta^s\mathcal{D}_0^{2s}\\ =&\Delta^{\frac{1}{2}}\mathcal{D}_{-s}(\Delta\mathcal{D}_{\frac{1}{2}}^\dagger\mathcal{D}_{-s}-4si\sigma r)\Delta^s\mathcal{D}_0^{2s}+4si\sigma\Delta^{s+\frac{1}{2}}\mathcal{D}_0^{2s}\\ =&\Delta^{\frac{1}{2}}\mathcal{D}_{-s}\big[\,\Delta(\mathcal{D}_{1-s}+(2s-1)\frac{r-M}{\Delta}-\frac{2iK}{\Delta})(\mathcal{D}_0^\dagger-2s\frac{r-M}{\Delta}+\frac{2iK}{\Delta})-4si\sigma r\,\big]\Delta^s\mathcal{D}_0^{2s}\\ &+4si\sigma\Delta^{s+\frac{1}{2}}\mathcal{D}_0^{2s}\\ =&\Delta^{\frac{1}{2}}\mathcal{D}_{-s}\big[\,(\Delta\mathcal{D}_{1-s}\mathcal{D}_0^\dagger-2(2s-1)i\sigma r)\Delta^s\mathcal{D}_0^{2s}+(2i\sigma r-2s-(r-M)\mathcal{D}_{-s})\Delta^s\mathcal{D}_0^{2s}\\ &+4si\sigma\Delta^s\mathcal{D}_0^{2s-1}\,\big]\end{aligned} \tag{10}$$

Note that

$$\begin{aligned}&\Delta^s\mathcal{D}_0^{2s}(2i\sigma r-(r-M)\mathcal{D}_0)\\ =&(2i\sigma r-(r-M)\mathcal{D}_{-s})\Delta^s\mathcal{D}_0^{2s}+\Delta^s(4si\sigma\mathcal{D}_0^{2s-1}-2s\mathcal{D}_0^{2s})\\ =&(2i\sigma r-2s-(r-M)\mathcal{D}_{-s})\Delta^s\mathcal{D}_0^{2s}+4si\sigma\Delta^s\mathcal{D}_0^{2s-1}\end{aligned} \tag{11}$$

Thus subtracting (10) from (9) and making use of (11) have

$$
\begin{aligned}
&\Delta^{s+\frac{1}{2}}\mathcal{D}_0^{2(s+\frac{1}{2})}\left[\,\Delta\mathcal{D}^{\dagger}_{1-(s+\frac{1}{2})}\mathcal{D}_0+2(\,2(s+\frac{1}{2})-1)i\sigma r\,\right]\\
&\quad-\left[\,\Delta\mathcal{D}_{1-(s+\frac{1}{2})}\mathcal{D}^{\dagger}_0-2(\,2(s+\frac{1}{2})-1)i\sigma r\,\right]\Delta^{s+\frac{1}{2}}\mathcal{D}_0^{2(s+\frac{1}{2})}\\
=&\Delta^{\frac{1}{2}}\mathcal{D}_{-s}\left[\,\Delta^s\mathcal{D}_0^{2s}(\Delta\mathcal{D}^{\dagger}_{1-s}\mathcal{D}_0+2(2s-1)i\sigma r)-(\Delta\mathcal{D}_{1-s}\mathcal{D}^{\dagger}_0-2(2s-1)i\sigma r)\Delta^s\mathcal{D}_0^{2s}\,\right]
\end{aligned}
\tag{12}
$$

A direct consequence of this result is that $\Delta\mathcal{D}_0^{2s}P_{-s}$ is a solution of the Teukolsky equation for P_{+s}. Similarly it may be proven that

$$
\Delta^s\mathcal{D}_0^{\dagger\,2s}\left[\,\Delta\mathcal{D}_{1-s}\mathcal{D}^{\dagger}_0-2(2s-1)i\sigma r\,\right]=\left[\,\Delta\mathcal{D}^{\dagger}_{1-s}\mathcal{D}_0+2(2s-1)i\sigma r\,\right]\Delta^s\mathcal{D}_0^{\dagger\,2s}
\tag{13}
$$

i.e. $\Delta^s\mathcal{D}_0^{\dagger\,2s}P_{+s}$ is a solution of the Teukolsky equation for P_{-s}. By suitable choice of the relative normalization of the functions we can write the following results

$$
\begin{aligned}
\Delta^s\mathcal{D}_0^{2s}P_{-s}&=D_sP_{+s}\\
\Delta^s\mathcal{D}_0^{\dagger\,2s}P_{+s}&=D_s^*P_{-s}
\end{aligned}
\tag{14}
$$

These are the Teukolsky-Starobinsky identities known to be true for $s=\frac{1}{2},1,\frac{3}{2},2$. For the variable θ we can prove a similar result

