

Structure and classification results for second order superintegrable systems

*From Darboux and Koenigs manifolds to quadratic
algebras to algebraic varieties*

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Abstract

A classical (or quantum) superintegrable system is an integrable n -dimensional Hamiltonian system with potential that admits $2n - 1$ functionally independent constants of the motion, polynomial in the momenta, the maximum possible. If the constants are all quadratic the system is second order superintegrable. Such systems have remarkable properties: multi-integrability and multi-separability, a quadratic algebra of symmetries whose representation theory yields spectral information about the Schrödinger operator, deep connections with special functions and with QES systems.

Abstract continued

For $n=2$ (and $n=3$ on conformally flat spaces with non-degenerate potentials) we have worked out the structure and (nearly) classified the possible spaces and potentials. The quadratic algebra closes at order 6 and there is a 1-1 classical-quantum relationship. All such systems are Stäckel transforms of systems on complex Euclidean space or the complex sphere.

2nd order superintegrability (classical)

Classical superintegrable system on an n -dimensional local Riemannian manifold:

$$\mathcal{H} = \sum_{ij} g^{ij} p_i p_j + V(\mathbf{x}).$$

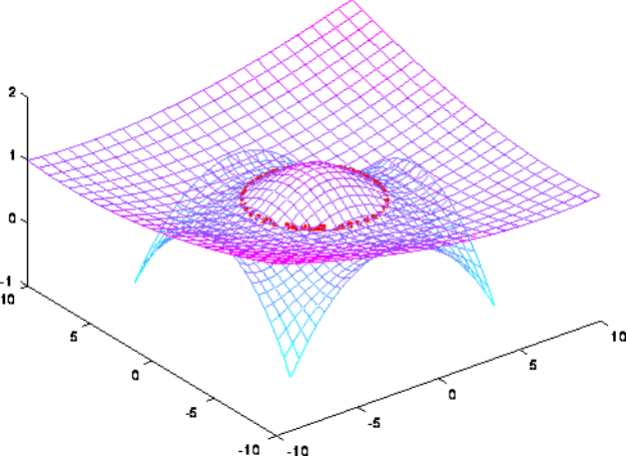
Require that Hamiltonian admits $2n - 1$ functionally independent 2nd-order symmetries $\mathcal{S}_k = \sum a_{(k)}^{ij}(\mathbf{x}) p_i p_j + W_{(k)}(\mathbf{x})$,
That is, $\{\mathcal{H}, \mathcal{S}_k\} = 0$ where $\{f, g\} = \sum_{j=1}^n (\partial_{x_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{x_j} g)$
is the Poisson bracket. Note that $2n - 1$ is the maximum possible number of functionally independent symmetries.

Significance

Generically, every trajectory $\mathbf{p}(t), \mathbf{x}(t)$, i.e., solution of the Hamilton equations of motion, is characterized (and parametrized) as a common intersection of the (constants of the motion) hypersurfaces

$$\mathcal{S}_k(\mathbf{p}, \mathbf{x}) = c_k, \quad k = 0, \dots, 2n - 2.$$

The trajectories can be obtained without solving the equations of motion. This is better than integrability.



2nd order superintegrability (quantum)

Schrödinger operator

$$H = \Delta + V(\mathbf{x})$$

where $\Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} g^{ij}) \partial_{x_j}$ is the Laplace-Beltrami operator on a Riemannian manifold, expressed in local coordinates x_j and S_1, \dots, S_n . Here there are $2n - 1$ second-order symmetry operators

$$S_k = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} a_{(k)}^{ij}) \partial_{x_j} + W_{(k)}, \quad k = 1, \dots, 2n - 1$$

with $S_1 = H$ and $[H, S_k] \equiv HS_k - S_kH = 0$.

Why second order?

This is the most tractable case due to the association with separation of variables. Special function theory can be applied and is relevant for the same reason.

Integrability and superintegrability

1. An integrable system has n functionally independent constants of the motion in involution. A superintegrable system has $2n - 1$ functionally independent constants of the motion. (Sometimes the definition of superintegrability also requires integrability. In this talk we prove it.)
2. Multiseparable systems yield many examples of superintegrability.
3. Superintegrable systems can be solved explicitly in multiple ways. It is the information gleaned from comparing the distinct solutions and expressing one solution set in terms of another that is a primary reason for their interest.

3D example:

The generalized anisotropic oscillator: Schrödinger equation $H\Psi = E\Psi$ or

$$H\Psi = -\frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Psi + V(x, y, z)\Psi = E\Psi.$$

The 4-parameter “nondegenerate” potential

$$V(x, y, z) = \frac{\omega^2}{2} (x^2 + y^2 + 4z^2 + \rho z) + \frac{1}{2} \left[\frac{\lambda_1}{x^2} + \frac{\lambda_2}{y^2} \right]$$

Separable coordinates

The corresponding Schrödinger equation has separable solutions in five coordinate systems: Cartesian coordinates, cylindrical polar coordinates, cylindrical elliptic coordinates, cylindrical parabolic coordinates and parabolic coordinates. The energy eigenstates for this equation are degenerate and important special function identities arise by expanding one basis of separable eigenfunctions in terms of another.

