Quantum and classical models for quadratic algebras associated with second order superintegrable systems

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Abstract. There are 13 equivalence classes of 2D second order quantum and classical superintegrable systems with nontrivial potential, each associated with a quadratic algebra of hidden symmetries. We study the finite and infinite irreducible representations of the quantum quadratic algebras though the construction of models in which the symmetries act on spaces of functions of a single complex variable via either differential operators or difference operators. In another paper we have already carried out parts of this analysis for the generic nondegenerate superintegrable system on the complex 2-sphere. Here we carry it out for a degenerate superintegrable system on the 2-sphere. We point out the connection between our results and a position dependent mass Hamiltonian studied by Quesne. We also show how to derive simple models of the classical quadratic algebras for superintegrable systems and then obtain the quantum models from the classical models, even though the classical and quantum quadratic algebras are distinct.

Key words: superintegrability; quadratic algebras; Wilson polynomials

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1 Introduction

A classical (or quantum) mth order superintegrable system is an integrable n-dimensional Hamiltonian system with potential that admits $2n-1$ functionally independent constants of the motion, the maximum possible, and such that the constants of the motion are polynomial of at most order m in the momenta. Such systems are of special significance in mathematical physics because the trajectories of the classical motions can be determined by algebraic means alone, whereas the quantum eigenvalues for the energy and the other symmetry operators can also be determined by algebraic methods. In contrast to merely integrable systems, they can be solved in multiple ways. The best known (and historically most important) examples are the classical Kepler system and the quantum Coulomb (hydrogen atom) system, as well as the isotropic oscillator. For these examples $m=2$ and the most complete classification and structure results are known for the second order case. There is an extensive literature on the subject [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], with a recent new burst of activity [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]. All such systems have been classified for real and complex Riemannian spaces with $n=2$ and their associated quadratic algebras of symmetries computed [15, 30, 31, 25, 32]. For nonconstant potentials there are 13 equivalence classes of such stems (under the Stäckel transform between manifolds), 7 with nondegenerate (3-parameter) potentials and 6 with degenerate (1-parameter) potentials.[45, 31]. The constants of the motion for each system generate a quadratic algebra that closes at order 6 in the nondegenerate case and at order 3 in the degenerate case.

The representation theory of such algebras is of great interest because it is this quadratic algebra "hidden symmetry" that accounts for the degeneracies of the energy levels of the quantum systems and the ability to compute all associated spectra of such systems by algebraic means alone. In principle, all of these quadratic algebras can be obtained from the quadratic algebra of a single generic 3-parameter potential on the complex two-sphere by prescribed limit operations and through Stäckel transforms. However, these limiting operations are not yet sufficiently understood. Each equivalence class has special properties, and each of the 13 cases is worthy of study in its own right. A powerful technique for carrying out this study is the use of "one variable models." In the quantum case these are realizations of the quadratic algebra (on an energy eigenspace) in terms of differential or difference operators acting on a space of functions of a single complex variable, and for which the energy eigenvalue is constant. Each model is adapted to the spectral decomposition of one of the symmetry operators, in particular, one that is associated with variable separation in the original quantum system. The possible unitary irreducible representations can be constructed on these spaces with appropriate function space inner products and intertwining operators to map the representation space to the solution space of the associated quantum system. (There have been several elegant treatments of the representation theory of some quadratic algebras, e.g. [16, 17, 18, 5, 6, 19, 20, 38]. However these have almost always been restricted to finite dimensional representations and the question of determining all one variable models has not been addressed.) In the classical case these are realizations of the quadratic algebra (restricted to a constant energy surface) by functions of a single pair of canonical conjugate variables.

In [40] we have already carried out parts of this analysis for the generic nondegenerate superintegrable system on the complex 2-sphere. There the potential was $V = a_1/s_1^2 + a_2/s_2^2 + a_3/s_3^2$ where $s_1^2+s_2^2+s_3^2 = 1$, and the one variable quantum model was expressed in terms of difference operators. It gave exactly the algebra that describes the Wilson and Racah polynomials in their full generality. In this paper we treat a superintegrable case with a degenerate potential. Our example is again on the complex 2-sphere, but now the potential is $V = \alpha/s_3^2$. Though this potential is a restriction of the generic potential, the degenerate case admits a Killing vector so the quadratic algebra structure changes dramatically. The associated quadratic algebra closes at level 3 and has a much richer representation theory than the nondegenerate case. Now we find one variable models for an irreducible representation expressed as either difference or differential operators, or sometimes both. We show that this system can occur in unobvious ways, such as in a position dependent mass Hamiltonian recently introduced by Quesne [38].

The second part of the paper concerns models of classical quadratic algebras. Here we inaugurate this study, in particular its relationship to quantum models of superintegrable systems. We first describe how these classical models arise out of standard Hamilton-Jacobi theory. In [45, 28] we have shown that for second order superintegrable systems in two dimensions there is a 1-1 relationship between classical quadratic algebras and quantum quadratic algebras, even though these algebras are not isomorphic. In this sense the quantum quadratic algebra, the spectral theory for its irreducible representations and its possible one variable models are already uniquely determined by the classical system. We make this concrete by showing explicitly how the possible classical models of the classical superintegrable system with potential $V = \alpha/s_3^2$ lead directly to the possible one variable differential or difference operator models for the quantum quadratic algebra. Then we repeat this analysis for the nondegenerate potential $V = a_1/s_1^2+a_2/s_2^2+a_3/s_3^2$ where the quantum model is essentially the Racah algebra QR(3) and its infinite dimensional extension to describe the Wilson polynomials. Our results show that the Wilson polynomial structure is already imbedded in the classical system with potential $V = a_1/s_1^2 + a_2/s_2^2 + a_3/s_3^2$, even though this potential admits no Lie symmetries. Thus the properties of the Wilson polynomials in their full generality could have been derived directly from classical mechanics!

This work is part of a long term project to study the structure and representation theory for quadratic algebras associated with superintegrable systems in n dimensions [24, 25, 26, 27, 28, 36, 37]. The analysis for $n = 3$ dimensions will be much more challenging, but also a good indication of behavior for general n.

2 The structure equations for S3

Up to a Stäckel transform, every 2D second order superintegrable system with nonconstant potential is equivalent to one of 13 systems [45]. There is a representative from each equivalence class on either the complex 2-sphere or complex Euclidean space. In several papers, in particular [15], we have classified all of the constant curvature superintegrable systems, and this paper focuses on two systems contained in that list: S9 and S3. The quadratic algebra of the generic nondegenerate system S9 was already treated in [40] and we will return to it again in this paper. First we study the quadratic algebra representation theory for the degenerate potential S3. This one-parameter potential 2-sphere system corresponds to the potential

$$V = \frac{\alpha}{s_3^2}$$

where $s_1^2 + s_2^2 + s_3^2 = 1$ is the imbedding of the sphere in Euclidean space. Thie quantum degenerate superintegrable system is

$$H = J_1^2 + J_2^2 + J_3^2 + V(x, y) = H_0 + V \tag{1}$$

where $J_3 = s_1\partial_{s_2} - s_2\partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of the indices $1, 2, 3$. The basis symmetries are

$$L_1 = J_1^2 + \frac{\alpha s_2^2}{s_3^2}, \quad L_2 = \frac{1}{2}(J_1J_2 + J_2J_1) - \frac{\alpha s_1 s_2}{s_3^2}, \quad X = J_3, \quad H = J_1^2 + J_2^2 + J_3^2 + V$$

where $J_3 = s_2\partial_{s_1} - s_1\partial_{s_2}$ plus cyclic permutations. They generate a quadratic algebra that closes at order 3. The quadratic algebra relations are $[H, X] = [H, L_j] = 0$ and

$$\begin{aligned} &[L_1, X] = 2L_2, \quad [L_2, X] = -X^2 - 2L_1 + H - \alpha, \\ &[L_1, L_2] = -(L_1X + XL_1) - (\frac{1}{2} + 2\alpha)X \end{aligned} \tag{2}$$

The Casimir relation is

$$\begin{aligned} \mathcal{C} \equiv &\frac{1}{3}\left(X^2L_1 + XL_1X + L_1X^2\right) + L_1^2 + L_2^2 - HL_1 + (\alpha + \frac{11}{12})X^2 - \frac{1}{6}H \\ &+(\alpha - \frac{2}{3})L_1 - \frac{5\alpha}{6} = 0. \end{aligned} \tag{3}$$

