Models for irreducible representations of quadratic algebras

Willard Miller (Joint with E.G. Kalnins and S. Post.)

miller@ima.umn.edu

University of Minnesota

Abstract 1

A quantum superintegrable system is an integrable n-dimensional Hamiltonian system with potential that admits $2n-1$ functionally independent constants of the motion, (finite order differential operators commuting with the Hamiltonian), the maximum number possible. If the operators are all second order, the system is second order superintegrable. In 2D and for 3D conformally flat spaces with nondegenerate potential, the algebra generated by the constants of the motion and their commutators has been proven to close at order 3 (the quadratic algebra).

Abstract 2

The representation theory of this algebra gives important information about the the energy eigenvalues and the spectra of the symmetries. In the 2D case we study possible realizations of the possible irreducible representations of the quadratic algebra by differential or difference operators in a single complex variable t acting on Hilbert spaces of analytic functions. Therse models greatly simplify the study of the representations and are also of considerable interest in their own right. In particular the Wilson polynomials emerge naturally in their full generality.

Abstract 3

We demonstrate that models of the classical superintegrable systems lead directly to models of the quantum systems, so that, for example, Wilson polynomials emerge directly from classical mechanics. The examples analyzed provide guidance concerning the models for higher dimensional superintegrable systems and may point the way towards a general structure theory for representations of quadratic algebras.

Basic 2D structure results 1

Theorem 1 *Let $\mathcal{H}$ be the Hamiltonian of a 2D superintegrable system with nondegenerate (i.e., 3 parameter) potential.*

1. *The space of second order symmetry operators is 3-dimensional.*
2. *The space of third order symmetry operators is 1-dimensional.*
3. *The space of fourth order symmetry operators is 6-dimensional and is spanned by symmetric quadratic polynomials in the second order symmetries.*
4. *The space of sixth order symmetry operators is 10-dimensional and is spanned by symmetric cubic polynomials in the second order symmetries.*

Basic 2D structure results 2

1. Every 2D superintegrable system with 1 or 2-parameter potential is a restriction of a nondegenerate potential.
2. However, for some 1-parameter potentials the structure of the quadratic algebra changes if the system admits a Killing vector, i.e., a first order symmetry operator. We will call these degenerate 1-parameter potentials.

Basic 2D structure results 3

1. In the degenerate 1-parameter potential case there is a 1-dimensional space of first order symmetry operators and a 4-dimensional space of second order symmetry operators.
2. The commutator of a first order and a second order symmetry operator is always expressible as a linear combination of second order symmetry operators.
3. The commutator of two second order symmetry operators is always expressible in terms of symmetric products of a first order and a second order symmetry.
4. There is a nontrivial quadratic symmetric polynomial relating the second order symmetry operators.

Basic 2D structure results 4

1. Thus the quadratic algebra generated by the symmetry operators always closes, at order 6 for nondegenerate potentials and at order 4 for degenerate (1-parameter) potentials.
2. Every 2D superintegrable system is Stäckel equivalent to a superintegrable system on a constant curvature space, either flat space E_2 or the complex 2-sphere S_2.
3. All superintegrable systems on E_2 and S_2 have been classified. There are 19 systems on E_2, 8 of them degenerate, and 9 systems on S_2, 3 of them degenerate.
4. Some of these systems are Stäckel equivalent. The numeber of distinct equivalence classes for all 2D superintegrable systems on possible manifolds is 13, 7 nondegenerate and 6 degenerate.

Basic 2D structure results 5

1. The quadratic algebras of two Stäckel equivalent systems are related by a simple permutation of the parameters in the potential and the enrgies. Thus They have the same abstract representation theory.
2. We conclude that the representation theory of quadratic algebras for second order superintegrable systems in 2D reduces to 13 distinct cases, 7 nondegenerate and 6 degenerate.

