Fine structure for second order superintegrable systems.

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Abstract

A classical (or quantum) superintegrable system is an integrable n-dimensional Hamiltonian system with potential that admits $2n-1$ functionally independent constants of the motion polynomial in the momenta, the maximum possible. If the constants are all quadratic the system is second order superintegrable. Such systems have remarkable properties: multi-integrability and multi-separability, a quadratic algebra of symmetries whose representation theory yields spectral information about the Schrödinger operator, deep connections with special functions and with QES systems. For n=2 we have worked out the structure and classified the possible spaces and potentials, and for n=3 on conformally flat spaces with nondegenerate potentials we determined the structure theory and made major progress on the classification.

The quadratic algebra closes at order 6 and there is a 1-1 classical-quantum relationship. All such systems are Stäckel transforms of systems on complex Euclidean space or the complex 3-sphere. We survey these results and announce a series of new results concerning the structure of superintegrable systems with degenerate potentials. In several cases the classification theory for such systems reduces to the study of polynomial ideals on which the symmetry group of the correspond manifold acts.

1 Introduction and examples

In this paper we report on recent work concerning the structure of second order superintegrable systems, both classical and quantum mechanical. We concentrate on the basic ideas; the details of the proofs can be (or will be) found elsewhere. The results on the quadratic algebra structure of 2D and 3D conformally flat systems with nondegenerate potential have appeared recently, but the results on fine structure of 3D superintegrable systems with degenerate potentials, and the relation of superintegrable systems to polynomial ideals are announced here.

Here we consider only superintegrable systems on complex conformally flat spaces. This is no restriction at all in two dimensions. An n-dimensional complex Riemannian space is conformally flat if and only if it admits a set of local coordinates $x_1, \cdots, x_n$ such that the contravariant metric tensor takes the form $g^{ij} = \delta^{ij}/\lambda(\mathbf{x})$. Thus the metric is $ds^2 = \lambda(\mathbf{x})(\sum_{i=1}^n dx_i^2)$. A classical superintegrable system $\mathcal{H} = \sum_{ij} g^{ij} p_i p_j + V(\mathbf{x})$ on the phase space of this manifold is one that admits $2n-1$ functionally independent generalized symmetries (or constants of the motion) $\mathcal{S}_k, \quad k = 1, \cdots, 2n-1$ with $\mathcal{S}_1 = \mathcal{H}$ where the $\mathcal{S}_k$ are polynomials in the momenta p_j. That is, $\{\mathcal{H}, \mathcal{S}_k\} = 0$ where

$$\{f, g\} = \sum_{j=1}^{n} (\partial_{x_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{x_j} g)$$

is the Poisson bracket for functions $f(\mathbf{x}, \mathbf{p}), g(\mathbf{x}, \mathbf{p})$ on phase space [1, 2, 3, 4, 5, 6, 7, 8]. It is easy to see that $2n-1$ is the maximum possible number of functionally independent symmetries and, locally, such (in general nonpolynomial) symmetries always exist.

Superintegrable systems can lay claim to be the most symmetric Hamiltonian systems though many such systems admit no group symmetry. Generically, every trajectory $\mathbf{p}(t), \mathbf{x}(t)$ in phase space, i.e., solution of the Hamilton equations of motion for the sytem, is obtained as the common intersection of the (constants of the motion) hypersurfaces

$$\mathcal{S}_k(\mathbf{p},\mathbf{x}) = c_k, \quad k = 0, \cdots, 2n-2.$$

The trajectories can be found without solving the equations of motion. This is better than integrability. The superintegrabilty of the Kepler-Coulomb two-body problem is the reason that Kepler was able to determine the planetary orbits before the invention of calculus.

A system is second order superintegrable if the $2n-1$ functionally independent symmetries can be chosen to be quadratic in the momenta. Usually a superintegrable system is also required to be integrable, i.e., it is assumed that n of the constants of the motion are in involution, although we do not make that assumption here. Sophisticated tools such as R-matrix theory can be applied to the general study of superintegrable systems, e.g., [9, 10, 11]. However, the most detailed and complete results are known for second order superintegrable systems because separation of variables methods for the associated Hamilton-Jacobi equations can be applied. Standard orthogonal separation of variables techniques are associated with second-order symmetries, e.g., [12, 13, 14, 15, 16, 17] and multiseparable Hamiltonian systems provide numerous examples of superintegrability. Here we concentrate on such systems, i.e., on those in which the symmetries take the form $\mathcal{S} = \sum a^{ij}(\mathbf{x})p_i p_j + W(\mathbf{x})$, quadratic in the momenta. However, the ultimate goal is to develop tools that will enable us to study the structure of superintegrable systems of all orders and to develop a classification theory.

There is an analogous definition for second-order quantum superintegrable systems with Schrödinger operator

$$H = \Delta + V(\mathbf{x}), \quad \Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i}(\sqrt{g} g^{ij})\partial_{x_j},$$

the Laplace-Beltrami operator plus a potential function, [12]. Here there are $2n-1$ second-order symmetry operators

$$S_k = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i}(\sqrt{g} a^{ij}_{(k)})\partial_{x_j} + W^{(k)}(\mathbf{x}), \quad k = 1, \cdots, 2n-1$$

with $S_1 = H$ and $[H, S_k] \equiv HS_k - S_k H = 0$. Again multiseparable systems yield many examples of superintegrability, though not all multiseparable systems are superintegrable and not all second-order superintegrable systems are multiseparable.

A basic motivation for studying these systems is that they can be solved explicitly and in multiple ways. It is the information gleaned from comparing the distinct solutions and expressing one solution set in terms of another that is a primary reason for their interest.

Two dimensional second order superintegrable systems have been studied and classified and the structure of three dimensional systems with nondegenerate potentials has been worked out in a recent series of papers [18, 19, 20, 21, 22]. We survey these results and announce a series of new results concerning the structure of superintegrable systems with degenerate potentials. In several cases the classification theory for such systems reduces to the study of polynomial ideals on which the symmetry group of the correspond manifold acts.