We know that the quantum Schrödinger equation separates in spherical coordinates, and that corresponding to a fixed energy eigenvalue H, the eigenvalues of X take the linear form

$$\lambda_n = \mathcal{A}n + \mathcal{B} \tag{4}$$

where n is an integer, so we will look for irreducible representations of the quadratic algebra such that the representation space has a basis of eigenvectors f_n with corresponding eigenvalues λ_n. We will use the abstract structure equations to list the corresponding representations and compute the action of L_1 and L_2 on an X basis. Thus, we assume that there is a basis $\{f_n\}$, for the representation space such that

$$Xf_n = \lambda_n f_n, \quad L_1 f_n = \sum_j C(j,n) f_j \quad L_2 f_n = \sum_j D(j,n) f_j. \tag{5}$$

Here, $\mathcal{A}, \mathcal{B}$ are not yet fixed. We do not impose any inner product space structure. From these assumptions we can compute the action of L_1 and L_2 on the basis. Indeed,

$$[L_1, L_2] f_n = \sum_{j,k} \left(C(j,k)D(k,n) - D(j,k)C(k,n)\right) f_j, \tag{6}$$

$$[L_1, X] f_n = \sum_j (\lambda_n - \lambda_j) C(j,n) f_j, \quad [L_2, X] f_n = \sum_j (\lambda_n - \lambda_j) D(j,n) f_j. \tag{7}$$

On the other hand, from the equations (2) we have

$$[L_1, X] f_n = 2 \sum_j D(j,n) f_j, \tag{8}$$

$$[L_2, X] f_n = -2 \sum_j C(j,n) f_j + (-\lambda_n^2 + H - \alpha) f_n, \tag{9}$$

$$[L_1, L_2] f_n = -\sum_j (\lambda_n + \lambda_j) C(j,n) f_j - (\frac{1}{2} + 2\alpha) \lambda_n f_n. \tag{10}$$

Now we equate equations (7) with (8) or (9). For $j = n$, equating coefficients of f_n in the resulting identities yields the conditions

$$D(n,n) = 0, \quad C(n,n) = \frac{-\lambda_n^2 + H - \alpha}{2} \tag{11}$$

Similarly, equating coefficients of f_j in the case $j \neq n$ yields

$$\mathcal{A}(n-j) D(j,n) = -2C(j,n), \quad \mathcal{A}(n-j) C(j,n) = 2D(j,n),$$

or

$$\left(\mathcal{A}^2 (n-j)^2 + 4\right) C(j,n) = 0, \quad j \neq n.$$

Thus, either $C(j,n$ and $D(j,n)$ vanish or $\mathcal{A}^2(n-j)^2 = -4$. We can scale $\mathcal{A}$ such that the smallest nonzero jump is for $j = n \pm 1$, in which case $\mathcal{A} = \pm 2i$. By replacing n by $-n$ if necessary, we can assume $\mathcal{A} = 2i$. (We also set $\mathcal{B} = i\mu$.) Thus the only possible nonzero values of $C(j,n), D(j,n)$ are for $j = n, n \pm 1$ and there are the relations

$$D(n+1,n) = -iC(n+1,n), \quad D(n-1,n) = iC(n-1,n). \tag{12}$$

Comparing (6) and (10) and equating coefficients of $f_{n\pm2}$, $f_{n\pm1}$, respectively, on both sides of the resulting identities, we do not obtain new conditions. However, equating coefficients of f_n results in the condition

$$F_{n+1} - F_n = \frac{1}{2}(2n + \mu)(4n^2 + 4\mu n + \mu^2 + H + \alpha + \frac{1}{2}), \tag{13}$$

where $F_n = C(n, n-1)C(n-1, n)$. The general solution of this difference equation is

$$F_n = n^4 + (2\mu - 2)n^3 + (\frac{3}{2}\mu^2 - 3\mu + \frac{H}{2} + \frac{\alpha}{2} + \frac{5}{3})n^2 \tag{14}$$
$$+(\frac{\mu^3}{2} - \frac{3\mu^2}{2} + [\frac{H}{2} + \frac{\alpha}{2} + \frac{5}{4}]\mu - \frac{H}{2} - \frac{\alpha}{2} - \frac{1}{4})n + \kappa$$

where κ is an arbitrary constant.

To determine κ we substitute these results into the Casimir equation (3) and set equal to zero the coefficients of f_j in the expression $\mathcal{C}f_n = 0$. For $j \neq n$ we get nothing new. However, $j = n$ we find

$$\kappa = \frac{1}{16}\mu^4 - \frac{1}{4}\mu^3 + \frac{H+\alpha+5/2}{8}\mu^2 - \frac{1/2+H+\alpha}{4}\mu \tag{15}$$
$$+\frac{1}{8}H + \frac{\alpha}{8} + \frac{H^2}{16} - \frac{H\alpha}{8} + \frac{\alpha^2}{16}.$$

Thus, $F_n = C(n, n-1)C(n-1, n)$ is an explicit 4th order polynomial in n. By factoring this polynomial in various ways, and re-normalizing the basis vectors f_n appropriately via $f_n \to c(n)f_n$, we can achieve a realization of the action of L_1 and L_2 such that

$$L_1 f_n = C(n+1, n)f_{n+1} + C(n, n)f_n + C(n-1, n)f_{n-1}, \tag{16}$$
$$L_2 f_n = D(n+1, n)f_{n+1} + D(n, n)f_n + D(n-1, n)f_{n-1}$$

and all of the coefficients are polynomials in n. The 4 roots of F_n are

$$\frac{1-\mu}{2} \pm \frac{1}{4}\sqrt{2 - 4(H+\alpha) \pm 2\sqrt{1 - 4(H+\alpha) + 16H\alpha}},$$

so a convenient factorization is

$$C(n, n-1) = (n + \frac{\mu-1}{2})^2 - \frac{1}{8}\left(1 - 2(H+\alpha) + \sqrt{1 - 4(H+\alpha) + 16H\alpha}\right), \tag{17}$$
$$C(n-1, n) = (n + \frac{\mu-1}{2})^2 - \frac{1}{8}\left(1 - 2(H+\alpha) - \sqrt{1 - 4(H+\alpha) + 16H\alpha}\right).$$

From these expressions and from

$$D(n, n) = 0,\ C(n, n) = \frac{(2n+\mu)^2 + H - \alpha}{2},\ D(n \pm 1, n) = \mp iC(n \pm 1, n) \tag{18}$$

we see that we can find C, D coefficients in which the the dependence on n is always as a polynomial.

There are raising and lowering operators

$$A^\dagger = L_1 + iL_2 + \frac{1}{2}(X^2 - H + \alpha),\ A = L_1 - iL_2 + \frac{1}{2}(X^2 - H + \alpha). \tag{19}$$

Indeed,

$$A^\dagger f_n = 2C(n+1, n)f_{n+1}, \quad Af_n = 2C(n-1, n)f_{n-1},$$

and $[A, A^\dagger]f_n = 2(F_{n+1} - F_n)f_n$, so $[A, A^\dagger]$ is a third order polynomial in X.

To get a one-variable model of the quadratic algebra in terms of second order differential operators, we can simply make the choices $f_n(t) = t^n$, $X = i(t\frac{d}{dt} + \mu)$ and define L_1 from expressions (16) via the prescription

$$L_1 f_n(t) = \left(tC(t\frac{d}{dt} + 1, t\frac{d}{dt}) + C(t\frac{d}{dt}, t\frac{d}{dt}) + t^{-1}C(t\frac{d}{dt} - 1, t\frac{d}{dt})\right) f_n(t), \tag{20}$$

with a similar procedure for L_2.