Degenerate Example: E14

1. The Hamiltonian is

$$H = \partial_x^2 + \partial_y^2 + \frac{a}{\bar{z}^2}$$

2. The symmetry operators are

$$L_1 = \frac{1}{2}\{M, p_-\} + \frac{\alpha}{i\bar{z}} \qquad L_2 = M^2 + \frac{\alpha z}{\bar{z}}$$

$$X = p_- = \partial_x - i\partial_y \qquad M = x\partial_y - y\partial_x$$

3. The algebra is

$$[L_1, L_2] = i\{X, L_2\} + \frac{i}{2}X,\ [L_1, X] = iX^2,$$

$$[L_2, X] = 2L_2,\ L_1^2 - L_2X^2 + \alpha H - \frac{11}{12}X^2 = 0$$

A model for E14

1.

$$X = \frac{1}{t}, \qquad L_1 = i\partial_t, \qquad L_2 = -t^2\partial_t^2 - 2t\partial_t + \alpha h - \frac{1}{4}t^2$$

2. $L_2\phi_\nu = -\nu^2\phi_\nu$,
   $\phi_\nu = \frac{1}{\sqrt{t}} J_\nu(\sqrt{-\alpha h}\,t)$, $J_\nu(t)$ a Bessel function
3. Action on a L_2 basis:

$$L_1\phi_\nu = \frac{i\sqrt{-\alpha h}}{4\nu}((2\nu-1)\phi_{\nu-1} - (1+2\nu)\phi_{\nu+1})$$

$$X\phi_\nu = \frac{\sqrt{-\alpha h}}{2\nu}(\phi_{\nu-1} + \phi_{\nu+1}).$$

4. X has continuous spectrum. No finite dimensional irreducible representions.

The nondegenerate system E1

1. The potential is

$$V = \omega^2(x^2+y^2) + \frac{1/4 - a^2}{x^2} + \frac{1/4 - b^2}{y^2}$$

2. Symmetries: $L_1 = (x\partial_y - y\partial_x)^2 + W_1$ and $L_2 = \partial_x^2 + W_2$.
3. The structure equations are $(R = [L_1, L_2])$:

$$[L_2, R] = -8L_2^2 + 8HL_2 - 16\alpha L_1 + 8\alpha,$$

$$[L_1, R] = -8HL_1 + 8\{L_2, L_1\} - 8(1+2\beta)H + 16(1+\beta+\gamma)L_2,$$

$$R^2 = -\frac{8}{3}\{L_2, L_2, L_1\} + 8H\{L_2, L_1\} - 4(3+4a)H^2 - 16(a+b-\frac{11}{3})$$

$$+16(2a+\frac{11}{3})HL_2 + \frac{176}{3}\omega L_2 + 16\omega(3a+3b+4ab+\frac{2\omega}{3})$$

A model for E1

1. Diagonalize L_1: $L_1 = 4i\omega t\partial_t + 2i\omega(1 - 2m - a)$
2. Find raising and lowering operators:

$$A = L_2 - \frac{R}{4i\omega} + \frac{L_1^2}{2\omega^2} - \frac{h}{2\omega^2}L_1 - \frac{1}{2} = t\partial_t^2 + (1+b)\partial_t$$

$$A^\dagger = L_2 + \frac{R}{4i\omega} + \frac{L_1^2}{2\omega^2} - \frac{h}{2\omega^2}L_1 - \frac{1}{2} =$$

$$64t^3\partial_t^2 + (192 - 64a - 128m)t^2\partial_t + (62m^2 + (64a - 128)m + 64 - 64a)t$$

3. If L_1 and L_2 are formally self-adjoint, then $i\omega$ must be real and $A^\dagger$ will be mutually adjoint.

The quantization condition

1. Assume there is a highest weight vector t^{m-1} for some positive integer m. Then $A^\dagger t^{m-1} = 0$.
2. This implies that the energy eigenvalue is given by $h = -2i\omega(2m + a + b)$
3. Imposing a finite dimensional representation we can require $At^0 = 0$.
4. Then the eigenvalue equation $L_2\psi_h = \lambda_k\psi_k$ has spectrum $\lambda_k = -3/2 - 2b - 2a - 4k - 2ba - 4bk - 4ak - 4k^2$ and the eigenfunctions are hypergeometric polynomials

$$\psi_k(t) = l_k(8t+1)^{m-1-k}{}_2F_1\left(\begin{matrix} -k, & -a-k \\ 1+b & \end{matrix} \middle| -8t\right)$$

$k = 0, \cdots, m-1$.

