

Planetary motion, the hydrogen atom, and superintegrability

History with hindsight

Willard Miller

milller@ima.umn.edu

University of Minnesota

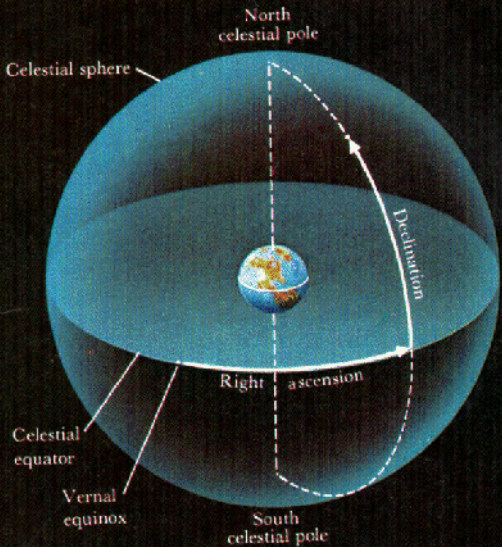
Abstract

Superintegrable systems are classical or quantum systems that exhibit the maximal symmetry possible. This is a brief, biased, history of classical and quantum mechanics that points out the critical role of superintegrable systems in the development of these subjects.

The start: Aristotle's astronomy

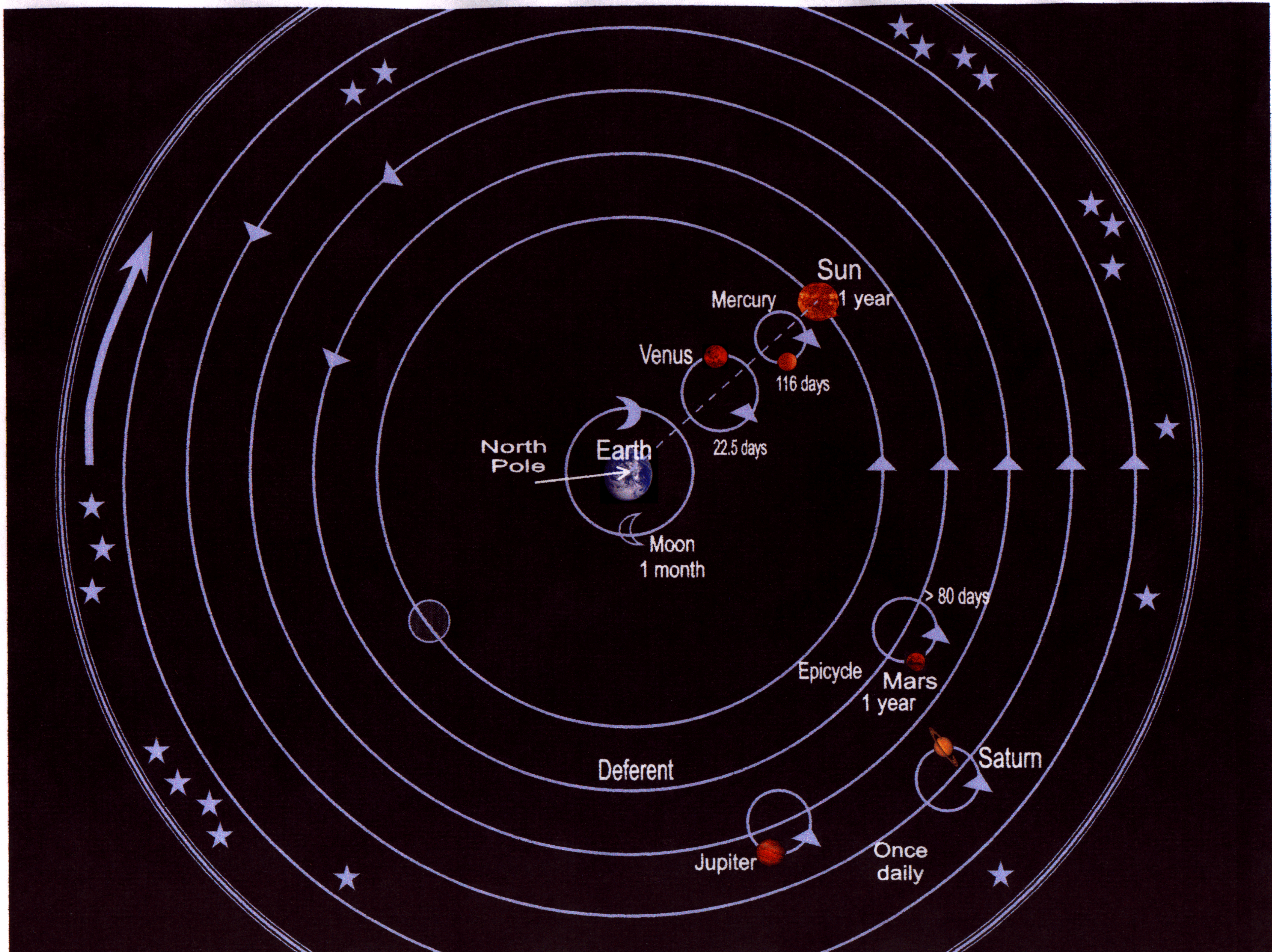
Aristotle taught:

1. The universe is spherical and finite, centered on the center of the earth.
2. Earth is spherical and motionless. The sun, moon, planets and stars are spheres.
3. The stars are fixed in the celestial sphere that rotates about the earth. The sun, moon and planets are attached to a series of interlocking smaller spheres that are rotating about the earth at various fixed rates.