We start with some simple 3D examples. (To make clearer the connection with quantum theory and Hilbert space methods we shall, for these examples alone, adopt standard physical normalizations, such as using the factor $-\frac{1}{2}$ in front of the free Hamiltonian.) Consider the Schrödinger equation $H\Psi = E\Psi$ or ($\hbar = m = 1, x_1 = x, x_2 = y, x_3 = z$)

$$H\Psi = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Psi + V(x,y,z)\Psi = E\Psi.$$

The generalized anisotropic oscillator corresponds to the 4-parameter potential

$$V(x,y,z) = \frac{\omega^2}{2}\left(x^2 + y^2 + 4z^2\right) + \frac{1}{2}\left[\frac{\alpha}{x^2} + \frac{\beta}{y^2}\right] + \gamma z$$

(This potential is "nondegenerate" in a precise sense that we will explain later.) The corresponding Schrödinger equation has separable solutions in five coordinate systems: Cartesian, cylindrical polar, cylindrical elliptic, cylindrical parabolic and parabolic coordinates. A basis for the second order symmetry operators is

$$M_1 = \partial_x^2 - \omega^2 x^2 + \frac{\alpha}{x^2}, \qquad M_2 = \partial_y^2 - \omega^2 y^2 - \frac{\beta}{y^2},$$

$$P = \partial_z^2 - 4\omega^2 (z+\rho)^2, \qquad L = L_{12}^2 - \alpha\frac{y^2}{x^2} - \beta\frac{x^2}{y^2} - \frac{1}{2},$$

$$S_1 = -\frac{1}{2}(\partial_x L_{13} + L_{13}\partial_x) + \rho\partial_x^2 + (z+\rho)\left(\omega^2 x^2 - \frac{\alpha}{x^2}\right),$$

$$S_2 = -\frac{1}{2}(\partial_y L_{23} + L_{23}\partial_y) + \rho\partial_y^2 + (z+\rho)\left(\omega^2 y^2 - \frac{\beta}{y^2}\right),$$

where $L_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}$. It can be verified that these symmetries generate a "quadratic algebra" that closes at level six. Indeed, the nonzero commutators of the above generators are

$$[M_1, L] = [L, M_2] = Q, \quad [L, S_1] = [S_2, L] = B, \quad [M_i, S_i] = A_i, \quad [P, S_i] = -A_i.$$

Nonzero commutators of the basis symmetries with Q (4th order symmetries) are expressible in terms of the second order symmetries:

$$[M_i, Q] = [Q, M_2] = 4\{M_1, M_2\} + 16\omega^2 L, \qquad [S_1, Q] = [Q, S_2] = 4\{M_1, M_2\},$$

$$[L, Q] = 4\{M_1, L\} - 4\{M_2, L\} + 16(\frac{3}{4} - \alpha)M_1 - 16(\frac{3}{4} - \beta)M_2.$$

There are similar expressions for commutators with B and the A_i. Also the squares of Q, B, A_i and products such as $\{Q, B\}$, (all 6th order symmetries) are all expressible in terms of 2nd order symmetries. Indeed

$$Q^2 = \frac{8}{3}\{L, M_1, M_2\} + 8\omega^2\{L, L\} - 16(\frac{3}{4} - \alpha)M_1^2 - 16(\frac{3}{4} - \beta)M_2^2$$

$$+\frac{64}{3}\{M_1, M_2\} - \frac{128}{3}\omega^2 L - 128\omega^2(\frac{3}{4} - \alpha)(\frac{3}{4} - \beta),$$

$$\{Q, B\} = -\frac{8}{3}\{M_2, L, S_1\} - \frac{8}{3}\{M_1, L, S_2\} + 16(\frac{3}{4} - \alpha)\{M_2, S_2\} + 16(\frac{3}{4} - \beta)\{M_1, S_1\}$$

$$-\frac{64}{3}\{M_1, S_2\} - \frac{64}{3}\{M_2, S_1\}.$$

Here $\{C_1, \cdots, C_j\}$ is the completely symmetrized product of operators $C_1, \cdots, C_j$. (For details see [23].) The point is that the algebra generated by products and commutators of the 2nd order symmetries closes at order 6. This is a remarkable fact, and ordinarily not the case for an integrable system. The algebra provides information about the spectra of the generators.

A counterexample to the existence of a quadratic algebra in Euclidean space is given by the Schrödinger equation with 3-parameter extended Kepler-Coulomb potential:

$$\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) + \left[2E + \frac{2\alpha}{\sqrt{x^2 + y^2 + z^2}} - \left(\frac{\beta}{x^2} + \frac{\gamma}{y^2}\right)\right]\Psi = 0 \quad (1)$$

This equation admits separable solutions in four coordinate systems: spherical, sphero-conical, prolate spheroidal and parabolic coordinates. Again the bound states are degenerate and important special function identities arise by expanding one basis of separable eigenfunctions in terms of another. However, the space of second order symmetries is only 5 dimensional and, although there are useful identities among the generators and commutators that enable one to derive spectral properties algebraically, there is no finite quadratic algebra structure. The key difference with our first example is, as we shall show later, that the 3-parameter Kepler-Coulomb potential is degenerate and cannot be extended to a 4-parameter potential.

In [20, 21] there are examples of superintegrable systems on the 3-sphere that admit a quadratic algebra structure. A more general set of examples arises from a space with metric

$$ds^2 = \lambda(A, B, C, D, E, \mathbf{x})(dx^2 + dy^2 + dz^2)$$

and classical potential $V = \lambda(\alpha, \beta, \gamma, \delta, \eta, \mathbf{x})/\lambda(A, B, C, D, E, \mathbf{x})$, where

$$\lambda = A(x+iy)+B\left(\frac{3}{4}(x+iy)^2 + \frac{z}{4}\right)+C\left((x+iy)^3 + \frac{1}{16}(x-iy) + \frac{3z}{4}(x+iy)\right)$$

$$+D\left(\frac{5}{16}(x+iy)^4 + \frac{z^2}{16} + \frac{1}{16}(x^2+y^2) + \frac{3z}{8}(x+iy)^2\right) + E.$$

If $A = B = C = D = 0$ this is a nondegenerate metric on complex Euclidean space. The quadratic algebra always closes, and for general values of A, B, C, D the space is not of constant curvature. As will be explained later, this is an example of a superintegrable system that is Stäckel equivalent to a system on complex Euclidean space.

