Exact and quasi-exact solvability of second order superintegrable quantum systems. II. Relation to separation of variables

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Abstract

We make explicit the intimate relationship between quasi-exact solvability, as expounded for example by Ushveridze, and the technique of separation of variables as it applies to specific superintegrable quantum Hamiltonians. It is the multiseparability of superintegrable systems that forces the existence of interesting families of polynomial solutions characteristic of quasi-exact solvability, that enables us to solve these systems in distinct ways, and that gives us the basis of a classification theory. This connection is generalized in terms of the understanding of the role of finite solutions of quantum Hamiltonians.

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1 Introduction

This paper is an extension of our study of the intimate connections between second order superintegrable quantum systems and quasi-exact solvability [1]. It also relies on the structure and classification theory for these systems, as presented in many articles [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

Here we understand a quasi-exactly solvable (QES) problem, in the sense of Turbiner and Ushveridze [2], [12], as a one-dimensional Schrödinger many parameter eigenvalue problem, where for certain values of the parameters, a finite subset of the energy eigenvalues and corresponding eigenfunctions may be found as solutions of algebraic equations. This means in practice that for these fixed parameters it is possible to write the solution as a special factor times a polynomial and to solve algebraic equations to find the energy one-dimensional oscillator with the same anharmonicity. Magyari [13] stated the theorem that the one-dimensional Schrödinger equation

$$\frac{d^2\psi(x)}{dx^2} + [E - V^{(N)}(x)]\psi(x) = 0 \tag{1}$$

with anharmonic potential

$$V^{(N)}(x) = \sum_{n=1}^{2N-1} b_n x^{2n}, \quad b_{2N-1} > 0, \qquad N = 1, 2, 3.... \tag{2}$$

admits square integrable even ($s = 0$) and odd ($s = 1$) parity eigenfunctions of the form

$$\psi^{(s)}(x) = \exp\left(-\frac{1}{2}\sum_{n=1}^{N} a_n x^{2n}\right) x^s P_M^{(s)}(x), \qquad a_N > 0 \tag{3}$$

where

$$P_M^{(s)}(x) = \sum_{m=0}^{M} c_m^{(s)} x^{2m}, \tag{4}$$

is a polynomial of degree M, $M = 0, 1, 2...$, if the first parameters $(b_1, b_2, ..., b_{N-1})$ satisfy a set of $N-1$ constraints. The last constrained parameter b_{N-1} is given by the following formula

$$b_{N-1} = -N(2N + 4M + 2s - 1)a_N + \sum_{i,j}^{i+j=N} ija_ia_j, \tag{5}$$

[2]In the previous article [1] we introduced another definition of quasi-exact solvability.

which ensures the existence of a polynomial solution of degree 2M, (4).

The numbers a_i, $(i = 1, 2, ..., N)$ formed from the N arbitrary constants $(b_N, b_{N+1}, ..., b_{2N-1})$; the energy E, the coefficients $c_m^{(s)}$ and constrained coupling constants $(b_1, b_2, ..., b_{N-1})$, are determined from the systems of algebraic equations (see for details [13]).

The change of variable $|y| = x^2$ or $x = \mathrm{sgn} y\sqrt{|y|}$ $(\mathbf{R} \to \mathbf{R})$ together with a wave function transformation $\psi(y) = |y|^{-1/4}\phi(y)$ reduces equation (1) to

$$\frac{d^2\phi(y)}{dy^2} + \left[\epsilon + \frac{\alpha}{|y|} - \frac{\nu(\nu-1)}{y^2} - \tilde{V}^{(N)}(y)\right]\phi(y) = 0 \tag{6}$$

where $\epsilon = -b_1/4$, $\alpha = E/4$, ν takes two values $\nu = 1/4$, $\nu = 3/4$ and

$$\tilde{V}^{(N)}(y) = \frac{1}{4}(b_2 y + b_3 y^2 + ... + b_{2N-1}y^{2N-2}) = \frac{1}{4}\sum_{n=1}^{2N-2} b_{n+1}y^n, \quad N = 2, 3, ... \tag{7}$$

and gives us a new class of QES systems which are dual to the first one. Indeed, in the equation (6) the energy E is fixed and plays the role of a coupling constant, the coupling constant b_1 is quantized, and the quantity $\epsilon = -b_1/4$ has the meaning of energy. Note, also that after the substitution $y = x^2 > 0$ $(\mathbf{R} \to \mathbf{R}^+)$ and with $\psi(y) = y^{-1/4}\phi(y)$ the equation (1) transforms to an equation having the same form as (6) but where y has the meaning of a radial variable for the three-dimensional Schrödinger equation.

Recently, an interesting extension of the QES family has been obtained in the article of Bender and Boettchter [14]. They introduced a new class of QES $\mathcal{PT}$ symmetric Hamiltonians (see also [15] for $\mathcal{PT}$ symmetric quantum mechanics) with a quartic polynomial potential, which formally can be considered as a particular case of eq. (6) for $N = 2$, $\alpha = 0$, $\nu = 0$ and imaginary coupling coefficients b_2 and b_4. The further generalization of QES systems in the complex plane is presented in articles of Znojil [16, 17, 18].

The main problem in subsequent investigations of QES systems is *to understand just which potentials and just which specializations of the parameters in these potentials allow explicit algebraic solutions.*

It has long been understood [12, 19] that one way of approaching such problems is to obtain the eigenfunction equation as one of the separation equations for a higher dimensional (PDE) Schrödinger equation that is separable in some set of coordinates. The parameters are the separation constants and the permissible values of these parameters and energy eigenvalues are obtained by solving the PDE eigenvalue problem.

In our work we make clear that quasi-exactly solvable equations are better treated as arising from higher dimensional superintegrable systems, in particular those that are multiseparable [1]. This enables us to solve these systems in distinct ways, thus better understanding the solutions, and gives us the basis of a classification theory [10, 20]. Our approach leads us to many new examples of QES systems.

To motivate our approach we start by reviewing some simple exactly solvable (ES) and QES systems. The solution of many of the partial differential equations of mathematical physics can be achieved via the method of separation of variables. Among the solutions found in this way are classical polynomials such as those of Legendre, Laguerre, Hermite and Jacobi [21]. As an example, Legendre polynomials $P_\ell(x)$ satisfy the differential equation

$$(1-x^2)\frac{d^2}{dx^2}P_\ell(x) - 2x\frac{d}{dx}P_\ell(x) + \ell(\ell+1)P_\ell(x) = 0, \qquad -1 \le x \le 1.$$

If we look for polynomial solutions $\Pi_{i=1}^{\ell}(x-\theta_i)$ of this equation we find as necessary and sufficient conditions that the zeros θ_i must satisfy the relations

$$(2\ell+1)\theta_i + 2\sum_{j\neq i}\frac{1}{\theta_j-\theta_i} = 0.$$

These are just the relations satisfied by the zeros of the Legendre polynomials $P_\ell(x)$. Now consider the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f = 0 \tag{8}$$

expressed in conical coordinates u, v, r $(r > 0, e_1 < u < e_2 < v < e_3)$:

$$x^2 = r^2\frac{(u-e_1)(v-e_1)}{(e_1-e_2)(e_1-e_3)}, \quad y^2 = r^2\frac{(u-e_2)(v-e_2)}{(e_2-e_1)(e_2-e_3)}, \quad z^2 = r^2\frac{(u-e_3)(v-e_3)}{(e_3-e_1)(e_3-e_2)}.$$

If we look for separated solutions of the form $f = r^\ell U(u)V(v)$ we see that the separation equations for the functions $U(u)$ and $V(v)$ are of Lamé type viz

$$\sqrt{P(\lambda)}\frac{d}{d\lambda}\sqrt{P(\lambda)}\frac{d}{d\lambda}\Lambda(\lambda) + [-\ell(\ell+1)\lambda + \mu]\Lambda(\lambda) = 0$$

where $P(x) = -4(x-e_1)(x-e_2)(x-e_3)$, and $\Lambda = U, V$ and $\lambda = u, v$. Requiring polynomial solutions

$$\Lambda(\lambda) = (\lambda-e_1)^p(\lambda-e_2)^q(\lambda-e_3)^r\Pi_{i=1}^{n}(\lambda-\theta_i)$$

where p, q, r are 0 or $\frac{1}{2}$ we see that this is possible if and only if

$$\frac{p+1}{\theta_i-e_1} + \frac{q+1}{\theta_i-e_2} + \frac{r+1}{\theta_i-e_3} + \sum_{j\neq i}\frac{2}{\theta_i-\theta_j} = 0$$

where $\ell = 2(n+p+q+r)$ and

$$\mu = e_1[-4(r+q+n)2 + 2p(2p-1)] + e_2[-4(r+p+s)2 + 2q(2q-1)]$$

$$+e_3[-4(p+q+n)2 + 2r(2r-1)] + [8(p+q+r+n) - 2]\sum_{i=1}^{n} \theta_i.$$

Such polynomial solutions can also be obtained for the Schrödinger equation $H\Psi = E\Psi$ in the presence of a potential. An example of this in three dimensions [22] is

$$H = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + [\omega^2(x^2+y^2+z^2) + \frac{p^2 - \frac{1}{4}}{x^2} + \frac{q^2 - \frac{1}{4}}{y^2} + \frac{r^2 - \frac{1}{4}}{z^2}]. \quad (9)$$