Ptolemy's epicycles

By two millennia ago it had become apparent (to astronomers who actually made observations) that Aristotle's theory was untenable. It simply did not fit the observed facts. For example it couldn't explain retrograde motion where a planet is observed to pause and then move backward in the sky, before resuming forward motion. The attempt to maintain the concept that uniform circular motion was primary while getting a better fit to actual observations lead to Ptolemy's theory of epicycles, or spheres rolling on spheres.



Epicycles 0

1. The epicycle procedure was extremely complicated and had to be revised frequently as new and more accurate data became available. By the early 1600s the Copernican notion that the planets orbited in epicycles about the Sun rather than the Earth had become popular, and this simplified the computations and improved accuracy somewhat. However, the much more accurate observations of Brahe showed that severe problems remained.
2. The greatest deficiency in the epicycle method was that it was merely descriptive; it provided no insight into the underlying natural processes that accounted for the motion.

Kepler 1

With the work of Kepler and the publication of his three laws of planetary motion, roughly 1615-1630, the epicycle edifice came crashing down. Kepler showed that the assumption that the planets followed elliptical orbits with the Sun at one focus gave a much simpler and more accurate means of fitting the observational data. Newton, 50 years later, used this breakthrough in his formulation of mechanics to determine the gravitational force, and to derive the planetary orbits from physical principles.

Newton's Gravitational force 1

Newton used Kepler's laws to demonstrate that the force exerted on a planet by the sun is

$$\mathbf{F} = -\frac{MmG}{r^2}\hat{\mathbf{r}}$$

where r is the distance from the center of the sun to the center of the planet, and $\hat{\mathbf{r}}$ is a unit vector pointing from the planet to the sun. Here, M is the mass of the sun and m is the mass of the planet. Newton's equation for the motion of the planet is

$$m\mathbf{r}'' = -\frac{mk}{r^2}\hat{\mathbf{r}},$$

where $k = MG$ and G is the gravitational constant.

Newton's Gravitational force 2

1. We know today that Newton's equation for planetary motion, the 2-body problem, can be solved explicitly, not just numerically, because it is of maximal symmetry. It admits 3 independent symmetries and this is the maximum possible in two dimensions. It is a very important example of a superintegrable system.
2. It also helps explain how Kepler found the trajectories of the planets without knowing Newton's equations or calculus.

Conservation laws 1

1. A basic principle here is that symmetries of a physical system lead to conservation laws obeyed by the system: quantities that do not change as the system evolves in time.
2. The precise general statement of this relationship is Noether's Theorem, named after Emmy Noether, one of the greatest female mathematicians. It also follows from Hamiltonian mechanics.