In general the irreducible representations that we have defined are infinite dimensional and the basis vectors f_n occur for all positive and negative integers n. We can obtain representations bounded below, and with lowest weight μ for $-iX$ and corresponding lowest weight vector f_0, simply by requiring $F_0 = 0$, which amounts to setting $\kappa = 0$. For convenience we set $\alpha = 1/4 - a^2$. Then we have

$$F_n = C(n, n-1)C(n-1, n) = n(n+\mu-1)(n+\mu-1+a)(n-a). \tag{21}$$

Since $\kappa = 0$, (15), H must be a solution of this quadratic equation:

$$H = -(\mu - 1 + a)^2 + \frac{1}{4}. \tag{22}$$

A convenient choice is

$$\begin{aligned} &C(n-1, n) = n(n+\mu-1+a),\ C(n+1, n) = (n+\mu)(n+1-a), \\ &C(n, n) = 2n^2 + 2n\mu - \mu a + a + \mu - \frac{1}{2}. \end{aligned} \tag{23}$$

If μ is not a negative integer then this bounded below representation is infinite dimensional. However, if there is a highest weight vector f_m then we must have $F_{m+1} = 0$, or $\mu = -m, m = 0, 1, \cdots$. Thus the finite dimensional representations are indexed by the nonnegative integer m and the eigenvalues of $-iX$ are $m, m-2, \cdots, -m$. The dimension of the representation space is $m+1$. A convenient representation of the finite dimensional model by second order differential operators is

$$\begin{aligned} L_1 = &\left(t^3 + 2t^2 + t\right)\frac{d^2}{dt^2} + \left((2-a-m)t^2 + 2(1-m)t + a - m\right)\frac{d}{dt} \\ &+ m(a-1)t + a(m+1) - m - \frac{1}{2}, \quad X = i(2t\frac{d}{dt} - m), \\ L_2 = &\ i\left(-t^3 + t\right)\frac{d^2}{dt^2} + i\left((a+m-2)t^2 + a - m\right)\frac{d}{dt} - im(a-1)t. \end{aligned} \tag{24}$$

This model is also correct for infinite dimensional bounded-below representations, except that now the lowest weight is $\mu = -m$ where $m \neq 0, 1, 2, \cdots$ is a complex number. The raising and lowering operators for the model are

$$A^\dagger = 2t^3\frac{d^2}{dt^2} + 2(2-a-m)t^2\frac{d}{dt} + 2m(a-1)t,\ A = 2t\frac{d^2}{dt^2} + 2(a-m)\frac{d}{dt}.$$

In the finite dimensional case, for example, the eigenvalues of L_1 are

$$\chi_n = a^2 - \frac{1}{4} - (n - a + \frac{1}{2})^2, \quad n = 0, 1, \cdots, m, \tag{25}$$

and the corresponding unnormalized eigenfunctions are

$$(1+t)^n {}_2F_1\left(\begin{array}{cc} n-a & n-m \\ a-m & \end{array}; -t\right).$$

Now, motivated by the quantum mechanical system on the real 2-sphere, we impose a Hilbert space structure on the irreducible representations such that L_1 and L_2 are self-adjoint and X is skew adjoint:

$$< L_j f_n, f_{n'} > = < f_n, L_j f_{n'} >,\ j = 1, 2; \quad < X f_n, f_{n'} > = - < f_n, X f_{n'} > .$$

Writing $\phi_n = k_n f_n$ where ϕ_n has norm 1, we have the recursion relation

$$k_n^2 = \frac{(n-1+\mu)(n-a)}{n(n-1+\mu+a)} k_{n-1}^2.$$

For infinite dimensional bounded below representations k_n^2 must be positive for all integers $n \geq 0$, and we normalize $k_0 = 1$. Thus

$$k_n^2 = \frac{(\mu)_n(1-a)_n}{n!(a+\mu)_n}.$$

For finite dimensional representations we have $\mu = -m$. Normalizing $k_0 = 1$, (possible for $a < 1$ or for $a > m$), we find that an orthonormal basis in the one variable model is given by $\phi_n(t) = k_n f_n(t) = k_n t^n$, $n = 0, 1, \cdots, m$ where

$$k_n = \sqrt{\frac{(-m)_n(1-a)_n}{n!(-m+a)_n}} = \sqrt{\frac{m!(1-a)_n(1-a)_{m-n}}{(1-a)_m n!(m-n)!}}.$$

Note the reflection symmetry $||f_n|| = ||f_{m-n}||$.

To derive a realization of the Hilbert space for the differential operator models of the finite dimensional and infinite dimensional bounded below unitary representations in terms of a function space inner product

$$< p, q >= K \int\int p(t)\overline{q(t)}\rho(t\bar{t})\ dt\ d\bar{t}$$

where p, q are polynomials and K is a normalization constant, we use the formal self- and skew-adjoint requirements and obtain a differential equation for the weight function:

$$(-\zeta^2+\zeta)\frac{d^2\rho(\zeta)}{d\zeta^2} + (-\mu - a + 1 + (-1+\mu-a)\zeta)\frac{d\rho(\zeta)}{d\zeta} + (-2+\mu-2a+a\mu)\rho(\zeta) = 0$$

where $\zeta = t\bar{t}$. The solution that vanishes at $\zeta = 1$ for $a < 1/2$ and is integrable at $\zeta = 0$ for $a + \mu > -1$ is

$$\rho_1(\zeta) = (1-\zeta)^{1-2a}{}_2F_1\left(\begin{array}{cc} -\frac{\mu+3a-Q}{2}+1, & -\frac{\mu+3a+Q}{2}+1 \\ 2-2a & \end{array};1-\zeta\right),$$

where $Q = \sqrt{a^2+(2\mu-8)a+\mu^2+4\mu-8}$. (Note that the integral is an even function of Q.) At $\zeta = 0$ this function has a branch point with behavior $\zeta^{a+\mu}$. We write $t = re^{i\theta}$, $\bar{t} = re^{-i\theta}$, $\zeta = r^2$ and choose our contours of integration for the inner product as the unit circle $|e^{i\theta}| = 1$, i.e., $0 \leq \theta \leq 2\pi$ and, in the complex ζ-plane, a contour that starts at $\zeta = 1$ and travels just above the real ζ-axis to circle $\zeta = 0$ once in the counterclockwise direction and returns to $\zeta = 1$ just below the real ζ-axis. We require that $< 1, 1 >= 1$. By choosing a regime where $a + \mu > -1$ we can shrink the ζ-contour about $\zeta = 0$ so that the norm takes the form

$$< 1, 1 >= -4\pi K_1 e^{i\pi(a+\mu)}\sin[\pi(a+\mu)]\int_0^1 \rho_1(\zeta)d\zeta$$

$$= \frac{-\pi K_1}{a-1}e^{i\pi(a+\mu)}\sin[\pi(a+\mu)]{}_2F_1\left(\begin{array}{cc} \frac{-\mu-3a+Q}{2}+1 & \frac{-\mu-3a-Q}{2}+1 \\ 3-2a & \end{array};1\right)$$

$$= 4\pi K_1 e^{i\pi(a+\mu)}\sin[\pi(a+\mu)]\frac{\Gamma(2-2a)\Gamma(a+\mu+1)}{\Gamma(2-\frac{a-\mu+Q}{2})\Gamma(2-\frac{a-\mu-Q}{2})},$$

where we have integrated term-by-term and then made use of Gauss' Theorem for the summation of ${}_2F_1(1)$. This gives us the value for K_1 such that $< 1, 1 >= 1$. Now, the result extends for the original contour by analytic continuation. This defines a pre-Hilbert space inner product that then can be extended to obtain a true Hilbert space.

The contour integral for the inner product obtained in the previous paragraph requires Re $a < 1$ for convergence, and this doesn't hold for some of the unitary irreducible representations defined above. Accordingly, we consider a second solution of the weight function equation. The solution that vanishes at $\zeta = 0$ for $a + \mu > 0$ and is integrable at $\zeta = 1$ for $a < 1$ is

$$\rho_2(\zeta) = \zeta^{\mu+a}{}_2F_1\left(\begin{matrix} -\frac{\mu+3a-Q}{2}, & -\frac{\mu+3a+Q}{2} \\ \mu+a+1 & \end{matrix};\zeta\right).$$

At $\zeta = 1$ this function has a branch point with behavior $(1-\zeta)^{1-2a}$. We write $t = re^{i\theta}$, $\overline{t} = re^{-i\theta}$, $\zeta = r^2$ and choose our contours of integration for the inner product as the unit circle $|e^{i\theta}| = 1$ and, in the complex ζ-plane, a contour that starts at $\zeta = 0$ and travels just below the real ζ-axis to circle $\zeta = 1$ once in the counterclockwise direction and returns to $\zeta = 0$ just above the real ζ-axis. This integral converges for Re $(a + \mu) > -1$. We require that $< 1, 1 >= 1$. By choosing a regime where $a < 1$ we can shrink the ζ-contour about $\zeta = 1$ so that the norm takes the form

$$\begin{aligned}
< 1, 1 > &= 4\pi K_2 e^{i\pi(2a-1)} \sin[\pi(2a-1)] \int_0^1 \rho_2(\zeta)d\zeta \\
&= \frac{\pi K_2}{\mu+a+1} e^{i\pi(2a-1)} \sin[\pi(2a-1)]{}_2F_1\left(\begin{matrix} \frac{\mu+3a+Q}{2} & \frac{\mu+3a-Q}{2} \\ a+\mu+2 & \end{matrix};1\right) \\
&= -4\pi K_2 e^{i\pi(2a-1)} \sin[\pi(2a-1)] \frac{\Gamma(2-2a)\Gamma(a+\mu+1)}{\Gamma(2-\frac{a-\mu+Q}{2})\Gamma(2-\frac{a-\mu-Q}{2})}.
\end{aligned}$$

This gives us the value for K such that $< 1, 1 >= 1$, and the result extends by analytic continuation to all values of a, μ for which the original contour integral converges.