If we again choose conical coordinates and look for solutions of the form $\Psi = R(r)U(u)V(v)$ we see that the functions $U(u)$ and $V(v)$ must satisfy the differential equation

$$\sqrt{P(\lambda)}\frac{d}{d\lambda}(\sqrt{P(\lambda)}\frac{d}{d\lambda}\Lambda(\lambda)) + [-\ell(\ell+1)\lambda + \mu + (e_1-e_2)(e_1-e_3)\frac{(p^2-\frac{1}{4})}{\lambda - e_1}$$

$$+(e_2-e_1)(e_2-e_3)\frac{(q^2-\frac{1}{4})}{\lambda-e_2} + (e_3-e_2)(e_3-e_1)\frac{(r^2-\frac{1}{4})}{\lambda-e_3}]\Lambda(\lambda) = 0$$

where $\Lambda = U, V$ and $\lambda = u, v$. Looking for solutions in exactly the same manner as we have done for solutions of Laplace's equation in these coordinates, we uncover the same relations obeyed by the zeros θ_i and the same expressions for ℓ and μ. The only difference is that p, q, and r can now be arbitrary. The radial equation satisfied by $R(r)$ is

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + [E - \omega^2 r^2 - \frac{\ell(\ell+1)}{r^2}]R = 0.$$

This equation admits polynomial solutions of the form

$$R(r) = r^{\ell}\exp(-\frac{\omega}{2}r^2)L_n^{\ell+\frac{1}{2}}(\omega r^2)$$

where the energy eigenvalues are $E_n = \omega(2n+p+q+r+3)$ for $n = 0, 1, \cdots$ and the zeros of the associated Laguerre polynomials $L_n^{\alpha}(z) \approx \Pi_{i=1}^{s}(z-\theta_i)$ satisfy the relations

$$-\alpha - 1 + \theta_i + 2\sum_{j\neq i}\frac{\theta_i}{\theta_j - \theta_i} = 0.$$

The above examples are all ES systems because the solutions can be given explicitly in terms of hypergeometric functions (for instance in spherical or Cartesian coordinates).

To motivate our more complicated results to follow we give one more example, a true QES system. This example is closely related to one treated our paper [1] but its ingredients are relevant to the present work. Consider the differential equation

$$\frac{d^2F}{d\ell^2}+\{-144k_1^2\ell^4-96k_1k_2\ell^3+16(2k_2^2+3k_1k_3)\ell^2+E\ell+\mu\}F=0, \quad (10)$$

a form of the triconfluent Heun equation that completely coincides, for $\ell = ix$, $k_1 = 1/12$, $4k_2 = a$, $4k_3 = 2b - 3a^2$, $\mu = -E$ and $J = N + 1$, with the $\mathcal{PT}$ symmetric QES example presented in [14]. It is known [23, 14] that in case of

$$E = 8[2k_2k_3 + 2\frac{k_2^3}{k_1} - 3k_1(n+1)], \quad n = 0, 1, 2... \quad (11)$$

the equation (10) has finite solutions

$$F = \exp\left(4k_1\ell^3 + 2k_2\ell^2 - 2(\frac{k_2^2}{k_1} + k_3)\ell\right)\prod_{i=0}^{n}(\ell - \theta_i)$$

where the θ_i satisfy the relations

$$-2k_1k_3 - 2k_2^2 + 4k_1k_2\sum_{i=1}^{n}\theta_i + 12k_1^2\sum_{i=1}^{n}\theta_i^2 + \sum_{j\neq i}\frac{k_1}{\theta_i - \theta_j} = 0$$

and the parameter μ is determined by the formulas

$$\mu = -4\frac{k_2^4}{k_1^2} - 8\frac{k_2^2k_3}{k_1} - 4k_3^2 - 4(1+2n)k_2 - 24\sum_{i=1}^{n}\theta_i.$$

Thus we have a true QES solution for a potential that is a polynomial of order 4. It is also possible to obtain such solutions by factoring out the expression $f = \exp(4k_1\ell^3 + 2k_2\ell^2 - 2(k_2^2/k_1 + k_3)\ell)$ and then looking directly for strictly polynomial solutions (see for instance [14]).

A natural question is: *How is this particular set of solutions related to superintegrability?* To answer this consider the quantum Hamiltonian

$$H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 36k_1^2[2(x-iy)^3 - 4(x^2+y^2)] + 24k_1k_2[3(x-iy)^2 - 2(x+iy)]$$
$$-16(2k_2^2 + 3k_1k_3)(x - iy).$$

The corresponding Schrödinger equation is superintegrable and multiseparable, [6]. If the coordinates x and y are related to semi-hyperbolic coordinates u, v via

$$x + iy = -\frac{1}{2}(u-v)^2, \quad x - iy = u + v \tag{12}$$

then the Schrödinger equation can be written as

$$H\Psi = \Big\{\frac{1}{u-v}\Big[\Big(\frac{\partial^2}{\partial u^2} - 144k_1^2u^4 - 96k_1k_2u^3 + 16(2k_2^2 + 3k_1k_3)u^2\Big)$$
$$-\Big(\frac{\partial^2}{\partial v^2} - 144k_1^2v^4 - 96k_1k_2v^3 + 16(2k_2^2 + 3k_1k_3)v^2\Big)\Big]\Big\}\Psi = E\Psi.$$

This can be solved for Ψ via the usual separation of variables ansatz $\Psi = U(u)V(v)$. In particular we can look for finite solutions of the form

$$\Psi = \exp[4k_1(u^3 + v^3) + 2k_2(u^2 + v^2) - 2(\frac{k_2^2}{k_1} + k_3)(u + v)]\prod_{i=1}^{n}(u - \theta_i)(v - \theta_i)$$

where the functions $U(u)$ and $V(v)$ satisfy the same separated differential equation as F above and the energy spectrum given the formula (11).

Another important question is: *How can we determine the values of μ for polynomial solutions of this type*? This is done by noting that μ is the eigenvalue of the symmetry operator

$$M = \frac{1}{u-v}\Big[v\Big(\frac{\partial^2}{\partial u^2} - 144k_1^2u^4 - 96k_1k_2u^3 + 16(2k_2^2 + 3k_1k_3)u^2\Big)$$
$$-u\Big(\frac{\partial^2}{\partial v^2} - 144k_1^2v^4 - 96k_1k_2v^3 + 16(2k_2^2 + 3k_1k_3)v^2\Big)\Big].$$

The eigenvalues can be determined by solving the eigenvalue equation $M\Psi = \mu\Psi$ in the form

$$\Psi = \exp(4k_1(u^3 + v^3) + 2k_2(u^2 + v^2) - 2(\frac{k_2^2}{k_1} + k_3)(u + v))S(u, v)$$

where $S(u, v)$ is a sum of symmetric polynomials

$$S_{ij} = S_{ji} = u^iv^j + u^jv^i, \quad 0 \le i + j \le 2n,$$

i.e., $S(u, v) = \sum_{i\le j} a_{ij}S_{ij}$, subject to the restriction $0 \le i + j \le 2n$. There are $\frac{1}{2}(n + 1)(n + 2)$ such functions. If we require that $M\Psi = \mu\Psi$ then we must solve a set of linear equations of the form

$$\sum_{k\le\ell}(C_{ij}^{k\ell} - \mu)a_{k\ell} = 0$$

where

$$\hat{M}S_{ij} = \sum_{k\leq\ell} C_{ij}^{k\ell} S_{k\ell},$$

and $\hat{M}$ is the induced action of M on $S(u,v)$. There is a determanental condition to be satisfied in order that these equations have nontrivial solutions. This condition must include the possibility of solutions of the form

$$\Psi = \exp[4k_1(u^3+v^3) + 2k_2(u^2+v^2) - 2(\frac{k_2^2}{k_1} + k_3)(u+v)]\Pi_{i=1}^s(u-\theta_i)(v-\theta_i)$$

where $0 \leq s \leq n$. This is clear from the observation that we have a linear system of dimension $\frac{1}{2}(n+1)(n+2)$. If we wish to determine the eigenvalues associated with solutions corresponding to $s = n$ only then we could take $S_{nn} = 1$. The rank of the system then becomes $n+1$, but the price we pay for this is that the determanental condition is more awkward to implement. As a illustration of this process, if we look for solutions of the form

$$S(u,v) = c_1 uv + c_2(u+v) + c_3$$

then μ satisfies the factorizable cubic equation

$$(\mu k_1^2 + 4k_1^2k_3^2 + 4k_2k_1^2 + 4k_2^4 + 8k_1k_3k_2^2)(k_1^4\mu^2 + 8k_1^2(2k_2k_1^2 + k_3^2k_1^2 + 2k_1k_3k_1^2 + k_2^4)\mu$$
$$-96k_3k_1^5 - 48k_2^2k_1^2 + 64k_3^2k_1^2k_2 + 16k_3^4k_1^4 + 128k_2^3k_3k_1^3 + 64k_3^3k_1^3k_2^2 + 64k_2^5k_1^2$$
$$+96k_1^2k_2^4k_3^2 + 64k_2^6k_1k_3 + 16k_2^8) = 0.$$

Thus we have used the superintegrable system to determine the values μ, E for which the separated equation is QES, and we have obtained the results in an alternate from that for the separated equations treated alone. Note further that if we perform a special ansatz in the nonseparable Cartesian coordinates to try to find polynomial solutions in x, y we will obtain exactly the same exponential phase factor as the product of the phase factors in u and in v. This means that the possible polynomial solutions in x, y for a fixed energy eigenspace (11) correspond exactly with the polynomial solutions symmetric in u and v. The polynomial energy eigenspace is degenerate but the diagonalization of M breaks the degeneracy.