THEOREM 2

If $s=\frac{1}{2},1,\ldots$ then

$$
\begin{aligned}
&\mathcal{L}_{1-s}\mathcal{L}_{2-s}\ldots\mathcal{L}_{s-1}\mathcal{L}_s\left[\,\mathcal{L}^{\dagger}_{1-s}\mathcal{L}_s+2(2s-1)\sigma a\cos\theta\,\right]\\
=&\left[\,\mathcal{L}_{1-s}\mathcal{L}^{\dagger}_s-2(2s-1)\sigma a\cos\theta\,\right]\mathcal{L}_{1-s}\mathcal{L}_{2-s}\ldots\mathcal{L}_{s-1}\mathcal{L}_s
\end{aligned}
\tag{15}
$$

PROOF

Again using induction we note that for $s=\frac{1}{2}$

$$
\mathcal{L}_{\frac{1}{2}}(\mathcal{L}^{\dagger}_{\frac{1}{2}}\mathcal{L}_{\frac{1}{2}})=(\mathcal{L}_{\frac{1}{2}}\mathcal{L}^{\dagger}_{\frac{1}{2}})\mathcal{L}_{\frac{1}{2}}
\tag{16}
$$

then

$$
\begin{aligned}
&\mathcal{L}_{1-(s+\frac{1}{2})}\mathcal{L}_{2-(s+\frac{1}{2})}\ldots\mathcal{L}_{(s+\frac{1}{2})-1}\mathcal{L}_{s+\frac{1}{2}}\left[\,\mathcal{L}^{\dagger}_{1-s}\mathcal{L}_s+2(\,2(s+\frac{1}{2})-1)\sigma a\cos\theta\,\right]\\
=&\frac{1}{\sqrt{\sin\theta}}\mathcal{L}_{-s}\mathcal{L}_{1-s}\ldots\mathcal{L}_{s-1}\mathcal{L}_s(\mathcal{L}^{\dagger}_s\mathcal{L}_s+4s\sigma a\cos\theta)\sqrt{\sin\theta}\\
=&\frac{1}{\sqrt{\sin\theta}}\mathcal{L}_{-s}\mathcal{L}_{1-s}\ldots\mathcal{L}_{s-1}\mathcal{L}_s\left[\,\mathcal{L}^{\dagger}_{1-s}\mathcal{L}_s+2(2s-1)\sigma a\cos\theta-\cot\theta\mathcal{L}_s+2\sigma a\cos\theta\,\right]\sqrt{\sin\theta}
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
&\left[\,\mathcal{L}_{1-(s+\frac{1}{2})}\mathcal{L}^{\dagger}_{s+\frac{1}{2}} - 2(\,2(s+\frac{1}{2})-1)\sigma a\cos\theta\,\right]\mathcal{L}_{1-(s+\frac{1}{2})}\mathcal{L}_{2-(s+\frac{1}{2})}\dots\mathcal{L}_{(s+\frac{1}{2})-1}\mathcal{L}_{(s+\frac{1}{2})}\\
=&\frac{1}{\sqrt{\sin\theta}}(\mathcal{L}_{-s}\mathcal{L}^{\dagger}_{s} - 4s\sigma a\cos\theta)\mathcal{L}_{-s}\mathcal{L}_{1-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s}\sqrt{\sin\theta}\\
=&\frac{1}{\sqrt{\sin\theta}}\mathcal{L}_{-s}(\mathcal{L}^{\dagger}_{s}\mathcal{L}_{-s} - 4s\sigma a\cos\theta)\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s}\sqrt{\sin\theta}\\
&\qquad - 4s\sigma a\sqrt{\sin\theta}\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s}\sqrt{\sin\theta}
\end{aligned}
\tag{18}
$$

Now note that we can write

$$
\begin{aligned}
&\mathcal{L}^{\dagger}_{s}\mathcal{L}_{-s} - 4s\sigma a\cos\theta\\
=&(\mathcal{L}_{1-s} + (2s-1)\cot\theta - 2Q)(\mathcal{L}^{\dagger}_{s} - 2s\cot\theta + 2Q) - 4s\sigma a\cos\theta\\
=&\mathcal{L}_{1-s}\mathcal{L}^{\dagger}_{s} - 2(2s-1)\sigma a\cos\theta - \cot\theta\mathcal{L}^{\dagger}_{s} + 2s\csc^2\theta - 2m\cot\theta\csc\theta
\end{aligned}
\tag{19}
$$