Basis for 2nd order symmetries

$$M_1 = \partial_x^2 - \omega^2 x^2 + \frac{\lambda_1}{x^2}, \quad M_2 = \partial_y^2 - \omega^2 y^2 - \frac{\lambda_2}{y^2},$$

$$P = \partial_z^2 - 4\omega^2(z + \rho)^2, \quad L = L_{12}^2 - \lambda_1 \frac{y^2}{x^2} - \lambda_2 \frac{x^2}{y^2} - \frac{1}{2},$$

$$S_1 = -\frac{1}{2}(\partial_x L_{13} + L_{13} \partial_x) + \rho \partial_x^2 + (z + \rho)(\omega^2 x^2 - \lambda_1 x^2),$$

$$S_2 = -\frac{1}{2}(\partial_y L_{23} + L_{23} \partial_y) + \rho \partial_y^2 + (z + \rho)(\omega^2 y^2 - \lambda_2 y^2),$$

where $L_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i}$.

Quadratic algebra closing at level 6

The nonzero commutators are

$$[M_1, L] = [L, M_2] = Q, \quad [L, S_1] = [S_2, L] = B,$$

$$[M_i, S_i] = A_i, \quad [P, S_i] = -A_i.$$

Nonzero commutators of the basis symmetries with Q (4th order symmetries) are expressible in terms of the second order symmetries, e.g.,

$$[M_i, Q] = [Q, M_2] = 4\{M_1, M_2\} + 16\omega^2 L,$$

$$[S_1, Q] = [Q, S_2] = 4\{M_1, M_2\},$$

$$[L, Q] = 4\{M_1, L\} - 4\{M_2, L\} - 16\lambda_1 M_1 + 16\lambda_2 M_2.$$

Level 6 closure

The squares of Q , B , A_i and products such as $\{Q, B\}$, (all 6th order symmetries) are all expressible in terms of 2nd order symmetries, e.g.,

$$Q^2 = \frac{8}{3}\{L, M_1, M_2\} + 8\omega^2\{L, L\} + 16\lambda_1 M_1^2 + 16\lambda_2 M_2^2$$

$$+ \frac{64}{3}\{M_1, M_2\} - \frac{128}{3}\omega^2 L - 128\omega^2 \lambda_1 \lambda_2,$$

$$\{Q, B\} = -\frac{8}{3}\{M_2, L, S_1\} - \frac{8}{3}\{M_1, L, S_2\} - 16\lambda_1\{M_2, S_2\}$$

$$- 16\lambda_2\{M_1, S_1\} - \frac{64}{3}\{M_1, S_2\} - \frac{64}{3}\{M_2, S_1\}.$$

Here $\{C_1, \dots, C_j\}$ is the completely symmetrized product of operators C_1, \dots, C_j .

Important properties-1

1. The algebra generated by the second order symmetries is *closed under commutation* in both the classical and operator cases. This is a remarkable, but typical of superintegrable systems with nondegenerate potentials.
2. Closure is at level 6, since we have to express the products of the 3rd order operators in terms of the basis of 2nd order operators.
3. The eigenfunctions of one separable system can be expanded in terms of the eigenfunctions of another, and this is the source of nontrivial special function expansion theorems.
4. The quadratic algebra identities allow us to relate eigenbases and eigenvalues of one symmetry operator to those of another.

Important properties-2

1. The representation theory of the abstract quadratic algebra can be used to derive spectral properties of the generators S_j , in a manner analogous to the use of Lie algebra representation theory to derive spectral properties of quantum systems that admit Lie symmetry algebras,
2. A common feature of quantum superintegrable systems is that after splitting off a gauge factor, the Schrödinger and symmetry operators are acting on a space of polynomials: MULTIVARIABLE ORTHOGONAL POLYNOMIALS.

Important properties-3

1. Closely related to the theory of QUASI-EXACTLY SOLVABLE SYSTEMS (QES). In many 2D and 3D examples the one-dimensional ODEs are quasi-exactly solvable and the eigenvalues that give polynomial solutions are easily obtained from the PDE superintegrable systems. Generalizes results of Ushveridze. Leads to new examples.

2D classical structure theory-1

Can assume Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{\lambda(x, y)}(p_1^2 + p_2^2) + V(x, y), \quad (x, y) = (x_1, x_2),$$

i.e., the complex metric is $ds^2 = \lambda(x, y)(dx^2 + dy^2)$.

Necessary and sufficient conditions that

$\mathcal{S} = \sum a^{ji}(x, y)p_j p_i + W(x, y)$ be a symmetry of \mathcal{H} are the Killing equations

$$a_i^{ii} = -\frac{\lambda_1}{\lambda}a^{i1} - \frac{\lambda_2}{\lambda}a^{i2}, \quad i = 1, 2$$

$$2a_i^{ij} + a_j^{ii} = -\frac{\lambda_1}{\lambda}a^{j1} - \frac{\lambda_2}{\lambda}a^{j2}, \quad i, j = 1, 2, \quad i \neq j,$$

2D classical structure theory-2

and the Bertrand-Darboux conditions on the potential
 $\partial_i W_j = \partial_j W_i$ or

$$(V_{22} - V_{11})a^{12} + V_{12}(a^{11} - a^{22}) =$$
$$\left[\frac{(\lambda a^{12})_1 - (\lambda a^{11})_2}{\lambda} \right] V_1 + \left[\frac{(\lambda a^{22})_1 - (\lambda a^{12})_2}{\lambda} \right] V_2.$$

Similar but more complicated conditions for the higher order symmetries.

