The Cheshire Cat effect in Lie theory: Variable separation and superintegrable systems

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1 Abstract

There are many physically important partial differential equations in quantum and classical mechanics that admit Lie symmetry groups, but for which the symmetry is destroyed by addition of symmetry breaking potentials or consideration of non-symmetric solutions. For superintegrable systems on constant curvature spaces there is a Cheshire Cat effect: the symmetry is gone but its influence lingers on and can be used to analyze the solution spaces. We describe this general approach to harmonic analysis and give a number of examples. In particular we show how this approach leads to new results for Lame' and Heun functions, and insight into the classical Niven transform.

Joint work with Ernie Kalnins, Jonathan Kress, George Pogosyan and Vadim Kuznetsov.

2 Introduction

Most functions commonly called " special " obey symmetry properties that are best described via group theory (the mathematics of symmetry. We have in mind:

1. Special functions as matrix elements of Lie group representations. (addition theorems, orthogonality relations, spherical functions)

2. Special functions as basis functions for Lie group representations (generating functions)

3. Special functions as Clebsch-Gordan coefficients for the reduction of tensor products of irreducible group representations (the motivation for Wilson polynomials).

There are, however, many physically important partial differential equations in quantum and classical mechanics that admit Lie symmetry groups, but for which the symmetry is destroyed by addition of symmetry breaking potentials or consideration of non-symmetric solutions. We will show that for superintegrable systems on constant curvature spaces there is a Cheshire Cat effect: the symmetry is gone but its influence lingers on and can be used to analyze the solution spaces. Our emphasis will be on

1. Special functions as solutions of Laplace-Beltrami eigenvalue problems on a Riemannian manifold (with potential) via separation of variables.

There is no general connection between the symmetry group and algebra of a manifold and the separable coordinate systems. However for constant curvature spaces this connection does exist, at the level of second-order elements in the universal enveloping algebra of the symmetry Lie algebra, and it may persist even when the symmetry is completely broken by a potential. We will describe these relations and give applications to special functions, such as Lame' and Heun functions, that directly have no Lie symmetry properties.

3 Symmetries of differential equations

Let D be a linear partial differential operator in n dimensions (with locally analytic coefficients). Let λ be a parameter.

Definition 1 *The linear partial differential operator S is a* **symmetry operator** *for the equation $D\Phi = \lambda\Phi$ if S maps local solutions Φ to local solutions $S\Phi$. This is basically equivalent to the requirement that $[S, D] = 0$.*

The linear partial differential operator $\tilde{S}$ is a **conformal symmetry operator** *for the equation $D\Phi = 0$ if $\tilde{S}$ maps local solutions Φ of $D\Phi = 0$ to local solutions $S\Phi$.*

The first order symmetry operators for $D\Phi = \lambda\Phi$ form a Lie algebra, the **symmetry algebra** of this equation. Indeed, if L_1, L_2 are first order symmetry operators, then the commutator $[L_1, L_1]$ is a first order symmetry operator. The associated local **Lie symmetry group** maps solutions to solutions. The first order conformal symmetry operators for $D\Phi = 0$ form a Lie algebra, the **conformal symmetry algebra** of this equation. The associated local **Lie conformal symmetry group** maps solutions to solutions.

4 Intrinsic characterization of variable separation

Let $\mathcal{V}_n$ be an n-dimensional Riemannian manifold and V be a potential function on $\mathcal{V}_n$. The **Laplace-Beltrami eigenvalue equation** (with potential), or **Schrödinger equation** for functions Ψ on V_n is

$$H\Psi(\mathbf{q}) = E\Psi(\mathbf{q}) \tag{1}$$

where in local coordinates $\{q^\ell\}$

$$H = \Delta_n + V(\mathbf{q}), \qquad \Delta_n = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial q^j} \cdot (\sqrt{g} g^{jk}) \frac{\partial}{\partial q^k} \tag{2}$$

and $g = \det(g^{jk})^{-1}$. The **Laplace equation** is

$$H\Psi(\mathbf{q}) = 0. \tag{3}$$

A particular set of orthogonal coordinates $\{x_\ell\}$ is R**-separable** for (1) if this equation admits solutions of the form

$$\Psi = \exp(R(\mathbf{x}))\Pi_{i=1}^{n}\Psi_i(x^i) = e^R\Theta,$$

where $R(\mathbf{x})$ is a fixed function, independent of parameters, and the factors $\Psi_i(x^i)$ are the solutions of n ODEs (the separation equations)

$$\Psi_i'' + g_i(x^i)\Psi_i' - \left(f_i(x^i) + \sum_{j=1}^{n} \lambda_j s_{ij}(x^i) \right) \Psi_i = 0, i = 1, \cdots, n \tag{4}$$

and $\lambda_1 = E$. The parameters λ_j are the **separation constants**. If $R \equiv 0$ we have *separation*, and if $R(x) = \sum_{i=1}^{n} R^{(i)}(x^i)$ we have *trivial* R-separation. Otherwise the R-separation is *nontrivial*. There is a corresponding definition of R-separation for the Laplace equation (3) with $E = 0$. The $n \times n$ matrix (the **Stäckel matrix**) is required to be nonsingular.

The basic result in the theory is

Theorem 1 *Necessary and sufficient conditions for the existence of an orthogonal R-separable coordinate system* $\{x^i\}$ *for the Schrödinger equation* $(\Delta_n + V)\Psi = E\Psi$ *on an n-dimensional Riemannian manifold are that there exists a linearly independent set* $\{S_1 = H = \Delta_n + V, S_2, \cdots, S_n\}$ *of second-order partial differential operators on the manifold such that:*

1. $[S_k, S_\ell] = 0, \quad 1 \le k, \ell \le n,$

2. *Each* S_h *is in self-adjoint form (in any local coordinate system):*

$$S_\ell = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial q^j} \cdot (\sqrt{g} a^{jk}_{(h)}(q^\ell)) \frac{\partial}{\partial q^k} + W_{(h)}(q^\ell), \quad a^{jk}_{(h)} = a^{kj}_{(h)}$$

3. *There is a basis* $\{\omega_{(j)} : 1 \le j \le n\}$ *of simultaneous eigenforms for the* $\{a^{jk}_{(h)}\}$.

If conditions (1)-(3) are satisfied then there exist functions $g_i(\mathbf{q})$ *such that:*

$$\omega_{(j)} = g_j dx^j, \quad j = 1, \cdots, n,$$

i.e., the eigenforms are proportional to the separable coordinates.

Corollary 1 *If conditions (1)-(3) are satisisfied then the R-separable solutions* $\Psi_{\lambda_1, \cdots, \lambda_n}(\mathbf{x}) = \exp(R(\mathbf{x})) \Pi_{i=1}^{n} \Psi_i(x^i)$ *are characterized as the simultaneous eigenfunctions of the commuting symmetry operators* S_h*:*

$$S_h \Psi_{\lambda_1, \cdots, \lambda_n} = \lambda_h \Psi_{\lambda_1, \cdots, \lambda_n}, \ \ h = 1, \cdots, n$$

REMARKS:

1. No necessary connection between the second-order differential symmetry operators S_j that describe separation on $\mathcal{V}_n$ (plus a potential $V(\mathbf{q})$ and the Lie algebra of first-order symmetries of $\mathcal{V}_n$. However, if $\mathcal{V}_n$ is a space of constant curvature and $V(\mathbf{q}) \equiv 0$ then all higher order symmetries of this space can be considered as elements of the universal enveloping algebra of the Lie symmetry algebra $\mathcal{G}_n$ of the manifold: the Euclidean Lie algebra $e(n, R)$ for Euclidean space, $so(n, R)$ for the n-sphere, etc. Similarly The conformal symmetry algebra for the Euclidian space Laplace equation is $so(n, 2)$.

