Completeness of multiseparable superintegrability in two dimensions

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Abstract

For complex Euclidean 2-space and the complex 2-sphere, we have found all classical and quantum superintegrable systems that correspond to nondegenerate potentials. These potentials have the property that associated with each of them is a quadratic algebra. Furthermore, each of these superintegrable systems admits separation of variables in more than one coordinate system. For degenerate superintegrable systems, both properties may be violated.

1 Introduction

The Hamilton Jacobi equation in two dimensions

$$H = p_x^2 + p_y^2 + V(x, y) = (\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2 + V(x, y) = E \tag{1}$$

can be solved by the method of separation of variables via the ansatz

$$S = U(u, E, \alpha) + V(v, E, \alpha)$$

for a suitable change of coordinates $u = u(x, y), v = v(x, y)$ and some separation constant α. Here, α can be characterized as the value of a classical constant of the motion of the form

$$A = a(x, y)p_x^2 + b(x, y)p_y^2 + c(x, y)p_x p_y + d(x, y). \tag{2}$$

The Hamiltonian H in (1) can admit at most three functionally independent 2nd order constants of the motion (2), of which one may be chosen as H itself. In this case the system is said to be *superintegrable* [1, 2, 3]. By definition, if a Hamiltonian admits two quadratic constants of the motion of the form

$$A_h = a_h(x, y)p_x^2 + b_h(x, y)p_y^2 + c_h(x, y)p_x p_y + d_h(x, y), h = 1, 2 \tag{3}$$

then each must satisfy the equation $\{H, A_h\} = 0$ with the usual Poisson bracket $\{\ ,\ \}$. It is also clear that $R = \{A_1, A_2\}$ is a 3rd order constant of the motion, i.e., $\{H, R\} = 0$, and is therefore functionally dependent on $A_0 = H, A_1$ and A_2: $R^2 = F(H, A_1, A_2)$.

Using the identity

$$\{K, G\} = \sum_{h=0}^{2} \{K, A_h\} \frac{\partial G}{\partial A_h},$$

we find the relations

$$\{A_1, R\} = \frac{1}{2}\frac{\partial F}{\partial A_2}, \{A_2, R\} = -\frac{1}{2}\frac{\partial F}{\partial A_1}, \tag{4}$$

which tells us that the constants of motion $\{A_1, R\}$ and $\{A_2, R\}$ are easily computed as functions of H, A_1, A_2 once F is known. We should also note that if F is polynomial in the invariants H, A_1, A_2 then so are $\{A_1, R\}$ and $\{A_2, R\}$. Of particular interest is the case when F is cubic in the generators, in which case these constants of the motion are quadratic in the basic invariants. The above relations then determine a *quadratic algebra*.

Note that for any such triple of invariants $A_i, i = 0, 1, 2$ we could always subject the system to a Euclidean motion where the coordinates x and y transform according to

$$x \to x\cos\alpha - y\sin\alpha + a, \qquad y \to x\sin\alpha + y\cos\alpha + b.$$

We regard systems so related as *equivalent*. Note that the invariants A_1 and A_2 can be expressed as

$$A_i = C_i^{\ell m} L_\ell L_m + d_i(x, y)$$

where $L_1 = p_x, L_2 = p_y$ and $L_3 = M = xp_y - yp_x$ are the generators of the Lie algebra of the Euclidean group. We have calculated [4] all the inequivalent superintegrable potentials V that are *nondegenerate* in the sense that they depend uniquely on four arbitrary parameters, i.e., one can prescribe the values of V, V_x, V_y, V_{yy} arbitrarily at any regular point (x_0, y_0) and these values determine V(x,y) uniquely. In so doing we obtained the following list (potential and invariants).

1. $$V = \frac{\alpha}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{x^2+y^2}}\left[\frac{\beta}{\sqrt{x^2+y^2+x}} + \frac{\gamma}{\sqrt{x^2+y^2-x}}\right],$$
$$A_1 = M^2 + d_1, A_2 = 2Mp_y + d_2.$$

   Separation of variables in parabolic, shifted elliptic and polar coordinates.