Thus we have an explicit pre-Hilbert function space inner product for each of our differential operator models. In the finite dimensional case we have the repoducing kernel function

$$\delta(t,\overline{s}) = \sum_{n=0}^{m} \phi_n(t)\overline{\phi}_n(s) = {}_2F_1\left(\begin{matrix} -m & 1-a \\ -m+a & \end{matrix};t\overline{s}\right).$$

In the infinite dimensional bounded below case we have the reproducing kernel function

$$\delta(t,\overline{s}) = \sum_{n=0}^{\infty} \phi_n(t)\overline{\phi}_n(s) = {}_2F_1\left(\begin{matrix} \mu & 1-a \\ \mu+a & \end{matrix};t\overline{s}\right)$$

which converges as an analytic function and in the Hilbert space norm for $|s| < 1$. Here,

$$||\delta(s,\overline{s})|| = {}_2F_1\left(\begin{matrix} \mu & 1-a \\ \mu+a & \end{matrix};|s|^2\right).$$

In each case $< f(t), \delta(t,\overline{s}) >= f(s)$ for f in the Hilbert space.

There is also a difference operator model for the representations of the S3 quadratic algebra. We first give the details for the finite dimensional representations indexed by the nonnegative integer m. Here the operator L_1 is diagonalized:

$$\begin{aligned}
L_1 &= -\lambda(t) + a - \frac{1}{2}, \quad \lambda(t) = t(t-2a+1), \\
-iX &= \frac{(t-2a+1)(t-m)}{2t-2a+1}T^1 - \frac{t(t+m-2a+1)}{2t-2a+1}T^{-1}, \\
L_2 &= \frac{(t-a+1)(t-2a+1)(t-m)}{2t-2a+1}T^1 + \frac{t(t-a)(t+m-2a+1)}{2t-2a+1}T^{-1},
\end{aligned} \tag{26}$$

where T^k is the difference operator $T^k f(t) = f(t+k)$. The basis functions are $f_n(t) = (-1)^n p_n(\lambda)$ where

$$p_n(\lambda(t)) = {}_3F_2\left(\begin{array}{ccc} -n & -t & t-2a+1 \\ -m & 1-a & \end{array}; 1\right).$$

Here f_n is a polynomial of order n in the variable $\lambda(t)$, a special case of the family of dual Hahn polynomials, [46], page 346. These polynomials are orthogonal with respect to a measure with support at the values $t = 0, 1, \cdots, m$, in agreement with equation (25) for the eigenvalues of L_1. Indeed, we have (for $a < 1$)

$$\sum_{t=0}^{m} \frac{(1-2a)_t(3/2-a)_t(-m-1)_t(-1)^t}{(1/2-a)_t(2+m-2a)_t t!} p_n(\lambda(t)) p_{n'}(\lambda(t))$$

$$= \frac{(2-2a)_m(a-m)_n n!}{(1-a)_m(1-a)_n(-m)_n}\delta_{nn'}.$$

For the infinite dimensional, bounded below, case we have

$$\begin{aligned} L_1 &= t^2 + a^2 - \frac{1}{4}, \\ -iX &= \frac{(1/2-a-it)(\mu+a-1/2-it)}{2t}T^i - \frac{(1/2-a+it)(\mu+a-1/2+it))}{2t}T^{-i}, \\ L_2 &= -i\frac{(1-2it)(1/2-a-it)(\mu+a-1/2-it)}{4t}T^i \\ &-i\frac{(1+2it)(1/2-a+it)(\mu+a-1/2+it))}{4t}T^{-i}. \end{aligned} \tag{27}$$

The basis functions are $f_n(t) = (-1)^n s_n(t^2)$ where

$$s_n(t^2) = {}_3F_2\left(\begin{array}{ccc} -n & \frac{1}{2}-a+it & \frac{1}{2}-a-it \\ \mu & 1-a & \end{array}; 1\right). \tag{28}$$

Here f_n is a polynomial of order n in the variable t^2, a special case of the family of continuous dual Hahn polynomials, [46], page 331. The orthogonality and normalization are given by

$$\begin{aligned} &\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(1/2-a+it)\Gamma(\mu+a-1/2+it)\Gamma(1/2+it)}{\Gamma(2it)}\right|^2 s_n(t^2)s_{n'}(t^2)\,dt \\ &= \frac{\Gamma(n+\mu)\Gamma(n+1-a)\Gamma(n+\mu+a)n!}{(\mu)_n^2|(1-a)_n|^2}\delta_{nn'}, \end{aligned} \tag{29}$$

where either 1) $\mu > 1/2 - a > 0$ or 2) $\mu > 0$ and $a = ((1-\mu)/2 + i\gamma$ is complex.

In summary, we have found the following possibilities for bounded below irreducible representations such that L_1, L_2 are self-adjoint and X is skew adjoint, together with associated one variable models. (Here, n_0 is a positive integer .)

representation	parameter range	model
finite dimensional	$\mu = -m,\ m = 0, 1, 2, \cdots$ either $a < 1$ or $a + \mu > 0$	differential operators difference operators
inf. dim. bdd. below	$\mu > 0$ $a < 1$ and $a + \mu > 0$	differential operators difference operators
inf, dim. bdd. below	$0 > \mu = -n_0 + t,\ t \in (0, 1)$ $a = n_0 + s,\ s \in (0, 1)$	differential operators
inf. dim. bdd. below	$0 > \mu = -n_0 + t,\ t \in (0, 1)$ $-t < a < 1 - t$	differential operators
inf, dim. bdd. below	$\mu > 0$ $a = (1 - \mu)/2 + i\gamma,\ \gamma \in R$	difference operators

In all but the last case, a, μ are real.

3 Quesne's position dependent mass (PDM) system in a two-dimensional semi-infinite layer.

In [38] Quesne considered a superintegrable exactly solvable position dependent mass (PDM) system in a two-dimensional semi-infinite layer. Her system is equivalent via a gauge transformation to a standard quantum mechanical problem on the real 2-sphere with potential of the form S3. Indeed, in Quesne's paper we are given the Hamiltonian

$$-H_Q = \cosh^2 qx(\partial_x^2 + \partial_y^2) + 2q \cosh qx \sinh qx \partial_x + q^2 \cosh^2 qx - \frac{q^2 k(k-1)}{\sinh^2 qx}.$$

We adopt coordinates on the unit sphere as

$$s_1 = \frac{\sin qy}{\cosh qx},\ s_2 = \frac{\cos qy}{\cosh qx},\ s_3 = \tanh qx,$$

where $s_1^2 + s_2^2 + s_3^2 = 1$ and the metric is $ds^2 = q^2(dx^2 + dy^2)/\cosh^2 qx$. The Laplacian becomes

$$\Delta_S = \frac{\cosh^2 qx}{q^2}(\partial_x^2 + \partial_y^2).$$

In these coordinates, the degenerate superintegrable system S3 becomes

$$H_S = \frac{\cosh^2 qx}{q^2}(\partial_x^2 + \partial_y^2) + \frac{\frac{1}{4} - \alpha^2}{\tanh^2 qx}.$$

By a gauge transform $H_O = (\cosh qx)^{-1} H_S \cosh qx$, we get

$$H_O = \frac{1}{q^2}\{\cosh^2 qx(\partial_x^2 + \partial_y^2) + 2q \cos qx \sinh qx \partial_x + q^2 \cosh^2 qx + q^2 \frac{\frac{1}{4} - \alpha^2}{\sinh^2 qx}\} + \frac{1}{4} - \alpha^2.$$

Thus we have $H_Q = -q^2 H_0 + q^2(1/4-\alpha^2) = -q^2(\cosh qx)^{-1} H_S \cosh qx + q^2(1/4-\alpha^2)$, with $1/4-\alpha^2 = -k(k-1)$ which has solutions $k = a+1/2$ or $k = -a+1/2$. Since k is assumed positive and a is required to be less than 1, we take $a < 0$.