The Hilbert space

1. This gives us the energy eigenvalues and the spectral decompositions for L_1, L_2 as well as the expansion of the L_2 eigenbasis in terms of the L_1 eigenbasis.
2. Assuming that L_1, L_2 are self-adjoint we can determine the orthonormal basis of L_1 eigenvectors $\phi_n(t) = c_n t^n$ where $c_n = \sqrt{\frac{(-m)_n(-m-a)_n}{n!(b)_n}}$
3. Reproducing kernel

$$\sum \phi_n(t)\phi_n(\overline{s}) = {}_2F_1\left(\begin{matrix} -m, & -m-a \\ b & \end{matrix} \middle| t\overline{s}\right)$$

4. Similarly we can define explicit function space inner products to realize the various finite and infinite dimmensional irreducible representations of the quadratic algebra

A degenerate system on the 2-sphere: S3

1.

$$H = J_1^2 + J_2^2 + J_3^3 + \frac{\alpha}{s_3^2}, \quad s_1^2 + s_2^2 + s_3^2 = 1, \quad J_3 = s_1 \partial_{s_2} - s_2 \partial_{s_1}$$

plus cyclic permutations.

2. Symmetries

$$L_1 = J_1^2 + \frac{\alpha s_2^2}{s_3^2}, \quad L_2 = \frac{1}{2}(J_1 J_2 + J_2 J_1) - \frac{\alpha s_1 s_2}{s_3^2}, \quad X = J_3.$$

S3 structure equations

1. $$[L_1, X] = 2L_2, \quad [L_2, X] = -X^2 - 2L_1 + H - \alpha,$$
2. $$[L_1, L_2] = -\{L_1, X\} - (2\alpha - \frac{1}{2})X,$$
3. $$\frac{1}{6}\{L_1, X, X\} - HL_1 + L_2^2 + L_1^2 + (\alpha + \frac{11}{12}) + (\alpha - \frac{2}{3})L_1 - \frac{H}{6} = \frac{5\alpha}{6},$$

where $\{a, b, c\}$ is the 6 term symmetrizer equal to $abc + acb + cba + cab + bca + bac$.

A recent appearance 1

1. Variable mass Hamiltonians, used in semiconductor research , quantum dots, nuclei, and quantum liquids, "Effective mass".
2. A general variable mass Hamiltonian in 2D:

$$H = \partial_x \frac{1}{M(x,y)} \partial_x + \partial_y \frac{1}{M(x,y)} \partial_y + V(x,y)$$

3. Potential introduced by Quesne (2007). $M(x,y) = -\frac{1}{\cosh^2 qx}$ and $V(x,y) = -q^2 \cosh^2 qx + \frac{q^2 k(k-1)}{\sinh^2 qx}$
4. The metric must be $d\mathbf{s}^2 = q^2 \frac{dx^2 + dy^2}{\cosh^2 qx}$, constant curvature.

A recent appearance 2

1. If we take the coordinates on the sphere to be

$$s_1 = \frac{\sin qy}{\cosh qx}, \quad s_2 = \frac{\cos qy}{\cosh qx}, \quad s_3 = \tanh qx$$

and perform a gauge transformation we get exactly $S3$.

The model 1

1. Diagonalize $X = i(2t\frac{d}{dt} - m)$
2. Write $h = -(-m + a - 1)^2 + \frac{1}{4}$. If $\mu = -m \quad m \in$ **N**, then our model is finite dimensional:

$$L_1 = \left(t^3 + 2t^2 + t\right)\frac{d^2}{dt^2} + \left((2 - a - m)t^2 + 2(1 - m)t\right.$$

$$\left.+a - m\right)\frac{d}{dt} + m(a - 1)t + a(m + 1) - m - \frac{1}{2},$$

$$L_2 = i\left(-t^3 + t\right)\frac{d^2}{dt^2} + i\left((a + m - 2)t^2 + a - m\right)\frac{d}{dt} - im(a - 1)t$$

3. If $m = -\mu$ for arbitrary complex μ then the model is infinite dimensional bounded below.
4. The finite dimensional representations gives us the quantization for the energy levels.

The model 2

1. We have raising and lowering operators

$$A^{\dagger} = L_1 + iL_2 + \frac{1}{2}(X^2 - H + \alpha), \; A = L_1 - iL_2 + \frac{1}{2}(X^2 - H + \alpha)$$

2. We can use these to find normalization coefficients for our eigenfunctions, $\phi_n = k_n t^n \quad n = 0, ..., m$ as well as a weight function and reproducing kernel.