We can exploit the fact that our superintegrable system is multiseparable: It also separates in shifted semi-hyperbolic coordinates, obtained formally from the standard semi-hyperbolic coordinates (12) through the transformation

$$x \rightarrow x + a, \quad y \rightarrow y - ia.$$

Then

$$V(x,y)-E\to V(x,y)-96ak_1[3k_1(x-iy)-k_2]-E=\bar{V}(x,y)-\bar{E}$$

which is again of the same form if we make the transformations

$$k_1\to k_1,\quad k_2\to k_2,\quad k_3\to 6ak_1+k_3,\quad E\to E+96k_1k_2a$$

of the parameters. From this observation we deduce that separation of variables can also be achieved in displaced coordinates for arbitrary a. Also the polynomial solutions in the u,v coordinates can be written as linear combinations of the polynomials in the shifted separable coordinates.

In addition to the possibility of solving the Schrödinger's equation via separation of variables there is the possibility of obtaining an explicit solution that corresponds to Lie form in the corresponding classical mechanical problem. This form can be calculated by writing the Schrödinger equation in complex conjugate coordinates, in which it becomes

$$\frac{\partial^2\Psi}{\partial z\partial\bar{z}}+(A(\bar{z})z+B(\bar{z}))\Psi=0,$$

not separable but solvable. We set $\Psi=e^{zC(\bar{z})+D(\bar{z})}$. This will be a solution provided that

$$C(\bar{z})^2=2\int A(\bar{z})d\bar{z},\quad D(\bar{z})=\int\frac{B(\bar{z})}{C(\bar{z})}d\bar{z}-\ln C(\bar{z}).$$

In our case we obtain the solution

$$C(\bar{z})=2\sqrt{3k_1}(3k_1\bar{z}^2+2k_2\bar{z}+\Lambda)^{1/2},$$

$$D(\bar{z})=-\frac{1}{8\sqrt{3k_1}}[\frac{1}{3}(-16k_1^3\Lambda+52k_1^2k_2\bar{z}-84k_1^2k_2^2-48k_1^2k_3+24k_1^4\bar{z}^2)(3k_1\bar{z}^2+2k_2\bar{z}+\Lambda)^{1/2}$$
$$+\frac{1}{k_1^2\sqrt{3k_1}}(28k_2^3-12k_1k_2\Lambda-Ek_1+16k_1k_2k_3)\ln[2\sqrt{3k_1}(3k_1\bar{z}+k_2)+$$
$$6k_1(3k_1\bar{z}^2+2k_2\bar{z}+\Lambda)^{1/2}]]-\frac{1}{2}\ln(3k_1\bar{z}^2+2k_2\bar{z}+\Lambda)$$

to within a constant. Can Λ and E can be chosen in such a way that there are "polynomial" solutions to within a factor of this type? We note here that the constant Λ has a definite meaning in terms of symmetries of H. One could in principle expand a given set of such solutions in terms of a separable set of solutions, but it is our intention to study this elsewhere.

Based on these examples, a natural question to ask is: *What is the most general way that we can obtain 1d QES eigenvalue problems and their solutions, related to separation of variables of a nondegenerate superintegrable system?*

2 Polynomial solutions from Euclidean space superintegrable systems.

A crucial observation to answer the preceding question is that the Hamiltonian H in our example is superintegrable and admits three second order symmetry operators: a quantum superintegrable system. To answer the question fully we need to recall some facts about generic elliptical coordinates in complex Euclidean n space and on the complex n sphere, and their relationship to superintegrable systems with nondegenerate potentials.

A quantum system of the form $H\Psi = E\Psi$ where $H = \Delta_n + V(\mathbf{x})$ (and Δ_n is the Laplace-Beltrami operator) on an n-dimensional real or complex pseudo-Riemannian manifold is second order **superintegrable** if there are $2n-1$ second order differential operators (expressed in local coordinates x_s), $L_j = \sum_{k,\ell} a^{k\ell}_{(j)} \partial^2_{k\ell} +$ lower order terms, $j = 1, \cdots, 2n-1$ that commute with $H = L_1$ and such that the corresponding quadratic forms $\mathcal{L}_j = \sum_{k,\ell} a^{k\ell}_{(j)} p_k p_\ell$ are functionally independent on an open set in the classical $2n$ dimensional phase space. In general, the potential V and the corresponding potential terms in the symmetry operators L_j need not be fixed but can range over a vector space of potentials. Suppose the pseudo-Riemannian space is conformally flat and the x_s are Cartesian-like coordinates with respect to which the metric tensor takes the diagonal form $g^{k\ell} = \delta^{k\ell}/\lambda(\mathbf{x})$. The potential is **nondegenerate** if it is the general solution of a system of the form

$$V_{jj} - V_{11} = \sum_{\ell=1}^{n} A^{jj,\ell}(\mathbf{x}) V_\ell, \quad j = 2, \cdots, n, \tag{13}$$

$$V_{kj} = \sum_{\ell=1}^{n} A^{kj,\ell}(\mathbf{x}) V_\ell, \quad 1 \le k < j \le n,$$

where all of the integrability conditions for this system of partial differential equations are identically satisfied, [8, 9]. (Here $V_j = \partial_j V$, etc. and the $A^{kj,\ell}$ are analytic in an open set of the n dimensional space with coordinates $\mathbf{x}$.) In this case there is an $n+2$ dimensional solution space of nondegenerate potentials V that can be parametrized by choosing the values $V(\mathbf{x}_0), V_{11}(\mathbf{x}_0), V_\ell(\mathbf{x}_0),\ \ell = 1, \cdots, n$ at some fixed point $\mathbf{x}_0$. One ordinarily refers to these nondegenerate potentials as $n+1$- parameter, neglecting the trivial constant that can be added to any potential. Equations (13) always arise from the Bertrand-Darboux equations for the potential of a superintegrable systems such that the $2n-1$ symmetries are linearly independent at a point [9], but in general the integrabilty conditions are not satisfied identically so that the space of solutions V has dimension $< n+2$. Thus the nondegenerate potentials are those with the maximum number of parameters possible.

Classification of the possible superintegrable systems with nondegenerate potentials in complex Euclidean n-space is a very difficult task that has been carried out only for $n = 2$, with significant results for $n = 3$, [8, 10]. However, there is an important subclass of such systems that can be constructed for all $n \geq 2$, based on their relationship to variable separation in generic Jacobi elliptic coordinates. The prototype superintegrable system which is nondegenerate in n dimensional flat space has the Hamiltonian [22]

$$H = \sum_{i=1}^{n}(\partial_i^2 + \alpha x_i^2 + \frac{\beta_i}{x_i^2}) + \delta, \quad \partial_i = \partial_{x_i}. \tag{14}$$

The system $H\Psi = E\Psi$ is superintegrable with nondegenerate potential and a basis of $n(n+1)/2$ second order symmetry operators given by

$$P_i = \partial_i^2 + \alpha x_i^2 + \frac{\beta_i}{x_i^2}, \quad J_{ij} = (x_i\partial_j - x_j\partial_i)^2 + \beta_i\frac{x_j^2}{x_i^2} + \beta_j\frac{x_i^2}{x_j^2}, \quad i \neq j.$$