Nondegenerate potentials-1

From the 3 second order constants of the motion we get 3 Bertrand-Darboux equations and can solve them to obtain fundamental PDEs for the potential of the form

$$V_{22} - V_{11} = A^{(22)}(\mathbf{x})V_1 + B^{22}(\mathbf{x})V_2, \quad V_{12} = A^{12}(\mathbf{x})V_1 + B^{12}(\mathbf{x})V_2.$$

If the B-D equations provide no further conditions on the potential and if the integrability conditions for the PDEs are satisfied identically, we say that the potential is *nondegenerate*. That means, at each regular point \mathbf{x}_0 where the A^{ij} , B^{ij} are defined and analytic, we can prescribe the values of V , V_1 , V_2 and V_{11} arbitrarily and there will exist a unique potential $V(\mathbf{x})$ with these values at \mathbf{x}_0 .

Nondegenerate potentials-2

Nondegenerate potentials depend on 3 parameters, in addition to the trival additive parameter. Degenerate potentials depend on < 3 parameters.

To recapitulate:

The system is **2nd order superintegrable with nondegenerate potential** if

1. it admits 3 functionally independent second-order symmetries (here $2N - 1 = 3$)
2. the potential is 3-parameter (in addition to the usual additive parameter) in the sense given above:

$$V(x, y) = \alpha_1 V^{(1)}(x, y) + \alpha_2 V^{(2)}(x, y) + \alpha_3 V^{(3)}(x, y).$$

Spaces of polynomial constants-3

THEOREM (Kalnins, Kress, Miller) : Let \mathcal{H} be the Hamiltonian of a 2D superintegrable (functionally linearly independent) system with nondegenerate potential.

1. The space of second order constants of the motion is exactly 3-dimensional.
2. The space of third order constants of the motion is at most 1-dimensional.
3. The space of fourth order constants of the motion is exactly 6-dimensional and spanned by products of second order symmetries.
4. The space of sixth order constants is at exactly 10-dimensional and spanned by cubics in the second order symmetries.
5. The system is multiseparable.

The Stäckel transform-1

ALL 2D SUPERINTEGRABLE NONDEGENERATE POTENTIALS IN EUCLIDEAN SPACE AND ON THE 2-SPHERE HAVE BEEN CLASSIFIED.

1. There are 11 families of nondegenerate potentials in flat space (4 in real Euclidean space).
2. There are 6 families of nondegenerate potentials on the complex 2-sphere (2 on the real sphere).

HOW TO GET ALL SUCH POTENTIALS ON GENERAL 2D MANIFOLDS?

The Stäckel transform-2

Suppose we have a (classical or quantum) superintegrable system

$$\mathcal{H} = \frac{1}{\lambda(x, y)}(p_1^2 + p_2^2) + V(x, y), \quad H = \frac{1}{\lambda(x, y)}(\partial_{11} + \partial_{22}) + V(x, y)$$

in local orthogonal coordinates, with nondegenerate potential $V(x, y)$ and suppose $U(x, y)$ is a particular case of the 3-parameter potential V , nonzero in an open set. Then the transformed systems

$$\tilde{\mathcal{H}} = \frac{1}{\tilde{\lambda}(x, y)}(p_1^2 + p_2^2) + \tilde{V}(x, y), \quad \tilde{H} = \frac{1}{\tilde{\lambda}(x, y)}(\partial_{11} + \partial_{22}) + \tilde{V}(x, y)$$

are also superintegrable, where $\tilde{\lambda} = \lambda U$, $\tilde{V} = \frac{V}{U}$.

The Stäckel transform-3

THEOREM:

1.

$$\{\tilde{\mathcal{H}}, \tilde{\mathcal{S}}\} = 0 \iff \{\mathcal{H}, \mathcal{S}\} = 0.$$

2.

$$\tilde{\mathcal{S}} = \sum_{ij} \frac{1}{\lambda U} p_i \left((a^{ij} + \delta^{ij} \frac{1 - W_U}{\lambda U}) \lambda U \right) p_j + \left(W - \frac{W_U V}{U} + \frac{V}{U} \right)$$

3.

$$[\tilde{H}, \tilde{S}] = 0 \iff [H, S] = 0.$$

4.

$$\tilde{S} = \sum_{ij} \frac{1}{\lambda U} \partial_i \left((a^{ij} + \delta^{ij} \frac{1 - W_U}{\lambda U}) \lambda U \right) \partial_j + \left(W - \frac{W_U V}{U} + \frac{V}{U} \right)$$

The Stäckel transform-4

COROLLARY: If $\mathcal{S}^{(1)}, \mathcal{S}^{(2)}$ are second order constants of the motion for \mathcal{H} , then

$$\{\tilde{\mathcal{S}}^{(1)}, \tilde{\mathcal{S}}^{(2)}\} = 0 \iff \{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}\} = 0.$$

If $S^{(1)}, S^{(2)}$ are second order symmetry operators for H , then

$$[\tilde{S}^{(1)}, \tilde{S}^{(2)}] = 0 \iff [S^{(1)}, S^{(2)}] = 0.$$

The Stäckel transform-5

This transform of one (classical or quantum) superintegrable system into another on a different manifold is called the **Stäckel transform**. Two such systems related by a Stäckel transform are called **Stäckel equivalent**.