Observed common features of superintegrable systems are that they are usually multiseparable and that the eigenfunctions of one separable system can be expanded in terms of the eigenfunctions of another. This is the source of nontrivial special function expansion theorems [24]. The symmetry operators are in formal self-adjoint form and suitable for spectral analysis. The representation theory of the abstract quadratic algebra can be used to derive spectral properties of the second order generators in a manner analogous to the use of Lie algebra representation theory to derive spectral properties of quantum systems that admit Lie symmetry algebras, [24, 25, 26, 27].

Another common feature of quantum superintegrable systems is that they can be modified by a gauge transformation so that the Schrödinger and symmetry operators are acting on a space of polynomials, [28]. This is closely related to the theory of exactly and quasi-exactly solvable systems, [29, 30]. The characterization of ODE quasi-exactly solvable systems as embedded in PDE superintegrable systems provides considerable insight into the nature of these phenomena [31].

The classical analogs of the above examples are obtained by the replacements $\partial_{x_i} \to p_{x_i}$ and modification of the potential by curvature terms. Commutators go over to Poisson brackets. The operator symmetries become second order constants of the motion. Symmetrized operators become products of functions. The quadratic algebra relations simplify: the highest order terms agree with the operator case but there are fewer nonzero lower order terms.

Many examples of 3D superintegrable systems are known, although they have not been classified, [32, 33, 34, 35, 36, 37]. Here, we employ theoretical methods based on integrability conditions to derive structure common to all such systems, with a view to complete classification. In each case we work out the classical problem first and then quantize. Finally, we exhibit a deep connection between 3D second order superintegrable systems with nondegenerate potential and a polynomial ideal in 10 variables.

2 2D classical structure theory

For any complex 2D Riemannian manifold we can always find local coordinates x, y such that the classical Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{\lambda(x,y)}(p_1^2 + p_2^2) + V(x,y), \qquad (x,y) = (x_1, x_2),$$

i.e., the complex metric is $ds^2 = \lambda(x,y)(dx^2 + dy^2)$. Necessary and sufficient conditions that $\mathcal{S} = \sum a^{ji}(x,y)p_j p_i + W(x,y)$ be a symmetry of $\mathcal{H}$ are the Killing equations

$$\begin{aligned} a_i^{ii} &= -\frac{\lambda_1}{\lambda}a^{i1} - \frac{\lambda_2}{\lambda}a^{i2}, \quad i = 1,2 \\ 2a_i^{ij} + a_j^{ii} &= -\frac{\lambda_1}{\lambda}a^{j1} - \frac{\lambda_2}{\lambda}a^{j2}, \quad i,j = 1,2,\ i \neq j, \end{aligned} \tag{2}$$

and the Bertrand-Darboux (B-D) conditions on the potential $\partial_i W_j = \partial_j W_i$ or

$$(V_{22} - V_{11})a^{12} + V_{12}(a^{11} - a^{22}) = [\frac{(\lambda a^{12})_1 - (\lambda a^{11})_2}{\lambda}]V_1 + [\frac{(\lambda a^{22})_1 - (\lambda a^{12})_2}{\lambda}]V_2.$$

There are similar but more complicated conditions for the higher order symmetries. From the 3 second order constants of the motion we get 3 B-D equations and can solve them to obtain fundamental PDEs for the potential of the form

$$V_{22} - V_{11} = A^{(22}(\mathbf{x})V_1 + B^{22}(\mathbf{x})V_2, \qquad V_{12} = A^{12}(\mathbf{x})V_1 + B^{12}(\mathbf{x})V_2. \tag{3}$$

If the B-D equations provide no further conditions on the potential and if the integrability conditions for these PDEs are satisfied identically, we say that the potential is **nondegenerate**. That means, at each regular point $\mathbf{x}_0$ where the A^{ij}, B^{ij} are defined and analytic, we can prescribe the values of $V,\ V_1,\ V_2$ and V_{11} arbitrarily and there will exist a unique potential $V(\mathbf{x})$

with these values at $\mathbf{x}_0$. Nondegenerate potentials depend on 3 parameters, in addition to the trivial additive parameter. Degenerate potentials depend on < 3 parameters.

To study the possible quadratic algebra structure it is important to compute the dimensions of the spaces of symmetries of these nondegenerate systems that are of orders 2,3,4 and 6. These symmetries are necessarily of a special type.

- The highest order terms in the momenta are independent of the parameters in the potential.
- The terms of order 2 less in the momenta less are linear in these parameters.
- Those of order 4 less are quadratic and those of order 6 less are cubic.

The system is **2nd order superintegrable** with **nondegenerate potential** if

- it admits 3 functionally independent second-order symmetries (here $2N-1=3$)
- the potential is 3-parameter (in addition to the usual additive parameter) in the sense that it satisfies equations (3) and their integrability conditions.

We say that n functionally independent symmetries are **functionally linearly independent** if at each regular point $\mathbf{x}_0$ the n matrices

$$a^{ij}_{(1)}(\mathbf{x}_0),\ a^{ij}_{(2)}(\mathbf{x}_0), \cdots\ a^{ij}_{(n)}(\mathbf{x}_0)$$

are linearly independent. This functional linear independence criterion splits superintegrable systems of all orders into two classes with different properties. In 2D there is essentially only one functionally linearly dependent superintegrable system, namely $\mathcal{H} = p_z p_{\bar{z}} + V(z)$, where $V(z)$ is an arbitrary function of z alone. This system separates in only one set of coordinates $z, \bar{z}$. For functionally linearly independent 2D systems the theory is much more interesting.

Theorem 1 *Let $\mathcal{H}$ be the Hamiltonian of a 2D superintegrable (functionally linearly independent) system with nondegenerate potential.*

- *The space of second order constants of the motion is 3-dimensional.*
- *The space of third order constants of the motion is 1-dimensional.*

- *The space of fourth order constants of the motion is 6-dimensional.*

- *The space of sixth order constants is 10-dimensional.*

Theorem 2 *Let $\mathcal{K}$ be a third order constant of the motion for a superintegrable system with nondegenerate potential V:*

$$\mathcal{K} = \sum_{k,j,i=1}^{2} a^{kji}(x,y)p_k p_j p_i + \sum_{\ell=1}^{2} b^{\ell}(x,y)p_\ell.$$

Then $b^\ell(x,y) = \sum_{j=1}^{2} f^{\ell,j}(x,y)\frac{\partial V}{\partial x_j}(x,y)$ with $f^{\ell,j} + f^{j,\ell} = 0, \quad 1 \le \ell, j \le 2$. The a^{ijk}, b^ℓ are uniquely determined by the number $f^{1,2}(x_0,y_0)$ at some regular point (x_0,y_0) of V.