The model 3

1. We can also diagonalize $L_1\psi - \lambda\psi = 0$ using hypergeometric functions. If we are in the finite dimensional model, we have the requirement that our hypergeometric functions be polynomials of order m; this gives us a quantization condition $\lambda = -(n - a + \frac{1}{2})^2 + a^2 - \frac{1}{4}$. The eigenfunctions become, for $n = 0, ..., m$

$$\psi_\nu(t) = l_n(t+1)^n {}_2F_1(\begin{pmatrix} -m+n & 1-a+n & \\ -m+a & & \end{pmatrix} | -t)$$

where l_n is a normalization coefficient.

Classical models 1

1. We can also find models of the classical quadratic algebras in terms of functions of two canonically conjugate variables c, β The analog of the one-variable quantum models.
2. The existence of such models follows easily from standard Hamilton-Jacobi theory for integrable systems.
3. Why bother? BECAUSE THE CLASSICAL MODELS TELL US THE FORMS OF THE POSSIBLE QUANTUM MODELS. Sometimes the possible quantum models will be in terms of differential operators, sometimes in terms of difference operators.

Classical models 2

1. Classical $S3$ constants of the motion

$$\mathcal{L}_1 = \mathcal{J}_1^2 + \alpha\frac{s_2^2}{s_3^2},\ \mathcal{L}_2 = \mathcal{J}_1\mathcal{J}_2 - \alpha\frac{s_1 s_2}{s_3^2},\ \mathcal{X} = \mathcal{J}_3$$

2. Structure relations

$$\{\mathcal{X}, \mathcal{L}_1\} = -2\mathcal{L}_2,\ \{\mathcal{X}, \mathcal{L}_2\} = 2\mathcal{L}_1 - \mathcal{H} + \mathcal{X}^2 + \alpha,$$

$$\{\mathcal{L}_1, \mathcal{L}_2\} = -2(\mathcal{L}_1 + \alpha)\mathcal{X}$$

3. Casimir relation

$$\mathcal{L}_1^2 + \mathcal{L}_2^2 - \mathcal{L}_1\mathcal{H} + \mathcal{L}_1\mathcal{X}^2 + \alpha\mathcal{X}^2 + \alpha\mathcal{L}_1 = 0.$$

Classical models 3

1. Require $X \equiv \mathcal{X}_h = c$ and $\mathcal{H} = h$ in the structure equations.
2. Result is

$$I: \; L_1 = \frac{1}{2}(E - c^2 - \alpha) + \frac{1}{2}\sqrt{c^4 - 2c^2(E+\alpha) + (E-\alpha)^2}\,\sin 2\beta,$$

3. Factor the term under the square root and set

$$\phi = \arctan\left(\frac{\sqrt{-4\alpha}}{c^2 - (E+\alpha)^2}\right).$$

Classical models 4

1. Now we let $2\beta \to 2\beta + \phi$ to obtain

$$L_1 = \frac{1}{2}(E - c^2 - \alpha) + \frac{1}{2}\left((c^2 - (E+\alpha)^2)\sin 2\beta + 2i\sqrt{\alpha}\cos 2\beta\right),$$

$$L_2 = \frac{1}{2}\left((c^2 - (E+\alpha)^2)\cos 2\beta - 2i\sqrt{\alpha}\sin 2\beta\right), \qquad X = c.$$

Classical models 5

1. In this form we can see that the symmetries are polynomial in c which suggest the substitution $\beta \to t, c \to -\partial_t$ leading to a quantum realization of L_1, L_2 by second order differential operators. This leads directly to the differential operator model of $S3$ that we have already exhibited.
2. For a second model we require $L_1 \equiv (\mathcal{L}_1)_h = c$ and proceed in a similar fashion. The result is

$$II: \; L_1 = c, \qquad L_2 = \sqrt{c(E-c-\alpha)}\,\sin(2\sqrt{c+\alpha}\beta),$$

$$X = \sqrt{\frac{c(E-c-\alpha)}{c+\alpha}}\cos(2\sqrt{c+\alpha}\beta).$$

Classical models 6

1. This model cannot produce finite order differential operator realizations of the quantum quadratic algebra, due to the intertwining of square root dependence for c and exponential dependence for β. However, it will produce a difference operator realization via Taylor's theorem: $e^{a\partial_t} f(t) = f(t+a)$.