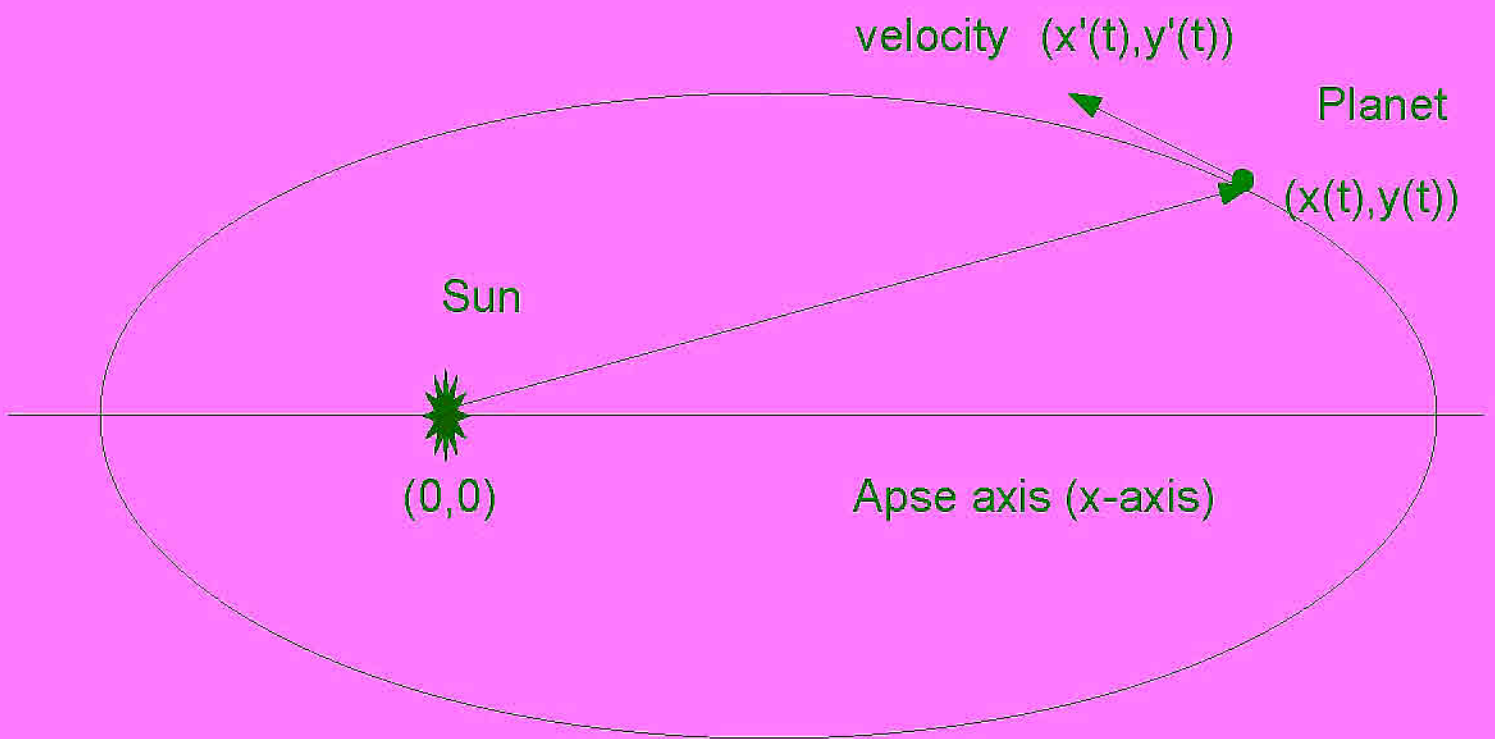
Conservation laws 2

For the 2-body problem,

1. Symmetry with respect to time translation \implies conservation of energy.
2. Symmetry with respect to rotation \implies conservation of angular momentum.
3. A higher-order symmetry \implies conservation of the Laplace vector.

The orbital plane

A planet orbiting the sun moves in a plane. Choose coordinates (x, y) in this plane such that the center of the sun is at the origin $(0, 0)$ and at time t the center of the planet is at the point $(x(t), y(t))$ and is moving with velocity $(x'(t), y'(t))$. The speed of the planet in its orbit is $s(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$.



Periapic coordinates

Hamilton's equations 1

Let q_1, q_2 be the position coordinates of a Hamiltonian system and let p_1, p_2 be the momenta. The Hamiltonian function $H(q_1, q_2, p_1, p_2)$ represents the energy of the system. Hamilton's equations give the time evolution of the system. They are

$$q'_k(t) = \frac{\partial H}{\partial p_k}, \quad p'_k(t) = -\frac{\partial H}{\partial q_k}, \quad k = 1, 2.$$

In our case,

$$q_1 = x, \quad q_2 = y, \quad p_1 = x', \quad p_2 = y', \quad H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{k}{r},$$

where $r = \sqrt{q_1^2 + q_2^2}$.

Hamilton's equations 2

Hamilton's equations are $q'_1 = x'$, $q'_2 = y'$, which are obvious, and

$$p'_1 = x'' = -\frac{kx}{r^3}, \quad p'_2 = y'' = -\frac{ky}{r^3},$$

which are just Newton's equations for the 2-body problem.

Hamilton's equations 3

A function $F(q_1, q_2, p_1, p_2)$ is a symmetry or *constant of the motion* if $F(q_1(t), q_2(t), p_1(t), p_2(t))$ remains constant as the system evolves in time. Thus F is a constant of the motion if and only if

$$\frac{d}{dt}F(q_1(t), q_2(t), p_1(t), p_2(t)) = 0.$$

From the chain rule,

$$\frac{d}{dt}F(t) = \frac{\partial F}{\partial q_1}q'_1 + \frac{\partial F}{\partial q_2}q'_2 + \frac{\partial F}{\partial p_1}p'_1 + \frac{\partial F}{\partial p_2}p'_2 =$$

$$\frac{\partial F}{\partial q_1} \frac{\partial H}{\partial p_1} + \frac{\partial F}{\partial q_2} \frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_1} \frac{\partial H}{\partial q_1} - \frac{\partial F}{\partial p_2} \frac{\partial H}{\partial q_2} \equiv \{H, F\},$$

where $\{H, F\}$ is the *Poisson bracket* of F and H .

Hamilton's equations 4

Thus F is a constant of the motion provided the Poisson bracket $\{H, F\} = 0$. Note that H itself (the energy) is always a constant of the motion. In our case, in addition to the energy, we have the following constants of the motion:

1. Angular momentum $L = q_1 p_2 - p_1 q_2$.

Proof:

$$\{H, L\} = p_2 p_1 - p_1 p_2 - \frac{k q_2 q_1}{r^3} + \frac{k q_2 q_1}{r^3} = 0.$$

Hamilton's equations 5

1. The first component of the Laplace vector

$$e_1 = p_2(q_1p_2 - q_2p_1) - \frac{kq_1}{r}.$$

2. The second component of the Laplace vector

$$e_2 = -p_1(q_1p_2 - q_2p_1) - \frac{kq_2}{r}.$$

Constants of the motion

Energy

$$\frac{1}{2} ((x')^2 + (y')^2) - \frac{k}{\sqrt{x^2 + y^2}} = E$$

Angular momentum

$$xy' - yx' = L$$

Laplace vector $\mathbf{e} = (e_1, e_2)$ where

$$y'(xy' - yx') - \frac{kx}{\sqrt{x^2 + y^2}} = e_1, \quad -x'(xy' - yx') - \frac{ky}{\sqrt{x^2 + y^2}} = e_2.$$

Relation

$$e_1^2 + e_2^2 = 2L^2E + k^2$$

The trajectories 1

By lining up the x, y coordinate system so that the x -axis is in the direction of the Laplace vector, we can assume $e_2 = 0$. (This means that the x -axis goes through the perihelion of the planet. It is called the apse axis in astronomy.) Then