This result enables us to choose standard bases for second and higher order symmetries. Indeed, given any 2×2 symmetric matrix $\mathcal{A}_0$, and any regular point (x_0,y_0) there exists one and only one second order symmetry (or constant of the motion) $\mathcal{S}$ such that $\{a^{kj}_{(i)}(x_0,y_0)\} = \mathcal{A}_0$ and $W(x_0,y_0) = 0$. Further, if

$$\mathcal{S}_1 = \sum a^{kj}_{(1)} p_k p_j + W_{(1)}, \quad \mathcal{S}_2 = \sum a^{kj}_{(2)} p_k p_j + W_{(2)}$$

are second order constants of the the motion and $\mathcal{A}_{(i)}(x,y) = \{a^{kj}_{(i)}(x,y)\}$, $i = 1,2$ are 2×2 symmetric matrix functions, then the Poisson bracket of these symmetries is given by

$$\{\mathcal{S}_1, \mathcal{S}_2\} = \sum_{k,j,i=1}^{2} a^{kji}(x,y)p_k p_j p_i + b^\ell(x,y)p_\ell$$

where

$$f^{k,\ell} = 2\lambda \sum_j (a^{kj}_{(2)} a^{j\ell}_{(1)} - a^{kj}_{(1)} a^{j\ell}_{(2)}).$$

Thus $\{\mathcal{S}_1, \mathcal{S}_2\}$ is uniquely determined by the skew-symmetric matrix

$$[\mathcal{A}_{(2)}, \mathcal{A}_{(1)}] \equiv \mathcal{A}_{(2)}\mathcal{A}_{(1)} - \mathcal{A}_{(1)}\mathcal{A}_{(2)},$$

hence by the constant matrix $[\mathcal{A}_{(2)}(x_0,y_0), \mathcal{A}_{(1)}(x_0,y_0)]$ evaluated at a regular point. There is a standard structure that allows the identification of the space of second order constants of the motion with the space of 2×2 symmetric matrices and identification of the space of third order constants of the motion with the space of 2×2 skew-symmetric matrices.

Let $\mathcal{E}^{ij}$ be the 2×2 matrix with a 1 in row i, column j and 0 for every other matrix element. Then the symmetric matrices

$$\mathcal{A}^{(ij)} = \frac{1}{2}(\mathcal{E}^{ij}+\mathcal{E}^{ji}) = \mathcal{A}^{(ji)}, \quad i,j=1,2$$

form a basis for the 3-dimensional space of symmetric matrices. Moreover,

$$[\mathcal{A}^{(ij)},\mathcal{A}^{(k\ell)}] = \frac{1}{2}\left(\delta_{jk}\mathcal{B}^{(i\ell)}+\delta_{j\ell}\mathcal{B}^{(ik)}+\delta_{ik}\mathcal{B}^{(j\ell)}+\delta_{i\ell}\mathcal{B}^{(jk)}\right)$$

where

$$\mathcal{B}^{(ij)} = \frac{1}{2}(\mathcal{E}^{ij}-\mathcal{E}^{ji}) = -\mathcal{B}^{(ji)}, \quad i,j=1,2.$$

Here $\mathcal{B}^{(ii)}=0$ and $\mathcal{B}^{(12)}$ forms a basis for the space of skew-symmetric matrices. This gives the commutation relations for the second order symmetries.

We define a standard set of basis symmetries $\mathcal{S}_{(k\ell)} = \sum a^{ij}(\mathbf{x})p_ip_j + W_{(k\ell)}(\mathbf{x})$ corresponding to a regular point $\mathbf{x}_0$ by

$$\left(\begin{array}{cc} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{array}\right)_{\mathbf{x}_0} = \lambda(\mathbf{x}_0)\left(\begin{array}{cc} a^{11} & a^{12} \\ a^{21} & a^{22} \end{array}\right)_{\mathbf{x}_0} = \lambda(\mathbf{x}_0)\mathcal{A}^{(k\ell)}, \quad W_{(k\ell)}(\mathbf{x}_0)=0.$$

These structure results and standard separation of variables theory [15] give us the tools to prove multiseparabilty of 2D systems.

Corollary 1 *Let V be a superintegrable nondegenerate potential and L be a second order constant of the motion with matrix function $\mathcal{A}(\mathbf{x})$. If at some regular point $\mathbf{x}_0$ the matrix $\mathcal{A}(\mathbf{x}_0)$ has 2 distinct eigenvalues, then H, L characterize an orthogonal separable coordinate system.*

Since a generic 2×2 symmetric matrix has distinct roots, it follows that any superintegrable nondegenerate potential is multiseparable.

We prove the existence of a quadratic algebra structure by demonstrating that polynomials in the basis symmetries span the space of fourth and sixth order symmetries:

Theorem 3 *The 6 distinct monomials*

$$(\mathcal{S}_{(11)})^2,\ (\mathcal{S}_{(22)})^2,\ (\mathcal{S}_{(12)})^2,\ \mathcal{S}_{(11)}\mathcal{S}_{(22)},\ \mathcal{S}_{(11)}\mathcal{S}_{(12)},\ \mathcal{S}_{(12)}\mathcal{S}_{(22)},$$

form a basis for the space of fourth order symmetries.

Theorem 4 *The 10 distinct monomials*

$$(\mathcal{S}_{(ii)})^3,\ (\mathcal{S}_{(ij)})^3,\ (\mathcal{S}_{(ii)})^2\mathcal{S}_{(jj)},\ (\mathcal{S}_{(ii)})^2\mathcal{S}_{(ij)},\ (\mathcal{S}_{(ij)})^2\mathcal{S}_{(ii)},$$

$\mathcal{S}_{(11)}\mathcal{S}_{(12)}\mathcal{S}_{(22)}$, *for $i,j=1,2,\ i\neq j$ form a basis for the space of sixth order symmetries.*

The analogous results for 5th order symmetries follow directly from the Jacobi equality.