THEOREM: Every nondegenerate second-order classical or quantum superintegrable system in two variables is Stäckel equivalent to a superintegrable system on a constant curvature space.

Classification results-1

BASIC RESULT: If $ds^2 = \lambda(dx^2 + dy^2)$ is the metric of a nondegenerate superintegrable system (expressed in coordinates x, y such that $\lambda_{12} = 0$) then $\lambda = \mu$ is a solution of the system

$$\mu_{12} = 0, \quad \mu_{22} - \mu_{11} = 3\mu_1(\ln a^{12})_1 - 3\mu_2(\ln a^{12})_2 + \left(\frac{a_{11}^{12} - a_{22}^{12}}{a^{12}}\right)\mu,$$

where either

$$I) \quad a^{12} = X(x)Y(y), \quad X'' = \alpha^2 X, \quad Y'' = -\alpha^2 Y,$$

or

Classification results-4

In a tour de force, Koenigs has classified all 2D manifolds (i.e., no potential) that admit exactly 3 second-order Killing tensors and listed them in two tables: Tableau VI and Tableau VII. Our methods show that these are exactly the spaces that admit superintegrable systems with nondegenerate potentials.

TABLEAU VI

$$[1] \quad ds^2 = \left[\frac{c_1 \cos x + c_2}{\sin^2 x} + \frac{c_3 \cos y + c_4}{\sin^2 y} \right] (dx^2 - dy^2)$$

$$[2] \quad ds^2 = \left[\frac{c_1 \cosh x + c_2}{\sinh^2 x} + \frac{c_3 e^y + c_4}{e^{2y}} \right] (dx^2 - dy^2)$$

$$[3] \quad ds^2 = \left[\frac{c_1 e^x + c_2}{e^{2x}} + \frac{c_3 e^y + c_4}{e^{2y}} \right] (dx^2 - dy^2)$$

$$[4] \quad ds^2 = \left[c_1(x^2 - y^2) + \frac{c_2}{x^2} + \frac{c_3}{y^2} + c_4 \right] (dx^2 - dy^2)$$

$$[5] \quad ds^2 = \left[c_1(x^2 - y^2) + \frac{c_2}{x^2} + c_3 y + c_4 \right] (dx^2 - dy^2)$$

$$[6] \quad ds^2 = \left[c_1(x^2 - y^2) + c_2 x + c_3 y + c_4 \right] (dx^2 - dy^2)$$

TABLEAU VII-1

$$\begin{aligned}
 [1] \quad ds^2 &= \left[c_1 \left(\frac{1}{\operatorname{sn}^2(x, k)} - \frac{1}{\operatorname{sn}^2(y, k)} \right) + c_2 \left(\frac{1}{\operatorname{cn}^2(x, k)} - \frac{1}{\operatorname{cn}^2(y, k)} \right) \right. \\
 &+ \left. c_3 \left(\frac{1}{\operatorname{dn}^2(x, k)} - \frac{1}{\operatorname{dn}^2(y, k)} \right) + c_4 (\operatorname{sn}^2(x, k) - \operatorname{sn}^2(y, k)) \right] \\
 &\times (dx^2 - dy^2)
 \end{aligned}$$

$$\begin{aligned}
 [2] \quad ds^2 &= \left[c_1 \left(\frac{1}{\sin^2 x} - \frac{1}{\sin^2 y} \right) + c_2 \left(\frac{1}{\cos^2 x} - \frac{1}{\cos^2 y} \right) \right. \\
 &+ \left. c_3 (\cos 2x - \cos 2y) + c_4 (\cos 4x - \cos 4y) \right] (dx^2 - dy^2)
 \end{aligned}$$

$$\begin{aligned}
 [3] \quad ds^2 &= [c_1 (\sin 4x - \sin 4y) + c_2 (\cos 4x - \cos 4y) \\
 &+ c_3 (\sin 2x - \sin 2y) + c_4 (\cos 2x - \cos 2y)] (dx^2 - dy^2)
 \end{aligned}$$

TABLEAU VII-2

$$[4] \ ds^2 = \left[c_1 \left(\frac{1}{x^2} - \frac{1}{y^2} \right) + c_2(x^2 - y^2) + c_3(x^4 - y^4) \right. \\ \left. + c_4(x^6 - y^6) \right] (dx^2 - dy^2)$$

$$[5] \ ds^2 = \left[c_1(x - y) + c_2(x^2 - y^2) + c_3(x^3 - y^3) + c_4(x^4 - y^4) \right] \\ \times (dx^2 - dy^2)$$

2D quantum systems-1

All of the Classical structure theory and classification results map to the quantum case in a 1-1 manner.

3D summary-1

1. We have classified all 2D superintegrable systems, including those with degenerate potentials.
2. The integrability condition approach that works for systems on 2D Riemannian manifolds extends to 3D conformally flat spaces ($2n-1=5$ functionally independent constants of the motion), with complications.
3. $5 \implies 6$ Theorem (Kalnins, Kress, Miller). 5 functionally independent second order symmetries for a nondegenerate superintegrable 3D system imply 6 linearly independent second order symmetries.
4. For 3D systems with nondegenerate potential the maximum possible dimensions of the spaces of second, third fourth and sixth order symmetries are 6,4,26 and 56, respectively, and these dimensions are achieved.