1.

$$e_2 = 0 \text{ and } L \text{ constant} \longrightarrow x' = -\frac{ky}{L\sqrt{x^2 + y^2}}$$

2.

$$e_1 \text{ and } L \text{ constant} \longrightarrow y' = \frac{e_1}{L} + \frac{kx}{L\sqrt{x^2 + y^2}}$$

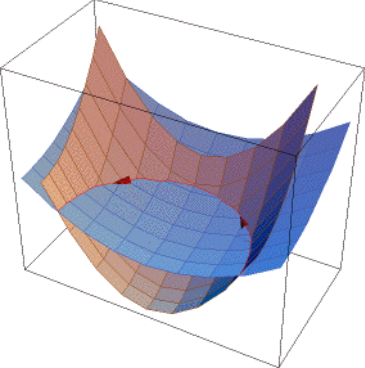
The trajectories 2

1. Substitute these expressions into the equation for L , and simplify to get the equation:

$$k\sqrt{x^2 + y^2} = L^2 - e_1x$$

2. Square and simplify to get the trajectory

$$\left(1 - \frac{e_1^2}{k^2}\right)x^2 + \frac{2L^2e_1}{k^2}x + y^2 = \frac{L^4}{k^2}$$



The paths are conic sections!

Set $e_1 = \epsilon k \geq 0$ where ϵ is called the eccentricity.

$\epsilon = 0$, Circle:

$$x^2 + y^2 = \left(\frac{L^2}{k}\right)^2$$

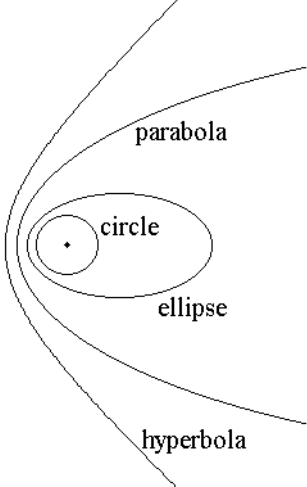
$0 < \epsilon < 1$, Ellipse:

$$(1 - \epsilon^2)x^2 + \frac{2\epsilon L^2}{k}x + y^2 = \left(\frac{L^2}{k}\right)^2$$

$\epsilon > 1$, Hyperbola:

$$(1 - \epsilon^2)x^2 + \frac{2\epsilon L^2}{k}x + y^2 = \left(\frac{L^2}{k}\right)^2$$

$\epsilon = 1$, Parabola: $\frac{2\epsilon L^2}{k}x + y^2 = \left(\frac{L^2}{k}\right)^2$