All 2D nondegenerate superintegrable systems in Euclidean space and on the 2-sphere have been classified [38, 39, 33]. There are 12 families of nondegenerate potentials in flat space (4 in real Euclidean space) and 6 families on the complex 2-sphere (2 on the real sphere). The principal tool in the classification of such systems on general Riemannian manifolds is the Stäckel transform. Suppose we have a (classical or quantum) superintegrable system

$$\mathcal{H} = \frac{1}{\lambda(x,y)}(p_1^2 + p_2^2) + V(x,y), \quad H = \frac{1}{\lambda(x,y)}(\partial_{11} + \partial_{22}) + V(x,y)$$

in local orthogonal coordinates, with nondegenerate potential $V(x,y)$ and suppose $U(x,y)$ is a particular case of the 3-parameter potential V, nonzero in an open set. Then the transformed systems

$$\tilde{\mathcal{H}} = \frac{1}{\tilde{\lambda}(x,y)}(p_1^2 + p_2^2) + \tilde{V}(x,y), \quad \tilde{H} = \frac{1}{\tilde{\lambda}(x,y)}(\partial_{11} + \partial_{22}) + \tilde{V}(x,y)$$

are also superintegrable, where $\tilde{\lambda} = \lambda U, \quad \tilde{V} = V/U$.

Theorem 5

$$\{\tilde{\mathcal{H}}, \tilde{\mathcal{S}}\} = 0 \iff \{\mathcal{H}, \mathcal{S}\} = 0.$$

$$\tilde{\mathcal{S}} = \sum_{ij} \frac{1}{\lambda U} p_i \left((a^{ij} + \delta^{ij} \frac{1 - W_U}{\lambda U}) \lambda U \right) p_j + \left(W - \frac{W_U V}{U} + \frac{V}{U} \right).$$

$$[\tilde{H}, \tilde{S}] = 0 \iff [H, S] = 0.$$

$$\tilde{S} = \sum_{ij} \frac{1}{\lambda U} \partial_i \left((a^{ij} + \delta^{ij} \frac{1 - W_U}{\lambda U}) \lambda U \right) \partial_j + \left(W - \frac{W_U V}{U} + \frac{V}{U} \right).$$

Corollary 2 *If $\mathcal{S}^{(1)}, \mathcal{S}^{(2)}$ are second order constants of the motion for $\mathcal{H}$, then*

$$\{\tilde{\mathcal{S}}^{(1)}, \tilde{\mathcal{S}}^{(2)}\} = 0 \iff \{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}\} = 0.$$

If $S^{(1)}, S^{(2)}$ are second order symmetry operators for H, then

$$[\tilde{S}^{(1)}, \tilde{S}^{(2)}] = 0 \iff [S^{(1)}, S^{(2)}] = 0.$$

This transform of one (classical or quantum) superintegrable system into another on a different manifold is called the **Stäckel transform**. Two such systems related by a Stäckel transform are called **Stäckel equivalent**.

Theorem 6 *Every nondegenerate second-order classical or quantum superintegrable system in two variables is Stäckel equivalent to a superintegrable system on a constant curvature space, i.e., flat space or the sphere.*

There is a fundamental duality between Killing tensors and the B-D equation (only in the 2D case) that enables us to get a complete classification of the possible manifolds.

Theorem 7 *If $ds^2 = \lambda(dx^2 + dy^2)$ is the metric of a nondegenerate superintegrable system (expressed in coordinates x, y such that $\lambda_{12} = 0$) then $\lambda = \mu$ is a solution of the system*

$$\mu_{12} = 0, \qquad \mu_{22} - \mu_{11} = 3\mu_1(\ln a^{12})_1 - 3\mu_2(\ln a^{12})_2 + (\frac{a^{12}_{11} - a^{12}_{22}}{a^{12}})\mu,$$

where either

$$I) \quad a^{12} = X(x)Y(y), \quad X'' = \alpha^2 X, \quad Y'' = -\alpha^2 Y,$$

or

$$II) \quad a^{12} = \frac{2X'(x)Y'(y)}{C(X(x) + Y(y))^2},$$

$$(X')^2 = F(X), \quad X'' = \frac{1}{2}F'(X), \quad (Y')^2 = G(Y), \quad Y'' = \frac{1}{2}G'(Y)$$

where

$$F(X) = \frac{\alpha}{24}X^4 + \frac{\gamma_1}{6}X^3 + \frac{\gamma_2}{2}X^2 + \gamma_3 X + \gamma_4,$$
$$G(Y) = -\frac{\alpha}{24}Y^4 + \frac{\gamma_1}{6}Y^3 - \frac{\gamma_2}{2}Y^2 + \gamma_3 Y - \gamma_4.$$

Conversely, every solution λ of one of these systems of equations defines a nondegenerate superintegrable system. If λ is a solution then the remaining solutions μ are exactly the nondegenerate superintegrable systems that are Stäckel equivalent to λ.

In a tour de force, Koenigs [40] has classified all 2D manifolds (i.e., zero potential) that admit exactly 3 second-order Killing tensors and listed them in two tables: Tableau VI and Tableau VII. Our methods show that these are exactly the spaces that admit superintegrable systems with nondegenerate potentials. (An alternate derivation can be found in [41].)

TABLEAU VI

$$
\begin{aligned}
[1]\ ds^2 &= \left[\frac{c_1\cos x + c_2}{\sin^2 x} + \frac{c_3\cos y + c_4}{\sin^2 y}\right](dx^2 - dy^2)\\
[2]\ ds^2 &= \left[\frac{c_1\cosh x + c_2}{\sinh^2 x} + \frac{c_3 e^y + c_4}{e^{2y}}\right](dx^2 - dy^2)\\
[3]\ ds^2 &= \left[\frac{c_1 e^x + c_2}{e^{2x}} + \frac{c_3 e^y + c_4}{e^{2y}}\right](dx^2 - dy^2)\\
[4]\ ds^2 &= \left[c_1(x^2-y^2) + \frac{c_2}{x^2} + \frac{c_3}{y^2} + c_4\right](dx^2 - dy^2)\\
[5]\ ds^2 &= \left[c_1(x^2-y^2) + \frac{c_2}{x^2} + c_3 y + c_4\right](dx^2 - dy^2)\\
[6]\ ds^2 &= \left[c_1(x^2-y^2) + c_2 x + c_3 y + c_4\right](dx^2 - dy^2)
\end{aligned}
$$