Classical models 7

1. To show this explicitly we make a coordinate change such that $2\sqrt{c+\alpha}\partial_c = \partial_{\mathcal{C}}$ which suggests realizations of the quantum operators in the form

$$L_1 f(t) = (t^2 - \alpha) f(t), \quad X f(t) = h(t) f(t+i) + m(t) f(t-i),$$

$$L_2 f(t) = -\frac{i}{2}(i+2t)h(t)f(t+i) + \frac{i}{2}(-i+2t)m(t)f(t-i).$$

2. A straightforward computation shows that the quantum algebra structure equations are satisfied if and only if

$$h(t)m(t+i) = \frac{1}{4}\frac{(\alpha - t^2 - it)(t^2 + it - E)}{t(t+i)}.$$

S3 difference model 1

1. Some manipulation yields the difference operator model

$$-iX = \frac{(1/2 - a - it)(\mu + a - 1/2 - it)}{2t} T^i$$

$$- \frac{(1/2 - a + it)(\mu + a - 1/2 + it))}{2t} T^{-i},$$

$$L_2 = -i \frac{(1 - 2it)(1/2 - a - it)(\mu + a - 1/2 - it)}{4t} T^i$$

$$-i \frac{(1 + 2it)(1/2 - a + it)(\mu + a - 1/2 + it))}{4t} T^{-i},$$

2. The basis functions are dual Hahn polynomials

$$f_n(t) = {}_3F_2 \left(\begin{matrix} -n & \frac{1}{2} - a + it & \frac{1}{2} - a - it \\ \mu & 1 - a & \end{matrix} ; 1 \right).$$

S3 difference model 2

1. The orthogonality and normalization are given by

$$\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(1/2-a+it)\Gamma(\mu+a-1/2+it)\Gamma(1/2+it)}{\Gamma(2it)}\right|^2 \times$$

$$s_n(t^2)s_{n'}(t^2)\,dt = \frac{\Gamma(n+\mu)\Gamma(n+1-a)\Gamma(n+\mu+a)n!}{(\mu)_n^2|(1-a)_n|^2}\delta_{nn'},$$

   where either 1) $\mu > 1/2 - a > 0$ or 2) $\mu > 0$ and $a = ((1-\mu)/2 + i\gamma$ is complex.

2. We can also use μ a negative integer to find finite dimensional difference operators representations with basis vectors of (not continuous) dual Hahn polynomials with a discrete measure.

The generic system S9 1

1. Potential
$$V = \frac{\frac{1}{4} - a^2}{s_1^2} + \frac{\frac{1}{4} - b^2}{s_2^2} + \frac{\frac{1}{4} - c^2}{s_3^2}$$
where $s_1^2 + s_2^2 + s_3^2 = 1$.
2. Hamiltonian
$$H = J_1^2 + J_2^2 + J_3^2 + V(x, y) = H_0 + V$$
where $J_3 = s_1 \partial_{s_2} - s_2 \partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutation.
3. Symmetries (symmetric form) L_1, L_2, L_3 where
$$L_1 = J_3^2 + W_1,\ L_2 = J_1^2 + W_2, L_3 = J_2^2 + W_3,$$
$$H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3.$$

The generic system S9 2

1. Structure equations

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8+16a_j)L_j + (8+16a_k)L_k + 8(a_j$$

$$R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1+12)L_1^2 - (16a_2+12)L_2^2 - (16a_3+12)L$$

$$+\frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1$$

$$+\frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3 + \frac{32}{3}(a_1 + a_2 + a_3)$$

$$+48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3.$$

Here i, j, k are chosen such that $\epsilon_{ijk} = 1$ where ϵ is the pure skew-symmetric tensor, and $R = [L_1, L_2]$.