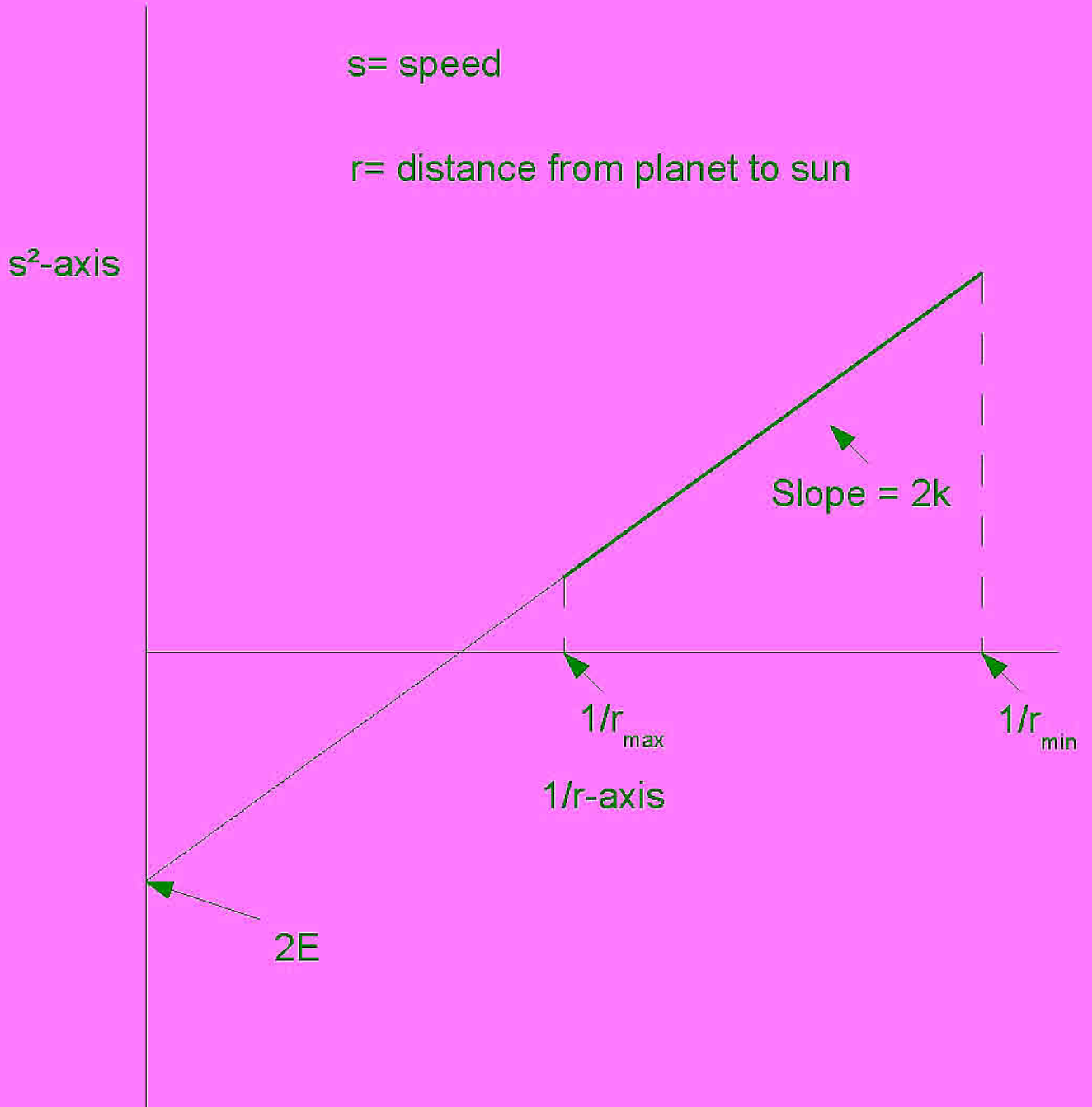


The trajectories 3

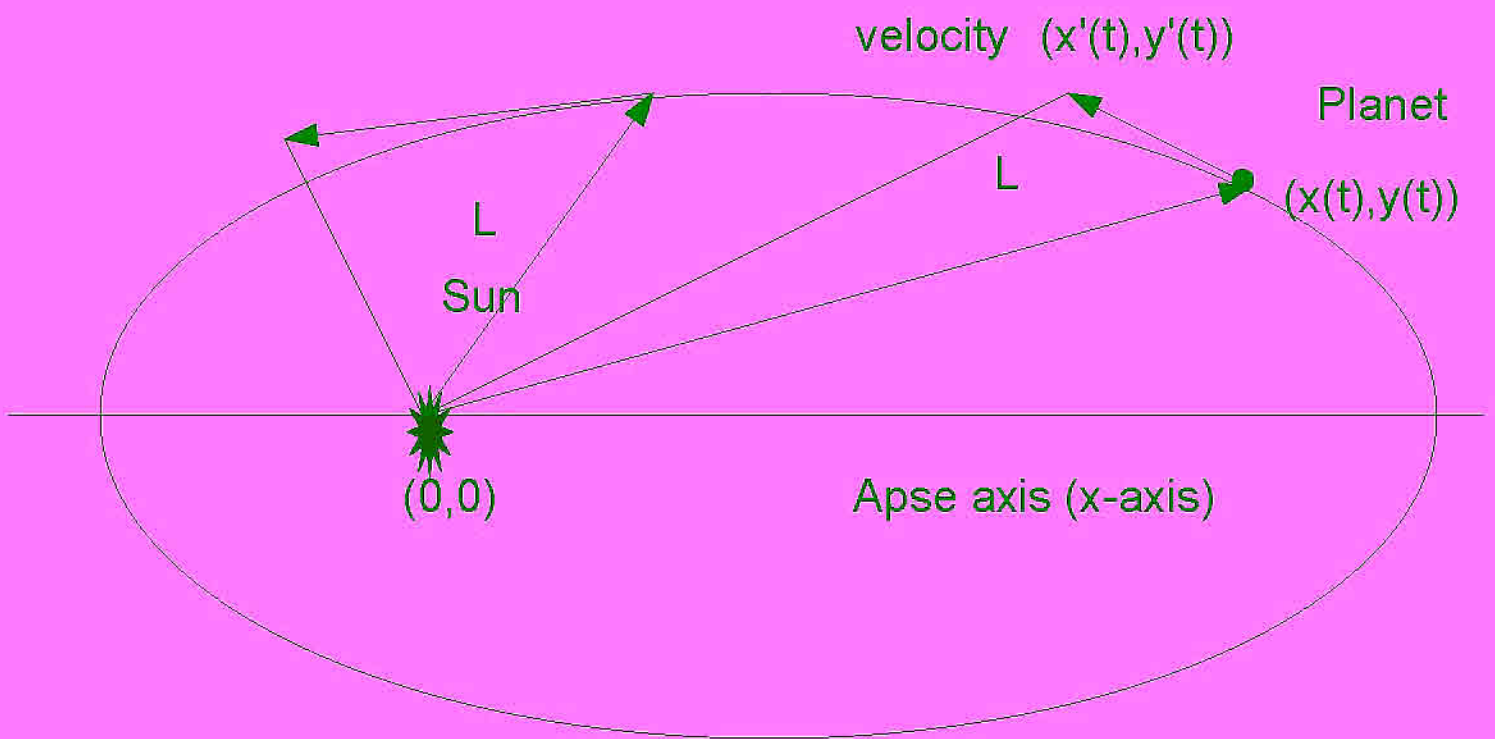
How could Kepler have found the equations of the trajectories? The coordinate system in the plane of the planetary orbit, with the origin at the center of the sun, and with the x -axis as apse axis is natural for the problem and used by Kepler. Kepler had lots of data about the positions of the planets, 1500 years worth. He could compute hundreds of data points (x, y, x', y') on each trajectory for the well known planets. Mars was his particular focus. Here is how he could discover the information provided by the constants of the motion.