Suppose we find an eigenvector for H_S with eigenvalue λ_S, call it v_{λ_S}. Then λ_Q will be the eigenvalue of v_{λ_Q} for H_Q. We have the transformations $v_{\lambda_Q} = v_{\lambda S}/\cosh qx$ and $\lambda_Q = -q^2\lambda_S + q^2(1/4-\alpha^2) = -q^2\lambda_S - q^2 k(k-1)$. Checking the two eigenvalues, we have $\lambda_S = -(\mu-1+a)^2+1/4$ and $\lambda_Q = q^2(N+2)(N+2k+1)$. We note that these two values coincide when $-\mu = m = N+1$ with m an integer.

Using the above calculations and the eigenfunctions given in the paper, we can obtain eigenfunctions for the S3 case as

$$v_{\lambda_S} = N^{(k)}_{n,\ell}(\tanh qx)^{-a+\frac{1}{2}}(\cosh qx)^{-\ell-1} P_n^{-a,\ell+1}(-\tanh^2 qx)\chi_\ell(y),$$

or in coordinates on the sphere

$$v_{\lambda_S} = N^{(k)}_{n,\ell}(s_3)^{-a+\frac{1}{2}}(s_1^2+s_2^2)^{\frac{\ell+1}{2}} P_n^{-a,\ell+1}(-s_3^2)\chi_\ell(y)$$

where $m = 2n+\ell+1$, and $\chi_\ell(y) = \sin[(\ell+1)qy]$ or $\cos[(\ell+1)qy]$. We can rewrite these by noting $1/\cosh^2 qx = 1-\tanh^2 qx$ so that we can write $\sin qy = s_1/\sqrt{1-s_3^2}$ and $\cos qy = s_2/\sqrt{1-s_3^2}$, then we obtain

$$\chi_\ell(y) = a_n T_{\ell+1}\left(\frac{s_2}{\sqrt{s_1^2+s_2^2}}\right) + b_n \frac{s_1}{\sqrt{s_1^2+s_2^2}} U_\ell\left(\frac{s_2}{\sqrt{s_1^2+s_2^2}}\right)$$

Where T_ℓ and U_ℓ are the Chebyshev polynomials of the first and second kind, respectively.

Quesne found the S3 quadratic algebra (which closes at order 3) but did not use it for spectral analysis purposes because her problem involved a boundary condition that broke the full quadratic algebra symmetry. Instead she considered her system as a special case of S9 and used the more complicated S9 symmetry algebra that closes at order 6 to find the finite dimensional representations. This point of view has merit, but it complicates the spectral analysis of the problem, since the only one-variable model is in terms of difference operators and Racah polynomials. From our vantage point of one one variable differential operator analysis for the model, Quesne's boundary conditions amount to decomposing an irreducible subspace corresponding to an m-dimensional representation into a direct sum of even and odd parity subspaces V^+, V^-. (Indeed her boundary conditions require choice of $\chi_\ell(y)$ in the cosine form for ℓ even and in the sine form for ℓ odd.) Let P be the operator $Pf(t) = t^m f(1/t)$. Since $P^2 = I$ and $||f_n|| = ||f_{m-n}||$, it is clear that P is unitary. We define unit vectors $\Phi^+_\ell = 2^{-1/2}(\phi_\ell + (-1)^m\phi_{m-\ell})$ and $\Phi^-_\ell = 2^{-1/2}(\phi_\ell - (-1)^m\phi_{m-\ell})$ for $\ell = 0, 1, \cdots, [m/2]$. Then for $m = 2k$ the vectors Φ^+_ℓ, $\ell = 0, \cdots, k$ form an ON basis for V^+_m and the vectors Φ^-_ℓ, $\ell = 0, \cdots, k-1$ form an ON basis for V^-_m. For $m = 2k-1$, the vectors Φ^+_ℓ, $\ell = 0, \cdots, k-1$ form an ON basis for V^+_m and the vectors Φ^-_ℓ, $\ell = 0, \cdots, k-1$ form an ON basis for V^-_m. These basis vectors are very easily expressible in terms of the one variable differential operator model, where they are sums of two monomials. The basis used by Quesne corresponds to the V^- subspaces.Thus our models can be used to carry out the spectral analysis for this PDM system, and they yield a simplification.

3.1 Classical models for S3

Now we describe how the methods of classical mechanics lead directly to the quantum models. The classical system S3 on the 2-sphere is determined by the Hamiltonian

$$\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2 + \frac{\alpha(s_1^2+s_2^2+s_3^2)}{s_3^2}, \tag{30}$$

where $\mathcal{J}_1 = s_2p_3 - s_3p_2$ and $\mathcal{J}_2, \mathcal{J}_3$ are cyclic permutations of this expression. For computational convenience we have imbedded the 2-sphere in Euclidean 3-space. Thus we use the Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\} = \sum_{i=1}^{3}(-\partial_{s_i}\mathcal{F}\partial_{p_i}\mathcal{G} + \partial_{p_i}\mathcal{F}\partial_{s_i}\mathcal{G})$$

for our computations, but at the end we restrict to the sphere $s_1^2 + s_2^2 + s_3^2 = 1$. The classical basis for the constants of the motion is

$$\mathcal{L}_1 = \mathcal{J}_1^2 + \alpha\frac{s_2^2}{s_3^2},\ \mathcal{L}_2 = \mathcal{J}_1\mathcal{J}_2 - \alpha\frac{s_1s_2}{s_3^2},\ \mathcal{X} = \mathcal{J}_3. \tag{31}$$

The structure relations are

$$\begin{aligned}&\{\mathcal{X}, \mathcal{L}_1\} = -2\mathcal{L}_2,\ \{\mathcal{X}, \mathcal{L}_2\} = 2\mathcal{L}_1 - \mathcal{H} + \mathcal{X}^2 + \alpha,\\ &\{\mathcal{L}_1, \mathcal{L}_2\} = -2(\mathcal{L}_1 + \alpha)\mathcal{X},\end{aligned} \tag{32}$$

and the Casimir relation is

$$\mathcal{L}_1^2 + \mathcal{L}_2^2 - \mathcal{L}_1\mathcal{H} + \mathcal{L}_1\mathcal{X}^2 + \alpha\mathcal{X}^2 + \alpha\mathcal{L}_1 = 0. \tag{33}$$

From the results of [47] we know that additive separation of variables in the Hamilton-Jacobi equation $\mathcal{H} = E$ is possible in subgroup type coordinates in which $\mathcal{X},\ \mathcal{L}_1$ or $\mathcal{S} = 2(\mathcal{L}_1 - i\mathcal{L}_2) - \mathcal{H} + \mathcal{X}^2$, respectively, are constants of separation. This corresponds to two choices of spherical coordinates and one of horospherical coordinates, respectively. We seek two variable models for the Poisson bracket relations (32),(33). There is also separation in ellipsoidal coordinates (i.e., non-subgroup type coordinates) but we will not make use of this here.

The justification for these models comes from Hamilton-Jacobi theory. The phase space for our problem is 4-dimensional. Thus it is possible to find canonical variables $\mathcal{H}, \mathcal{I}, \mathcal{Q}, \mathcal{P}$ such that $\{\mathcal{I}, \mathcal{H}\} = \{\mathcal{P}, \mathcal{Q}\} = 1$ and all other Poisson brackets vanish. In terms of $\mathcal{H}$ and the other canonical variables the Poisson bracket can be expressed as

$$\{\mathcal{F}, \mathcal{G}\} = -\partial_{\mathcal{H}}\mathcal{F}\partial_{\mathcal{I}}\mathcal{G} + \partial_{\mathcal{I}}\mathcal{F}\partial_{\mathcal{H}}\mathcal{G} - \partial_{\mathcal{Q}}\mathcal{F}\partial_{\mathcal{P}}\mathcal{G} + \partial_{\mathcal{P}}\mathcal{F}\partial_{\mathcal{Q}}\mathcal{G}. \tag{34}$$