2. $$V = \frac{\alpha z}{\sqrt{c^2 - z^2}} + \frac{\beta}{\sqrt{(c-z)(c+z)}} + \frac{\gamma}{\sqrt{(c+z)(c+\bar{z})}}$$

   where $z = x + iy, \bar{z} = x - iy$, and

$$A_1 = M^2 + c^2 p_x^2 + d_1, \qquad A_2 = M(p_x + ip_y) + d_2.$$

   Separation in elliptic coordinates as well as shifted eliptic coordinates.

3.

$$V = \frac{\alpha}{\sqrt{\bar{z}(z+2)}} + \frac{\beta}{\sqrt{\bar{z}(z-2)}} + \frac{\gamma z}{\sqrt{z^2-4}},$$

$$A_1 = M^2 + (p_x - ip_y)^2 + d_1, \qquad A_2 = M(p_x + ip_y) + d_2.$$

Separation in hyperbolic and shifted elliptic coordinates.

4.

$$V = \frac{\alpha}{z^2} + \frac{\beta}{\sqrt{z^3(\bar{z}+2)}} + \frac{\gamma}{\sqrt{z(\bar{z}+2)}},$$

$$A_1 = M^2 + (p_x + ip_y)^2 + d_1, \qquad A_2 = (M + 2i(p_x + ip_y))^2 + (p_x + ip_y)^2 + d_2.$$

Separation in hyperbolic and displaced hyperbolic coordinates.

5.

$$V = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2},$$

$$A_1 = M^2 + d_1, A_2 = p_x^2 + d_2.$$

Separation in elliptic, shifted elliptic, polar and cartesian coordinates.

6.

$$V = \alpha\frac{z\bar{z}}{z^4} + \beta\frac{1}{z^2} + \gamma z\bar{z},$$

$$A_1 = M^2 + d_1, A_2 = (p_x + ip_y)^2 + d_2,$$

Separation in polar and hyperbolic coordinates.

7.

$$V = \frac{\alpha}{\sqrt{z^2-c^2}} + \frac{\beta\bar{z}}{\sqrt{z^2-c^2}(z+\sqrt{z^2-c^2})^2} + \gamma z\bar{z},$$

$$A_1 = M^2 + c^2 p_x^2 + d_1, \qquad A_2 = (p_x + ip_y)^2 + d_2.$$

Separation in elliptic and hyperbolic coordinates

8.

$$V = \frac{\alpha}{\sqrt{x^2+y^2}} + \beta\frac{(\sqrt{x^2+y^2}+x)^{\frac{1}{2}}}{\sqrt{x^2+y^2}} + \gamma\frac{(\sqrt{x^2+y^2}-x)^{\frac{1}{2}}}{\sqrt{x^2+y^2}},$$

$$A_1 = 2Mp_x + d_1, \qquad A_2 = 2Mp_y + d_2.$$

Separation in two types of parabolic coordinates.

9.

$$V = \alpha(4x^2 + y^2) + \beta x + \frac{\gamma}{y^2},$$

$$A_1 = p_x^2 + d_1, \qquad A_2 = 2Mp_y + d_2.$$

Separation in cartesian and parabolic coordinates.

10.

$$V = \alpha\bar{z} + \beta(z - \frac{3}{2}\bar{z}^2) + \gamma(z\bar{z} - \frac{1}{2}\bar{z}^3),$$

$$A_1 = 2M(p_x - ip_y) + (p_x + ip_y)^2 + d_1, \qquad A_2 = (p_x - ip_y)^2 + d_2.$$

Separation in semihyperbolic and shifted semihyperbolic coordinates.

11.

$$V = \frac{\alpha}{\sqrt{z}} + \beta x + \gamma\frac{x + z}{\sqrt{z}},$$

$$A_1 = p_y(p_x + ip_y) + d_1, \qquad A_2 = Mp_y + d_2.$$

Separation in parabolic and displaced parabolic coordinates.

12.

$$V = \alpha z + \frac{\beta z}{\sqrt{\bar{z}}} + \frac{\gamma}{\sqrt{\bar{z}}},$$

$$A_1 = (p_x - ip_y)^2 + d_1, \qquad A_2 = M(p_x + ip_y) + d_2.$$

Separation in semihyperbolic and shifted semihyperbolic coordinates.