The trajectories 4

1. Plot s^2 versus $1/r$ on a graph. The result is a straight line through $2E$ on the vertical axis and with slope $2k$.
2. Compute the area of the triangle with vertices $(0, 0)$, (x, y) , $(x + x', y + y')$ for each data point on the planetary orbit. The result is always the same, the constant L .
3. Plot x'/y versus $1/r$ on a graph. The result is a straight line through the origin and with slope $-k/L$.
4. Plot y' versus x/r on a graph. The result is a straight line through e_1/L on the vertical axis and with slope k/L .



How Kepler could discover conservation of energy



Periaptic coordinates

How Kepler could discover conservation of angular momentum

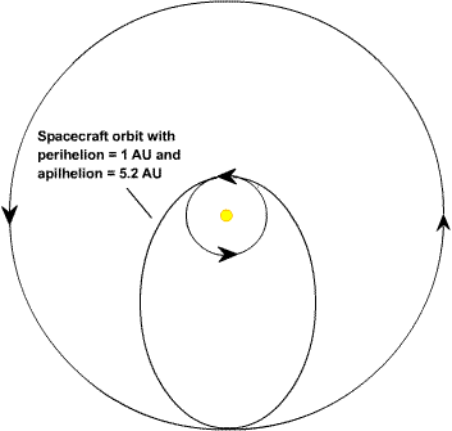
Impulse maneuvers

1. Position and velocity (x_0, y_0, x'_0, y'_0) at a single instant determines the trajectory: Just compute the constants of the motion E, L, e_1, e_2 at the instant and they in term uniquely define the trajectory.
2. This is the basis for impulse maneuvers in rocket science. The Hohmann transfer.