(As follows from standard theory [48] one can construct a set of such canonical variables from a complete integral of the Hamilton-Jacobi equation. Our 2D second order superintegrable systems are always multiseparable, and each separable solution of the Hamilton-Jacobi equation provides a complete integral. Thus we can find these canonical variables in several distinct ways.) Now we restrict our attention to the algebra of constants of the motion. This algebra is generated by $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_1, \mathcal{X}$, subject to the relation (33). Thus, considered as functions of the canonical variables, the constants of the motion are independent of $\mathcal{H}$. If we further restrict the system to the constant energy space $\mathcal{H} = E$ then we can consider $\mathcal{H}$ as non varying and every constant of the motion $\mathcal{F}$ can be expressed in the form $\mathcal{F}(E, \mathcal{Q}, \mathcal{P})$. This means that the Poisson bracket of two constants of the motion, $\mathcal{F}, \mathcal{G}$ can be computed as

$$\{\mathcal{F}, \mathcal{G}\} = -\partial_{\mathcal{Q}}\mathcal{F}\partial_{\mathcal{P}}\mathcal{G} + \partial_{\mathcal{P}}\mathcal{F}\partial_{\mathcal{Q}}\mathcal{G}.$$

Thus all functions depend on only two canonically conjugate variables $\mathcal{Q}, \mathcal{P}$ and the parameter E. This shows the existence and the form of two variable models of conjugate variables. However the proof is not constructive and, furthermore, it is not unique. Two obtain constructive results we will use the strategy of setting $\mathcal{Q}$ equal to one of the constants of the motion that corresponds to separation of variables in some coordinate system, and then use (34) for the Poisson bracket and require that relations (32),(33) hold. In order to make clear that we are computing on the constant energy hypersurface expressed in canonical variables we will use a different notation. We will set $\mathcal{Q}_E = c, \mathcal{P}_E = \beta$ so, $\mathcal{F}(\mathcal{H},\mathcal{Q},\mathcal{P}) = f(c,\beta)$, $\mathcal{G}(\mathcal{H},\mathcal{Q},\mathcal{P}) = g(c,\beta)$, and

$$\{\mathcal{F},\mathcal{G}\}_E = \{f,g\} = -\partial_c f \partial_\beta g + \partial_\beta f \partial_c g.$$

For our first model we require $X \equiv \mathcal{X}_E = c$. Substituting this requirement and $\mathcal{H} = E$ into the structure equations we obtain the result

$$\begin{aligned} I: \qquad & L_1 = \frac{1}{2}(E - c^2 - \alpha) + \frac{1}{2}\sqrt{c^4 - 2c^2(E+\alpha) + (E-\alpha)^2}\sin 2\beta, \\ X = c,\ & L_2 = \frac{1}{2}\sqrt{c^4 - 2c^2(E+\alpha) + (E-\alpha)^2}\cos 2\beta. \end{aligned} \tag{35}$$

In this model, and in all other classical models, β is not uniquely determined: we can replace it by $\beta' = \beta + k(c)$ for any function $k(c)$ and the variables c and β' remain canonically conjugate.

For a second model we require $L_1 \equiv (\mathcal{L}_1)_E = c$ and proceed in a similar fashion. The result is

$$\begin{aligned} II: \qquad & L_1 = c, \qquad L_2 = \sqrt{c(E-c-\alpha)}\sin(2\sqrt{c+\alpha}\beta), \\ X = & \sqrt{\frac{c(E-c-\alpha)}{c+\alpha}}\cos(2\sqrt{c+\alpha}\beta). \end{aligned} \tag{36}$$

For the third and last model we need to diagonalize the symmetry $\mathcal{S} = 2(\mathcal{L}_1 - i\mathcal{L}_2) - \mathcal{H} + \mathcal{X}^2$ corresponding to separation in horospherical coordinates. For this it is convenient to rewrite the structure equations (32),(33) in terms of the new basis $\mathcal{S}, \mathcal{L}_1 + i\mathcal{L}_2, \mathcal{X}$:

$$\begin{aligned} & \{\mathcal{S},\mathcal{X},\} = 2i(\mathcal{S}+\alpha),\ \{\mathcal{S},\mathcal{L}_1+i\mathcal{L}_2\} = -2i\mathcal{X}(\mathcal{S} - 2\mathcal{X}^2 + 2\mathcal{H} + 3\alpha), \\ & \{\mathcal{L}_1 + i\mathcal{L}_2, \mathcal{X}\} = -i\left(\mathcal{X}^2 + 2(\mathcal{L}_1 + i\mathcal{L}_2) - \mathcal{H} + \alpha\right). \end{aligned} \tag{37}$$

The Casimir relation is

$$\begin{aligned} & -2\mathcal{S}(\mathcal{L}_1 + i\mathcal{L}_2) - \mathcal{S}\mathcal{X}^2 + \mathcal{X}^4 + \mathcal{H}\mathcal{S} - 2\mathcal{X}^2\mathcal{H} + \mathcal{H}^2 \\ & -\alpha\left(2(\mathcal{L}_1 + i\mathcal{L}_2) + \mathcal{S} + 3\mathcal{X}^2 + \mathcal{H}\right) = 0. \end{aligned} \tag{38}$$

For model 3 we set $S = c$ and obtain

$$\begin{aligned} III: \qquad & S = c, \qquad X = -2i(c+\alpha)\beta, \\ & L_1 + iL_2 = 8(c+\alpha)^3\beta^4 + 2(c+\alpha)(3\alpha + c + 2E)\beta^2 - \frac{(c+E)(\alpha - E)}{2(c+\alpha)}. \end{aligned} \tag{39}$$

3.2 Classical model → quantum model

What have we achieved with these classical models? For one thing they show us how to parameterize the constants of the motion and exhibit their functional dependence. More important for our purposes, they give us a rational means to derive the possible one-variable quantum models. This may seem surprising. How can classical mechanics determine quantum mechanics uniquely? How can structures such as the Wilson family of orthogonal polynomials, containing the Hahn polynomials, be derived directly from classical mechanics? The point is that the structures we are studying are second order superintegrable systems in 2D. In papers [31, 45, 28] it has been shown that there is a 1-1 relationship between the quantum and classical versions for such systems, for all 2D Riemannian spaces. (Similarly there is a 1-1 relationship in 3D for nondegenerate potentials on conformally flat spaces.) The structures are not identical, since as we can see from the examples in this paper, the structure relations in the classical and quantum cases are not identical; there are quantum modifications of the classical equations. Although we know of no direct prescription for their determination, nonetheless the quantum structure equations are uniquely determined by the classical structure equations. Given a second system of second order constants of the motion we write down the corresponding quantum system via the usual correspondence, where products of classical functions are replaced by symmetrized quantum operators, and generate the quadratic algebra by taking repeated commutators. Even order classical symmetries correspond to formally self-adjoint quantum symmetries, and odd order classical symmetries correspond to formally skew-adjoint quantum symmetries. (This relationship no longer holds for third order superintegrable systems, [22, 41].) We will demonstrate here how to get quantum models from the classical ones that we have derived.

The basic prescription for the transition from the classical case to the operator case is to replace a pair of canonically conjugate variables c, β by $c \to t, \beta \to \partial_t$. (There is no obstruction to quantization for second order superintegrable systems.) Once an appropriate choice of β is made in a classical model, we can use this prescription to go to a differential operator model of the quantum structure equations. In particular model III above suggests a operator model such that S is multiplication by c, X is a first order differential operator in c and $L_1 + iL_2$ is a fourth order differential operator. The result, whose existence is implied by the 1-1 classical/quantum relationship for second order superintegrable systems, is

$$\begin{aligned} &III: \qquad S = t, \qquad X = -2i(t+\alpha)\partial_t + 2i, \\ &L_1 + iL_2 = 8(t+\alpha)^3\partial_t^4 + 2(t+\alpha)(3\alpha + t + 2E + 9)\partial_t^2 - 2(t + 5\alpha + 4E + 18)\partial_t \\ &+(2 + \frac{E}{2} - \frac{\alpha}{2}) + \frac{E^2 + 2E(9-\alpha) + (\alpha+12)(\alpha+6)}{2(t+\alpha)}. \end{aligned} \tag{40}$$

The leading order differential operators terms agree with the classical case but there are lower order correction terms needed to correct for the noncommutivity of t and ∂_t. We can realize various irreducible representations of the quadratic algebra by choosing subspaces of functions of t on which the operators act. This model agrees with (20),(21), (22) in the case where $C(n-1, n) = 1$ and $C(n+1, n)$ is fourth order. However, there we had a space spanned by a countable number of eigenvectors of the skew-adjoint symmetry X whereas here we want the spectral decomposition of the self-adjoint symmetry S to govern the model. This forces L_2 to be skew-adjoint and X to be self-adjoint. Thus, though the differential operators are formally the same, the Hilbert spaces and the spectral analysis are different. All the representations are infinite-dimensional. One class can be realized by closing the dense subspace of C^∞ functions with compact support on $0 < t < \infty$ where the measure is dt/t. The the spectrum of S is continuous and runs over the positive real axis. Here X also has continuous real spectra covering the full real axis. In particular the generalized eigenfunction of X with real eigenvalue λ is proportional to $t^{-i\lambda}$, and μ is pure imaginary. Thus the spectral analysis of X is given by the Mellin transform. There is a similar irreducible representation defined on $-\infty < t < 0$. By a canonical transformation we can also get models of these representations in which both $C(n-1, n) = 1$ and $C(n+1, n)$ are second order. (We shall illustrate this explicitly for Model I.) Then the spectral decomposition of S is given by the Hankel transform. Since these particular eigenspaces of H admit no discrete spectrum for any of the symmetries of interest, we shall not analyze them further.