observing the identities

$$
\begin{aligned}
\mathcal{L}_{a}\mathcal{L}_{a+1}\dots\mathcal{L}_{b-1}\mathcal{L}_{b}\cos\theta =& \cos\theta\mathcal{L}_{a}\mathcal{L}_{a+1}\dots\mathcal{L}_{b-1}\mathcal{L}_{b}\\
&- (b-a+1)\sin\theta\mathcal{L}_{a+1}\mathcal{L}_{a+2}\dots\mathcal{L}_{b-1}\mathcal{L}_{b}
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
\mathcal{L}_{a}\mathcal{L}_{a+1}\dots\mathcal{L}_{b-1}\mathcal{L}_{b}\cot\theta =& \cot\theta\mathcal{L}_{a-1}\mathcal{L}_{a}\dots\mathcal{L}_{b-2}\mathcal{L}_{b-1}\\
&- (b-a+1)\mathcal{L}_{a}\mathcal{L}_{a+1}\dots\mathcal{L}_{b-2}\mathcal{L}_{b-1}
\end{aligned}
\tag{21}
$$

and noting that

$$
\begin{aligned}
&\frac{1}{\sqrt{\sin\theta}}\mathcal{L}_{-s}\Big[\,\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s}(-\cot\theta\mathcal{L}_{s} + 2\sigma a\cos\theta)\\
&- (-\cot\theta\mathcal{L}^{\dagger}_{s} + 2s\csc^2\theta - 2m\cot\theta\csc\theta)\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s}\,\Big]\sqrt{\sin\theta}\\
&+ 4s\sigma a\sqrt{\sin\theta}\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s}\sqrt{\sin\theta}\\
=&\frac{1}{\sqrt{\sin\theta}}\mathcal{L}_{-s}\Big[-\cot\theta\mathcal{L}_{-s}\mathcal{L}_{1-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s} + 2s\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s}\\
&+ 2\sigma a\cos\theta\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s} - 4s\sigma a\sin\theta\mathcal{L}_{2-s}\mathcal{L}_{3-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s}\\
&- (-\cot\theta\mathcal{L}_{s} + 2Q\cot\theta + 2s\csc^2\theta - 2m\cot\theta\csc\theta)\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s}\,\Big]\sqrt{\sin\theta}\\
&+ 4s\sigma a\sqrt{\sin\theta}\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_{s}\sqrt{\sin\theta}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{\sqrt{\sin\theta}}\mathcal{L}_{-s}\big[-\cot\theta\mathcal{L}_{-s}+2s+2\sigma a\cos\theta\\
&+\cot\theta\mathcal{L}_s-2Q\cot\theta-2s\csc^2\theta+2m\cot\theta\csc\theta\big]\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_s\sqrt{\sin\theta}\\
=&0
\end{aligned}
\tag{22}
$$

we have established that

$$
\begin{aligned}
&\mathcal{L}_{1-(s+\frac{1}{2})}\mathcal{L}_{2-(s+\frac{1}{2})}\dots\mathcal{L}_{(s+\frac{1}{2})-1}\mathcal{L}_{s+\frac{1}{2}}\big[\mathcal{L}^\dagger_{1-s}\mathcal{L}_s+2(2(s+\frac{1}{2})-1)\sigma a\cos\theta\big]\\
&-\big[\mathcal{L}_{1-(s+\frac{1}{2})}\mathcal{L}^\dagger_{s+\frac{1}{2}}-2(2(s+\frac{1}{2})-1)\sigma a\cos\theta\big]\mathcal{L}_{1-(s+\frac{1}{2})}\mathcal{L}_{2-(s+\frac{1}{2})}\dots\mathcal{L}_{(s+\frac{1}{2})-1}\mathcal{L}_{(s+\frac{1}{2})}\\
=&\frac{1}{\sqrt{\sin\theta}}\mathcal{L}_{-s}\big[\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_s(\mathcal{L}^\dagger_{1-s}\mathcal{L}_s+2(2s-1)\sigma a\cos\theta)\\
&-(\mathcal{L}_{1-s}\mathcal{L}^\dagger_s-2(2s-1)\sigma a\cos\theta)\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_s\big]\sqrt{\sin\theta}
\end{aligned}
\tag{23}
$$

and the result is proven. In this case the Teukolsky equations are defined as

$$
\begin{aligned}
(\mathcal{L}^\dagger_{1-s}\mathcal{L}_s+2(2s-1)\sigma a\cos\theta)S_{+s}&=-\lambda S_{+s}\\
(\mathcal{L}_{1-s}\mathcal{L}^\dagger_s-2(2s-1)\sigma a\cos\theta)S_{-s}&=-\lambda S_{-s}
\end{aligned}
\tag{24}
$$

Consequently Theorem 2 tells us that we can upon suitable renormalization, find a constant $\mathcal{C}_s$ such that

$$
\mathcal{L}_{1-s}\mathcal{L}_{2-s}\dots\mathcal{L}_{s-1}\mathcal{L}_sS_{+s}=\mathcal{C}_sS_{-s}
\tag{25}
$$