TABLEAU VII

$$
\begin{aligned}
[1]\ ds^2 &= \left[c_1(\frac{1}{\mathrm{sn}^2(x,k)} - \frac{1}{\mathrm{sn}^2(y,k)} + c_2(\frac{1}{\mathrm{cn}^2(x,k)} - \frac{1}{\mathrm{cn}^2(y,k)})\right.\\
&+ \ c_3(\frac{1}{\mathrm{dn}^2(x,k)} - \frac{1}{\mathrm{dn}^2(y,k)}) + c_4(\mathrm{sn}^2(x,k) - \mathrm{sn}^2(y,k)\ ]\\
&\times \ (dx^2 - dy^2)\\
[2]\ ds^2 &= \left[c_1(\frac{1}{\sin^2 x} - \frac{1}{\sin^2 y}) + c_2(\frac{1}{\cos^2 x} - \frac{1}{\cos^2 y})\right.\\
&+ \ c_3(\cos 2x - \cos 2y)\ c_4(\cos 4x - \cos 4y)]\,(dx^2 - dy^2)\\
[3]\ ds^2 &= [c_1(\sin 4x - \sin 4y) + c_2(\cos 4x - \cos 4y)\\
&+ \ c_3(\sin 2x - \sin 2y)\ c_4(\cos 2x - \cos 2y)]\,(dx^2 - dy^2)\\
[4]\ ds^2 &= \left[c_1(\frac{1}{x^2} - \frac{1}{y^2}) + c_2(x^2 - y^2) + c_3(x^4 - y^4)\right.\\
&+ \ \left. c_4(x^6 - y^6)\right](dx^2 - dy^2)\\
[5]\ ds^2 &= \left[c_1(x-y) + c_2(x^2-y^2) + c_3(x^3-y^3) + c_4(x^4-y^4)\right]\\
&\times \ (dx^2 - dy^2)
\end{aligned}
$$

The quantization of our 2D results is relatively straightforward. For a manifold with metric $ds^2 = \lambda(x,y)(dx^2 + dy^2)$ the Hamiltonian system

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{\lambda(x,y)} + V(x,y)$$

is replaced by the Hamiltonian (Schrödinger) operator with potential

$$H = \frac{1}{\lambda(x,y)}(\partial_{11} + \partial_{22}) + V(x,y).$$

A second-order symmetry of the Hamiltonian system

$$\mathcal{S} = \sum_{k,j=1}^{2} a^{kj}(x,y)p_k p_j + W(x,y),$$

with $a^{kj} = a^{jk}$, corresponds to the formally self-adjoint operator

$$S = \frac{1}{\lambda(x,y)} \sum_{k,j=1}^{2} \partial_k(a^{kj}(x,y)\lambda(x,y)\partial_j) + W(x,y), \quad a^{kj} = a^{jk}.$$

Lemma 1

$$\{\mathcal{H}, \mathcal{S}\} = 0 \iff [H, S] = 0.$$

This result is not generally true for higher dimensional manifolds.

Through use of the self-adjointness of even order operator symmetries and the skew adjointness of odd order symmetries, it follows that the main classical results for symmetries corresponding to a nondegenerate potential can be taken over with little change [22]. In particular there is closure of the quadratic algebra for 2D quantum superintegrable potentials: All fourth order and sixth order symmetry operators can be expressed as symmetric polynomials in the second order symmetry operators.

3 Conformally flat spaces in three dimensions

The 3D results are considerably more complicated, but essential features are preserved. We limit ourselves to conformally flat spaces. For each such space there always exists a local coordinate system x, y, z and a nonzero function $\lambda(x,y,z) = \exp G(x,y,z)$ such that the Hamiltonian is

$$\mathcal{H} = (p_1^2 + p_2^2 + p_3^2)/\lambda + V(x,y,z).$$

A quadratic constant of the motion (or generalized symmetry)

$$\mathcal{S} = \sum_{k,j=1}^{3} a^{kj}(x,y,z)p_k p_j + W(x,y,z) \equiv \mathcal{L} + W, \quad a^{jk} = a^{kj}$$

must satisfy $\{\mathcal{H},\mathcal{S}\}=0$. i.e.,

$$\begin{array}{lcl} a_i^{ii} & = & -G_1a^{1i}-G_2a^{2i}-G_3a^{3i} \\ 2a_i^{ij}+a_j^{ii} & = & -G_1a^{1j}-G_2a^{2j}-G_3a^{3j}, \quad i\neq j \\ a_k^{ij}+a_j^{ki}+a_i^{jk} & = & 0, \quad i,j,k \text{ distinct} \end{array} \tag{4}$$

and

$$W_k=\lambda\sum_{s=1}^{3}a^{sk}V_s, \quad k=1,2,3. \tag{5}$$

(Here a subscript j denotes differentiation with respect to x_j.) The requirement that $\partial_{x_\ell}W_j=\partial_{x_j}W_\ell,\ \ell\neq j$ leads from (5) to the second order B-D partial differential equations for the potential.

$$\sum_{s=1}^{3}\Big[V_{sj}\lambda a^{s\ell}-V_{s\ell}\lambda a^{sj}+V_s\Big((\lambda a^{s\ell})_j-(\lambda a^{sj})_\ell\Big)\Big]=0. \tag{6}$$

For second order superintegrabilty in 3D there must be five functionally independent constants of the motion (including the Hamiltonian itself). Thus the Hamilton-Jacobi equation admits four additional constants of the motion:

$$\mathcal{S}_h=\sum_{j,k=1}^{3}a_{(h)}^{jk}p_kp_j+W_{(h)}=\mathcal{L}_h+W_{(h)}, \qquad h=1,\cdots,4.$$