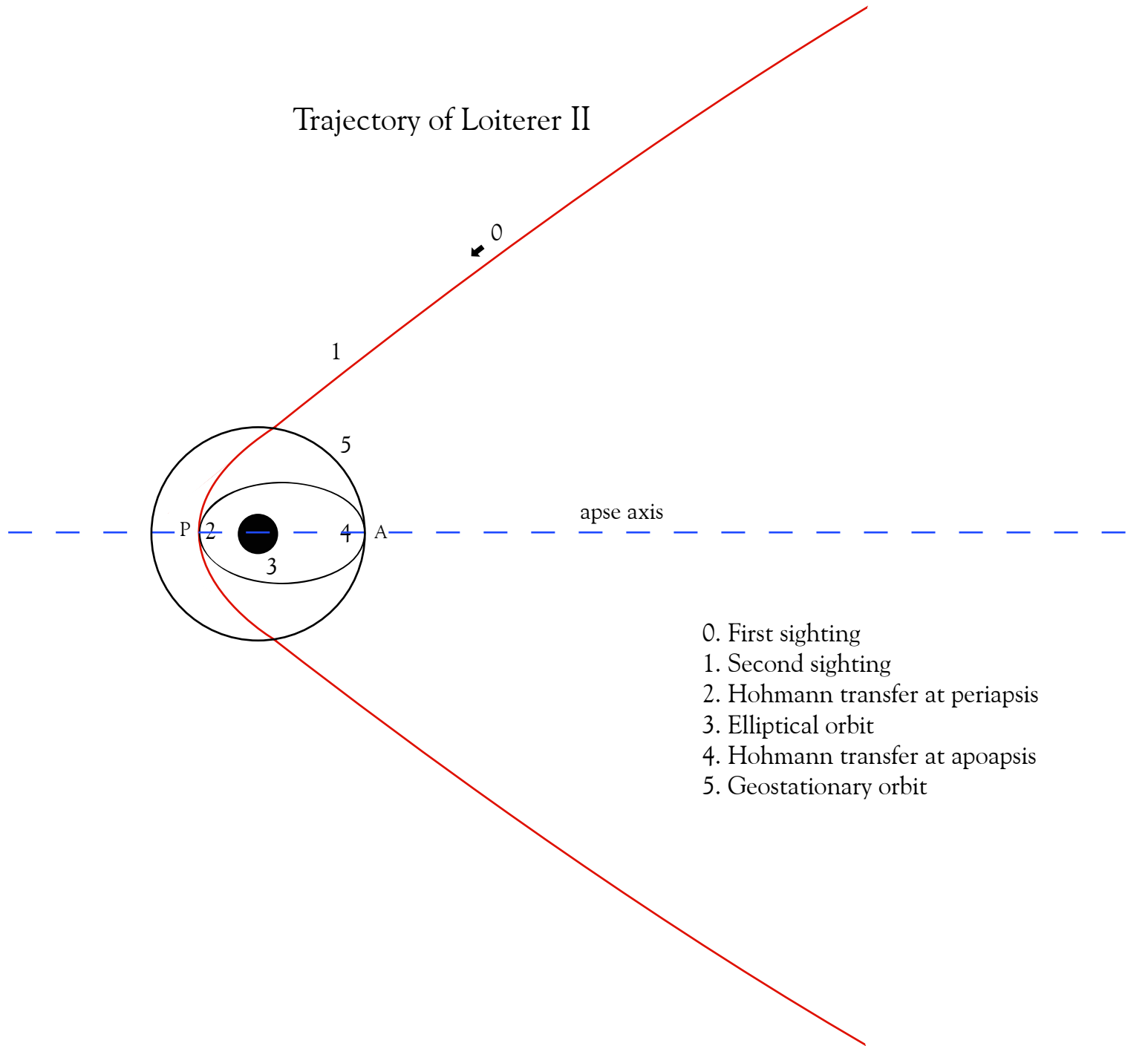
The Hohmann transfer

1. Space ship (with engines turned off) on trajectory with constants of the motion E, L, e_1, e_2 .
2. At time t_0 ship has position and velocity (x_0, y_0, x'_0, y'_0) .
3. Turn on the engine, for an instant, at time t_0 : impulse propulsion. This pulse changes the velocity of the ship instantaneously, but not the position.
4. Immediately after the impulse the ship has position and velocity $(x_0, y_0, \tilde{x}'_0, \tilde{y}'_0)$.
5. This gives us the new trajectory with constants of the motion $\tilde{E}, \tilde{L}, \tilde{e}_1, \tilde{e}_2$.
6. The change of trajectories is determined by simple algebra.

**Spacecraft orbit with
perihelion = 1 AU and
aphelion = 5.2 AU**



Trajectory of Loiterer II



- 0. First sighting
- 1. Second sighting
- 2. Hohmann transfer at periapsis
- 3. Elliptical orbit
- 4. Hohmann transfer at apoapsis
- 5. Geostationary orbit

Properties of the Poisson bracket

1. $\{F, G\} = \sum_{i=1}^2 \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right) = -\{G, F\}$
2. $\{F, G_1 G_2\} = G_1 \{F, G_2\} + \{F, G_1\} G_2$
3. Jacobi identity:
 $\{F, \{G_1, G_2\}\} = \{\{F, G_1\}, G_2\} + \{G_1, \{F, G_2\}\}$

The algebra of symmetries

1. The symmetries H, L, e_1, e_2 of the Kepler systems generate an algebra of symmetries closed under the Poisson bracket:
2. The product and the Poisson bracket of two symmetries F, G are symmetries, i.e., $\{H, FG\} = 0$ and $\{H, \{F, G\}\} = 0$.
3. Structure equations for symmetries:
 $\{L, e_1\} = e_2, \{L, e_2\} = -e_1, \{e_1, e_2\} = -2LH$
4. Casimir: $e_1^2 + e_2^2 - 2L^2H = k^2$

The quantum Coulomb problem 1

The Hamiltonian and the the constants of the motion are replaced by differential operators: We make the formal replacement $p_1 \rightarrow \partial_{q_1}$ and handle the ambiguity of replacements $q_i p_i \rightarrow q_i \partial_{q_i}$, $\partial_{q_i} q_i$ by symmetrizing:
 $q_i p_i \rightarrow \frac{1}{2}(q_i \partial_{q_i} + \partial_{q_i} q_i)$.

$$H = \frac{1}{2}(\partial_{q_1}^2 + \partial_{q_2}^2) - \frac{k}{\sqrt{x^2 + y^2}}$$

$$L = q_1 \partial_{q_2} - q_2 \partial_{q_1}, \quad e_1 = \frac{1}{2}(L \partial_{q_2} + \partial_{q_2} L) - \frac{kx}{\sqrt{x^2 + y^2}}$$

$$e_2 = -\frac{1}{2}(L \partial_{q_1} + \partial_{q_1} L) - \frac{ky}{\sqrt{x^2 + y^2}}$$

The quantum Coulomb problem 2

The Poisson bracket $\{F, G\}$ is replaced by the operator commutator $[F, G] = FG - GF$ and the constants of the motion are differential operators that commute with H .

$$[H, L] = [H, e_1] = [H, e_2] = [H, H] = 0$$

.

Important properties of the Commutator

1. $[F, G] = -[G, F]$, linear in both arguments
2. $[F, G_1 G_2] = G_1 [F, G_2] + [F, G_1] G_2$
3. **Jacobi identity:** $[F, [G_1, G_2]] = [[F, G_1], G_2] + [G_1, [F, G_2]]$

The quantum Coulomb problem 3

1. Structure equations for symmetries:

$$[L, e_1] = e_2, [L, e_2] = -e_1, [e_1, e_2] = -2LH$$

2. Casimir: $e_1^2 + e_2^2 - 2L^2H + \frac{1}{2}H = k^2$

3. Structure equations slightly different in quantum case.
Note that this is NOT a Lie algebra.

Transition to Quantum Mechanics 0

The second order symmetry operators, H, e_1, e_2 are formally self adjoint and the first order symmetry L is skew adjoint with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mathbf{q}) \overline{g(\mathbf{q})} dq_1 dq_2.$$

$$\langle Sf, g \rangle = \langle f, Sg \rangle, \quad S = H, e_1, e_2, \quad \langle Lf, g \rangle = - \langle f, Lg \rangle.$$

Transition to Quantum Mechanics 1

Bound states are eigenfunctions of H that are square integrable:

$$H\Psi = E\Psi, \quad \langle \Psi, \Psi \rangle = 1.$$

We look for irreducible representations of the structure equations. Necessarily, the Hamiltonian is constant in these cases, i.e., we are restricted to an eigenspace of H : $H\Psi = E\Psi$.

Transition to Quantum Mechanics 2

1. The classical energy, angular momentum and Laplace vector become operators and the eigenvalues of the operators are the quantized values of the constants of the motion. In appropriate units the possible states of energy E and angular momentum L have values

$$E_n = \frac{4k^2}{(n+1)^2}, \quad L_m = im/2,$$

$$m = n, n-1, n-2, \dots, 1-n, -n$$

where n is an integer. These values can be derived through symmetry considerations alone.