Similarly one may prove the identity

$$
\begin{aligned}
&\mathcal{L}^\dagger_{1-s}\mathcal{L}^\dagger_{2-s}\dots\mathcal{L}^\dagger_{s-1}\mathcal{L}^\dagger_s\big[\mathcal{L}_{1-s}\mathcal{L}^\dagger_s-2(2s-1)\sigma a\cos\theta\big]\\
=&\big[\mathcal{L}^\dagger_{1-s}\mathcal{L}_s+2(2s-1)\sigma a\cos\theta\big]\mathcal{L}^\dagger_{1-s}\mathcal{L}^\dagger_{2-s}\dots\mathcal{L}^\dagger_{s-1}\mathcal{L}^\dagger_s
\end{aligned}
\tag{26}
$$

from which we can write

$$
\mathcal{L}^\dagger_{1-s}\mathcal{L}^\dagger_{2-s}\dots\mathcal{L}^\dagger_{s-1}\mathcal{L}^\dagger_sS_{-s}=\mathcal{C}_sS_{+s}
\tag{27}
$$

These are the Teukolsky-Starobinsky identities known to be true for $s=\frac{1}{2},1,\frac{3}{2},2$. The question we now ask is what if any significance do the Teukolsky functions have for general s. Before giving a covariant equation which works for general s let us recapitulate how things work in the case of the Rarita-Schwinger field. The Rarita-Schwinger equation written in spinor notation is

$$
\nabla^{AA'}F_{AB}{}^{B'}=0
\tag{28}
$$

where $F_{ABB'} = F_{(AB)B'}$ We can constuct a coupled system of equations as follows. Let

$$h_{ABC} = \nabla_{(AA'}F_{BC)}{}^{A'} \tag{29}$$

then h_{ABC} satisfies a first order equation as follows

$$\begin{aligned} \nabla^{AA'}h_{ABC} &= \nabla^{AA'}\nabla_{(AB'}F_{BC)}{}^{B'} \\ &= \frac{1}{3}\nabla^{AA'}\nabla_{AB'}F_{BC}{}^{B'} + \frac{2}{3}\nabla^{AA'}\nabla_{(BB'}F_{C)A}{}^{B'} \end{aligned} \tag{30}$$

Using the Rarita-Schwinger equation and the symmetry in the indices B and C we write

$$\begin{aligned} \frac{1}{3}\nabla^{AA'}\nabla_{AB'}F_{BC}{}^{B'} &= \frac{1}{3}\nabla^{AA'}\nabla_{BB'}F_{AC}{}^{B'} + \frac{1}{3}\nabla_{B}{}^{A'}\nabla_{AB'}F^{A}{}_{C}{}^{B'} \\ &= \frac{1}{3}\nabla^{AA'}\nabla_{(BB'}F_{C)A}{}^{B'} \end{aligned} \tag{31}$$

Consequently (30) becomes

$$\begin{aligned} \nabla^{AA'}h_{ABC} &= \nabla^{AA'}\nabla_{(BB'}F_{C)A}{}^{B'} \\ &= \nabla_{(BB'}\nabla^{AA'}F_{C)A}{}^{B'} + [\nabla^{AA'}, \nabla_{(BB'}]F_{C)A}{}^{B'} \\ &= [\nabla^{AA'}, \nabla_{(BB'}]F_{C)A}{}^{B'} \\ &= -\varepsilon^{A'}{}_{B'}\Psi^{A}{}_{BC}{}^{M}F_{MA}{}^{B'} \\ &= \Psi_{BC}{}^{AM}F_{AM}{}^{A'} \end{aligned} \tag{32}$$

The pair of equations (29) , (32) when written in Newman-Penrose notation become

$$(D - \rho)h_{111} - (\delta^* + 3\pi + \alpha)h_{110} = \Psi_2 F_{110'} \tag{33}$$

$$(D - 2\rho)h_{110} - (\delta^* + 2\pi - \alpha)h_{100} = -\Psi_2 F_{100'} \tag{34}$$

$$(D - 3\rho)h_{100} - (\delta^* + \pi - 3\alpha)h_{000} = \Psi_2 F_{000'} \tag{35}$$

$$(\delta + 3\beta - \tau)h_{111} - (\Delta + \gamma + 3\mu)h_{110} = \Psi_2 F_{111'} \tag{36}$$

$$(\delta + \beta - 2\tau)h_{110} - (\Delta - \gamma + 2\mu)h_{100} = -\Psi_2 F_{101'} \tag{37}$$

$$(\delta - \beta - 3\tau)h_{100} - (\Delta + \mu - 3\gamma)h_{000} = \Psi_2 F_{001'} \tag{38}$$