We assume that the four functions $\mathcal{S}_h$ together with $\mathcal{H}$ are functionally linearly independent in the six-dimensional phase space. In [20] it is shown that the matrix of the 15 B-D equations for the potential has rank at least 5, hence we can solve for the second derivatives of the potential in the form

$$\begin{array}{lclcl} V_{22} & = & V_{11} & + & A^{22}V_1+B^{22}V_2+C^{22}V_3, \\ V_{33} & = & V_{11} & + & A^{33}V_1+B^{33}V_2+C^{33}V_3, \\ V_{ij} & = & & & A^{ij}V_1+B^{ij}V_2+C^{ij}V_3, \end{array} \tag{7}$$

where $1\leq i<j\leq 3$. If the matrix has rank >5 then there will be additional conditions on the potential and it will depend on fewer parameters. $D^1_{(s)}V_1+D^2_{(s)}V_2+D^3_{(s)}V_3=0$. Here the $A^{ij},B^{ij},C^{ij},D^i_{(s)}$ are functions of x, symmetric in the superscripts, that can be calculated explicitly. Suppose now that the superintegrable system is such that the rank is exactly 5 so that the relations are only (7). Further, suppose the integrability conditions for system (7) are satisfied identically. In this case we say that the potential is *nondegenerate*. Otherwise the potential is *degenerate*. If V is nondegenerate then at any point $\mathbf{x}_0$, where the A^{ij},B^{ij},C^{ij} are defined and analytic, there is a unique solution $V(\mathbf{x})$ with arbitrarily prescribed values of $V_1(\mathbf{x}_0),V_2(\mathbf{x}_0),V_3(\mathbf{x}_0),V_{11}(\mathbf{x}_0)$ (as well as the value of $V(\mathbf{x}_0)$ itself.) The points $\mathbf{x}_0$ are called *regular*. The points of singularity for the A^{ij}, B^{ij}, C^{ij} form a manifold of dimension < 3. Degenerate potentials depend on fewer parameters. For example, it may be that the rank of the B-D equations is exactly 5 but the integrability conditions are not satisfied identically. This occurs for the generalized Kepler-Coulomb potential.

Assuming that V is nondegenerate, we substitute the requirement for a nondegenerate potential (7) into the B-D equations (6) and obtain three equations for the derivatives a_i^{jk}, the first of which is

$$
\begin{aligned}
(a_3^{11} - a_1^{31})V_1 + (a_3^{12} - a_1^{32})V_2 + (a_3^{13} - a_1^{33})V_3 \qquad (8)\\
+a^{12}(A^{23}V_1 + B^{23}V_2 + C^{23}V_3) - (a^{33} - a^{11})(A^{13}V_1 + B^{13}V_2 + C^{13}V_3)\\
-a^{23}(A^{12}V_1 + B^{12}V_2 + C^{12}V_3) + a^{13}(A^{33}V_1 + B^{33}V_2 + C^{33}V_3)\\
= (-G_3a^{11} + G_1a^{13})V_1 + (-G_3a^{12} + G_1a^{23})V_2 + (-G_3a^{13} + G_1a^{33})V_3,
\end{aligned}
$$

and the other two are obtained in a similar fashion.

Since V is a nondegenerate potential we can equate coefficients of V_1, V_2, V_3, V_{11} on each side of the conditions $\partial_1 V_{23} = \partial_2 V_{13} = \partial_3 V_{12}$, $\partial_3 V_{23} = \partial_2 V_{33}$, etc., to obtain integrability conditions, the simplest of which are

$$
A^{23} = B^{13} = C^{12},\ B^{12} - A^{22} = C^{13} - A^{33},\ B^{23} = A^{31} + C^{22},\ C^{23} = A^{12} + B^{33}. \qquad (9)
$$

In general, the integrability conditions satisfied by the potential equations can be expressed as follows. We introduce the vector $\mathbf{w} = (V_1, V_2, V_3, V_{11})^{\mathrm{T}}$, and the matrices $\mathbf{A}^{(j)}$, $j = 1, 2, 3$, such that

$$
\partial_{x_j}\mathbf{w} = \mathbf{A}^{(j)}\mathbf{w} \qquad j = 1, 2, 3. \qquad (10)
$$

The integrability conditions for this system are

$$
A_i^{(j)} - A_j^{(i)} = A^{(i)}A^{(j)} - A^{(j)}A^{(i)} \equiv [A^{(i)}, A^{(j)}]. \qquad (11)
$$

The integrability conditions (9) and (11) are analytic expressions in x_1, x_2, x_3 and must hold identically. Then the system has a solution V depending on 4 parameters (plus an arbitrary additive parameter).

Using the nondegenerate potential condition and the B-D equations we can solve for all of the first partial derivatives a_i^{jk} of a quadratic symmetry to obtain

$$
\begin{aligned}
a_1^{11} &= -G_1a^{11} - G_2a^{12} - G_3a^{13} \qquad (12)\\
a_2^{22} &= -G_1a^{12} - G_2a^{22} - G_3a^{23},
\end{aligned}
$$