Importance of models 1

Consider the 1-variable model for irreducible representations:

$$L = it \frac{d}{dt} + iD, \quad e_1 = -\frac{i}{2}(E + t^2) \frac{d}{dt} - iDt,$$

$$e_2 = \frac{1}{2}(E - t^2) \frac{d}{dt} - Dt, \quad \text{where } D = \frac{1}{2} - k/\sqrt{E}.$$

This model satisfies the structure equations identically. It is designed to give a simple spectral resolution for L .

Importance of models 2

Set $f_\lambda(t) = t^\lambda$. Then

$$Lf_\lambda = i(\lambda + D)f_\lambda$$

Note that $a^\pm = -e_2 \pm ie_1$ act like raising and lowering operators for the f_λ :

$$a^+ f_\lambda = (\lambda + 2D)f_{\lambda+1}, \quad a^- f_\lambda = (-E\lambda)f_{\lambda-1}$$

The lowest weight vector f_0 is annihilated by a^- . Suppose we have a finite dimensional representation. Then it will contain a highest weight vector f_n annihilated by a^+ . This means that $n = -2D$ for integer n so $n + 1 = 2k/\sqrt{E}$ or

$$E_n = \frac{4k^2}{(n + 1)^2}$$

Importance of models 3

The requirement that iL, e_1, e_2 be self-adjoint with respect to an inner product $\langle u, v \rangle$ leads to the recurrence relation for the norms

$$(\lambda - n) \|f_{\lambda+1}\|^2 = -(\lambda + 1) E_n \|f_\lambda\|^2$$

Normalizing $\|f_0\| = 1$ we get the ON basis

$$g_\lambda = \sqrt{n!} \left(\frac{n+1}{2k} \right)^\lambda \frac{t^\lambda}{\sqrt{(\lambda)!(n-\lambda)!}}$$

We can find an explicit function space inner product.

Importance of models 4

We can also compute the ON basis of eigenfunctions of e_2 in the model. They are

$$G_k(t) = K(t - \sqrt{E_n})^k (t + \sqrt{E_n})^{n-k}$$

with corresponding eigenvalues $\xi = (k - n/2)\sqrt{E_n}$ for $k = 0, 1, \dots, m$, and computable normalization constant K . The expansion of G_k in powers of t gives the expansion of the e_2 basis in terms of the L basis.

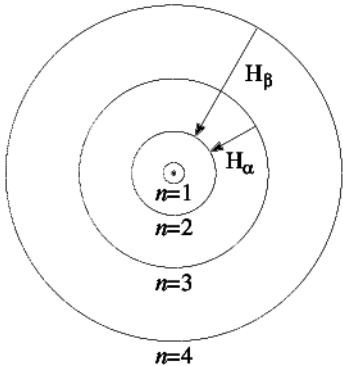
Importance of models 5

In the original quantum mechanical problem the L basis eigenfunctions are just the separable eigenfunctions in spherical (or polar coordinates), the Spherical harmonics, while the e_2 basis functions are those that separate in parabolic coordinates, products of Laguerre polynomials. Thus the result is an interbasis expansion yielding a special function identity.

Transition to Quantum Mechanics 3

1. The attractive Coulomb force between a positively charged particle and a negatively charged particle is proportional to the gravitational force between these particles so the same analysis can be applied to the hydrogen atom with one proton (sun) orbited by one electron (planet). The energy levels are given by $-E_n$.
2. If a hydrogen atom with energy $-E_n$ jumps to a state with lower energy $-E_{n'}$ it emits a photon with energy

$$-E_n + E_{n'} = -4k^2 \left(\frac{1}{(n+1)^2} - \frac{1}{(n'+1)^2} \right).$$



Transition to Quantum Mechanics 4

1. Similarly, a hydrogen atom with energy $E_{n'}$ can absorb a photon of energy $-E_n + E_{n'}$ and jump to a state with higher energy $-E_n$. These energy differences can be detected and measured through spectroscopy. This is a “quantum Hohmann transfer”.

Balmer series 1

Balmer (about 1885 A.D.) knew from experimental evidence that the lowest wavelengths in the visible spectrum of light from hydrogen occur at 410 nm, 434 nm, 486 nm, 656 nm, \dots . He observed that the inverses of these wavelengths were proportional to

$$\frac{1}{(n+1)^2} - \frac{1}{(m+1)^2}$$

for $m = 1$ and $n = 2, 3, 4, 5, \dots$. This is called the Balmer series. Just as with Kepler, the consequences of the super-integrability of the Kepler-Coulomb problem were observed and utilized before they were derived from theory.

Balmer series 2

Balmer's astonishingly simple observation demonstrated that the classical planetary model of the atom had to be abandoned and a new structure discovered to explain this regularity.

Balmer was a Ph. D. in mathematics and a math teacher in a secondary school for girls. He published the Balmer series at the age of 60.

Prelude

This is the tip of the iceberg! Superintegrable systems occur in all dimensions $n \geq 2$ and for a great variety of conformally flat spaces and potentials on these spaces. They are always associated with noncommuting (non-Lie) algebras of higher order symmetries and the representation theory of the algebras yields crucial information about the spectral theory of the symmetry operators. In the following lectures we discuss the structure and classification of these systems and the representation theory of their symmetry algebras.