and

$$(D - \rho^*)F_{00}{}^{0'} + (\delta - \alpha^* - 2\beta + \pi^*)F_{00}{}^{1'} = h_{000} \tag{39}$$

$$(\delta^* + 2\alpha + \beta^* - \tau^*)F_{11}{}^{0'} + (\Delta + 2\gamma - \gamma^* + \mu^*)F_{11}{}^{1'} = h_{111} \tag{40}$$

$$2[(D-\rho^*)F_{10}{}^{0'}+(\delta-\alpha^*)F_{10}{}^{1'}-\pi F_{00}{}^{0'}-\mu F_{00}{}^{1'}]$$
$$+[(\delta^*+\beta^*-2\alpha-\tau^*)F_{00}{}^{0'}+(\boldsymbol{\Delta}+\mu^*-\gamma^*-2\gamma)F_{00}{}^{1'}+2\tau F_{10}{}^{1'}+2\rho F_{10}{}^{0'}]=3h_{100} \quad (41)$$
$$2[(\delta^*+\beta-\tau^*)F_{10}{}^{0'}+(\boldsymbol{\Delta}-\gamma^*+\mu^*)F_{10}{}^{1'}+\rho F_{11}{}^{0'}+\tau F_{11}{}^{1'}]$$
$$+[(D-\rho^*)F_{11}{}^{0'}+(\delta-\alpha^*+2\beta+\pi^*)F_{11}{}^{1'}-2\mu F_{01}{}^{1'}-2\pi F_{10}{}^{0'}]=3h_{110} \quad (42)$$

Considering equations (35) , (38) and (39) and putting

$$\begin{aligned} h_{100} &= \frac{1}{\sqrt{2}\,\tilde{\rho}^*}H_1 \quad , \quad h_{000}=H_0 \\ F_{000'} &= \frac{1}{\sqrt{2}\,\tilde{\rho}^*}G_{000'} \quad , \quad F_{001'}=\frac{1}{2\rho^2}G_{001'} \end{aligned} \quad (43)$$

we obtain

$$\begin{aligned} &(\mathcal{L}^\dagger_{-\frac{1}{2}}+\frac{2ia\sin\theta}{\tilde{\rho}^*})H_1+\Delta(\mathcal{D}^\dagger_{\frac{3}{2}}-\frac{2}{\tilde{\rho}^*})H_0=\Psi_2 G_{001'} \\ &(\mathcal{D}_0+\frac{2}{\tilde{\rho}^*})H_1-(\mathcal{L}_{\frac{3}{2}}-\frac{2ia\sin\theta}{\tilde{\rho}^*})H_0=\Psi_2 G_{000'} \\ &(\mathcal{D}_0-\frac{1}{\tilde{\rho}^*})G_{001'}-(\mathcal{L}^\dagger_{-\frac{1}{2}}-\frac{ia\sin\theta}{\tilde{\rho}^*})G_{000'}=2\rho^2 H_0 \end{aligned} \quad (44)$$

These equations imply that H_0 satisfies the separable equation

$$(\Delta\mathcal{D}_1\mathcal{D}^\dagger_{\frac{3}{2}}+\mathcal{L}^\dagger_{-\frac{1}{2}}\mathcal{L}_{\frac{3}{2}}-4i\sigma\tilde{\rho})H_0=0 \quad (45)$$

admitting solutions $H_0=P_{+\frac{3}{2}}S_{+\frac{3}{2}}$. Similarly if the equations (33) , (36) and (40) are considered then putting

$$\begin{aligned} h_{110} &= \frac{1}{2\tilde{\rho}^{*2}}H_2 \quad , \quad h_{111}=\frac{1}{2\sqrt{2}\,\tilde{\rho}^{*3}}H_3 \quad , \\ F_{110'} &= \frac{1}{2\sqrt{2}\,\tilde{\rho}^{*3}}G_{110'} \quad , \quad F_{111'}=\frac{1}{4\rho^2\tilde{\rho}^{*2}}G_{111'} \end{aligned} \quad (46)$$

we obtain

$$\begin{aligned} &(\mathcal{D}_0-\frac{2}{\tilde{\rho}^*})H_3-(\mathcal{L}_{-\frac{1}{2}}+\frac{2ia\sin\theta}{\tilde{\rho}^*})H_2=\Psi_2 G_{110'} \\ &(\mathcal{L}^\dagger_{\frac{3}{2}}-\frac{2ia\sin\theta}{\tilde{\rho}^*})H_3+\Delta(\mathcal{D}^\dagger_{\frac{3}{2}}+\frac{2}{\tilde{\rho}^*})H_2=\Psi_2 G_{111'} \\ &(\mathcal{L}_{-\frac{1}{2}}-\frac{ia\sin\theta}{\tilde{\rho}^*})G_{111'}+\Delta(\mathcal{L}^\dagger_{-\frac{1}{2}}-\frac{1}{\tilde{\rho}^*})G_{110'}=2\rho^2 H_3 \end{aligned} \quad (47)$$