Although there appear to be "too many" symmetry operators, all are functionally dependent on a subset of $2n-1$ functionally independent symmetries. A crucial observation is that the corresponding equation $H\Psi = E\Psi$ admits multiplicative separation in n generic elliptical coordinates $x_i^2 = c^2\Pi_{j=1}^n(u_j - e_i)/\Pi_{k\neq i}(e_k - e_i)$, simultaneously *for all* values of the parameters with $e_i \neq e_j$ if $i \neq j$ and $i, j = 1, \cdots, n$. If we were dealing with real Euclidean space then we could assume that the inequalities $e_1 < e_2 < \cdots < e_n$ and $e_1 < u_1 < e_2 < \cdots < e_n < u_n$ hold. Thus the equation is multiseparable and separates in a continuum of elliptic coordinate systems (and in many others besides). The n commuting symmetries characterizing a fixed elliptic separable system are polynomial functions of the e_i, and requiring separation for all e_i simultaneously sweeps out the full $n(n+1)/2$ space of symmetries and uniquely determines the nondegenerate potential. The infinitesimal distance in Jacobi elliptical coordinates u_j has the form

$$ds^2 = -\frac{c^2}{4}\sum_{i=1}^{n}\frac{\Pi_{j\neq i}(u_i - u_j)}{P(u_i)}du_i^2, \tag{15}$$

where $P(\lambda) = \Pi_{k=1}^n(\lambda - e_k)$. However, it is well known that (15) is a flat space metric for any polynomial $P(\lambda)$ of order $\leq n$ and that each choice of such a $P(\lambda)$ defines an elliptic type multiplicative separable solution of the Laplace - Beltrami eigenvalue problem (with constant potential) in complex Euclidean n-space, [24]. The distinct cases are labeled by the degree of the polynomial and the multiplicities of its distinct roots. If for each distinct case we determine the most general potential that admits separation for all e_i compatible with the multiplicity structure of the roots, we obtain a unique superintegrable system with nondegenerate potential and $n(n+1)/2$ second order symmetries, [20, 10]. These are the generic complex superintegrable systems. (Thus, for $n=3$ there are 7 distinct cases for $-\frac{1}{4}\ P(\lambda)$:

$$(\lambda-e_1)(\lambda-e_2)(\lambda-e_3),\ (\lambda-e_1)(\lambda-e_2)^2,\ (\lambda-e_1)^3,$$

$$(\lambda-e_1)(\lambda-e_2),\ (\lambda-e_1)^2,\ (\lambda-e_1),\ 1,$$

where $e_i \neq e_j$ for $i \neq j$. The first case corresponds to Jacobi elliptic coordinates.) The number of distinct generic superintegrable systems for each integer $n \geq 2$ is

$$\sum_{j=0}^{n} p(j),$$

where $p(j)$ is the number of integer partitions of j, given by the Euler generating function

$$\frac{1}{\prod_{k=1}^{\infty}(1-t^k)} = \sum_{j=0}^{\infty} p(j)t^j.$$

All of the generic separable systems, their potentials and their defining symmetries can be obtained from the basic Jacobi elliptic system in n dimensions by a complicated but well defined set of limit processes [20, 10]. Although we cannot write down master canonical expressions for all such generic systems in Cartesian coordinates, it is easy to take these limits and write down a master equation for the separated ordinary differential equations in the elliptic coordinates, u_i. The separation equation for each of the coordinates $x = u_i$ is essentially the same:

$$\sqrt{P(x)}\frac{d}{dx}(\sqrt{P(x)}\frac{dF}{dx}) + (a_{n+p_0}x^{n+p_0} + \cdots + a_n x^n + \hat{a}_{n-1}x^{n-1} + \cdots + \hat{a}_0 \tag{16}$$

$$+\frac{b_1^{(1)}}{(x-e_1)} + \frac{b_2^{(1)}}{(x-e_1)^2} + \cdots + \frac{b_{p_1}^{(1)}}{(x-e_1)^{p_1}} + \cdots + \frac{b_1^{(r)}}{(x-e_r)} + \frac{b_2^{(r)}}{(x-e_r)^2} + \cdots$$

$$+\frac{b_{p_r}^{(r)}}{(x-e_r)^{p_r}})F = 0, \quad P(x) = -4(x-e_1)^{p_1}\cdots(x-e_r)^{p_r} \tag{17}$$

and $p_0 + p_1 + \cdots + p_r = n$. Here the n constants $\hat{a}_s,\ s = 0, \cdots, n-1$ are the separation constants for the superintegrable system in these elliptic coordinates. In particular, $E = \hat{a}_{n-1}$. The other $n+1$ constants depend on the $n+1$ parameters in the potential and can be assigned arbitrarily by specifying the appropriate potential.

First we look for explicit solutions of the single equation (16), ignoring the fact that it is a separation equation for a superintegrable system. To obtain polynomials, we perform a gauge transformation and look for solutions of the form

$$\Psi = \exp\left(\ell_1 x + \ell_2 x^2 + \cdots + \ell_{p_0+1}x^{p_0+1} + \frac{c_1^{(1)}}{(x-e_1)} + \frac{c_2^{(1)}}{(x-e_1)^2} + \cdots + \right.$$

$$\left. \frac{c_{p_1-1}^{(1)}}{(x-e_{p_1})^{p_1-1}} + \cdots + \frac{c_1^{(r)}}{(x-e_r)} + \frac{c_2^{(r)}}{(x-e_r)^2} + \cdots + \frac{c_{p_r-1}^{(r)}}{(x-e_r)^{p_r-1}}\right)$$

$$\times \Pi_{t=1}^{r}(x-e_t)^{q_t}\Pi_{i=1}^{s}(x-\theta_i) = f(x)\Pi_{i=1}^{s}(x-\theta_i) = f(x)\Phi(x). \qquad (18)$$

We require that the differential equation satisfied by $\Phi(x)$ is of the form

$$(r_{n-p_0}x^{n-p_0}+\cdots+r_0)\frac{d^2\Phi}{dx^2}+(s_nx^n+\cdots+s_0)\frac{d\Phi}{dx}+(t_{n-1}x^{n-1}+t_0)\Phi = 0. \quad (19)$$

This can be achieved by choosing values for $a_{n+p_0},\cdots,a_n$ to make the coefficients of powers of x ($x^i,\ i=n+p_0,\cdots,n$) zero in the expression for the polynomial multiplying Φ in (19). Similarly by choosing the coefficients $b_j^{(J)} j = 1,\cdots,p_J$ all singular terms in this equation can be removed. Now, polynomial solutions of (19) can be sought and there are just enough conditions to determine $\hat{a}_{n-1},\cdots,\hat{a}_0$ and $\theta_i, i=1,\cdots,s$, in terms of a solution $\Phi = \Pi_{i=1}^{s}(x-\theta_i)$. We will not write down these equations for the separation constants in the general case, but note that they exist. The zeros θ_i of these polynomials satisfy relations of the form

$$\sum_{j\neq i}\frac{1}{\theta_i-\theta_j} + \ell_1\theta_i + 2\ell_2\theta_i^2 + \cdots + (p_0+1)\ell_{p_0+1}\theta_i^{p_0} + \frac{\frac{1}{4}p_1+q_1}{(\theta_i-e_1)} - \frac{c_1^{(1)}}{(\theta_i-e_1)^2}$$

$$+\frac{(p_1-1)c_{p_1-1}^{(1)}}{(\theta_i-e_1)^{p_1}}\cdots + \frac{\frac{1}{4}p_r+q_r}{(\theta_i-e_r)} - \frac{c_r^{(1)}}{(\theta_i-e_r)^2} + \cdots + \frac{(p_r-1)c_{p_r-1}^{(r)}}{(\theta_i-e_r)^{p_r}} = 0.$$

This is easy to see from the ansatz (18). Indeed if we evaluate the differential equation using (18) at $x=\theta_i$, these equations are readily obtained.

Next we relate these finite solutions to the superintegrable system in n dimensional Euclidean space from which they are obtained by separation of variables. To begin this process we first label Cartesian type coordinates by $y_j^J,\ j=1,\cdots,p_J,\ J=0,\cdots,r$. In terms of the separable coordinates $x=u_1,\cdots,u_n$ the Cartesian type coordinates are specified by the relations [25, 20]

$$y_j^0 = \frac{1}{j!}(\frac{\partial}{\partial\epsilon})^j\left[\frac{(\epsilon u_1-1)\cdots(\epsilon u_n-1)}{(1-\epsilon e_1)^{p_1}\cdots(1-\epsilon e_r)^{p_r}}\right]^{\frac{1}{2}}|_{\epsilon=0}\ .$$

For coordinates corresponding to y_k^K, $k=1,\cdots,p_K$ and $K\neq 0$ we choose

$$y_k^K=\frac{1}{(k-1)!}(\frac{\partial}{\partial e_K})^{k-1}\left[\frac{(u_1-e_K)\cdots(u_n-e_K)}{\Pi_{J\neq K}(e_K-e_J)^{p_J}}\right]^{\frac{1}{2}}.$$

These coordinates can be identified with the standard Cartesian coordinates via the quadratic form

$$\sum_{s+t=p_0+1} y_s^0 y_t^0+\sum_{J=1}^{r}\sum_{s+t=p_J} y_s^J y_t^J=x_1^2+\cdots.+x_n^2.$$