3D summary-2

1. The 3D quadratic algebra always closes at level 6 and there is a standard structure for the algebra.
2. The passage from the 3D conformally flat classical superintegrable systems to quantum superintegrable systems is still straightforward, but requires modifying the quantum potential by an additive term that is the scalar curvature.
3. All 3D conformally flat superintegrable systems with nondegenerate potential are Stäckel equivalent to superintegrable systems in 3D Euclidean space and on the 3-sphere.
4. All 3D conformally flat superintegrable systems have “essentially” been classified.

0.2 Canonical Forms for Superintegrable Potentials

2D, nondegenerate

$$\begin{aligned} V_{22} &= V_{11} + A^{22}V_1 + B^{22}V_2 \\ V_{12} &= A^{12}V_1 + B^{12}V_2, \end{aligned}$$

The integrability conditions for these equations are required to be satisfied identically. Thus a solution V is uniquely determined by specifying the values of V, V_1, V_2, V_{11} at a single point. The solution space is 4-dimensional.

3D, nondegenerate

$$\begin{aligned} V_{22} &= V_{11} + A^{22}V_1 + B^{22}V_2 + C^{22}V_3, \\ V_{33} &= V_{11} + A^{33}V_1 + B^{33}V_2 + C^{33}V_3, \\ V_{12} &= A^{12}V_1 + B^{12}V_2 + C^{12}V_3 \\ V_{13} &= A^{13}V_1 + B^{13}V_2 + C^{13}V_3 \\ V_{23} &= A^{23}V_1 + B^{23}V_2 + C^{23}V_3 \end{aligned}$$

The integrability conditions for these equations are required to be satisfied identically. Thus a solution V is uniquely determined by specifying the values of V, V_1, V_2, V_3, V_{11} at a single point. The solution space is 5-dimensional. Simple consequences of the integrability conditions are the linear relations

$$\begin{aligned} A^{23} = B^{13} = C^{12}, \quad C^{13} = A^{33} - A^{22} + B^{12}, \\ B^{23} = A^{13} + C^{22}, \quad C^{23} = A^{12} + B^{33} \end{aligned}$$

There is no obstruction to extending the 5-dimensional space of symmetries to a 6-dimensional space.

3D, 3 parameter

$$V_{ij} = A^{ij}V_1 + B^{ij}V_2 + C^{ij}V_3, \quad 1 \leq i \leq j \leq 3$$

The integrability conditions for these equations are required to be satisfied identically. Thus a solution V is uniquely determined by specifying the values of V, V_1, V_2, V_3 at a single point. The solution space is 4-dimensional. There is an obstruction so the 5-dimensional space of symmetries cannot be extended to a 6-dimensional space.

0.1 3D Nondegenerate Systems (classical)

$$\mathcal{H} = \frac{1}{\lambda}(p_1^2 + p_2^2 + p_3^2) + V(\mathbf{x}), \quad ds^2 = \lambda(x, y, z)(dx^2 + dy^2 + dz^2)$$

2nd order symmmetries:

$$\mathcal{S} = \sum_{i,j=1}^3 a^{ij} p_i p_j + W$$

18 symmetry equations:

$$\partial_k a^{ij} = f(D^{\ell m}, a^{hs}, \lambda), \quad D = A, B, C$$

Integrability conditions for symmetry equations:

$$(*) \quad \partial_\ell(\partial_k a^{ij}) = \partial_k(\partial_\ell a^{ij})$$

Integrability conditions for the potential:

$$(**) \quad \mathbf{v} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_{11} \end{pmatrix}, \quad \partial_i \mathbf{v} = \mathbf{A}^{(i)} \mathbf{v},$$

$$\partial_j(\partial_i \mathbf{v}) = \partial_i(\partial_j \mathbf{v}) \iff \partial_j \mathcal{A}^{(i)} - \partial_i \mathcal{A}^{(j)} = [\mathcal{A}^{(i)}, \mathcal{A}^{(j)}]$$

(*) and (**) \implies

$$\partial_k D^{\ell m} = g^{(k\ell m)}(D^{rs}, \lambda)$$

where g is a third order polynomial in the D^{rs} , as well as 5 quadratic identities for the $D^{\ell m}$

Example: No quadratic algebra-1

The Euclidean space Schrödinger equation with 3-parameter extended Kepler-Coulomb potential:

$$\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) + \left[2E + \frac{2\alpha}{\sqrt{x^2 + y^2 + z^2}} - \left(\frac{\lambda_1}{x^2} + \frac{\lambda_2}{y^2} \right) \right] \Psi = 0$$

Example: No quadratic algebra-2

This equation admits separable solutions in the four coordinates systems: spherical, sphero-conical, prolate spheroidal and parabolic coordinates. Again the bound states are degenerate and important special function identities arise by expanding one basis of separable eigenfunctions in terms of another.

Example: No quadratic algebra-3

However, the space of second order symmetries is only 5 dimensional and, although there are useful identities among the generators and commutators that enable one to derive spectral properties algebraically, there is no finite quadratic algebra structure.