2. If $\mathcal{V}_n$ is a space of constant curvature and $V(\mathbf{q}) \neq 0$ then the symmetry operators (or conformal symmetry operators) describing variable separation take the form $S_j = \mathcal{S}_j + W_j$ where $\mathcal{S}_j$ is a second-order element in the enveloping algebra of $\mathcal{G}_n$ and W_j is a scalar (potential) function.

3. For constant curvature spaces there is no nontrival R-separation for the equation $H\Psi = E\Psi$, i.e., we can set $R = 0$ and only consider pure separation . However, nontrivial R-separation does occur for the Lapace equation $H\Psi = 0$.

4. There is a natural (coordinate independent) inner product on the space $L_2(\mathcal{V}_n)$ of square integrable functions F, G, on $\mathcal{V}_n$, given by

$$< F, G >= \int F(\mathbf{q})\overline{G(\mathbf{q})}\sqrt{g}\ d\mathbf{q}.$$

   The second-order symmetry operators S_j are formally symmetric with respect to this inner product: $< S_\ell F, G >=< F, S_\ell G >$.

How does one find all orthogonal separable coordinate systems $\mathbf{q}$ of $H\Psi = E\Psi$ for a given space $\mathcal{V}_n$ for zero potential, $V \equiv 0$? This is a difficult problem in differential geometry. For real n-dimensional Euclidean space, the n-sphere, and the n-hyperboloid of two sheets, Kalnins and the author have a graphical procedure to classify and construct all possibilities. Once the zero potential separation problem is solved, it is straightforward to compute the possible separable systems for any nonzero potential, as a restriction of zero-potential separable systems.

In real Euclidean 3-space there are 11 separable systems see Table 1. Here the equation $H\Psi = E\Psi$ is

$$(\partial_X^2 + \partial_Y^2 + \partial_Z^2)\Psi = -\omega^2\Psi.$$

A basis for the symmetry algebra is : $P_X, P_Y, P_Z, M_{ZY}, M_{XZ}, M_{YX}$ where

$$P_X = \partial_X, \cdots \quad M_{ZY} = Z\partial_Y - Y\partial_Z = -M_{YZ}, \cdots,$$

the commutation relations are

$$[M_{YZ}, M_{ZX}] = M_{YX}, \quad [M_{ZY}, P_Y] = P_Z, \quad [P_X, P_Y] = 0$$

and the modifications of these obtained by cyclically permuting X, Y, Z. We define

$$\mathbf{P}\cdot\mathbf{P} = P_X^2 + P_Y^2 + P_Z^2, \quad \mathbf{M}\cdot\mathbf{M} = M_{ZY}^2 + M_{XZ}^2 + M_{YZ}^2.$$

For the Laplace equation

$$(\partial_X^2 + \partial_Y^2 + \partial_Z^2)\Psi = 0$$

orthogonal separation is possible in the 11 Helmholtz separable systems and nontrivial R- separation in 6 additional systems.

5 Hilbert space models for the solution space of the Helmoltz equation

We return to the 3D Euclidean space Helmholtz equation with zero potential:

$$(\partial_X^2 + \partial_Y^2 + \partial_Z^2)\Psi = -\omega^2\Psi.$$

We can impose a Hilbert space structure on the solution space of this equation such that the operators defining variable separation are self-adjoint. This is done by taking the Fourier transform. Consider the mapping I from functions on the two-sphere to the solution space:

$$\Psi(\mathbf{X}) = \int\int_{S_2} e^{i\omega\mathbf{X}\cdot\mathbf{K}} h(\mathbf{K}) d\Omega(\mathbf{K}) = I(h),$$

$$\mathbf{X} = (X, Y, Z), \quad \mathbf{K} = (k_1, k_2, k_3), \quad \mathbf{K}\cdot\mathbf{K} = 1.$$

Here $\mathbf{K}$ is a unit vector that runs over the unit two-sphere S_2, $d\Omega$ is the usual solid-angle measure on the sphere. The set $L_2(S_2)$ of measurable functions h is a Hilbert space with inner product

$$< h_1, h_2 >= \int\int_{S_2} h_1(\mathbf{K})\overline{h_2(\mathbf{K})} d\Omega(\mathbf{K}).$$

In spherical coordinates on S_2 we have

$$\mathbf{K} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad d\Omega(\mathbf{K}) = \sin\theta\ d\theta\ d\phi.$$

The Lie algebra generators on $L_2(S_2)$ induced by the symmetries on the solution space of the Helmholtz equations are

$$\mathcal{P}_X = i\omega k_1, \quad \mathcal{P}_Y = i\omega k_2, \quad \mathcal{P}_X = i\omega k_3, \quad k_1^2 + k_2^2 + k_3^2 = 1,$$

$$\mathcal{M}_{ZY} = k_3\partial_{k_2} - k_2\partial_{k_3} = \sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi,$$

$$\mathcal{M}_{XZ} = k_1\partial_{k_3} - k_3\partial_{k_1} = -\cos\phi\partial_\theta + \sin\phi\cot\theta\partial_\phi, \quad \mathcal{M}_{YX} = k_2\partial_{k_1} - k_1\partial_{k_2} = -\partial_\phi.$$

We can use the two-variable model $L_2(S_2)$ to work out the precise spectral resolutions for pairs of formally commuting symmetry operators S, S' that define a separation of variables for the Helmholtz equation. That is, we compute a basis of eigenfunctions:

$$\mathcal{S}f_{\lambda\mu} = \lambda f_{\lambda\mu}, \quad \mathcal{S}' f_{\lambda\mu} = \mu f_{\lambda\mu}, \quad < f_{\lambda\mu}, f_{\lambda'\mu'} >= \delta(\lambda - \lambda')\delta(\mu - \mu').$$

Then the functions $\Psi_{\lambda\mu}(\mathbf{X}) = I(f_{\lambda\mu})$ will form a corresponding basis on the solution space of the Helmholtz equation.