Indeed if we label the Cartesian coordinates by x_j^J, $j=1,\cdots,p_J$, $J=0,\cdots,r$ then we can make the choices

$$y_j^J=\frac{1}{\sqrt{2}}(x_j^J+ix_{2m+1-j}^J),\quad y_{j+m}^J=\frac{1}{\sqrt{2}}(x_j^J-ix_{2m+1-j}^J)$$

where $J=2m,\ j=1,\cdots,m$, and

$$y_j^J=\frac{1}{\sqrt{2}}(x_j^J+ix_{2m+2-j}^J),\quad y_{j+m+1}^J=\frac{1}{\sqrt{2}}(x_j^J-ix_{2m+2-j}^J),\quad y_m^J=x_{m+1}^J$$

where $J=2m+1,\ j=1,\cdots m$. The infinitesimal distance is given by

$$ds^2=\sum_{I=0}^{r}\sum_{i=1}^{p_I}(dx_i^I)^2=\sum_{i=1}^{n}\frac{\Pi_{j\neq i}(u_i-u_j)}{P(u_i)}du_i^2.$$

If we take the equations (17) and allow x to have the values $u_1,\cdots,u_n$ and $F=f(u_1,..,u_n)$ then we can calculate each of the expressions $\hat{a}_j, j=n-1,\cdots,0$ in terms of the general coordinates y_i^I. In permitting x to take on these values we are identifying the resulting ordinary differential equations as the separation equations of a separable Schrödinger equation with potential in n dimensions. To do this calculation we need to make use of the symmetric functions S_j, $j=1,\cdots,n$, defined by the expressions

$$S_0^i=1,\quad S_1^i=S_1^i[u_1,\cdots,u_{i-1},u_{i+1}\cdots,u_n]=\sum_{j=1}^{n}u_j,\ j\neq i,$$

$$S_2^i=S_2^i[u_1,\cdots,u_{i-1},u_{i+1}\cdots u_n]=\sum_{j\neq k}^{n}u_k u_j+\cdots,\quad k,j\neq i,$$

$$\cdots,\quad S_{n-1}^i=S_{n-1}^i[u_1,\cdots,u_{i-1},u_{i+1}\cdots,u_n]=\Pi_{j=1}^n u_j,\ j\neq i.$$

Previously we considered $\hat{a}_j$, $j = 0, \cdots, n-1$ as an eigenvalue. Now we consider it as the symmetry operator with this eigenvalue.) Solving the n equations of type (17) we find

$$\hat{a}_j = \sum_{i=1}^{n} \frac{1}{\Pi_{k\neq i}(u_i - u_k)} S^i_{n-1-j} P_i$$

where

$$P_i = \sqrt{P(u_i)}\frac{d}{du_i}(\sqrt{P(u_i)}\frac{d}{du_i}) + a_{n+p_0}u_i^{n+p_0} + \cdots + a_n u_i^n +$$

$$\frac{b_1^{(1)}}{(u_i - e_1)} + \frac{b_2^{(1)}}{(u_i - e_1)^2} + \cdots + \frac{b_p^{(1)}}{(u_i - e_1)^{p_1}} + \cdots + \frac{b_1^{(r)}}{(u_i - e_r)} + \frac{b_2^{(r)}}{(u_i - e_r)^2}$$

$$+ \cdots + \frac{b_p^{(r)}}{(u_i - e_r)^{p_r}}.$$

If we look for an eigenfunction F of the form

$$F = \Pi_{j=1}^n f(u_j)\Pi_{r=1}^s (u_j - \theta_r)$$

then the expression for the second part of this viz $\Pi_{j=1}^n \Pi_{r=1}^s (u_j - \theta_r)$ can be written in terms of the y_i^I via the identity

$$S_0^0\lambda^{p_0-1} + S_1^0\lambda^{p_0-2} + \cdots + S_{p_0-1}^0 + \sum_{J=1}^{r}\sum_{j=1}^{p_J} \frac{S^J_{p_J+2-j}}{(\lambda - e_J)^j} = \frac{\Pi_{j=1}^n (u_j - \lambda)}{P(\lambda)}$$

where $S_j^J = \sum_{\ell+k=s} y_\ell^J y_k^J$.

The product function $\Pi_{j=1}^n f(u_j)$ can be written in terms of the y_i^I variables via our implicit evaluation of symmetric functions $S_j[u_1, \cdots, u_n]$ in terms of the y_i^I. We can determine that $\hat{a}_{n-1}$ has the form of an n dimensional Laplacian plus a potential which is rational in the coordinates y_i^I.

We now take a closer look at $H = \hat{a}_{n-1}$. The differential operator part of this term can be identified with the Laplacian of the operator associated with the metric ds^2. The various parts of the potential can be associated with the calculation of Stäckel multiplier terms of the form

$$\sum_{i=1}^{n} \frac{1}{\Pi_{k\neq i}(u_i - u_k)} Q_i$$

where $Q_i = u_i^s$ or $Q_i(u_i - e_J)^{-t}$ corresponding to the coefficients a_s or $b_t^{(J)}$ of (17). We can calculate a basis for these Stäckel multipliers in the y_i^I

coordinates. For Stäckel multipliers with terms of the form $Q_i = u_i^s$ we observe the identity

$$\frac{\Pi_{I=1}^r(1-\epsilon e_I)}{\Pi_{i=1}^n(\epsilon u_i - 1)} = \sum_{i=1}^n \frac{u_i^{p_0-1}\Pi_{J=1}^r(u_i-e_J)^{p_J}}{\Pi_{j\neq i}(u_j-u_i)(\epsilon u_i-1)}.$$

From this identity we can calculate combinations of Stäckel multipliers of the form with $Q = u_i^s$ and $s > n-1$. Indeed we differentiate according to

$$(\frac{\partial}{\partial\epsilon})^t\left[\frac{\Pi_{I=1}^r(1-\epsilon e_I)}{\Pi_{i=1}^n(\epsilon u_i-1)}\right]\Bigg|_{\epsilon=0} = \sum_{i=1}^n(-1)^t t!\frac{u_i^{p_0+t-1}\Pi_{J=1}^r(u_i-e_J)^{p_J}}{\Pi_{j\neq i}(u_j-u_i)(\epsilon u_i-1)}\Bigg|_{\epsilon=0}$$

where $1 \leq t \leq p_0+1$.

From this formula we see that we can we can construct Stäckel multipliers in which the Q_i are polynomials in u_i where the maximal order is $n+t-1$. Further, we can explicitly calculate these in terms of the coordinates y_i^0. This can be seen from the formulas for S_s^0. If we define

$$\frac{1}{(\tilde{y}_0^0)^2} = \frac{\Pi_{I=1}^r(1-\epsilon e_I)}{\Pi_{i=1}^n(\epsilon u_i-1)} = \sum_{i=1}^n\frac{u_i^{p_0-1}\Pi_{J=1}^r(u_i-e_J)^{p_J}}{\Pi_{j\neq i}(u_j-u_i)(\epsilon u_i-1)}$$

then the terms that we need for our Stäckel multipliers are

$$\tilde{Y}_t^0 = (\frac{\partial}{\partial\epsilon})^t\left[\frac{1}{(\tilde{y}_0^0)^2}\right]|_{\epsilon=0}, \quad t = 1,\cdots,p_0+1\ ,$$

where we can use the recurrence formula $\partial/\partial\epsilon(\tilde{y}_j^0) = (j+1)\tilde{y}_{j+1}^0$ and, once all the calculations are done, we put $\epsilon \to 0$ and then $\tilde{y}_j^0 \to y_j^0$.

In this way we can construct basis elements for those parts of our potential which are entirely polynomial in the u_i variables. For the parts that are singular in these variables we use the identity

$$\frac{\Pi_{J\neq K}(e_K-e_J)^{p_J}}{\Pi_{i=1}^n(u_i-e_K)} = \sum_{i=1}^n\frac{\Pi_{J\neq K}(u_i-e_J)^{p_J}}{(u_i-e_K)\Pi_{j\neq i}(u_i-u_j)}.$$

The right hand side of this expression is in the form of a Stäckel multiplier with $Q_i = \Pi_{J\neq K}(u_i-e_J)^p/(u_i-e_K)$ which can be written as a polynomial in u_i plus a term $1/(u_i-e_K)$. We see immediately that Stäckel multipliers containing terms of the form $1/(u_i-e_k)^t$ where $t \geq 2$ can be generated by differentiating this identity $t-1$ times with respect to e_K. Indeed we recognize that

$$\frac{\Pi_{J\neq K}(e_K-e_J)^{p_J}}{\Pi_{i=1}^n(u_i-e_K)} = \frac{1}{(y_1^K)^2}.$$

The additional terms that arise in the potential and complete our basis have the form

$$Y_t^K = (\frac{\partial}{\partial e_k})^t \left[\frac{1}{(y_1^K)^2}\right], \quad t = 0, \cdots, p_K - 1.$$

These expressions can be determined in terms of the y_s^K via the recursion relation

$$\frac{\partial}{\partial e_K}(y_k^K) = (k+1) y_{k+1}^K.$$

Indeed we could establish the eigenvalues of the operators $\hat{a}_j$ by acting on the functions $\Phi(u_1, \cdots, u_n)$ when expressed in terms of a basis of symmetric polynomials in the variables u_i. In general these expressions are of a different form than those obtained from the single separated equation, and are easier to calculate.