The key difference with our first example is that the 3-parameter Kepler-Coulomb potential is **degenerate** and it cannot be extended to a 4-parameter potential.

Example not of constant curvature-1

Space with metric

$$ds^2 = \lambda(A, B, C, D, E, \mathbf{x})(dx^2 + dy^2 + dz^2),$$

$$\lambda = A(x + iy) + B \left(\frac{3}{4}(x + iy)^2 + \frac{z}{4} \right)$$

$$+ C \left((x + iy)^3 + \frac{1}{16}(x - iy) + \frac{3z}{4}(x + iy) \right)$$

$$+ D \left(\frac{5}{16}(x + iy)^4 + \frac{z^2}{16} + \frac{1}{16}(x^2 + y^2) + \frac{3z}{8}(x + iy)^2 \right) + E.$$

The nondegenerate potential is

$$V = \lambda(\alpha, \beta, \gamma, \delta, \epsilon, \mathbf{x}) / \lambda(A, B, C, D, E, \mathbf{x}).$$

Example not of constant curvature-2

If $A = B = C = D = 0$ this is a classical superintegrable system on complex Euclidean space, and it extends to a quantum superintegrable system.

The quadratic algebra always closes, and for general values of A, B, C, D, E the space is not of constant curvature. This is an example of a superintegrable system that is Stäckel equivalent to a system on complex Euclidean space.

Fine structure

For fine structure of superintegrable systems we drop the requirement of nondegeneracy and study the various possibilities for systems with potentials depending on fewer parameters. For 2D systems the structure is very simple.

THEOREM: Every 2D system with a one- or two-parameter potential and 3 functionally linearly independent second-order symmetries is the restriction of some nondegenerate (three-parameter) potential.

Fine 3D structure 0

For 3D systems the results are much more complicated and have not yet been fully determined. We first consider those systems that just fail to be nondegenerate in the sense that the four functions \mathcal{S}_h together with \mathcal{H} are functionally linearly independent in the six-dimensional phase space but that the associated potential functions V span only a 3 dimensional subspace of the 4 dimensional space of solutions of the potential equations, ignoring the trivial added constant. In particular, we stipulate that we can arbitrarily prescribe V_1, V_2, V_3 at a regular point, but not V_{11} independently of these.

Fine 3D structure 1

This circumstance can occur in only two ways: either the potential is a 3-parameter restriction of a nondegenerate potential, or the integrability conditions for the potential equations are not satisfied identically and an additional condition is imposed.

Fine 3D structure 2

In either case the canonical potential equations are replaced by the 6 equations

$$V_{ij} = \tilde{A}^{ij}V_1 + \tilde{B}^{ij}V_2 + \tilde{C}^{ij}V_3, \quad i \leq j,$$

whose integrability conditions are satisfied identically. The canonical equations still hold, but with the identifications

$$D^{ij} = \tilde{D}^{ij}, \quad 1 \leq i < j \leq 3, \quad D^{kk} = \tilde{D}^{kk} - \tilde{D}^{11}, \quad k = 2, 3,$$

where $D = A, B, C$. For short, we will call the solutions of the potential equations **3-parameter potentials**.

Fine 3D structure 3

In analogy to the nondegenerate potential case we can compute the full set of integrability conditions satisfied by the potential, and we can use the 10 second order Killing tensor equations and the $3 \times 3 = 9$ conditions for the derivatives $a_h^{\ell m}$ that result from substituting the potential relations into the 3 Bertrand-Darboux equations and equating coefficients of V_1, V_2, V_3 . There are 19 conditions for the 18 derivatives $a_h^{\ell m}$. We get exactly the standard symmetry equations and the remaining condition

$$a^{11}(\tilde{C}^{12} - \tilde{B}^{13}) + a^{22}(\tilde{A}^{23} - \tilde{C}^{12}) + a^{33}(\tilde{B}^{13} - \tilde{A}^{23}) + a^{12}(\tilde{A}^{13} + \tilde{C}^{22} - \tilde{C}^{11}) \\ + a^{13}(\tilde{C}^{23} + \tilde{B}^{11} - \tilde{B}^{33} - \tilde{A}^{12}) + a^{23}(\tilde{B}^{12} + \tilde{A}^{33} - \tilde{A}^{22} - \tilde{C}^{13}) = 0,$$

which we can regard as an obstruction.

Fine 3D structure 4

The analogous obstruction equation appears for the nondegenerate potential case, but there the linear integrability conditions for the nondegenerate potential cause the obstruction to vanish identically. By exploitation of the integrability conditions for the potential and for the symmetry equations we have obtained the following results:

THEOREM: A 3D 3-parameter potential is a restriction of a nondegenerate potential if and only if the obstruction vanishes identically. If the obstruction doesn't vanish then the space of second order symmetries is 5 dimensional and the system is uniquely determined by the values of $\tilde{D}^{ij}, i \leq j$, $D = A, B, C$ at a single regular point.

Fine 3D structure 5

The extended Kepler-Coulomb system is an example of a 3-parameter potential with obstruction, as are two other real Euclidean space potentials in Evans' list. Another example is defined by the potential

$$V = \frac{\alpha}{\sqrt{x^2 + y^2 + z^2}} + \frac{\beta}{(x + iy)^2} + \frac{\gamma(x - iy)}{(x + iy)^3}.$$

These are true 3-parameter potentials in the sense that they cannot be extended to nondegenerate potentials.