The function H_3 satisfies the separable equation

$$(\Delta\mathcal{D}^\dagger_{-\frac{1}{2}}\mathcal{D}_0+\mathcal{L}_{-\frac{1}{2}}\mathcal{L}^\dagger_{\frac{3}{2}}+4i\sigma\tilde{\rho})H_3=0 \quad (48)$$

admitting solutions $H_3 = \Delta^{\frac{3}{2}} P_{-\frac{3}{2}} S_{-\frac{3}{2}}$. Two solutions to the equations (44) and (47) can be found

$$
\begin{aligned}
&(1) \quad h_{000} = P_{+\frac{3}{2}} S_{+\frac{3}{2}} \\
&\quad\ \ F_{000'} = \frac{-1}{\sqrt{2}\,\tilde{\rho}^* \Psi_2} (\mathcal{L}_{\frac{3}{2}} - \frac{2ia\sin\theta}{\tilde{\rho}^*}) P_{+\frac{3}{2}} S_{+\frac{3}{2}} \\
&\quad\ \ F_{001'} = \frac{1}{2\rho^2 \Psi_2} \Delta (\mathcal{D}^\dagger_{\frac{3}{2}} - \frac{2}{\tilde{\rho}^*}) P_{+\frac{3}{2}} S_{+\frac{3}{2}} \\
\text{and} \quad &(2) \quad h_{111} = \frac{1}{2\sqrt{2}\,\tilde{\rho}^{*3}} \Delta^{\frac{3}{2}} P_{-\frac{3}{2}} S_{-\frac{3}{2}} \\
&\quad\ \ F_{110'} = \frac{1}{2\sqrt{2}\,\tilde{\rho}^{*3} \Psi_2} (\mathcal{D}_0 - \frac{2}{\tilde{\rho}^*}) \Delta^{\frac{3}{2}} P_{-\frac{3}{2}} S_{-\frac{3}{2}} \\
&\quad\ \ F_{111'} = \frac{\Delta}{4\rho^2 \tilde{\rho}^{*2} \Psi_2} (\mathcal{L}^\dagger_{\frac{3}{2}} - \frac{2ia\sin\theta}{\tilde{\rho}^*}) \Delta^{\frac{3}{2}} P_{-\frac{3}{2}} S_{-\frac{3}{2}}
\end{aligned} \tag{49}
$$

In the case of each solution we have given only the non-zero components. It is interesting to note that the spinor $F_{ABA'}$ satisfies the equation

$$
\nabla^A{}_{C'} \nabla_{(AA'} F_{BC)}{}^{A'} = \Psi^{AM}{}_{BC} F_{AMC'} \tag{50}
$$

The above choices do not however satisfy the Rarita-Schwinger equation (28) . It is possible to extend these equations to a set which have solutions in terms of Teukolsky functions for general s. If we consider the equations

$$
\begin{aligned}
&\phi \nabla_{(A_1 A'} F^{A'}{}_{A_2 \ldots A_{2s})} + \frac{1}{6}(2s-3)(\nabla_{(A_1 A'}\phi) F^{A'}{}_{A_2 \ldots A_{2s})} = \phi h_{A_1 \ldots A_{2s}} \\
&\nabla^{AA'} h_{AA_2 \ldots A_{2s}} = (2s-1)(s-1)\Psi_{(A_2 A_3}{}^{BC} F^{A'}{}_{A_4 \ldots A_{2s})BC}
\end{aligned} \tag{51}
$$

where $\phi = 2I = \Psi_{ABCD}\Psi^{ABCD}$ [8] then these equations admit analogous solutions viz.