Table 1 Separable coordinates	in 3-D real Euclidean space.
Coordinate System	Coordinates and Operators $S_1,\ S_2$
I. Cartesian $x, y, z \in \mathbf{R}$	x, y, z $S_1 = P_X^2,\ S_2 = P_Y^2$
II. Cylindrical polar $\rho > 0,\ \varphi \in [0, 2\pi)$	$x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z$ $S_1 = M_{YX}^2, S_2 = P_Z^2$
III. Cylindrical elliptic $z \in \mathbf{R},\ d > 0$	$x = d\cosh\alpha\cos\beta,\quad y = d\sinh\alpha\sin\beta,\quad z$ $S_1 = M_{YX}^2 + d^2P_X^2,\ S_2 = P_Z^2$
IV. Cylindrical parabolic $\xi, x \in \mathbf{R},\ \eta \geq 0$	$z,\ y = \xi\eta,\quad x = \frac{1}{2}(\xi^2 - \eta^2)$ $S_1 = M_{YX}P_Y + P_YMYX,\ S_2 = P_Z^2$
V. Spherical $r > 0,\ \theta \in [0, \pi],\quad \varphi \in [0, 2\pi)$	$x = r\sin\theta\cos\varphi,\quad y = r\sin\theta\sin\varphi,\quad z = r\cos\theta$ $S_1 = \mathbf{M}\cdot\mathbf{M},\ S_2 = M_{YX}^2$
VI. Prolate spheroidal $a > 0,$ $\varphi \in [0, 2\pi)$	$x = a\sinh\eta\sin\alpha\cos\varphi,\quad y = a\sinh\eta\sin\alpha\sin\varphi,$ $z = a\cosh\eta\cos\varphi$ $S_1 = \mathbf{M}\cdot\mathbf{M} - a^2(P_X^2 + P_Y^2),\ S_2 = M_{YX}^2$
VII. Oblate spheroidal $a > 0,$ $\varphi \in [0, 2\pi)$	$x = a\cosh\eta\sin\alpha\cos\varphi,\quad y = a\cosh\eta\sin\alpha\sin\varphi$ $z = a\sinh\eta\cos\varphi$ $S_1 = \mathbf{M}\cdot\mathbf{M} + a^2(P_X^2 + P_Y^2),\ S_2 = M_{YX}^2$
VIII. Sphero-conical $r \geq 0$ $0 < b < 1 < \rho_2 < \frac{1}{b},\ 0 < \rho_1 < 1$	$x^2 = r^2\frac{(b\rho_1-1)(b\rho_2-1)}{1-b},\quad y^2 = r^2\frac{b(\rho_1-1)(\rho_2-1)}{b-1}$ $z^2 = r^2b\rho_1\rho_2$ $S_1 = \mathbf{M}\cdot\mathbf{M},\ S_2 = M_{ZY}^2 + bM_{XZ}^2$
IX. Parabolic $\xi, \eta \geq 0,\quad \varphi \in [0, 2\pi)$	$x = \xi\eta\cos\varphi,\quad y = \xi\eta\sin\varphi,\quad z = \frac{1}{2}(\xi^2 - \eta^2)$ $S_1 = M_{ZY}P_Y + P_YM_{ZY} - M_{XZ}P_X + P_XM_{XZ},$ $S_2 = M_{YX}^2$
X. Ellipsoidal $0 < u_1 < 1 < u_2 < a < u_3$	$x^2 = \frac{(u_1-a)(u_2-a)(u_3-a)}{a(a-1)},\quad y^2 = \frac{(u_1-1)(u_2-1)(u_3-1)}{1-a}$ $z^2 = \frac{u_1u_2u_3}{a}$ $S_1 = P_X^2 + aP_Y^2 + (a+1)P_Z^2 + \mathbf{M}\cdot\mathbf{M},$ $S_2 = M_{XZ}^2 + a(M_{ZY}^2 + P_Z^2)$
XI. Paraboloidal $0 < c,\ o \leq \alpha,\ \varphi \in [0, 2\pi)$	$x = 2c\cosh\alpha\cos\beta\sinh\gamma,\quad y = 2c\sinh\alpha\sin\beta\cosh\gamma$ $z = \frac{c}{2}(\cosh 2\alpha + \cos 2\beta - \cosh 2\gamma)$ $S_1 = M_{YX}^2 - c^2P_Z^2 +$ $c(M_{XZ}P_X + P_XM_{XZ} + M_{ZY}P_Y + P_YM_{ZY})$ $S_2 = c(P_Y^2 - P_X^2) +$ $M_{XZ}P_X + P_XM_{XZ} - M_{ZY}P_Y - P_YM_{ZY}$

Examples:

1. Spherical system.

$$f_m^{(\ell)}(\theta,\phi) = Y_\ell^m(\theta,\phi), \quad \Psi_m^{(\ell)}(\mathbf{X}) = 4\pi i^\ell j_\ell(\omega r) Y_\ell^m(\theta,\phi),$$

$$\pm m = 0,1,\cdots,\ell, \quad \ell = 0,1,\cdots \quad S_1 = -\ell(\ell+1)E, \quad S_2 = -m^2 E.$$

Here, $< f_m^{(\ell)}, f_{m'}^{(\ell')} >= \delta_{mm'}\delta_{\ell\ell'}$, Y_ℓ^m is a spherical harmonic and j_ℓ is a spherical Bessel function.

2. Parabolic system.

$$f_{\lambda m}(\theta,\phi) = \frac{[\tan(\theta/2)]^{-i\lambda}}{2\pi \sin\theta} e^{im\phi},$$

$$\Psi_{\lambda,m}(\mathbf{X}) = \frac{i^m\sqrt{2}}{\xi\eta\omega}\Gamma(\frac{1-m+i\lambda}{2})\Gamma(\frac{1-m-i\lambda}{2})\mathcal{M}_{i\lambda/2,-m/2}[\frac{e^{-i\pi/2}\omega\xi^2}{\sqrt{2}}]$$

$$\times\mathcal{M}_{i\lambda/2,-m/2}[\frac{e^{i\pi/2}\omega\eta^2}{\sqrt{2}}]e^{im\phi},$$

$$\pm m = 0,1,\cdots \quad -\infty < \lambda < \infty \quad S_1 = 2\lambda\omega E, \quad S_2 = -m^2 E.$$

Here, $< f_{\lambda,m}, f_{\lambda',m'} >= \delta_{\lambda-\lambda'}\delta_{mm'}$, and $\mathcal{M}_{\alpha,\mu}$ is a Whittaker function.

3. Ellipsoidal system.

$$\mathrm{el}p_n{}^m(\eta,\psi) = \mathrm{el}_n^m(\eta)\mathrm{el}_n^m(\psi), \quad \Psi_{nm}(\mathbf{X})) = K_n{}^m(\omega,k)\mathrm{el}_n^m(\alpha)\mathrm{el}_n^m(\beta)\mathrm{el}_n^m(\gamma),$$

$$\pm m = 0,1,\cdots,n \quad n = 0,1,\cdots \quad S_1 = \lambda_{mn}E, \quad S_2 = \mu_{mn}E.$$

Here, $< \mathrm{el}p_n{}^m, \mathrm{el}p_{n'}{}^{m'} >= \delta_{nn'}\delta_{mm'}$, and el_n^m is an ellipsoidal wave function with modulus $k = a^{-1/2}$.

The expressions $\Psi = I(f)$ are nontrivial integral representations for the separated solutions. However, the integrals are easy to evaluate because we know in advance that they separate in the corresponding coordinates. Thus we need only calculate normalization constants.

6 Hilbert space expansion formulas

To expand one type of separable basis in terms of another, it is most convenient to compute the expansion coefficients in the sphere model. For example, the overlap integrals between the spherical basis and the parabolic basis can be shown to be

$$< f_m^{(\ell)}, f_{\lambda,m'} >= \delta_{mm'} \frac{(-1)^{\frac{m+|m|}{2}}}{(|m|!)^2} \sqrt{\frac{(2\ell+1)(\ell+|m|)!}{4\pi(\ell-|m|)!}} \Gamma(\frac{i\lambda+|m|+1}{2})$$

$$\times\Gamma(\frac{-i\lambda+|m|+1}{2})_3F_2\left(\begin{array}{ccc} |m|-\ell, & |m|+\ell+1, & \frac{i\lambda+|m|+1}{2} \\ |m|+1, & |m|+1 & \end{array} |1\right),$$

$\pm m = 0, 1, \cdots, \ell$. This allows us the expand a spherical basis function as an integral over products of Whittaker functions.

7 Conformal symmetries of the Laplace equation and the lower-variable models

Basic facts concerning the solution space of the 3-variable Laplace equation in Euclidean space

$$(\partial_X^2 + \partial_Y^2 + \partial_Z^2)\Psi = 0. \tag{5}$$

The conformal symmetry algebra of this equation is 10-dimensional, with basis

$$P_X = \partial_X, \quad P_Y = \partial_Y, \quad P_Z = \partial_Z, \quad M_{YX} = -M_{XY} = Y\partial_X - X\partial_Y,$$

$$M_{XZ} = Z\partial_Z - Z\partial_X, \quad M_{ZY} = Z\partial_Y - Y\partial_Z, \quad D = -(\frac{1}{2} + X\partial_X + Y\partial_Y + Z\partial_Z),$$

$$K_X = -2XD - R^2\partial_X, \quad K_Y = -2YD - R^2\partial_Y, \quad K_Z = -2ZD - R^2\partial_Z,$$

where $R^2 = X^2 + Y^2 + Z^2$. Note the relation

$$\mathbf{M} \cdot \mathbf{M} \equiv M_{YX}^2 + M_{XZ}^2 + M_{ZY}^2 = \frac{1}{4} - D^2$$

that holds only on the solution space of (5).

