Math 2243, Practice Final exam. You will need a photo ID for the final!!!
Please let me know if you find any errors in these solutions.

Name:_______________________________ Instructor:________________________

Remember to show all your work. Without it, a correct answer may be given no credit.

(1) **Problem 1**.

Find the general solution $y(t)$ to $ty' + 2y = 6t$.

Solution:

First we divide by t to put this ODE into the standard form

$$y' + p(t)y = f(t):$$

$$y' + (2/t)y = 6.$$

The integrating factor for this equation is $\mu = e^{\int p dt} = e^{\int 2/t dt} = e^{ln t^2 + C} = \tilde{C}t^2$. The homogeneous solution is $y_h = 1/\mu = C_1 t^{-2}$, and a particular solution is $y_p = \frac{1}{\mu}\int \mu f(t) dt = t^{-2}\int t^2 6 dt = 2t$. So the general solution is $y_h + y_p = C_1 t^{-2} + 2t$.

(2) **Problem 2**.

Find the solution to the initial value problem $y'' - 4y' + 13y = 0$, $y(0) = 0$, $y'(0) = 1$.

Solution:

First we find the general solution by considering the characteristic equation $r^2 - 4r + 13$. By the quadratic formula, the roots of the characteristic equation are $r_{1,2} = 2 \pm 3i$. This tells us that $e^{2t}\sin 3t$ and $e^{2t}\cos 3t$ span the solution space, so $y(t) = C_1 e^{2t}\cos 3t + C_2 e^{2t}\sin 3t$.

The initial conditions give us linear equations for the C_i: $y(0) = 0 = C_1$ and $y'(0) = 1 = C_1 2e^0 \cos 0 + C_2 2e^0 \sin 0 - C_1 3e^0 \sin 0 + 3C_2 e^0 \cos 0 = 2C_1 + 3C_2$. So $C_1 = 0$ and $C_2 = 1/3$, and the solution is $y(t) = e^{2t}\sin 3t/3$.

(3) **Problem 3.**

Show that the polynomials $p_1 = t^2 + 2t + 3$, $p_2 = 4t^2 + 5t + 6$, $p_3 = 7t^2 + 8t + 9$ are not linearly independent.

Solution:

One way to do this is to show the Wronskian

$$\det \begin{bmatrix} p_1 & p_2 & p_3 \\ p_1' & p_2' & p_3' \\ p_1'' & p_2'' & p_3'' \end{bmatrix}$$

is zero; usually this is a good idea but here that calculation is somewhat messy.

For quadratic polynomials, we can use the basis $\{t^2, t, 1\}$ to write the polynomials as their coefficient vectors, $(1, 2, 3)$, $(4, 5, 6)$, and $(7, 8, 9)$. Since

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0$$

the polynomials are not linearly independent.

(4) **Problem 4**.

Find the kernel and range of the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by the matrix $A = \begin{bmatrix} 3 & 9 & 10 \\ 3 & -1 & 0 \\ -3 & -1 & -2 \end{bmatrix}$.

Solution:

This problem is a little too difficult for a test question.

The characteristic polynomial of A is $det(A - \lambda I) = -\lambda^3 + 4\lambda = -\lambda(\lambda^2 - 4)$ so the eigenvalues are 0, -2, and 2. The kernel, or nullspace, will be one-dimensional and consists of the span of the 0-eigenvector. We find that eigenvector by solving $Ax = 0$, by row-reducing A. One way to do that is:

Start by adding row 1 to row 3,

$$\begin{bmatrix} 3 & 9 & 10 \\ 3 & -1 & 0 \\ 0 & 8 & 8 \end{bmatrix} \rightarrow$$

then subtract row 1 from row 2,

$$\begin{bmatrix} 3 & 9 & 10 \\ 0 & -10 & -10 \\ 0 & 8 & 8 \end{bmatrix} \rightarrow$$

divide each row by its leftmost nonzero entry,

$$\begin{bmatrix} 1 & 3 & 10/3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow$$

subtract row 2 from row 3,

$$\begin{bmatrix} 1 & 3 & 10/3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$$

and finally subtract 3 times row 2 from row 1,

$$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From this we can read off that the span of the 0-eigenvector will be $(-a/3, -a, a)$.

The image will be the span of the other two eigenvectors, which can be computed by solving $Ax = 2x$ and $Ax = -2x$. The eigenvector for $\lambda = 2$ is $(-a, -a, a)$, and the eigenvector for $\lambda = 2$ is $(5a, -15a, 11a)$. So the image is $span(\{(-1, -1, 1), (5, -15, 11)\})$.

(5) **Problem 5**.

The matrix $B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ has eigenvalues 0, 1, and -1, with respective eigenvectors v_0, v_1 and v_{-1}. What is the effect of multiplying those eigenvectors by the matrix $c = B^4 + I_3$ (where I_3 is the 3 by 3 identity matrix)?

Solution:

Since $Bv_0 = (0, 0, 0)$, we find that $cv_0 = B^4 v_0 + v_0 = v_0$ so v_0 is unchanged by c.

Any even power of B will leave v_1 and v_{-1} unchanged, so $cv_{\pm 1} = v_{\pm 1} + v_{\pm 1} = 2v_{\pm 1}$; i.e. c doubles the length of the other two eigenvectors.

(6) **Problem 6.**

Find the general solution to the linear system $x' = \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix} x$.

Solution:

Since the equation for the second component of x is just $x_2' = x_2$, we could solve that seperately ($x_2 = C_2 e^t$) and plug it into the equation for the second component, giving us $x_1' = -2x_1 + 2x_2 = -2x_1 + 2C_2 e^t$, a nonhomogeneous first order equation for x_1. This could be solved using variation of parameters or integrating factor approach.

Alternatively we can solve it using the techniques of chapter 6. We can calculate the characteristic polynomial $det(A - \lambda I) = (-2 - \lambda)(1 - \lambda)$ which is already factored since the matrix is upper-triangular. The eigenvalues are 1 and -2. The eigenvectors are found by row-reducing $A - I$ and $A + 2I$ and we find that they are $v_1 = (2, 3)$ and $v_{-2} = (1, 0)$. So the solution is $C_1 e^t v_1 + c_2 e^{-2t} v_{-2}$, or in components $x_1 = 2C_1 e^t + C_2 e^{-2t}$ and $x_2 = 3C_1 e^t$.