This procedure is best illustrated by a specific example. If we choose $P(x) = (x-1)^2 x$ and $n = 4$ then the corresponding separated equation is

$$\{\sqrt{P(\lambda)}\frac{d}{d\lambda}(\sqrt{P(\lambda)}\frac{d}{d\lambda} \tag{20}$$

$$+(a_5\lambda^5 + a_4\lambda^4 + \hat{a}_3\lambda^3 + \hat{a}_2\lambda^2 + \hat{a}_1\lambda + \hat{a}_0 + \frac{b_1}{\lambda - 1} + \frac{b_2}{(\lambda-1)^2} + \frac{c_1}{\lambda})\}f(\lambda) = 0.$$

If we consider this equation in isolation then we can find polynomial solutions as follows. Making the assignments

$$a_5 = -4\ell_4^2,\ a_4 = 4\ell_4(2\ell_4 - \ell_5),\ b_1 = \ell_6(-\ell_6 + 2p - 1),\ b_2 = -\ell_6^2,\ c_1 = -\frac{1}{2}q(2q-1),$$

we can look for solutions of the form

$$f(\lambda) = \exp(\ell_4\lambda^2 + \ell_5\lambda + \frac{\ell_6}{\lambda - 1})(\lambda-1)^p\lambda^q\Pi_{r=1}^s(\lambda - \theta_r)$$

where the θ_j satisfy the relations

$$4\ell_5 + 8\ell_4\theta_i + \frac{2(2p+1)}{\theta_i - 1} - \frac{6\ell_6}{(\theta_i - 1)^2} + \frac{4q+1}{\theta_i} - \sum_{j \neq i}\frac{4}{\theta_i - \theta_j} = 0.$$

The values of the constants $\hat{a}_j$, $j = 0, \cdots, 3$ can be represented in terms of the θ_i via the formulas

$$\begin{aligned}
\hat{a}_3 &= -\ell_5^2 - 4\ell_4^2 + 8\ell_4\ell_5 - [4(p+q+s)+5]\ell_4, \\
\hat{a}_2 &= -2\ell_5^2 - 4\ell_4\ell_5 + 4\ell_4\ell_6 - [2(p+q+s) + \frac{3}{2}]\ell_5 + 4(p+2q+2s+2)\ell_4 - 4\ell_4\sum_{i=1}^s\theta_i,
\end{aligned}$$

$$
\begin{aligned}
\hat{a}_1 &= -\ell_5^2 + 2\ell_5\ell_6 + 2[p+1+2q+2s]\ell_5 - [4s+4q+3]\ell_4 - (p+q)(p+q+\frac{1}{2}) \\
&- s(s+\frac{1}{2}) - 2(p+q)s + (-2\ell_5 + 8\ell_4)\sum_{i=1}^{s}\theta_i - 4\ell_4\sum_{i=1}^{s}\theta_i^2, \\
\hat{a}_0 &= -(2q+2s+\frac{1}{2})\ell_5 + (2p+2q+2s-\frac{1}{2})\ell_6 + [4\ell_5 - 4\ell_4 - 2p - 2q \\
&- \frac{1}{2}(4s-1))\sum_{i=1}^{s}\theta_i + (-2\ell_5 + 8\ell_4)\sum_{i=1}^{s}\theta_i^2 - 4\ell_4\sum_{i=1}^{s}\theta_i^3.
\end{aligned}
$$

We now show the superintegrable context of these observations. Consider the coordinates in four dimensions given by the formulas

$$y_1^2 = -4(s-1)(t-1)(u-1)(v-1),\ y_4 = -2(s+t+u+v-2),$$

$$2y_1y_2 = 4(3-2(s+t+u+v)+(st+su+sv+tu+tv+uv)-stuv),\ y_3^2 = -4stuv,$$

where we identify four dimensional Cartesian coordinates as

$$y_1 = \frac{1}{\sqrt{2}}(x+iy), \quad y_2 = \frac{1}{\sqrt{2}}(x-iy), \quad y_3 = z, \quad y_4 = t.$$

The appropriate superintegrable system in four dimensions is

$$H = \Delta + \frac{\alpha}{y_1^2} + \frac{\beta y_2}{y_1^3} + \frac{\gamma}{y_3^2} + \delta(4y_4^2 + 2y_1y_2 + y_3^2) + \mu y_4$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_t^2$. In these coordinates the separation equations for $H\Psi = E\Psi$ have exactly the form (20) with the identifications

$$\alpha = i4(1-2p),\ \beta = -8\ell_6^2,\ \gamma = -2q(2q-1),\ \delta = -\ell_4^2,\ \mu = 4\ell_4\ell_5,$$

and identification of $\hat{a}_3$ with E. Indeed, if we use coordinates y_1, y_2, y_3 and y_4 in the Schrödinger equation $\Delta\Psi = E\Psi$ and employ the relationship

$$\frac{2y_1y_2}{L-1} + \frac{y_1^2}{(L-1)^2} + \frac{y_3^2}{L} - 2y_4 - L = -\frac{(u-L)(v-L)(s-L)(t-L)}{L(L-1)^2}$$

we can find finite solutions of the form

$$\Psi = \exp(\frac{1}{2}\ell_4(2y_1y_2 + y_3^2 + 2y_4^2) - \ell_5 y_4 + 2i\ell_6\frac{y_2}{y_1})y_1^{p/2}y_3^{q/2}\times$$

$$\Pi_{r=1}^{s}(\frac{2y_1y_2}{\theta_r - 1} + \frac{y_1^2}{(\theta_r-1)^2} + \frac{y_3^2}{\theta_r} - 2y_4 - \theta_r)$$

where the same equations for the zeros are satisfied. This method of derivation is in complete analogy with that used by Niven [21]. The $\hat{a}_2, \hat{a}_1, \hat{a}_0$ are then separation constants. A useful feature of this observation is the availability of direct algebraic methods to determine the eigenvalues of these parameters. Indeed in this case if we write (in operator form)

$$\hat{a}_i = \frac{1}{(s-t)(s-u)(s-v)} P_s^i + \frac{1}{(t-u)(t-s)(t-v)} P_t^i + \frac{1}{(u-t)(u-s)(u-v)} P_u^i$$
$$+ \frac{1}{(v-t)(v-u)(v-s)} P_v^i$$

then typically

$$P_s^3 = \sqrt{P(s)}\frac{\partial}{\partial s}(\sqrt{P(s)}\frac{\partial}{\partial s}) + (a_1 s^5 + a_2 s^4 + \frac{b_1}{s-1} + \frac{b_2}{(s-1)^2} + \frac{c_1}{s}),$$

$$P_s^4 = (t+u+v)P_s^3, \quad P_s^5 = (tu+tv+uv)P_s^3, \quad P_s^6 = tuvP_s^3.$$

Since the products of separated eigenfunctions are symmetric in s, t, u, v we see that eigenvalues of $\hat{a}_i$ can be obtained by acting on a basis of symmetric functions. We consider the symmetric functions

$$S_0 = 1, \ S_1 = s+t+u+v, \ S_2 = st+su+sv+tu+tv+uv,$$

$$S_3 = stu + tuv + suv + stv, \ S_4 = stuv.$$

We can, as a particular example, look for solutions of the form

$$\begin{aligned} \Psi &= \exp(\ell_4(s^2+t^2+u^2+v^2) + \ell_5(s+t+u+v) + \ell_6(\frac{1}{s-1} + \frac{1}{t-1} + \frac{1}{u-1} \\ &+ \frac{1}{v-1})[(s-1)(t-1)(u-1)(v-1)]^p (stuv)^q [B_0 S_0 + B_1 S_1 + B_2 S_2 + B_3 S_3]. \end{aligned}$$

If we look for eigenvalues of $\hat{a}_i$, $i = 0, \cdots, 3$ we obtain the determanental equations for the corresponding eigenvalues a_i. In the simplified case for which $\ell_4 = \ell_5 = \ell_6 = p = q = 1$ we find

$$(a_3-18)(a_3-14)4 = 0, \ (2a_2-33)(4a_2^4-1344a_2^3+42472a_2^2-596992a_2+3141805) = 0,$$

$$(a_1-3)(4a_1^4 - 92a_1^3 + 1005a_1^2 - 6318a_1 + 15641) =$$
$$(2a_0-11)(4a_0^4 - 120a_0^3 + 1309a_0^2 - 6321a_0 + 11723) = 0.$$

3 Polynomial solutions from superintegrable systems on the complex n sphere.

Applications of superintegrable systems on the complex n-sphere proceed in analogy with the approach in complex Euclidean n-space. There is an important subclass of such systems with nondegenerate potentials that can be constructed for all $n \geq 2$, based on their relationship to variable separation in generic Jacobi elliptic coordinates. The prototype superintegrable system which is nondegenerate on the n dimensional sphere has the Hamiltonian

$$H = \sum_{i=0}^{n}(\partial_i^2 + \frac{\beta_i}{s_i^2}) + \delta, \quad \partial_i = \partial_{s_i}, \tag{21}$$

where $\sum_{i=0}^{n} s_i^2 = 1$. The system $H\Psi = E\Psi$ has a basis of $n(n+1)/2$ second order symmetry operators given by

$$M_{ij} = M_{ji} = (s_i\partial_j - s_j\partial_i)^2 + \beta_i\frac{s_j^2}{s_i^2} + \beta_j\frac{s_i^2}{s_j^2}, \quad i \neq j.$$

Again, all are dependent on a subset of $2n-1$ functionally independent symmetries. The corresponding equation $H\Psi = E\Psi$ admits multiplicative separation in the generic n dimensional elliptical coordinates.

$$s_i^2 = \Pi_{j=1}^{n}(u_j - e_i)/\Pi_{k\neq i}(e_k - e_i)$$

simultaneously *for all* values of the parameters with $e_i \neq e_k$ if $i \neq k$ and $i, k = 0, \cdots, n$. If we were dealing with the real sphere then we could assume that the inequalities $e_1 < e_2 < \cdots < e_n$ and $e_0 < u_1 < e_1 < \cdots < e_{n-1} < u_n < e_n$ hold. Thus the equation is multiseparable and separates in a continuum of elliptic coordinate systems (and in others besides).