Every separable or R-separable solution set for (5), as well as the corresponding separable coordinates, is characterized by a pair of second-order commuting conformal symmetry operators for (5). For ellipsoidal coordinates ($R = 0$) the operators can be chosen as

$$\mathbf{M} \cdot \mathbf{M} + (a-1)P_Y^2 + aP_Z^2, \quad M_{XZ}^2 + aM_{YZ}^2 - aP_Z^2,$$

whereas for conical coordinates ($R = 0$) the operators are

$$\mathbf{M} \cdot \mathbf{M}, \quad M_{XZ}^2 + aM_{YZ}^2.$$

THE MODEL: We represent solutions $\Psi(X,Y,Z)$ of the Laplace equation in the integral form

$$\Psi(X,Y,Z) = \int_{C_1} d\beta \int_{\tilde{C}_2} d\varphi\ h[\beta,\varphi] \exp\left[\beta(iX\cos\varphi + iY\sin\varphi - Z)\right]$$
$$\equiv I(h).$$

The action of the scale Euclidean symmetries on the solution space of the Laplace equation corresponds to the model operators

$$\mathcal{P}_X = i\beta w_1, \quad \mathcal{P}_Y = i\beta w_2, \quad \mathcal{P}_Z = -\beta, \quad D = \beta\partial_\beta + \frac{1}{2}, \quad \mathcal{M}_{XY} = -w_2\partial_{w_1},$$
$$\mathcal{M}_{ZX} = iw_1\beta\partial_\beta + iw_2^2\partial_{w_1}, \quad \mathcal{M}_{ZY} = iw_2\beta\partial_\beta - iw_1w_2\partial_{w_1}. \tag{6}$$

where $w_1 = \cos\varphi,\ w_2 = \sin\varphi$, so $w_1^2 + w_2^2 = 1$.

Integral representation for solutions Ψ of the Laplace equation that are eigenfunctions of the dilation operator D with eigenvalue $-\ell - \frac{1}{2}$:

$$D\Psi = (-\ell - \frac{1}{2})\Psi.$$

Choose C_1 and C_2 as unit circles in the β and t complex planes, respectively. Setting $\mathcal{D}h = (-\ell - \frac{1}{2})h$ we find

$$h(\beta,t) = \beta^{-\ell-1}j(t), \quad j(t) = \sum_{m=-\ell}^{\ell} a_m t^m, \quad t = e^{i\phi}.$$

Then we evaluate the β integral by residues to obtain

$$\Psi(X,Y,Z) = I(h) = \int_0^{2\pi} \left[X\cos\varphi + Y\sin\varphi + iZ\right]^\ell j(e^{i\varphi})d\varphi.$$

For $j(t) = t^m$, $-\ell \le m \le \ell$ we have $M^0\Psi = m\Psi$ so Ψ must be a multiple of the solid harmonic $R^\ell Y_\ell^m(\theta,\phi)$, expressed in spherical coordinates.

8 Non-Hilbert space expansion formulas

Eigenvalue equations for the separable parabolic system of the Laplace equation. In the two-variable model these eigenfunctions are elementary

$$f_{\lambda,m}(\beta,t)=\frac{1}{\beta}e^{-\frac{\lambda}{2\beta}}t^m=\sum_k \frac{(-\lambda/2)^k}{k!}\beta^{-k-1}t^m. \tag{7}$$

Choosing the contours C_1, C_2 and mapping back to the solution space of the Laplace equation, we find

$$\begin{aligned}\Psi_{\lambda,m}(X,Y,Z) &= -2\pi\int_0^{2\pi} J_0\left[i\sqrt{2\lambda(Z-iX\cos\alpha-iY\sin\alpha}\right]e^{im\alpha}d\alpha\\ &= -4\pi^2 J_m(-i\sqrt{\lambda}\xi)J_m(\sqrt{\lambda}\eta)e^{im\phi},\end{aligned}$$

where $X=\xi\eta\cos\phi$, $Y=\xi\eta\sin\phi$, $Z=(\xi^2-\eta^2)/2$. The fact that the variables must separate in parabolic coordinates enables us to compute the integral almost immediately. Furthermore, applying the transformation I successively to each term in the series (7), we find the expansion

$$\Psi_{\lambda,m}(X,Y,Z)=-\sum_{\ell=|m|}^{\infty}\frac{16\pi^3 i^m(\lambda\rho/2)^\ell}{\ell!\sqrt{4\pi(2\ell+1)(\ell-m)!(\ell+m)!}}Y_\ell^m(\theta,\phi)$$

of products of Bessel functions in terms of spherical harmonics.

9 Models in higher dimensions

Extend this analysis to Laplace equations in n dimensions. For $n = 4$:

$$(\partial_X^2 + \partial_Y^2 + \partial_Z^2 + \partial_T^2)\Psi = 0. \tag{8}$$

The conformal symmetry algebra of this equation is 15-dimensional, but for simplicity we consider only the 11-dimensional scale Euclidean subalgebra with basis

$$P_X = \partial_X, \quad P_Y = \partial_Y, \quad P_Z = \partial_Z, \quad P_T = \partial_T \quad M_{YX} = -M_{XY} = Y\partial_X - X\partial_Y,$$

$$M_{XZ} = Z\partial_Z - Z\partial_X, \quad M_{ZY} = Z\partial_Y - Y\partial_Z, \quad M_{TX} = T\partial_X - X\partial_T,$$

$$M_{TY} = T\partial_Y - Y\partial_T, \quad M_{TZ} = T\partial_Z - Z\partial_T, \quad D = -(X\partial_X + Y\partial_Y + Z\partial_Z + T\partial_T).$$

Note the relation

$$\mathbf{M}\cdot\mathbf{M} \equiv M_{YX}^2 + M_{XZ}^2 + M_{ZY}^2 + M_{TX}^2 + M_{TY}^2 + M_{TZ}^2 = -D(D-2)$$

holding only on the solution space of (8).

The various spherical and ellipsoidal separable solution sets for (8), are each characterized by a triplet of second-order commuting scale Euclidean symmetry operators for (8). Computations involving separable solutions of the Laplace equation are simplified by making use of a 3 variable model for the solution space.

Represent solutions $\Psi(X,Y,Z,T)$ in the integral form

$$\Psi(X,Y,Z,T) = \int_{C_1} d\beta \int\int_{\mathcal{D}} \exp\left[\beta(w_1X + w_2Y + w_3Z + iT)\right] h(\beta,\mathbf{w}) \frac{dw_1\, dw_2}{w_3} \tag{9}$$

$$\equiv I(h),$$

where $w_1^2 + w_2^2 + w_3^2 = 1$.

Integrating by parts, we find that the action of the symmetries $P_X, \cdots, D$ on the solution space of the Laplace equation corresponds to the operators

$$\mathcal{P}_X = \beta w_1, \quad \mathcal{P}_Y = \beta w_2, \quad \mathcal{P}_Z = \beta w_3, \quad \mathcal{P}_T = i\beta, \quad \mathcal{D} = \beta \partial_\beta \tag{10}$$

$$\begin{gathered}
\mathcal{M}_{XY} = w_1 \partial_{w_2} - w_2 \partial_{w_1}, \quad \mathcal{M}_{XZ} = -w_3 \partial_{w_1}, \quad \mathcal{M}_{YZ} = -w_3 \partial_{w_2}, \\
\mathcal{M}_{TX} = -iw_1 + iw_1 \beta \partial_\beta + i(1 - w_1^2)\partial_{w_1} - iw_1 w_2 \partial_{w_2}, \\
\mathcal{M}_{TY} = -iw_2 + iw_2 \beta \partial_\beta - iw_2 w_1 \partial_{w_1} + i(1 - w_2^2)\partial_{w_2}, \\
\mathcal{M}_{TZ} = -iw_3 + iw_3 \beta \partial_\beta - iw_3 w_1 \partial_{w_1} - iw_3 w_2 \partial_{w_2}.
\end{gathered}$$

Integral representation for solutions Ψ of the Laplace equation that are eigenfunctions of the dilation operator D with eigenvalue $-\ell$: $D\Psi = -\ell\Psi$. Proceeding as in the $n = 3$ case, we find

$$\Psi(X, Y, Z, T) = I(h) = \int\int_{\mathcal{D}} [w_1 X + w_2 Y + w_3 Z + iT]^\ell \, j(\mathbf{w}) \frac{dw_1 \, dw_2}{w_3}. \tag{11}$$

10 Niven operators

Niven constructed an operator that maps harmonic functions into ellipsoidal solutions. Indeed, it maps a conical coordinate solution to an ellipsoidal solution. A detailed technical proof is given by Whittaker and Watson. Here we give a much simpler proof, from the viewpoint of Lie theory. We treat an example of the case $n = 3$ for simplicity.