$$
\begin{aligned}
&(1) \quad h_{0\ldots 0} = P_{+s} S_{+s} \\
&\quad\ \ F_{0\ldots 01'} = \frac{1}{2\rho^2 (2s-1)(s-1)\Psi_2} \Delta (\mathcal{D}^\dagger_s - \frac{(2s-1)}{\tilde{\rho}^*}) P_{+s} S_{+s} \\
&\quad\ \ F_{0\ldots 00'} = \frac{-1}{\sqrt{2}\,\tilde{\rho}^* (2s-1)(s-1)\Psi_2} (\mathcal{L}_s - \frac{(2s-1)ia\sin\theta}{\tilde{\rho}^*}) P_{+s} S_{+s} \\
\text{and} \quad &(2) \quad h_{1\ldots 1} = \frac{1}{(\sqrt{2}\,\tilde{\rho}^*)^{2s}} \Delta^s P_{-s} S_{-s} \\
&\quad\ \ F_{1\ldots 10'} = \frac{1}{(\sqrt{2}\,\tilde{\rho}^*)^{2s} (2s-1)(s-1)\Psi_2} (\mathcal{D}_0 - \frac{(2s-1)}{\tilde{\rho}^*}) \Delta^s P_{-s} S_{-s} \\
&\quad\ \ F_{1\ldots 11'} = \frac{\Delta}{2\tilde{\rho}(\sqrt{2}\,\tilde{\rho}^*)^{2s} (2s-1)(s-1)\Psi_2} (\mathcal{L}^\dagger_s - \frac{(2s-1)ia\sin\theta}{\tilde{\rho}^*}) \Delta^s P_{-s} S_{-s}
\end{aligned} \tag{52}
$$

where $P_{+s}S_{+s}$ are separable solutions of

$$(\Delta\mathcal{D}_1\mathcal{D}_s^\dagger + \mathcal{L}_{1-s}^\dagger\mathcal{L}_s - 2(2s-1)i\sigma\tilde{\rho})P_{+s}S_{+s} = 0$$

and $P_{-s}S_{-s}$ separable solutions of

$$(\Delta\mathcal{D}_1^\dagger\mathcal{D}_s + \mathcal{L}_{1-s}\mathcal{L}_s^\dagger + 2(2s-1)i\sigma\tilde{\rho})P_{-s}S_{-s} = 0$$

Equation (51) is a generalization of the Rarita-Schwinger equation although it does not in itself have obvious physical significance for $s > \frac{3}{2}$. We also note that the method of proof for the Teukolsky-Starobinsky identities can be successfully used in the general context of Torres del Castillo [5]. Indeed we have the result

THEOREM 3

If the operators $\mathcal{D}_n$ and $\mathcal{D}_n^\dagger$ are defined by

$$\begin{aligned} \mathcal{D}_n &= \frac{\partial}{\partial r} + i\frac{q}{Q} + n\frac{Q^{(1)}}{Q} = Q^{-n}\mathcal{D}_0 Q^n \\ &\text{and} \\ \mathcal{D}_n^\dagger &= \frac{\partial}{\partial r} - i\frac{q}{Q} + n\frac{Q^{(1)}}{Q} = Q^{-n}\mathcal{D}_0^\dagger Q^n \end{aligned} \tag{53}$$

with the functions q and Q polynomials such that $q^{(3)} = 0$ and $Q^{(5)} = 0$ then for all integer s

$$\begin{aligned} &Q^s\mathcal{D}_0^{\dagger\,2s}\left[\,Q\mathcal{D}_{1-s}\mathcal{D}_0^\dagger - (2s-1)iq^{(1)} + \frac{1}{6}(s-1)(2s-1)Q^{(2)}\,\right] \\ &=\left[\,Q\mathcal{D}_{1-s}^\dagger\mathcal{D}_0 + (2s-1)iq^{(1)} + \frac{1}{6}(s-1)(2s-1)Q^{(2)}\,\right]Q^s\mathcal{D}_0^{\dagger\,2s} \end{aligned} \tag{54}$$

with a similar complex conjugate identity also holding.

This theorem applies to all non-null orbit, type D vacuum metrics given as for example by Torres del Castillo [5].

Appendix

Here we list the value of the Teukolsky-Starobinsky constant $|D_s|^2$ for a number of values of s where $\tilde{\alpha}^2 = a^2 + ma/\sigma$

$$|D_{\frac{1}{2}}|^2 = \lambda$$

$$|D_1|^2 = \lambda^2 - 4\sigma^2\tilde{\alpha}^2$$

$$|D_{\frac{3}{2}}|^2 = \lambda^2(\lambda+1) - 16\sigma^2(\lambda\tilde{\alpha}^2 - a^2)$$

$$|D_2|^2 = \lambda^2(\lambda+2)^2 - 8\sigma^2\tilde{\alpha}^2\lambda(5\lambda+6) + 96\sigma^2 a^2\lambda + 144\sigma^4\tilde{\alpha}^4 + 144\sigma^2 M^2$$

$$\begin{aligned}|D_{\frac{5}{2}}|^2 =& \lambda^2(\lambda+3)^2(\lambda+4) - 16\sigma^2\tilde{\alpha}^2\lambda(\lambda+3)(5\lambda+8) + 48\sigma^2a^2\lambda(7\lambda+12) + 1024\sigma^4\tilde{\alpha}^4(\lambda+1)\\ & - 3072\sigma^4\tilde{\alpha}^2a^2 + 1152\sigma^2M^2(\lambda+2)\end{aligned}$$