Now we consider model I, (35). Due to the presence of trigonometric terms in β we cannot realize this as a finite order differential operator model. However, we can perform a hodograph transformation, i.e. use the prescription $\beta \to t, c \to -\partial_t$ to realize the model. This would seem to make no sense due to the appearance of functions of c under the square root sign. However, before using the prescription we can make use of the freedom to make a replacement $\beta' = \beta + g(c)$ which preserves canonical variables. We choose

$$e^{-2i\beta} \to e^{-2i\beta}/\sqrt{c^4 - 2c^2(E+\alpha) + (E-\alpha)^2}$$

but leave c unchanged. Then we find

$$L_1 = \frac{1}{2}(E - c^2 - \alpha) - \frac{i}{4}\left[\left(c^4 - 2c^2(E+\alpha) + (E-\alpha)^2\right)e^{2i\beta} - 1\right], \tag{41}$$
$$L_2 = -\frac{i}{4}\left[\left(c^4 - 2c^2(E+\alpha) + (E-\alpha)^2\right)e^{2i\beta} + 1\right],$$

with X as before. Now we apply the quantization prescription $\beta \to t, c \to -\partial_t$ and obtain a model in which both L_1 and L_2 are fourth order and X is a first order differential operator. This is, in fact, identical to within a coordinate change to model (40). One might also try to obtain a difference operator model from (41) with the replacement $c \to t, \beta \to \partial_t$, so that $e^{2i\beta}$ would become a difference operator. However, this difference operator quantum model is equivalent to what would get from the $\beta \to t, c \to -\partial_t$ model by taking a Fourier transform. Thus we don't regard it as new.

There is an alternate way to obtain a quantum realization from model I. We use the fact that

$$c^4 - 2c^2(E+\alpha) + (E-\alpha)^2 = (c^2 - (E+\alpha))^2 - 4\alpha$$

and set

$$\phi = \arctan\left(\frac{\sqrt{-4\alpha}}{c^2 - (E+\alpha)^2}\right).$$

Now we let $2\beta \to 2\beta + \phi$ to obtain

$$L_1 = \frac{1}{2}(E - c^2 - \alpha) + \frac{1}{2}\left((c^2 - (E+\alpha)^2)\sin 2\beta + 2i\sqrt{\alpha}\cos 2\beta\right),$$

$$L_2 = \frac{1}{2}\left((c^2 - (E+\alpha)^2)\cos 2\beta - 2i\sqrt{\alpha}\sin 2\beta\right), \quad X = c.$$

Now the prescription $\beta \to t, c \to -\partial_t$ leads to a quantum realization of L_1, L_2 by second order differential operators. Indeed

$$L_1 = \frac{1}{2}(\cos(2t) - 1)\partial_t^2 - 8i\xi\sin(2t)\partial_t + (-\frac{E}{2} + 64\xi^2 + 8i\xi - \frac{1}{4} - \frac{\alpha}{2})\cos(2t) + \frac{E-\alpha}{2},$$

$$L_2 = \frac{1}{2}\sin(2t)\partial_t^2 + 8i\xi\cos(2t)\partial_t + (-\frac{E}{2} + 64\xi^2 + 8i\xi - \frac{1}{4} - \frac{\alpha}{2})\cos(2t) + \frac{E-\alpha}{2}, \; X = \partial_t.$$

Here ξ is arbitrary and can be removed via a gauge transformation. The change of variable $\tau = e^{2it}$ reduces this model to the form (24). This shows that the flexibility we had in constructing differential operator models from the abstract representation theory by renormalizing our basis vectors f_n is replaced in the classical model case by appropriate canonical transformations $c \to c,\ \beta \to \beta + g(c)$. In either case there is essentially only one differential operator model that can be transformed in various ways.

It is clear that Model II cannot produce finite order differential operator realizations of the quantum quadratic algebra, due to the intertwining of square root dependence for c and exponential dependence for β. However, it will produce a difference operator realization via

Taylor's theorem: $e^{a\partial_t}f(t)=f(t+a)$.To show this explicitly we make a coordinate change such that $2\sqrt{c+\alpha}\partial_c=\partial_{\mathcal{C}}$ in (36), which suggests realizations of the quantum operators in the form

$$L_1f(t)=(t^2-\alpha)f(t),\quad Xf(t)=h(t)f(t+i)+m(t)f(t-i),$$
$$L_2f(t)=-\frac{i}{2}(i+2t)h(t)f(t+i)+\frac{i}{2}(-i+2t)m(t)f(t-i). \tag{42}$$

A straightforward computation shows that the quantum algebra structure equations are satisfied if and only if

$$h(t)m(t+i)=\frac{1}{4}\frac{(\alpha-t^2-it)(t^2+it-E)}{t(t+i)}. \tag{43}$$

Since $\alpha=-a^2+\frac14$ and $E=-(\mu-1+a)^2+\frac14$ for bounded below representations, we can factor (43) simply to obtain

$$h(t)m(t+i)=-\frac{1}{4t(t+i)}(t+\frac{i}{2}+ia)(t+\frac{i}{2}-ia)(t+\frac{i}{2}+i\mu+ia)(t+\frac{3i}{2}-i\mu-ia). \tag{44}$$

Note that only the product (43) is determined, not the individual factors. Thus we can choose $h(t)$, say, as an arbitrary nonzero function and then determine $m(t)$ from (43). All these modifications of the factors are accomplished by gauge transformations on the representation space: $\tilde{f}(t)=\rho(t)f(t)$ where $\rho(t)$ is the gauge function. If we choose the factors in the form

$$h(t)=i\frac{(\frac12-a-it)(\mu+a-\frac12-it)}{2t},\ m(t)=-i\frac{(\frac12-a+it)(\mu+a-\frac12+it)}{2t},$$

then we we get exactly the model (27). The finite dimensional model is related by the simple change of variables $t\to i(t-a+1/2)$, $\mu=-m$. In any case, there is only a single solution of these equations, up to a gauge transformation.

3.3 The classical model for S9

This is the system on the complex sphere, with nondegenerate potential

$$V=\frac{a_1}{s_1^2}+\frac{a_2}{s_2^2}+\frac{a_3}{s_3^2},$$

where $s_1^2+s_2^2+s_3^2=1$. The classical S9 system has a basis of symmetries

$$\mathcal{L}_1=\mathcal{J}_1^2+a_2\frac{s_3^2}{s_2^2}+a_3\frac{s_2^2}{s_3^2},\ \mathcal{L}_2=\mathcal{J}_2^2+a_3\frac{s_1^2}{s_3^2}+a_1\frac{s_3^2}{s_1^2},\ \mathcal{L}_3=\mathcal{J}_3^2+a_1\frac{s_2^2}{s_1^2}+a_2\frac{s_1^2}{s_2^2}, \tag{45}$$

where $\mathcal{H}=\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3+a_1+a_2+a_3$ and the $\mathcal{J}_i$ are defined by $\mathcal{J}_3=s_1p_{s_2}-s_2p_{s_1}$ and cyclic permutation of indices. The classical structure relations are

$$\{\mathcal{L}_1,\mathcal{R}\}=8\mathcal{L}_1(\mathcal{H}+a_1+a_2+a_3)-8\mathcal{L}_1^2-16\mathcal{L}_1\mathcal{L}_2-16a_2\mathcal{L}_2+16a_3(\mathcal{H}+a_1+a_2+a_3-\mathcal{L}_1-\mathcal{L}_2),$$
$$\{\mathcal{L}_2,\mathcal{R}\}=-8\mathcal{L}_2(\mathcal{H}+a_1+a_2+a_3)+8\mathcal{L}_2^2+16\mathcal{L}_1\mathcal{L}_2+16a_1\mathcal{L}_1-16a_3(\mathcal{H}+a_1+a_2+a_3-\mathcal{L}_1-\mathcal{L}_2), \tag{46}$$

with $\{\mathcal{L}_1,\mathcal{L}_2\}=\mathcal{R}$ and

$$\mathcal{R}^2-16\mathcal{L}_1\mathcal{L}_2(\mathcal{H}+a_1+a_2+a_3)+16\mathcal{L}_1^2\mathcal{L}_2+16\mathcal{L}_1\mathcal{L}_2^2+16a_1\mathcal{L}_1^2+16a_2\mathcal{L}_2^2+16a_3(\mathcal{H}+a_1+a_2+a_3)^2$$
$$-32a_3(\mathcal{H}+a_1+a_2+a_3)(\mathcal{L}_1+\mathcal{L}_2)+16a_3\mathcal{L}_1^2+32a_3\mathcal{L}_1\mathcal{L}_2+16a_3\mathcal{L}_2^2-64a_1a_2a_3.$$