Let H_ℓ be the space of solutions of the Laplace equation that are homogeneous of degree ℓ. There is an operator F_ℓ, the Niven operator, such that the identities

$$\begin{aligned} \left(\mathbf{M}\cdot\mathbf{M} + (a-1)P_Y^2 + aP_Z^2\right) F_\ell &= F_\ell(\mathbf{M}\cdot\mathbf{M}) \\ \left(M_{XZ}^2 + aM_{YZ}^2 + aP_Z^2\right) F_\ell &= F_\ell(M_{XZ}^2 + aM_{YZ}^2) \end{aligned} \tag{12}$$

hold on H_ℓ. The operator can be chosen in the form

$$F_\ell = {}_0F_1\left(\begin{matrix} - \\ -\ell - \frac{1}{2} \end{matrix}; \frac{1}{4}((a-1)P_Y^2 + aP_Z^2)\right).$$

Thus the Niven operator is an intertwining operator on H_ℓ: If Θ satisfies the conical eigenvalue equations $(M_{XZ}^2 + aM_{YZ}^2)\Theta = \lambda_1\Theta$ and $\mathbf{M}\cdot\mathbf{M}\Theta = -\ell(\ell+1)\Theta$, then $\Psi = F_\ell\Theta$ satisfies the ellipsoidal eigenvalue equations

$$\left(M_{XZ}^2 + aM_{YZ}^2 + aP_Z^2\right)\Psi = \lambda_1\Psi, \tag{13}$$

$$\left(\mathbf{M}\cdot\mathbf{M} + (a-1)P_Y^2 + aP_Z^2\right)\Psi = -\ell(\ell+1)\Psi. \tag{14}$$

We verify (12) using the model, on the space $\mathcal{H}_\ell$ of functions $h(\beta,t) = \beta^{-\ell-1}j(t)$. Setting $t = e^{i\varphi}$ we have $(a-1)\mathcal{P}_Y^2 + a\mathcal{P}_Z^2 = \beta^2(a\sin^2\varphi + \cos^2\varphi)$. Set $\mathcal{F}_\ell = \mathcal{F}_\ell(x)$, where $x = \beta^2(a\sin^2\varphi + \cos^2\varphi)$. Thus on $\mathcal{H}_\ell$ the Niven operator is just multiplication by an ordinary analytic function of x. The first equation (12) on $\mathcal{H}_\ell$ then reduces to a second-order ODE for $\mathcal{F}_\ell$:

$$4x\mathcal{F}_\ell'' + (-4\ell+2)\mathcal{F}_\ell' - \mathcal{F}_\ell = 0. \tag{15}$$

The solution bounded at 0 is

$$\mathcal{F}_\ell = {}_0F_1\left(\begin{matrix} - \\ -\ell - \frac{1}{2} \end{matrix} ; \frac{1}{4}x \right).$$

Transferring this operator over to the solution space of the Laplace equation via $F_\ell \Psi = I(\mathcal{F}_\ell h)$ we obtain the required result.

A classical result of Whittaker and Watson is that if $L_\ell^m(\alpha)$ are Lamé polynomials then

$$L_\ell^m(\alpha) L_\ell^m(\beta) L_\ell^m(\gamma) = \int_{-2K}^{2K} P_\ell(\mu) L_\ell^m(\delta) d\delta$$

where

$$\mu = k^2 \mathrm{sn}\alpha\ \mathrm{sn}\beta\ \mathrm{sn}\gamma\ \mathrm{sn}\delta - (k^2/k'^2)\mathrm{cn}\alpha\ \mathrm{cn}\beta\ \mathrm{cn}\gamma\ \mathrm{cn}\delta - (1/k'^2)\mathrm{dn}\alpha\ \mathrm{dn}\beta\ \mathrm{dn}\gamma\ \mathrm{dn}\delta$$

and $P_\ell(z)$ is a Legendre polynomial. This result is a direct consequence of the fact proved above that if

$$H_\ell^m(X, Y, Z) = \int_{-\pi}^{\pi} (X\cos t + Y \sin t + iZ)^\ell f(t) dt,$$

then $G_\ell(X, Y, Z)$ is also a solution of the Laplace equation, where (transforming the action of the Niven operator from the model to the solution space of the Laplace equation)

$$G_\ell^m(X, Y, Z) = \Gamma(\frac{1}{2} - \ell)(\frac{D}{2})^{\ell + \frac{1}{2}} I_{-\ell - \frac{1}{2}}(D) H_\ell^m(X, Y, Z)$$

$$= \left[1 + \sum_{r=1}^{\frac{1}{2}\ell} \frac{(-1)^r D^{2r}}{2 \cdot 4 \cdots 2r \cdot (2\ell - 1) \cdot (2\ell - 3) \cdots (2\ell - 2r + 1)} \right] H_\ell^m(X, Y, Z)$$

and $D^2 = \partial_Y^2 + a\partial_Z^2$.

11 Lamé polynomials.

If $\Psi(X,Y,Z)$ satisfies Laplace's equation and is homogeneous of degree ℓ (an integer). Choose spheroconical coordinates

$$X = Rk\,\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k), \quad Y = R\frac{ik}{k'}\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k), \quad Z = R\frac{i}{k'}\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k). \tag{16}$$

If we set $\Psi = R^\ell U(\mu)V(\nu)$ and substitute into the Laplace equation, we obtain a solution if $U(\mu)$ satisfies the equation

$$(\partial_\mu^2 - k^2\mathrm{sn}^2(\mu,k) + \lambda)\hat{U}(\mu) = 0,$$

with a similar result for $V(\nu)$. The operator which characterizes λ is

$$\Lambda = \frac{1}{k^2}M_{XY}^2 + M_{XZ}^2.$$

We now look for eigenfunctions of the same operator $L = \frac{1}{k^2}\mathcal{M}_{XY}^2 + \mathcal{M}_{XZ}^2$ in the two variable model. In this representation they must satisfy

$$\left[(\frac{1}{k^2} - \sin^2\varphi)\partial_\varphi^2 - (2\ell+3)\sin\varphi\cos\varphi\partial_\varphi - (\ell+1)(\ell+2)\cos^2 t + \ell + 1 - \lambda\right] f(\varphi) = 0,$$

which can be transformed to the Lamé equation. Mapping from the two variable model to the solution space of the Laplace equation we obtain the product formula

$$L(\mu,k)L(\nu,k) = \int [-ikk'\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k)\mathrm{sn}(\omega,k) - \frac{k}{k'}\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k)\mathrm{cn}(\omega,k) +$$

$$\frac{i}{k}\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k)\mathrm{dn}(\omega,k)]^\ell L(\omega,k)d\omega,$$

where $L(z,k)$ is a solution of the Lamé equation

$$(\partial_z^2 - k^2\ell(\ell+1)\mathrm{sn}^2(z,k) - \lambda)L(z,k) = 0.$$

12 A general Heun type product formula.

Consider the Laplace equation in $p+q+r+3$ variables. We choose coordinates of the form

$$X = R\sqrt{\frac{uv}{a}}s_p, \quad Y = R\sqrt{\frac{(u-1)(v-1)}{1-a}}s_q, \quad Z = R\sqrt{\frac{(u-a)(v-a)}{a(a-1)}}s_r \tag{17}$$

where the s_p can be expressed in terms of coordinates on a p dimensional sphere. We look for solutions of the $p+q+r+3$ dimensional Laplace equation in these coordinates, i.e.,

$$(\partial_X^2 + \partial_Y^2 + \partial_Z^2)\Psi = 0.$$

If we take out the angular dependence on the spheres and write $\Phi(R,u,v) = R^t U(u)V(v)$ then variables separate. Using the two variable model we obtain a general product formula

$$(\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k))^{-p/2}\times$$

$$(\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k))^{-q/2}(\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k))^{-r/2}L(\mu)L(\nu)P_\ell(s_1)P_m(s_2)P_n(s_3)$$

$$= \int_{s_1}\int_{s_2}\int_{s_3}[\frac{1}{k'_2}\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k)\mathrm{dn}(\omega,k)s_3\cdot s'_3 - \frac{k}{k'_2}\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k)\mathrm{cn}(\omega,k)s_2\cdot s'_2$$

$$-ik\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k)\mathrm{sn}(\omega,k)s_1\cdot s'_1]^\ell$$

$$\mathrm{sn}(\omega,k)^{p/2}\mathrm{cn}(\omega,k)^{q/2}\mathrm{dn}(\omega,k)^{r/2}P_\ell(s'_1)P_m(s'_2)P_n(s'_3)ds'_1ds'_2ds'_3d\omega.$$