$$\begin{aligned}|D_3|^2 =& \lambda^2(\lambda+4)^2(\lambda+6)^2 - 4\sigma^2\tilde{\alpha}^2\lambda(\lambda+4)(35\lambda^2+252\lambda+360) + 128\sigma^2a^2\lambda(\lambda+4)(7\lambda+15)\\ & + 16\sigma^4\tilde{\alpha}^4(259\lambda^2+1140\lambda+900) - 2560\sigma^4\tilde{\alpha}^2a^2(11\lambda+15) + 25600\sigma^4a^4 - 14400\sigma^6\tilde{\alpha}^6\\ & + 576\sigma^2M^2((3\lambda+10)^2 - 100\sigma^2\tilde{\alpha}^2)\end{aligned}$$

$$\begin{aligned}|D_{\frac{7}{2}}|^2 =& \lambda^2(\lambda+5)^2(\lambda+8)^2(\lambda+9) - 32\sigma^2\tilde{\alpha}^2\lambda(\lambda+5)(\lambda+8)(7\lambda^2+63\lambda+108)\\ & + 288\sigma^2a^2\lambda(\lambda+5)(7\lambda^2+65\lambda+120) + 256\sigma^4\tilde{\alpha}^4(49\lambda^3+549\lambda^2+1728\lambda+1296)\\ & - 4608\sigma^4\tilde{\alpha}^2a^2(31\lambda^2+175\lambda+180) + 57600\sigma^4a^4(5\lambda+9) - 147456\sigma^6\tilde{\alpha}^6(\lambda+3)\\ & + 884736\sigma^6\tilde{\alpha}^4a^2 - 92160\sigma^4M^2(7\tilde{\alpha}^2\lambda + 30\tilde{\alpha}^2 - 15a^2)\\ & + 5760\sigma^2M^2(3\lambda^3+45\lambda^2+220\lambda+360)\end{aligned}$$

$$\begin{aligned}|D_4|^2 =& \lambda^2(\lambda+6)^2(\lambda+10)^2(\lambda+12)^2 - 48\sigma^2\tilde{\alpha}^2\lambda(\lambda+6)(\lambda+10)(7\lambda^3+154\lambda^2+996\lambda+1680)\\ & + 576\sigma^2a^2\lambda(\lambda+6)(\lambda+10)(7\lambda^2+78\lambda+168)\\ & + 96\sigma^4\tilde{\alpha}^4(329\lambda^4+7372\lambda^3+55484\lambda^2+156240\lambda+117600)\\ & - 4608\sigma^4\tilde{\alpha}^2a^2(115\lambda^3+1592\lambda^2+6216\lambda+5880) + 9216\sigma^4a^4(191\lambda^2+1344\lambda+1764)\\ & - 256\sigma^6\tilde{\alpha}^6(3229\lambda^2+31010\lambda+63700) + 64512\sigma^6\tilde{\alpha}^4a^2(169\lambda+630) - 25288704\sigma^6\tilde{\alpha}^2a^4\\ & + 2822400\sigma^8\tilde{\alpha}^8 + 28224000\sigma^6M^2\tilde{\alpha}^4 + 25401600\sigma^4M^4\\ & - 11520\sigma^4M^2(341\tilde{\alpha}^2\lambda^2 + 4242\tilde{\alpha}^2\lambda + 12740\tilde{\alpha}^2 - 1596a^2\lambda - 8232a^2)\\ & + 1440\sigma^2M^2(33\lambda^4+924\lambda^3+9436\lambda^2+42000\lambda+70560)\end{aligned}$$

References

1 S. A. Teukolsky. Rotating black holes. Separable wave equations for gravitational and electromagnetic perturbations. *Physical Review Letters.* 29, 1114, (1972)

2 E. T. Newman and R. Penrose. An approach to gravitational radiation by a method of spin coefficents. *Journal of Mathematical Physics.* 3, 566, (1962)

3 R. Güven. The solution of Dirac's equation in a class of type D backgrounds. *Proc. Roy. Soc. Lond.* A356, 465, (1977)

4 R. Güven. Black holes have no superhair. *Physical Review.* D10, 2327, (1980)

5 G. F. Torres del Castillo. The Teukolsky-Starobinsky identities in type D vacuum backgrounds with cosmological constant. *Journal of Mathematical Physics.* (In the press.)

6 R. Penrose and W. Rindler. *Spinors and Spacetime Vol. 1. Two spinor calculus and relativistic fields.* C.U.P. (1984)

7 S. Chandrasekhar. *The Mathematical Theory of Black Holes.* Oxford U.P. (1984)

8 D. Kramer, H. Stephani, E. Herlt and M. MacCallum. *Exact Solutions of Einstein's Field Equations* C.U.P. (1980)