(We have recently proved similar results for nondegenerate potentials on the complex 2-sphere, [5]. See §2 for a list of the superintegrable potentials.)

What are the distinguishing features of these 12 potentials and the quadratic invariants that describe them as superintegrable systems? If we calculate $R = \{A_1, A_2\}$ for each potential we can directly verify that

$$R^2 = a^{ijk}A_iA_jA_k + b^jA_j + c.$$

As a consequence, a quadratic algebra can be generated from the relations (4). Indeed if we turn off the potential and look for all quadratic algebras that have these properties up to group equivalence, we find that the equivalence classes are in one to one correspondence with the potentials presented above.

Furthermore, in each of the cases, solutions via separation of variables are possible in more than one coordinate system.

If we do not have the criterion of nondegeneracy satisfied we may not have a quadratic algebra or multiseparability. For example, consider the potential $V = \alpha(x+iy)^2$. The corresponding Hamiltonian admits the constants of the motion $(p_x + ip_y)^2$ and $(xp_y - yp_x)(p_x + ip_y) - \frac{i}{3}\alpha(x+iy)^3$, and separates in light cone coordinates only. This degenerate system is superintegrable but not multiseparable.

2 Nondegenerate potentials on the complex 2-sphere.

(Here $x^2 + y^2 + z^2 = 1, w = x + iy, \bar{w} = x - iy$.)

1.
$$V = \frac{\alpha}{x^2} + \frac{\beta}{y^2} + \frac{\gamma}{z^2}$$

2.
$$V = \frac{\alpha}{z^2} + \frac{\beta}{w^2} + \frac{\gamma\bar{w}}{w^3}$$

3.
$$V = \frac{\alpha}{w^2} + \frac{\beta z}{\sqrt{x^2+y^2}} + \frac{\gamma}{\sqrt{(x^2+y^2)w}}$$

4.
$$V = \frac{\alpha z}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{x^2+y^2}}[\frac{\beta}{\sqrt{x^2+y^2}+x} + \frac{\gamma}{\sqrt{x^2+y^2}-x}]$$

5.
$$V = \frac{\alpha}{w^2} + \frac{\beta z}{w^3} + \frac{\gamma(1-4z^2)}{w^4}$$

6.
$$V = \frac{\alpha(w + c^2\bar{w})}{\sqrt{(c^2\bar{w} - w^2 - 4c^2z^2))}} + \frac{\beta(w - c^2\bar{w})}{z^2\sqrt{(c^2\bar{w} - w^2 - 4c^2z^2))}} + \gamma\frac{(x^2+y^2)}{z^2}$$

Appendix. Separable coordinates on $E_{2,C}$.

(Here the coordinates are related to cartesian coordinates and to the operator that characterizes the separation constant in the free Hamiltonian)

1. Cartesian coordinates
$$x, y, \qquad A = p_x^2$$

2. Lightlike coordinates
$$x = \xi + i\eta, \quad y = \xi - i\eta, \qquad A = (p_x + ip_y)^2$$

3. Polar coordinates
$$x = r cos\theta, \quad y = r sin\theta, \qquad A = M^2 = (xp_y - yp_x)^2$$

4. Parabolic coordinates
$$x_P = \frac{1}{2}(\xi^2 - \eta^2), \quad y_P = \xi\eta, \qquad A = Mp_y$$

5. Elliptic coordinates
$$x_E^2 = c^2(u-1)(v-1), \quad y_E^2 = -c^2uv, \qquad A = M^2 + c^2p_x^2$$

6. Hyperbolic coordinates
$$x_H = c\frac{r^2 + s^2 + r^2s^2}{2rs}, \quad y_H = ic\frac{r^2 + s^2 - r^2s^2}{2rs}, \qquad A = M^2 + c^2(p_x + ip_y)^2$$

7. Semi-Hyperbolic coordinates
$$x_{SH} = -\frac{c}{4}(w-u)^2 + \frac{c}{2}(u+w), \quad y_{SH} = -i(-\frac{c}{4}(w-u)^2 - \frac{c}{2}(u+w)),$$
$$A = 4iM(p_x - ip_y) + c(p_x + ip_y)^2$$

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