Here, the P_ℓ are spherical harmonics and the L are Heun functions.

13 Second-order superintegrability and multiseparability

The maximal possible number of linearly independent 2nd-order symmetry operators for a Schrödinger equation $H\Psi = E\Psi$, where $H = \Delta_n + V$, is $2n-1$. This maximum is achieved only for a few remarkable manifolds and potentials.

Definition 2 *The system $H\Psi = E\Psi$ is second-order superintegrable if there are $2n-1$ linearly independent second-order partial differential symmetry operators: $S_\ell, \quad \ell = 0, 1, \cdots, 2n-2, \quad S_0 = H, \quad [S_\ell, H] = 0.$*

Let's compare the concepts of second-order superintegrability and separability on an n-dimensional Riemannian manifold.

Superintegrable: There are $2n-1$ linearly independent 2nd order constants of the motion $S_0 = H, S_1, S_2, \cdots, S_{2n-2}$: $[H, S_j] = 0, \qquad j = 1, \cdots, 2n-2.$

Separable: There are n linearly independent 2nd order symmetries $S_0 = H, S_1, S_2, \cdots, S_{n-1},$: $[S_j, S_k] = 0, 0 \le j, k \le n-1.$ The symmetries must also satisfy eigenform conditions.

One of the most effective methods of finding superintegrable systems is to search for systems that are *multiseparable.*

Example : Real Euclidean 2-space. Here, $n = 2$, $2n-1 = 3$, so each separable system yields one new symmetry. Consider the Schrödinger equation with potential

$$V(x, y) = \frac{1}{2}\left(\omega^2(x^2+y^2) + \frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}\right),$$

i.e.,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi - \left(\omega^2(x^2+y^2) + \frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}\right)\Psi = -2E\Psi.$$

This equation separates in three systems: *Cartesian* coordinates (x, y); *polar* coordinates $x = r\cos\theta$, $y = r\sin\theta$, and *elliptical* coordinates

$$x^2 = c^2 \frac{(u_1 - e_1)(u_2 - e_1)}{(e_1 - e_2)}, \qquad y^2 = c^2 \frac{(u_1 - e_2)(u_2 - e_2)}{(e_2 - e_1)}.$$

The bound state energies are given by $E_n = \omega(2n + 2 + k_1 + k_2)$ for integer n. The corresponding wave functions are

1) *Cartesian*:

$$\Psi_{n_1,n_2}(x, y) = 2\omega^{\frac{1}{2}(k_1+k_2+2)} \sqrt{\frac{n_1! n_2!}{\Gamma(n_1 + k_1 + 1)\Gamma(n_2 + k_2 + 1)}}\, x^{(k_1+\frac{1}{2})} y^{(k_2+\frac{1}{2})}$$

$$e^{-\frac{\omega}{2}(x^2+y^2)} L_{n_1}^{k_1}(\omega x^2) L_{n_2}^{k_2}(\omega y^2), \qquad n = n_1 + n_2,$$

and the $L_n^k(x)$ are Laguerre polynomials.

2) *polar*:

$$\Psi(r, \theta) = \Phi_q^{(k_1,k_2)}(\theta)\omega^{\frac{1}{2}(2q+k_1+k_2+1)} \sqrt{\frac{2m!}{\Gamma(m + 2q + k_1 + k_2 + 1)}}$$

$$e^{(-\omega r^2/2)} r^{(2q+k_1+k_2+1)} L_m^{2q+k_1+k_2+1}(\omega r^2), \quad n = m + q,$$

$$\Phi_q^{(k_1,k_2)}(\theta) = \sqrt{2(2q + k_1 + k_2 + 1)\frac{q!\Gamma(k_1 + k_2 + q + 1)}{\Gamma(k_2 + q + 1)\Gamma(k_1 + q + 1)}}$$

$$\times(\cos\theta)^{k_1+(1/2)}(\sin\theta)^{k_2+(1/2)} P_q^{(k_1,k_2)}(\cos 2\theta),$$

and the $P_q^{(k_1,k_2)}(\cos 2\theta)$ are Jacobi polynomials .

3) *elliptical*:

$$\Psi = e^{-\omega(x^2+y^2)} x^{k_1+\frac{1}{2}} y^{k_2+\frac{1}{2}} \prod_{m=1}^{n} \left(\frac{x^2}{\theta_m - e_1} + \frac{y^2}{\theta_m - e_2} - c^2 \right)$$

where

$$\frac{x^2}{\theta - e_1} + \frac{y^2}{\theta - e_2} - c^2 = -c^2 \frac{(u_1 - \theta)(u_2 - \theta)}{(\theta - e_1)(\theta - e_2)}.$$

A basis for the second order symmetry operators is

$$L_1 = \partial_x^2 + \frac{(\frac{1}{4} - k_1^2)}{x^2} - \omega^2 x^2, \qquad L_2 = \partial_y^2 + \frac{(\frac{1}{4} - k_2^2)}{y^2} - \omega^2 y^2$$

$$M^2 = (x\partial_y - y\partial_x)^2 + (\frac{1}{4} - k_1^2)\frac{y^2}{x^2} + (\frac{1}{4} - k_2^2)\frac{x^2}{y^2} - \frac{1}{2}.$$

(Note that $H = L_1 + L_2$.) The separable solutions are eigenfunctions of the symmetry operators L_1, M^2 and $M^2 + e_2 L_1 + e_1 L_2$. The algebra constructed by repeated commutators is

$$[L_1, M^2] = [M^2, L_2] \equiv R, \qquad [L_i, R] = -4\{L_i, L_j\} + 16\omega^2 M^2, \qquad i \neq j,$$

$$[M^2, R] = 4\{L_1, M^2\} - 4\{L_2, M^2\} + 8(1 - k_2^2)L_1 - 8(1 - k_1^2)L_2,$$

$$R^2 = \frac{8}{3}\{M^2, L_1, L_2\} + \frac{64}{3}\{L_1, L_2\} + 16\omega^2 M^4 - 16(1 - k_2^2)L_1^2$$

$$-16(1 - k_1^2)L_2^2 - \frac{128}{3}\omega^2 M^2 - 64\omega^2(1 - k_1^2)(1 - k_2^2).$$

Note that these relations are quadratic. Here $\{A, B\} = AB + BA$ is a symmetrizer. The important fact to observe about the algebra generated by L_1, L_2, M^2, R is that it is *closed under commutation.*

One can use the structure of such closed quadratic algebras to compute the spectra of L_1, L_2, M^2, R algebraically!