Fine 3D structure 6

THEOREM: Let V be a superintegrable true 3-parameter potential on a conformally flat space. Then the space of third order constants of the motion is 3-dimensional and is spanned by Poisson brackets of the second order constants of the motion. The Poisson bracket of two second order constants of the motion is uniquely determined by the matrix commutator of the second order constants at a regular point.

THEOREM: Let V be a superintegrable true 3-parameter potential in a 3D conformally flat space. Then V defines a multiseparable system.

Fine 3D structure 7

THEOREM: Every superintegrable system with true 3-parameter potential on a 3D conformally flat space is Stäckel equivalent to a superintegrable system on either 3D flat space or the 3-sphere.

Although the spaces of higher order symmetries for true 3-parameter systems have an interesting structure, the quadratic algebra doesn't close.

THEOREM: For a superintegrable system with true 3-parameter potential on a 3D conformally flat space there exist two second order constants of the motion $\mathcal{S}_1, \mathcal{S}_2$ such that $\{\mathcal{S}_1, \mathcal{S}_2\}^2$ is not expressible as a cubic polynomial in the second order constants of the motion.

N=2, $2N-1=3$

2D

nondegenerate potential

N=3, $2N-1=5$

3D

nondegenerate potential

# of linearly independent 2 nd order symmetries	3	5 => 6
# of linearly independent 3 rd order symmetries	1	4
# of linearly independent 4 th order symmetries	6	26
# of linearly independent 6 th order symmetries	10	56
Quadratic algebra closes?	Yes	Yes
# of Euclidean systems	11	10
# of sphere systems	6	>6
Multiseparable?	Yes	Yes
Stäckel equivalent to flat space or sphere?	Yes	Yes

N=3, 2N-1=5
 3D,
 true 3-parameter potential

5	# of linearly independent 2 nd order symmetries
3	# of linearly independent 3 rd order symmetries
?	# of linearly independent 4 th order symmetries
?	# of linearly independent 6 th order symmetries
No	Quadratic algebra closes?
?	# of Euclidean systems
?	# of sphere systems
Yes	Multiseparable?
Yes	Stäckel equivalent to flat space or sphere?

Polynomial ideals 1

We introduce a very different way of studying and classifying superintegrable systems, through polynomial ideals. Here we confine our analysis to 3D Euclidean superintegrable systems with nondegenerate potentials, though the approach is also effective in 2D and for spheres. The canonical potential equations are just

$$V_{jj} - V_{11} = A^{jj}V_1 + B^{jj}V_2 + C^{jj}V_3, \quad j = 2, 3$$

$$V_{k\ell} = A^{k\ell}V_1 + B^{k\ell}V_2 + C^{k\ell}V_3, \quad 1 \leq k \leq \ell \leq 3.$$

Polynomial ideals 2

All of the functions A^{ij}, B^{ij}, C^{ij} can be expressed in terms of the 10 basic terms

$$(A^{12}, A^{13}, A^{22}, A^{23}, A^{33}, B^{12}, B^{22}, B^{23}, B^{33}, C^{33}).$$

Since the symmetry equations admit 6 linearly independent solutions a^{hk} the integrability conditions $\partial_i a_\ell^{hk} = \partial_\ell a_i^{hk}$ for these equations must be satisfied identically. These conditions plus the integrability conditions for the potential allow us to compute the 30 derivatives $\partial_\ell D^{ij}$ of the 10 basic terms. Each is a quadratic polynomial in the 10 terms.

Polynomial ideals 3

In addition there are 5 quadratic conditions $I^{(j)} = 0$:

$$I^{(a)} = -A^{23}B^{33} - A^{12}A^{23} + A^{13}B^{12} + B^{22}A^{23} + B^{23}A^{33} - A^{22}B^{23}$$

$$I^{(b)} = (A^{33})^2 + B^{12}A^{33} - A^{33}A^{22} - B^{33}A^{12} - C^{33}A^{13} + B^{22}A^{12} \\ - B^{12}A^{22} + A^{13}B^{23} - (A^{12})^2$$

$$I^{(c)} = B^{23}C^{33} + B^{12}A^{33} + (B^{12})^2 + B^{22}B^{33} - (B^{33})^2 - A^{12}B^{33} - (B^{23})^2$$

$$I^{(d)} = -B^{12}A^{23} - A^{33}A^{23} + A^{13}B^{33} + A^{12}B^{23}$$

$$I^{(e)} = -B^{23}A^{23} + C^{33}A^{23} + A^{22}B^{33} - A^{33}B^{33} + B^{12}A^{12}$$

Polynomial ideals 4

These 5 polynomials determine an ideal Σ' . Already we see that the values of the 10 terms at a fixed regular point must uniquely determine a superintegral system. However, choosing those values such that the 5 quadratic conditions are satisfied will not guarantee the existence of a solution, because the conditions may be violated for values of (x, y, z) away from the chosen regular point.