Taking $\mathcal{L}_1 = c$, $\mathcal{H} = E$ with c, β as conjugate variables, we find the model

$$\mathcal{L}_2 = \frac{1}{2}(a_1 + 2a_2 + E - c) - \frac{(a_2 - a_3)(a_1 + 2a_2 + 2a_3 + E)}{2(c + a_2 + a_3)} + \tag{47}$$
$$\frac{\sqrt{(4a_1a_2 + 4a_1a_3 + 2c(E + a_1 + a_2 + a_3) + 4ca_1 - (E + a_1 + a_2 + a_3)^2 - c^2)(4a_2a_3 - c^2)}}{2(a_2 + a_3 + c)}$$
$$\times \cos(4\beta\sqrt{a_2 + a_3 + c}).$$

This suggests a difference operator realization of the quantum model.

In the quantum case the symmetry operators L_1, L_2, L_3 are obtained from the corresponding classical constants of the motion (45) through the replacements $\mathcal{J}_k \to J_k$ where the angular momentum operators J_k are defined by $J_3 = x_1\partial_{x_2} - x_2\partial_{x_1}$ and cyclic permutation of indices. Here $H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3$. The quantum structure relations can be put in the symmetric form

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k), \tag{48}$$

$$R^2 = \frac{8}{6}\{L_1L_2L_3\} + -(16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 \tag{49}$$
$$+\frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1$$
$$+\frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3 + \frac{32}{3}(a_1 + a_2 + a_3)$$
$$+48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3.$$

Here i, j, k are chosen such that $\epsilon_{ijk} = 1$ where ϵ is the pure skew-symmetric tensor, $R = [L_1, L_2]$ and $\{L_1, L_j\} = L_iL_j + L_jL_i$ with an analogous definition of $\{L_1, L_2, L_3\}$ as a sum of 6 terms. In practice we will substitute $L_3 = H - L_1 - L_2 - a_1 - a_2 - a_3$ into these equations.

Proceeding exactly as in the S3 case (42), (43), (44), we find that the difference operator analogy of (47) for the quantum quadratic algebra is

$$L_1 = 4t^2 - \frac{1}{2} + \beta^2 + \gamma^2, \tag{50}$$
$$L_2 = h(t)T^i + m(t)T^{-i} + \ell(t) =$$
$$\frac{[-4\alpha^2 - 8\alpha - 4 + 4\mathcal{E}^2 + 16i(\alpha + 1)t + 16t^2](\beta + 1 + \gamma - 2it)(\beta - 1 - \gamma + 2it)}{1024t(t + i)(2t + i)^2}$$
$$\times[-4\alpha^2 - 4 + 8\alpha + 4\mathcal{E}^2 + 16i(1 - \alpha)t + 16t^2](\beta + 1 - \gamma - 2it)(\beta - 1 + \gamma + 2it)T^i + T^{-i}$$
$$+[-2t^2 - \frac{1}{2}\mathcal{E}^2 - \frac{1}{2}\beta^2 + \frac{1}{2}\alpha^2 + \frac{1}{2}\gamma^2 + \frac{(\gamma^2 - \beta^2)(-4\alpha^2 + 4\mathcal{E}^2)}{8(1 + 4t^2)},$$

where

$$a_1 = \frac{1}{4} - \alpha^2,\ a_2 = \frac{1}{4} - \beta^2,\ a_3 = \frac{1}{4} - \gamma^2,\ H = \frac{1}{4} - \mathcal{E}^2.$$

The quadratic terms factor into simple linear terms, and just as in the S3 case, it is only $\ell(t)$ and the product $h(t)m(t + i)$ that is uniquely determined. We can change the individual factors $h(t), m(t)$ by a gauge transformation. With the change of variable $t = i\tau$ and a gauge transformation to an operator with maximal symmetry in τ, we obtain the standard model

$$h(t) = \tilde{h}(\tau) = \frac{(A + \tau)(B + \tau)(C + \tau)(D + \tau)}{4\tau(\tau + 1/2)},$$

$$m(t) = \tilde{m}(\tau) = \frac{(A-\tau)(B-\tau)(C-\tau)(D-\tau)}{4\tau(\tau-1/2)},$$

$$A = \frac{\mathcal{E}+\alpha+1}{2}, \ B = \frac{\mathcal{E}-\alpha+1}{2}, \ C = \frac{\beta+\gamma+1}{2}, \ D = \frac{\beta-\gamma+1}{2}.$$

It follows that $L_2 = \tilde{h}(\tau)E^{+1} + \tilde{m}(\tau)E^{-1} + \tilde{\ell}(\tau)$ is a linear combination of L_1 and the difference operator whose eigenfunctions are the Wilson polynomials, just as found in [40]. Here $E^s f(\tau) = f(\tau+s)$.

4 Conclusions and prospects

This paper consists of two related parts. In the first part we have studied the representation theory for the quadratic algebra associated with a 2D second order quantum superintegrable system with degenerate potential, namely S3. We have classified the possible finite-dimensional representations and infinite dimensional bounded below representations, i.e., those with a lowest weight vector. Then we have constructed the possible Hilbert space models for these representations, in terms of differential operators or of difference operators acting on spaces of functions of one complex variable. These models make it easy to find raising and lowering operators for the representations and to uncover relationships between the algebras and families of orthogonal polynomials. Here S3 has been treated as an example of a degenerate potential superintegrable system. The example S9 of a nondegenerate potential was treated in [40]. In 2D there are 13 equivalence classes of superintegrable systems with nontrivial potentials: 7 nondegenerate and 6 degenerate. Results for all of these cases will be included in the thesis of the third author.

In the second part of this work we have taken up the study of models of the quadratic algebras associated with the classical second order superintegrable systems. In each model there is only a single pair of canonically conjugate variables, rather than the 2 pairs in the original classical system. We showed, based on classical Hamilton-Jacobi theory, that such models always exist. Then we described a procedure to derive the one variable models for the quantum quadratic algebras from the models for the classical quadratic algebras. Since it is known that there is a 1-1 relationship between classical and quantum second order superintegrable systems (even though the algebras are not the same), it is not too surprising that one should be able to compute the quantum models from the classical models. However, we have made this explicit. We applied this procedure not only to obtain the differential and difference operator models for system S3, but also for the generic system S9. For S9 we showed that there is a difference operator model associated with general Wilson polynomials, but no differential operator model. This construction demonstrates that the theory of general Wilson polynomials is imbedded in classical mechanics in a manner quite different from the usual group theory (Racah polynomial) approach.

There is much more work to be done. Once the models are worked out and the corresponding functional Hilbert spaces are constructed, usually Hilbert spaces with kernel function, then one needs to find intertwining operators that map the model space to the space on which the quantum Schrödinger operator is defined. Also,we have demonstrated how to determine the classical models and show how they quantize in a unique fashion. A puzzle here is that we are finding classical models corresponding to non-hypergeometric type variable separation. These classical models typically involve elliptic functions. We do not yet understand how they can be quantized. They clearly do not lead to differential or ordinary difference operator quantum models.

Another part of our effort is to study the structure of quadratic algebras corresponding to 3D nondegenerate superintegrable systems, and to find two variable models for them. This is a much more difficult problem than in 2D, where it led to general Wilson and Racah polynomials, among other models. The quadratic algebra still closes at order 6 but now there are 6 linearly second order symmetries, rather than 3, and they are functionally dependent, satisfying a polynomial relation of order 8. There are 4 commutators, instead of 1. For the models we expect to find multivariable extensions of Wilson polynomials, among many other constructs.

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