In real Euclidean two-space there are precisely four potentials that have the multiseparation property. The second potential is

$$V(x, y) = \omega^2(4x^2 + y^2) - \frac{(\frac{1}{4} - k_2^2)}{y^2}.$$

The corresponding Schrödinger equation is separable in *Cartesian* coordinates and *parabolic* coordinates $x = \frac{1}{2}(\xi^2 - \eta^2), y = \xi\eta$. The third potential is

$$V(x, y) = -\frac{\alpha}{\sqrt{x^2 + y^2}} + \frac{B_1}{4}\frac{\sqrt{\sqrt{x^2 + y^2} + x}}{\sqrt{x^2 + y^2}} + \frac{B_2}{4}\frac{\sqrt{\sqrt{x^2 + y^2} - x}}{\sqrt{x^2 + y^2}},$$

separable in *parabolic* and *parabolic coordinates of the second type* $x = \mu\nu, \;\; y = \frac{1}{2}(\mu^2 - \nu^2)$. The fourth potential is

$$V(x, y) = -\frac{\alpha}{\sqrt{x^2 + y^2}} + \frac{1}{4\sqrt{x^2 + y^2}}\left(\frac{(k_1^2 - \frac{1}{4})}{\sqrt{x^2 + y^2} + x} + \frac{(k_2^2 - \frac{1}{4})}{\sqrt{x^2 + y^2} - x}\right),$$

separable in *polar*, *parabolic* and *modified elliptic* coordinates.

For $n = 2$, a complete classification of all second-order superintegrable potentials (classical and quantum) has been given for real and complex Euclidean space and real and complex spheres. All these cases share the same basic features:

1. The potential V permits separability of the Schrödinger equation $H\Psi = E\Psi$ in at least two coordinate systems, characterized by symmetry conditions $L_1\Psi = \lambda_1\Psi$, $L_2\Psi = \lambda_2\Psi$. *Superintegrability implies multiseparability.*

2. One can obtain alternate spectral resolutions $\{\Psi_j^{(1)}\}, \{\Psi_k^{(2)}\}$ for the multiply-degenerate eigenspaces of H, $L_1\Psi_j^{(1)} = \lambda_1^{(1)}\Psi_j^{(1)}$, $L_2\Psi_k^{(2)} = \lambda_2^{(2)}\Psi_k^{(2)}$. These alternate resolutions resolve the bound state degeneracy problem.

3. The interbasis expansions $\Psi_k^{(2)} = \sum_j a_{jk}\Psi_j^{(1)}$ yield important special function identities.

4. The operators H, L_1, L_2 generate a quadratic algebra. With $R = [L_1, L_2]$, R^2 is a polynomial of order 3 in H, L_1, L_2, whereas $[L_1, R]$ and $[L_2, R]$ are of order 2 in H, L_1, L_2. Closure of these algebras **is a remarkable property**, and is false for general symmetries. The quadratic algebra structure can be used to compute the interbase expansion coefficients.

14 Example of superintegrabiltiy in n dimensions: A "magic" potential on the n-sphere

Introduce Cartesian coordinates $z_0, z_1, \cdots, z_n$ in $n+1$ dimensional Euclidean space and restrict these coordinates by the conditions

$$z_0^2 = 1 - \sum_{i=1}^{n} x_i \equiv x_{n+1}, \quad z_1^2 = x_1,\ z_2^2 = x_2,\ \cdots, z_n^2 = x_n.$$

Note that $z_0^2 + z_1^2 + \cdots + z_n^2 = 1$. Define a metric by $ds^2 = \sum_{m=0}^{n}(dz_m)^2$. The space corresponds to a portion of the n-sphere S^n. Consider the Laplace-Beltrami equation

$$\textbf{MODEL ONE}:\ (\Delta_n + V_n(\mathbf{x}))\Psi = -M(M+G-1)\Psi,$$

where

$$V_n = -\frac{1}{4}\sum_{i=1}^{n+1}\frac{(\gamma_i - \frac{1}{2})(\gamma_i - \frac{3}{2})}{x_i} + \frac{1}{4}\left[(1-G)^2 - 1 - \frac{(n-3)(n+1)}{4}\right].$$

and the γ_i are real constants with $G = \sum_{i=1}^{n+1}\gamma_i$ and $M = 0, 1, \cdots$. This equation has a natural metric

$$d\omega = g^{1/2}dx_1\cdots dx_n = x_1^{-1/2}\cdots x_n^{-1/2}x_{n+1}^{-1/2}dx_1\cdots dx_n.$$

Set $H = \Delta_n + \Lambda_n$ and

$$\rho^{-1} = x_1^{\gamma_1/2-1/4}\cdots x_n^{\gamma_n/2-1/4}x_{n+1}^{\gamma_{n+1}/2-1/4}, \quad \Lambda_n = \sum_{j=1}^{n}[\gamma_j - \frac{1}{2} + (\frac{n+1}{2} - G)x_j]\partial_{x_j}.$$

Then with $\Phi(\mathbf{x}) = \rho(\mathbf{x})\Psi(\mathbf{x})$ we see that

$$\textbf{MODEL TWO}:\ (\Delta_n + \Lambda_n)\Phi = -M(M+G-1)\Phi$$
$$\Longleftrightarrow (\Delta_n + V_n(\mathbf{x}))\Psi = -M(M+G-1)\Psi.$$

Furthermore the eigenspace of H corresponding to eigenvalue $-M(M+G-1)$ consists entirely of polynomials in the x_j. This induces an inner product on the space of polynomial functions $\Phi(\mathbf{x}) = \rho\Psi$, with respect to which H is self-adjoint:

$$(\Phi_1, \Phi_2) \equiv <\Psi_1, \Psi_2> = \int\cdots\int_{x_i>0}\Phi_1\overline{\Phi_2}\ d\tilde{\omega},$$
$$d\tilde{\omega} = x_1^{\gamma_1-1}\ldots x_n^{\gamma_n-1}x_{n+1}^{\gamma_{n+1}-1}dx_1\ldots dx_n,$$
$$(H\Phi_1, \Phi_2) = (\Phi_1, H\Phi_2).$$

REMARKS:

1. If $\gamma_1 = \gamma_2 = \cdots = \gamma_{n+1} = 1/2$ then $H = \Delta_n$ and that the Lie algebra of real symmetry operators of Δ_n is so$(n+1)$, with dimension $n(n+1)/2$ and a basis of the form $\{L_{\ell k}\}$ where $0 \le \ell < k \le n$, and $L_{\ell k} = -L_{k\ell}$. Explicitly,
$$L_{\ell k} = z_\ell \partial_{z_k} - z_k \partial_{z_\ell},$$
and
$$\begin{array}{rl} L_{ij} = & 2\sqrt{x_i x_j}(\partial_{x_j} - \partial_{x_i}), \quad 1 \le i,j \le n \\ L_{0i} = & 2\sqrt{x_i x_{n+1}}\partial_{x_i}, \quad 1 \le i \le n. \end{array}$$

2. For $\gamma_1, \ldots, \gamma_{n+1}$ arbitrary, the only first order symmetry is multiplication by a constant. However, there are second order symmetries (Chesire cat effect):
$$S_{ij} \equiv 4x_i x_j(\partial_{x_i} - \partial_{x_j})^2 + 4(\gamma_i x_j - \gamma_j x_i)(\partial_{x_i} - \partial_{x_j})$$
$$= L_{ij}^2 + 4[(\gamma_i - \frac{1}{2})x_j - (\gamma_j - \frac{1}{2})x_i](\partial_{x_i} - \partial_{x_j}) = S_{ji}, \qquad 1 \le i < j \le n,$$
$$S_{0i} \equiv 4x_i x_{n+1}\partial_{x_i}^2 + 4[\gamma_i x_{n+1} - \gamma_{n+1} x_i]\partial_{x_i}$$
$$= L_{0i}^2 + 4[(\gamma_i - \frac{1}{2})x_{n+1} - (\gamma_{n+1} - \frac{1}{2})x_i]\partial_{x_i} = S_{i0}, \qquad 1 \le i \le n.$$

14.1 Orthogonal bases of separable solutions

All separable coordinates on the n-sphere are known, i.e., all separable coordinates for the Laplace-Beltrami eigenvalue equation $\Delta_n \Psi = \lambda \Psi$. They can be constructed by a graphical procedure. We know that:

- For $n = 2$ there are two separable systems (ellipsoidal and spherical coordinates), while for $n = 3$ there are 6 systems. The number of separable systems grows rapidly with n, but all systems can be constructed through a simple graphical procedure.