Polynomial ideals 5

To test this we compute the derivatives $\partial_i \Sigma'$ and obtain a single new condition, the quadratic expression

$$\begin{aligned} I^{(f)} = & A^{13}C^{33} + 2A^{13}B^{23} + B^{22}B^{33} - (B^{33})^2 + A^{33}A^{22} - (A^{33})^2 \\ & + 2A^{12}B^{22} + (A^{12})^2 - 2B^{12}A^{22} + (B^{12})^2 + B^{23}C^{33} \\ & - (B^{23})^2 - 3(A^{23})^2 = 0. \end{aligned}$$

This polynomial extends the ideal.

Polynomial ideals 6

Let Σ be the ideal generated by the 6 quadratic polynomials. It can be verified that $\partial_i \Sigma \subseteq \Sigma$, so that the system is closed under differentiation! This leads us to a fundamental result.

THEOREM: Choose the 10-tuple at a regular point, such that the 6 polynomial identities are satisfied. Then there exists one and only one Euclidean superintegrable system with nondegenerate potential that takes on these values at a point.

Polynomial ideals 7

We see that all possible nondegenerate 3D Euclidean superintegrable systems are encoded into the 6 quadratic polynomial identities. These identities define an algebraic variety that generically has dimension 6, though there are singular points, such as the origin $(0, \dots, 0)$, where the dimension of the tangent space is greater. This result gives us the means to classify all superintegrable systems.

Polynomial ideals 8

An issue is that many different 10-tuples correspond to the same superintegrable system. How do we sort this out? The key is that the Euclidean group $E(3, C)$ acts as a transformation group on the variety and gives rise to a foliation. The action of the translation subgroup is determined by the derivatives $\partial_k D^{ij}$ that we have already determined. The action of the rotation subgroup on the D^{ij} can be determined from the behavior of the canonical equations under rotations. The local action on a 10-tuple is then given by 6 Lie derivatives that are a basis for the Euclidean Lie algebra $e(3, C)$.

Polynomial ideals 9

For “most” 10-tuples D_0 on the 6 dimensional variety the action of the Euclidean group is locally transitive with isotropy subgroup only the identity element. Thus the group action on such points sweeps out a solution surface homeomorphic to the 6 parameter $E(3, C)$ itself. This corresponds to the generic Jacobi elliptic system with potential

$$V = \alpha(x^2 + y^2 + z^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2}.$$

At the other extreme the isotropy subgroup of the origin $(0, \dots, 0)$ is $E(3, C)$ itself, i.e., the point is fixed under the group action. This corresponds to the isotropic oscillator with potential

$$V = \alpha(x^2 + y^2 + z^2) + \beta x + \gamma y + \delta z.$$

Polynomial ideals 10

More generally, the isotropy subgroup at D_0 will be H and the Euclidean group action will sweep out a solution surface homeomorphic to the homogeneous space $E(3, C)/H$ and define a unique superintegrable system. For example, the isotropy subalgebra formed by the translation and rotation generators $\{P_1, P_2, P_3, J_1 + iJ_2\}$ determines the system with potential

$$\alpha ((x - iy)^3 + 6(x^2 + y^2 + z^2)) + \beta ((x - iy)^2 + 2(x + iy)) + \gamma(x - iy)$$

Polynomial ideals 11

Indeed, each class of Stäckel equivalent Euclidean superintegrable systems is associated with a unique isotropy subalgebra of $e(3, C)$, although not all subalgebras occur. (Indeed, there is no isotropy subalgebra conjugate to $\{P_1, P_2, P_3\}$.) Thus to find all superintegrable systems we need to determine a list of all subalgebras of $e(3, C)$, defined up to conjugacy, and then for each subalgebra to determine if it occurs as an isotropy subalgebra.

Polynomial ideals 12

We have not been able to classify the Euclidean superintegrable systems by pure algebraic geometry and group theory methods, but have succeeded by using the fact that every such system is multiseparable and that we know all Euclidean separable systems. Using the simplification provided by the Euclidean action on the variety we can examine each separable system and determine to which superintegrable systems it could belong. We have found the 10 possible Euclidean superintegrable systems.

Outlook 1

We have given an overview of some of the tools used and results obtained in the study of second order superintegrable systems. The basic problems for 2D systems have been solved, and the extension of these methods to complete the fine structure analysis for 3D systems appears relatively straightforward.

Outlook 2

The 3D fine structure analysis can be extended to analyze 2 parameter and 1 parameter potentials with 5 functionally linearly independent second order symmetries. Here first order PDEs for the potential appear, as well as second order, and Killing vectors may occur. The other class of 3D superintegrable systems is that for which the 5 functionally independent symmetries are functionally linearly dependent. This class contains the Calogero potential and necessarily leads to first order PDEs for the potential, as well as second order.

Outlook 3

The integrability condition methods discussed here should be able to handle this class with no special difficulties. On a deeper level, we think that algebraic geometry methods can be extended to classify the possible superintegrable systems in all these cases.

Outlook 4

Whereas 2D superintegrable systems are very special, the 3D systems seem to be good guides to the structure of general n D systems, and we intend to proceed with this analysis. Finally, the ultimate aim is to understand the structure of superintegrable systems in general.

Outlook 5

Finally, the algebraic geometry related results that we have sketched suggest strongly that there is an underlying geometric structure to superintegrable systems that is not apparent from the usual presentations of these systems.