The infinitesimal distance in Jacobi elliptical coordinates u_j has the form

$$ds^2 = \sum_{i=1}^{n}\frac{\Pi_{j\neq i}(u_i - u_j)}{P(u_i)}du_i^2, \tag{22}$$

where $P(\lambda) = -4\Pi_{k=0}^{n}(\lambda - e_k)$. Here $e_j \neq e_k$ for $j \neq k$. However, it is well known that (22) is a metric on the sphere for any polynomial $P(\lambda)$ of order $n+1$ and that each choice of such a $P(\lambda)$ defines an elliptic type multiplicative separable solution of the Laplace - Beltrami eigenvalue problem (with constant potential) on the complex n-sphere. The distinct cases are labeled by the multiplicities of the $n+1$ roots. If for each distinct case we determine the most general potential that admits separation for all e_j

compatible with the multiplicity structure of the roots, we determine a unique superintegrable system with nondegenerate potential and $n(n+1)/2$ second order symmetries, [25, 10]. These are the generic superintegrable systems on the sphere. (Thus, for $n = 3$ there are 5 distinct cases for $-\frac{1}{4}P(\lambda)$:

$$(\lambda - e_0)(\lambda - e_1)(\lambda - e_2)(\lambda - e_3),\ (\lambda - e_0)(\lambda - e_1)(\lambda - e_2)^2,\ (\lambda - e_0)^2(\lambda - e_1)^2,$$

$$(\lambda - e_0)(\lambda - e_1)^3,\ (\lambda - e_0)^4,$$

where $e_j \neq e_k$ for $j \neq k$. The first case corresponds to Jacobi elliptic coordinates.) The number of distinct generic superintegrable systems for each integer $n \geq 2$ is $p(n+1)$ where $p(j)$ is the number of integer partitions of j.

As in the Euclidean case, all of the generic separable systems, their potentials and their defining symmetries can be obtained from the basic Jacobi elliptic system in n dimensions by a complicated but well defined set of limit processes [25]. Although we cannot write down master canonical expressions for all such generic systems in Cartesian-like coordinates s_k, it is easy to take these limits and write down a master equation for the separated ordinary differential equations in the elliptic coordinates, u_i. The separation equation for each of the coordinates $x = u_i$ is essentially of the same form:

$$\sqrt{P(x)}\frac{d}{dx}(\sqrt{P(x)}\frac{dF}{dx}) + \Big(a_{n-1}x^{n-1} + \cdots + \hat{a}_0 \tag{23}$$

$$\frac{b_1^{(1)}}{(x-e_1)} + \frac{b_2^{(1)}}{(x-e_1)^2} + \cdots + \frac{b_{p_1}^{(1)}}{(x-e_1)^{p_1}} + \cdots + \frac{b_1^{(r)}}{(x-e_r)} + \frac{b_2^{(r)}}{(x-e_r)^2}$$

$$+ \cdots + \frac{b_{p_r}^{(r)}}{(x-e_r)^{p_r}}\Big) F = 0$$

where $P(x) = -4(x-e_1)^{p_1} \cdots (x-e_r)^{p_r}$ and $p_1 + \cdots + p_r = n+1$, given that there are now n variables $u_1, \cdots, u_n$. Much of what has already been done for Euclidean space coordinates carries over, with the essential difference that there is no singularity at infinity. If we look for finite solutions we can proceed as before. We seek solutions of the form

$$\begin{aligned}
\Psi &= \exp\left(\frac{c_1^{(1)}}{(x-e_1)} + \frac{c_2^{(1)}}{(x-e_1)^2} + \cdots + \frac{c_{p_1-1}^{(1)}}{(x-e_{p_1})^{p_1-1}} + \cdots\right. \\
&+ \left.\frac{c_1^{(r)}}{(x-e_r)} + \frac{c_2^{(r)}}{(x-e_r)^2} + \cdots + \frac{c_{p_r-1}^{(r)}}{(x-e_r)^{p_r-1}}\right) \Pi_{t=1}^{r}(x-e_t)^{q_t}\Pi_{i=1}^{s}(x-\theta_i) \\
&= g(x)\Pi_{i=1}^{s}(x-\theta_i)
\end{aligned}$$

for which the zeros of the polynomials satisfy

$$
\begin{aligned}
0 &= \sum_{j\neq i}\frac{1}{\theta_i-\theta_j}+\cdots+\frac{\frac14 p_1+q_1}{(\theta_i-e_1)}-\frac{c_1^{(1)}}{(\theta_i-e_1)^2}+\cdots\\
&+ \frac{(p_1-1)c_{p_1-1}}{(\theta_i-e_1)^{p_1}}\cdots+\frac{\frac14 p_r+q_r}{(\theta_i-e_r)}-\frac{c_r^{(1)}}{(\theta_i-e_r)^2}+\cdots+\frac{(p_r-1)c_{p_r-1}}{(\theta_i-e_r)^{p_r}}
\end{aligned}
$$

To relate these finite solutions to superintegrable systems on the sphere we introduce a set of coordinates that we again call y_k^K, $k=1,\cdots,p_k$, $K=1,\cdots,r$ and which are given by the formulas

$$y_k^K=\frac{1}{(k-1)!}(\frac{\partial}{\partial e_K})^{k-1}\left[\frac{(u_1-e_K)\cdots(u_n-e_K)}{\Pi_{J\neq K}(e_K-e_J)^{p_J}}\right]^{\frac12}.$$

These coordinates can be identified with the usual coordinates on a sphere via the relation

$$\sum_{K=1}^{r}\sum_{s+t=p_K}y_s^K y_t^K=1.$$

This is done via the assignment

$$y_j^J=\frac{1}{\sqrt2}(s_j^J+is_{2m+1-j}^J),\ y_{j+m}^J=\frac{1}{\sqrt2}(s_j^J-is_{2m+1-j}^J),\quad J=2m,\ j=1,\cdots,m,$$

$$y_j^J=\frac{1}{\sqrt2}(s_j^J+is_{2m+2-j}^J),\ y_{j+m+1}^J=\frac{1}{\sqrt2}(s_j^J-is_{2m+2-j}^J),\quad y_m^J=s_{m+1}^J$$

where

$$\sum_{K=1}^{r}\sum_{k=1}^{p_k}(s_k^K)^2=1$$

The infinitesimal distance is given by

$$ds^2=\sum_{K=1}^{r}\sum_{k=1}^{p_K}(ds_k^K)^2=\sum_{i=1}^{n}\frac{\Pi_{j\neq i}(u_i-u_j)}{P(u_i)}du_i^2.$$

Solving the n equations of type (23), we find the operator expressions

$$\hat a_j=\sum_{i=1}^{n}\frac{1}{\Pi_{k\neq i}(u_i-u_k)}S_{n-1-j}^{i}P_i$$

where

$$
\begin{aligned}
P_i &= \sqrt{P(u_i)}\frac{d}{du_i}(\sqrt{P(u_i)}\frac{d}{du_i})+\left(\frac{b_1^{(1)}}{(u_i-e_1)}+\frac{b_2^{(1)}}{(u_i-e_1)^2}+\cdots+\frac{b_p^{(1)}}{(u_i-e_1)^{p_1}}\right.\\
&+ \left.\cdots+\frac{b_1^{(r)}}{(u_i-e_r)}+\frac{b_2^{(r)}}{(u_i-e_r)^2}+\cdots+\frac{b_p^{(r)}}{(u_i-e_r)^{p_r}}\right).
\end{aligned}
$$

We look for an eigenfunction of the form

$$F = \Pi_{j=1}^{n} g(u_j) \Pi_{r=1}^{s} (u_j - \theta_r).$$

Then the second part of this expression viz $\Pi_{j=1}^{n}\Pi_{r=1}^{s}(u_j - \theta_r)$ can be written in terms of the y_i^I via the identity

$$\sum_{J=1}^{r} \sum_{j=1}^{p_J} \frac{S^J_{p_J+2-j}}{(\lambda - e_J)^j} = \frac{\Pi_{j=1}^{n}(u_j - \lambda)}{P(\lambda)}.$$