- The equation $(\Delta_n + V_n)\Psi = \lambda \Psi$ where the scalar potential takes the special form

$$V_n = \sum_{i=1}^{n} \frac{\alpha_i}{z_i^2} + \frac{\alpha_0}{z_0^2}, \quad \alpha_0, \alpha_1, \ldots, \alpha_n \text{ constants.},$$

  is separable in *all* the coordinate systems in which the Laplace-Beltrami eigenvalue equation is separable. Indeed, the equation with this potential is separable in general ellipsoidal coordinates and their limits.

EXAMPLE: Spherical coordinates $\{u_i\}$ on S^n

$$\begin{array}{ll}
z_0^2 = & 1 - x = 1 - u_n \\
z_1^2 = & x_1 = u_1 u_2 \ldots u_n \\
z_2^2 = & x_2 = (1 - u_1) u_2 \ldots u_n \\
\vdots & \\
z_{n-1}^2 = & x_{n-1} = (1 - u_{n-2}) u_{n-1} u_n \\
z_n^2 = & x_n = (1 - u_{n-1}) u_n .
\end{array}$$

(Note that in terms of angles $\{\theta_i\}$ one usually sets $u_i = \sin^2 \theta_i$.)

14.2 Relations between bases on the sphere

Take the case $n = 2$ in MODEL TWO. Then

$$H\Phi = -j(j + G - 1)\Phi,$$

where

$$H = \sum_{i,k=1}^{2} (x_i\delta_{ik} - x_i x_k)\frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{2}(\gamma_i - Gx_i)\frac{\partial}{\partial x_i}.$$

Here $G = \gamma_1 + \gamma_2 + \gamma_3$.

FACTS:

- H maps polynomials of maximum degree m_i in x_i to polynomials of the same type.
- The polynomial eigenfunctions of H form a basis for the space of all polynomials $f(x_1, x_2)$.
- The spectrum of H acting on this space is exactly $\{-j(j+G-1) : j = 0, 1, \ldots\}$.
- H is self-adjoint with respect to the inner product

  $$(f_1, f_2) = \int \cdots \int_{x_1, x_2 > 0, 1 - x_1 - x_2 > 0} f_1(\mathbf{x})\overline{f_2(\mathbf{x})} d\omega$$

  where

  $$dw = x_1^{\gamma_1 - 1} x_2^{\gamma_2 - 1}(1 - x_1 - x_2)^{\gamma_3 - 1} dx_1 dx_2, \quad (Hf_1, f_2) = (f_1, Hf_2),$$

  and $\gamma_1, \gamma_2, \gamma_3$ are positive real numbers. Here f_1, f_2 are polynomials in $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2})$.
- There are exactly two separable coordinate systems for this equation: spherical coordinates and ellipsoidal coordinates.

Spherical coordinates:
For fixed j the polynomials

$$\begin{aligned}\psi_{jm}(x_1, x_2) &= (x_1 + x_2)^m P_{j-m}^{\gamma_1+\gamma_2+2m-1, \gamma_3 - 1}(2x_1 + 2x_2 - 1) \\ &\quad \times P_m^{\gamma_2 - 1,\ \gamma_1 - 1}\left(\frac{2x_1}{x_1 + x_2} - 1\right),\ m = 0, 1, \ldots, j\end{aligned}$$

form an orthogonal basis for the eigenspace corresponding to eigenvalue $-j(j+G-1)$. Note that variables separate in spherical coordinates.

Ellipsoidal coordinates:

For ellipsoidal coordinates $\{x, y\}$ we have

$$z_i^2 = \frac{(x-e_i)(y-e_i)}{(e_j-e_i)(e_k-e_i)}, \quad i=1,2,3, \quad i,j,k \text{ pairwise distinct.}$$

The separation equations are

$$[-(\lambda-e_1)(\lambda-e_2)(\lambda-e_3)\left[\tfrac{d^2}{d\lambda^2}+\left(\tfrac{\gamma_1}{\lambda-e_1}+\tfrac{\gamma_2}{\lambda-e_2}\right.\right. \\ \left.\left.+\tfrac{\gamma_3}{\lambda-e_3}\right)\tfrac{d}{d\lambda}\right]+j(j+G-1)\lambda+q]\Phi^{\epsilon}_{jq}(\lambda)=0$$

where $\lambda = x, y$ according as $\epsilon = 1, 2$, respectively. **This is Heun's equation, the Fuchsian equation of second order with four singularities.**

The solutions for the functions $\Phi^{\epsilon}_{jq}(\lambda)$ are Heun polynomials which for fixed j will form a complete set of basis functions once the eigenvalues q have been calculated. To calculate the eigenvalues it is convenient to observe that in the coordinate system x_1, x_2 the operator $\mathcal{M}$ whose eigenvalues u are

$$u = -4q - (e_1+e_2+e_3)j(j+G-1)$$

is given by

$$\mathcal{M} = (e_1+e_2)S_{12} + (e_2+e_3)S_{23} + (e_1+e_3)S_{13}$$

where the S_{ik} are the symmetry operators. That is, $\mathcal{M}$ is the second order symmetry operator for the Laplacian ($[\mathcal{M}, \Delta] = 0$) which corresponds to the separable coordinates x, y, and the Heun basis $\psi = \Phi^1_{jq}(x)\Phi^2_{jq}(y)$ is characterized as the set of eigenfunctions $\mathcal{M}\psi = u\psi$.

Expanding the Heun basis $\Phi^1_{jq}(x)\Phi^2_{jq}(y)$ in terms of the Jacobi polynomial basis:

$$\psi = \Phi^1_{jq}(x)\Phi^2_{jq}(y) = \sum_{m=0}^{j} \xi_m \psi_{jm}[\theta, \phi].$$

Three term recurrence relations for the expansion coefficients ξ_m can be deduced by requiring that

$$\mathcal{M}\psi = u\psi.$$

To carry out the computation we need the action of the various pieces S_{ik} of $\mathcal{M}$ on the Jacobi bases $\psi_{jm}[\theta, \phi]$. We find

$$\mathcal{M}\psi_{jm}[\theta, \phi] = \sum_{r=-1}^{+1} X_r \psi_{j,m+r}[\theta, \phi],$$

$$X_1(m, j) = \frac{4(e_1 - e_2)(\gamma_1 + \gamma_2 + \gamma_3 + m + j - 1)(\gamma_3 - m + j - 1)(m + 1)}{(\gamma_1 + \gamma_2 + 2m - 1)(\gamma_1 + \gamma_2 + 2m)}(\gamma_1+\gamma_2+m-1),$$

with similar expressions for $X_{-1}(m, j), X_0(m, j)$. Substituting this expansion into the eigenvalue equation $\mathcal{M}\psi = u\psi$ we find the three term recurrence relation

$$X_1(m - 1, j)\xi_{m-1} + (X_0(m, j) - u)\,\xi_m + X_{-1}(m + 1, j)\xi_{m+1} = 0$$

where $m = 0, 1, \ldots j$. Consequently the $j + 1$ independent eigenvalues q are calculated from the determinant

$$\begin{vmatrix} X_0(j,j) - u & X_1(j-1,j) & & \\ X_{-1}(j,j) & X_0(j-1,j) - u & X_1(j-2,j) & \\ \ddots & \ddots & \ddots & \\ & & & \\ & & X_{-1}(1,j) & X_0(0,j) - u \end{vmatrix}$$

$$= 0.$$

MAIN ISSUES:

1. The importance of models.
2. Classification of superintegrable systems.
3. The relationship between superintegrabiltiy and multiseparability.
4. The structure of quadratic algebras.