2. In practice we substitute $L_3 = H - L_1 - L_2 - a_1 - a_2 - a_3$ into these equations

Abstract representation theory for S9 1

1. Assume the existence of a discrete set of eigenvectors f_n for the symmetry operator L_1
2. Only possibility is $L_1 f_n = \lambda_n f_n$ where

$$\lambda_n = -[2n+B]^2 + \mathcal{K}, \quad n = 0, 1, \cdots, m.$$

$$L_1 f_n = (\mathcal{K} - [2n+B]^2) f_n, \quad L_2 f_n = \sum_\ell C(\ell, n) f_\ell.$$

3. Structure equations give

$$H = -\frac{1}{4}(-4\mu + 2a + 2b + 2c + 5)(-4\mu + 2a + 2b + 2c + 3).$$

Abstract representation theory for S9 2

1. and

$$C(n,n) = \frac{1}{2}(2n+a+b+2)(2n+a+b)$$

$$-\frac{1}{2}\left[(-2\mu+a+b+c+2)^2+a^2-b^2-c^2-1\right]$$

$$+\frac{1}{2}\frac{(a^2-b^2)(a+b-2\mu+2)(a+b+2c-2\mu+2)}{(2n+a+b+2)(2n+a+b)},$$

$$C(n,n+1)C(n+1,n) =$$

$$16(n+1)(n+\mu)(n-c+\mu)(n+b+1)(n+a+1)(n+a+b+1)\times$$

$$\frac{(n-\mu+a+b+2)(n-\mu+a+b+c+2)}{(2n+a+b+3)(2n+a+b+2)^2(2n+a+b+1)}.$$

Abstract representation theory for S9 3

1. Here μ is an arbitrary complex parameter but if $\mu = -m, \quad m \in \mathbf{N}$ the representation becomes finite dimensional.
2. Only the coefficients $C(n, N)$ where $N = n, n \pm 1$ are nonzero.
3. Only the product $C(n, n+1)C(n+1, n)$ is determined, the individual factors can be modified via gauge transformation.

A model for for S9 4

1. The action of L_2 on the L_1 basis yields the general three-term recurrence relation for the Wilson polynomials p_n:

$$p_n(t^2) \equiv p_n(t^2, \alpha, \beta, \gamma, \delta) = (\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n \times$$

$$ {}_4F_3\left(\begin{matrix} -n, & \alpha+\beta+\gamma+\delta-n-1, & \alpha-t, & \alpha+t \\ \alpha+\beta, & \alpha+\gamma, & \alpha+\delta \end{matrix} ; 1\right)$$

with the identification

$$\alpha = -\frac{a+c+1}{2}+\mu,\ \beta = \frac{a+c+1}{2},\ \gamma = \frac{a-c+1}{2},\ \delta = \frac{a+c-}{2}$$

2. In the particular case that the representation is finite dimensional, we obtain instead the Racah polynomials.

Classical S9

1. The classical model with $\mathcal{H} = h$ and c, β conjugate variables, gives $\mathcal{L}_1 = c$ and

$$\mathcal{L}_2 = \frac{1}{2}(a_1 + 2a_2 + E - c) - \frac{(a_2 - a_3)(a_1 + 2a_2 + 2a_3 + E)}{2(c + a_2 + a_3)} +$$

$$\frac{\sqrt{(4a_1a_2 + 4a_1a_3 + 2c(E + a_1 + a_2 + a_3) + 4ca_1 - (E + a_1 + a}}{2(a_2 + a_3 + c)}$$

$$\times \cos(4\beta\sqrt{a_2 + a_3 + c}).$$

   which suggests a difference operator model.

2. We quantize so that $\beta = \partial_c$ and then take a change of coordinates so that $4\sqrt{a_2 + a_3 + c}\partial_c = \partial_t$ and obtain an ansatz difference model. Plugging this into the quantum algebra relations we recover exactly the recursion relations for the Wilson and Racah polynomials.

3D nondegenerate systems

1. $2n - 1 = 5$ but there are 6 linearly independent second order symmetries.
2. The quadratic algebra generated by the second order symmetries closes at order 6 again, but there are 4 independent commutators $R_1, \cdots, R_4$ and 10 relations expressing $R_i R_j$ as symmetric cubic polynomials in the 6 second order symmetries.
3. The 6 second order symmetries obey a quartic polynomial relation.
4. The quantum models will be in terms of 2 complex variables t_1, t_2.

Outlook

1. It appears that the 3D nondegenerate case will lead to 2-variable Wilson polynomials and their special cases. The theory is much more complicated but the quadratic algebra structure is very restricted.
2. Since Wilson polynomials extend naturally to Askey-Wilson polynomials, this suggests the possibility of a q-theory of superintegrability.
3. The general definition and representation theory for quadratic algebras of all orders is an important future project.