The product function $\Pi_{j=1}^{n} g(u_j)$ can be written in terms of the y_i^I variables via our implicit evaluation of symmetric functions $S_j[u_1, \cdots, u_n]$ in terms of the y_i^I. We can determine that $\hat{a}_{n-1}$ has the form of an n dimensional Laplacian plus a potential which is rational in the coordinates y_i^I. If we look at $H = \hat{a}_{n-1}$, we can identify the differential operator part with the Laplacian of the operator associated with the metric ds^2. The various parts associated with the potential require the calculation of the Stäckel multiplier terms of the form

$$\sum_{i=1}^{n} \frac{1}{\Pi_{k \neq i}(u_i - u_k)} Q_i$$

where $Q_i = (u_i - e_J)^{-t}$ corresponding to the coefficients $b_t^{(J)}$ of (23). We can readily calculate a basis for these Stäckel multipliers in terms of the y_i^I coordinates. For the parts that are singular in these variables we proceed as previously. We first observe the identity

$$\frac{\Pi_{J \neq K}(e_K - e_J)^{p_J}}{\Pi_{i=1}^{n}(u_i - e_K)} = \sum_{i=1}^{n} \frac{\Pi_{J \neq K}(u_i - e_J)^{p_J}}{(u_i - e_K)\Pi_{j \neq i}(u_i - u_j)}.$$

Note that the right hand side of this expression is in the form of a Stäckel multiplier with $Q_i = \Pi_{J \neq K}(u_i - e_J)^p/(u_i - e_K)$ which can be written as a polynomial in u_i plus a term $1/(u_i - e_K)$. We see immediately that Stäckel multipliers containing terms of the form $1/(u_i - e_k)^t$ where $t \geq 2$ can be generated by differentiating this identity $t - 1$ times with respect to e_K. Indeed,

$$\frac{\Pi_{J \neq K}(e_K - e_J)^{p_J}}{\Pi_{i=1}^{n}(u_i - e_K)} = \frac{1}{(y_1^K)^2}.$$

The additional terms that arise in the potential, and complete our basis, have the form

$$Y_t^K = \left(\frac{\partial}{\partial e_k}\right)^t \left[\frac{1}{(y_1^K)^2}\right], \quad t = 0, \cdots, p_K - 1.$$

These expressions can be determined in terms of the y^K_s via the recursion relation

$$\frac{\partial}{\partial e_K}(y^K_k) = (k+1)y^K_{k+1}.$$

We could now establish the eigenvalues of the operators $\hat{a}_j$ by acting on the functions $\Phi(u_1,\cdots,u_n)$ expressed in terms of a basis of symmetric polynomials in the variables u_i. The observations we have made are best illustrated by a specific example. If we choose $P(x)=(x-1)^2x$ and $n=2$ then the ordinary differential equation is

$$\{\sqrt{P(\lambda)}\frac{d}{d\lambda}(\sqrt{P(\lambda)}\frac{d}{d\lambda}+(\hat{a}_1\lambda+\hat{a}_0+\frac{b_1}{\lambda-1}+\frac{b_2}{(\lambda-1)^2}+\frac{c_1}{\lambda})\}f(\lambda)=0. \quad (24)$$

If we consider this equation in isolation then we can find polynomial solutions as follows. We make the assignments

$$c_1=-p(p+\frac{1}{2}) \quad b_1=-\ell(\ell-1)+2\ell q, \quad b_2=-\ell^2,$$

and look for solutions of the form

$$f(\lambda)=\exp(\frac{\ell}{\lambda-1})(\lambda-1)^q\lambda^p\Pi^s_{r=1}(\lambda-\theta_r)$$

where the θ_j satisfy the relations

$$\frac{2(3+2q)}{\theta_i-1}-\frac{\ell^2}{(\theta_i-1)^2}+\frac{3+4q}{\theta_i}+\sum_{j\neq i}\frac{4}{\theta_i-\theta_j}=0.$$

The values of the constants $\hat{a}_j$, $j=0,1$, can be represented in terms of the θ_i via the formulas

$$\hat{a}_1=-\frac{1}{2}(p+q+s+2)(2p+2q+2s+3),$$

$$\hat{a}_0=2s(2p+q+s+2)+\ell(2p+2q+2s+\frac{5}{2})+2p(p+q)+4p+\frac{3}{2}q+2$$
$$+\{-2(p+q)-\frac{1}{2}(5+4s)\}\sum_{r=1}^{s}\theta_r.$$

To display the superintegrable context of these observations we employ the coordinates on the 2-sphere given by the formulas

$$s_1^2+s_2^2=1-uv, \quad (s_1+is_2)^2=(u-1)(v-1), \quad s_3^2=uv,$$

and consider the superintegrable system [10]

$$H = p_{s_1}^2 + p_{s_2}^2 + p_{s_3}^2 + \alpha \frac{1}{(s_1 + is_2)^2} + \beta \frac{s_1 - is_2}{(s_1 + is_2)^3} + \frac{\gamma}{s_3^2}.$$

In these coordinates the separation equations for $H\Psi = E\Psi$ have exactly the form (24) with

$$\alpha = \ell(2q - 1), \quad \beta = -2\ell^2, \quad \gamma = -p(p + \frac{1}{2})$$

and the identification of $\hat{a}_1$ with E. Indeed, using coordinates y_1, y_2 and y_3 in the Schrödinger equation $\Delta\Psi = E\Psi$ and the relationship

$$\frac{2(s_1^2 + s_2^2)}{L - 1} + \frac{(s_1 + is_2)^2}{(L - 1)^2} + \frac{s_3^2}{L} = \frac{(u - L)(v - L)}{L(L - 1)^2}$$

we can find finite solutions of the form

$$\Psi = \exp[-2\ell(\frac{s_1 + is_2}{s_1 - is_2})] s_3^{p/2} (s_1 + is_2)^{q/2} \Pi_{r=1}^{s} (\frac{2(s_1^2 + s_2^2)}{\theta_r - 1} + \frac{(s_1 + is_2)^2}{(\theta_r - 1)^2} + \frac{s_3^2}{\theta_r})$$

where the same equations for the zeros are satisfied. This method of derivation is in complete analogy with that used by Niven. The $\hat{a}_1, \hat{a}_0$ are separation constants. A useful feature of this observation is the availability of direct algebraic methods to determine the eigenvalues of these parameters. We write

$$\hat{a}_1 = \frac{1}{(u - v)}[P_u - P_v], \quad \hat{a}_0 = \frac{1}{(u - v)}[vP_u - uP_v]$$

where

$$P_\lambda = \sqrt{P(\lambda)} \frac{\partial}{\partial \lambda} (\sqrt{P(\lambda)}\} \frac{\partial}{\partial \lambda}) + (a_1 \lambda + a_0 + \frac{b_1}{s - 1} + \frac{b_2}{(s - 1)^2} + \frac{c_1}{s}).$$

We can now demonstrate that eigenvalues of $\hat{a}_i$ can be obtained by acting on a basis of symmetric functions. As a particular example, if we look for solutions of the form

$$\Psi = \exp(\frac{\ell}{u - 1} + \frac{\ell}{v - 1})[(u - 1)(v - 1)]^q (uv)^p (k_1 uv + k_2(u + v) + k_3),$$

we find that the corresponding eigenvalues of a_1 and a_0 are determined by

$$(2a_1 + 45)^2 (a_1 + 14) = 0, \quad (a_0 - 18)(4a_0^2 - 200a_0 + 2423) = 0,$$

where we have made the simplifying assumptions $p = 1$, $q = 1$, $\ell = 1$.

Note further that if we perform a gauge transformation in the nonseparable Cartesian coordinates to try to find polynomial solutions in s_1^2, s_2^2 we will obtain exactly the same exponential phase factor as the product of the phase factors in u and in v. This means that the possible polynomial solutions in s_1^2, s_2^2 for a fixed energy eigenspace correspond exactly with the polynomial solutions symmetric in u and v. The polynomial energy eigenspace is degenerate but the diagonalization of a_0 breaks the degeneracy. For a dramatic example of similar behavior on the n-sphere see [26].

4 Conclusion.

We have demonstrated for a large class of superintegrable potentials that the method of separation of variables can be used to solve the corresponding Schrödinger equation in a specific form of elliptical coordinates. In these coordinates we have demonstrated that finite polynomial solutions are possible up to multiplication by an explicit function. This provides a generalization for the property of quasi-exact solvability as given by Ushveridze [12]. It also consolidates further the close connection with classical separation of variables concepts of families of polynomial solutions. In addition, the problem of finding all the algebraic equations determining the separation parameters has its natural place in the multivariable interpretation of these equations. The very fact that these polynomial solutions arise from superintegrable systems and that these systems have polynomial algebras associated with them implies that there are relations among the various solutions of the Schrödinger equation. This is a matter that we will pursue.

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