$$
\begin{aligned}
a_3^{33} &= -G_1a^{13} - G_2a^{23} - G_3a^{33},\\
3a_1^{12} &= a^{12}A^{22} - (a^{22} - a^{11})A^{12} - a^{23}A^{13} + a^{13}A^{23}\\
&+ G_2a^{11} - 2G_1a^{12} - G_2a^{22} - G_3a^{23},\\
3a_2^{11} &= -2a^{12}A^{22} + 2(a^{22} - a^{11})A^{12} + 2a^{23}A^{13} - 2a^{13}A^{23}\\
&- 2G_2a^{11} + G_1a^{12} - G_2a^{22} - G_3a^{23},\\
3a_3^{13} &= -a^{12}C^{23} + (a^{33} - a^{11})C^{13} + a^{23}C^{12} - a^{13}C^{33}\\
&- G_1a^{11} - G_2a^{12} - 2G_3a^{13} + G_1a^{33},\\
3a_1^{33} &= 2a^{12}C^{23} - 2(a^{33} - a^{11})C^{13} - 2a^{23}C^{12} + 2a^{13}C^{33}\\
&- G_1a^{11} - G_2a^{12} + G_3a^{13} - 2G_1a^{33},\\
3a_2^{23} &= a^{23}(B^{33} - B^{22}) - (a^{33} - a^{22})B^{23} - a^{13}B^{12} + a^{12}B^{13}\\
&- G_1a^{13} - 2G_2a^{23} - G_3a^{33} + G_3a^{22},\\
3a_3^{22} &= -2a^{23}(B^{33} - B^{22}) + 2(a^{33} - a^{22})B^{23} + 2a^{13}B^{12} - 2a^{12}B^{13}\\
&- G_1a^{13} + G_2a^{23} - G_3a^{33} - 2G_3a^{22},\\
3a_1^{13} &= -a^{23}A^{12} + (a^{11} - a^{33})A^{13} + a^{13}A^{33} + a^{12}A^{23}\\
&- 2G_1a^{13} - G_2a^{23} - G_3a^{33} + G_3a^{11},\\
3a_3^{11} &= 2a^{23}A^{12} + 2(a^{33} - a^{11})A^{13} - 2a^{13}A^{33} - 2a^{12}A^{23}\\
&+ G_1a^{13} - G_2a^{23} - G_3a^{33} - 2G_3a^{11},\\
3a_2^{33} &= -2a^{13}C^{12} + 2(a^{22} - a^{33})C^{23} + 2a^{12}C^{13} - 2a^{23}(C^{22} - C^{33})\\
&- G_1a^{12} - G_2a^{22} + G_3a^{23} - 2G_2a^{33},\\
3a_3^{23} &= a^{13}C^{12} - (a^{22} - a^{33})C^{23} - a^{12}C^{13} - a^{23}(C^{33} - C^{22})\\
&- G_1a^{12} - G_2a^{22} - 2G_3a^{23} + G_2a^{33},\\
3a_2^{12} &= -a^{13}B^{23} + (a^{22} - a^{11})B^{12} - a^{12}B^{22} + a^{23}B^{13}\\
&- G_1a^{11} - 2G_2a^{12} - G_3a^{13} + G_1a^{22},\\
3a_1^{22} &= 2a^{13}B^{23} - 2(a^{22} - a^{11})B^{12} + 2a^{12}B^{22} - 2a^{23}B^{13}\\
&- G_1a^{11} + G_2a^{12} - G_3a^{13} - 2G_1a^{22},\\
3a_1^{23} &= a^{12}(B^{23} + C^{22}) + a^{11}(B^{13} + C^{12}) - a^{22}C^{12} - a^{33}B^{13}\\
&+ a^{13}(B^{33} + C^{23}) - a^{23}(C^{13} + B^{12})\\
&- 2G_1a^{23} + G_2a^{13} + G_3a^{12}.\\
3a_3^{12} &= a^{12}(-2B^{23} + C^{22}) + a^{11}(C^{12} - 2B^{13}) - a^{22}C^{12} + 2a^{33}B^{13}\\
&+ a^{13}(-2B^{33} + C^{23}) + a^{23}(-C^{13} + 2B^{12})\\
&- 2G_3a^{12} + G_2a^{13} + G_1a^{23}.\\
3a_2^{13} &= a^{12}(B^{23} - 2C^{22}) + a^{11}(B^{13} - 2C^{12}) + 2a^{22}C^{12} - a^{33}B^{13}\\
&+ a^{13}(B^{33} - 2C^{23}) + a^{23}(2C^{13} - B^{12})\\
&- 2G_2a^{13} + G_1a^{23} + G_3a^{12}.
\end{aligned}
$$

plus the linear relations (9). Using the linear relations we can express $C^{12}, C^{13}, C^{22}, C^{23}$ and B^{13} in terms of the remaining 10 functions.

Since the above system of first order partial differential equations is involutive the general solution for the 6 functions a^{jk} can depend on at most 6 parameters, the values $a^{jk}(\mathbf{x}_0)$ at a fixed regular point $\mathbf{x}_0$. For the integrability conditions we define the vector-valued function

$$\mathbf{h}(x,y,z) = \left(a^{11}, a^{12}, a^{13}, a^{22}, a^{23}, a^{33}\right)^{\mathrm{T}}$$

and directly compute the 6×6 matrix functions $\mathcal{A}^{(j)}$ to get the first-order system

$$\partial_{x_j}\mathbf{h} = \mathcal{A}^{(j)}\mathbf{h} \qquad j = 1,2,3.$$

The integrability conditions for this system are are

$$\mathcal{A}^{(j)}_i\mathbf{h} - \mathcal{A}^{(i)}_j\mathbf{h} = \mathcal{A}^{(i)}\mathcal{A}^{(j)}\mathbf{h} - \mathcal{A}^{(j)}\mathcal{A}^{(i)}\mathbf{h} \equiv [\mathcal{A}^{(i)}, \mathcal{A}^{(j)}]\mathbf{h}. \tag{13}$$

By assumption we have 5 functionally linearly independent symmetries, so at each regular point the solutions sweep out a 5 dimensional subspace of the 6 dimensional space of symmetric matrices. However, from the conditions derived above there seems to be no obstruction to construction of a 6 dimensional space of solutions. Indeed in [20] we show that this construction can always be done.

Theorem 8 ($5 \Longrightarrow 6$) *Let V be a nondegenerate potential corresponding to a conformally flat space in 3 dimensions that is superintegrable, i.e., suppose V satisfies the equations (7) whose integrability conditions hold identically, and there are 5 functionally independent constants of the motion. Then the space of second order symmetries for the Hamiltonian $\mathcal{H} = (p_x^2+p_y^2+p_z^2)/\lambda(x,y,z)+V(x,y,z)$ (excluding multiplication by a constant) is of dimension $D = 6$.*

Corollary 3 *If $\mathcal{H}+V$ is a superintegrable conformally flat system with nondegenerate potential, then the dimension of the space of 2nd order symmetries*

$$\mathcal{S} = \sum_{k,j=1}^{3} a^{kj}(x,y,z)p_k p_j + W(x,y,z)$$

is 6. At any regular point (x_0, y_0, z_0), and given constants $\alpha^{kj} = \alpha^{jk}$, there is exactly one symmetry $\mathcal{S}$ (up to an additive constant) such that $a^{kj}(x_0,y_0,z_0) = \alpha^{kj}$. Given a set of 5 functionally independent 2nd order symmetries $\mathcal{L} = \{\mathcal{S}_\ell : \ell = 1, \cdots 5\}$ associated with the potential, there is always a 6th second order symmetry $\mathcal{S}_6$ that is functionally dependent on $\mathcal{L}$, but linearly independent.

It appears that the functional relationship between these 6 symmetries is always expressible in terms of a polynomial of order 8 in the momenta, but we do not yet have a proof.

As in the 2D case, the key to understanding the structure of the space of constants of the motion for 3D superintegrable systems with nondegenerate potential is an investigation of third order constants of the motion. We have

$$\mathcal{K} = \sum_{k,j,i=1}^{3} a^{kji}(x,y,z)p_k p_j p_i + b^\ell(x,y,z)p_\ell, \tag{14}$$

which must satisfy $\{\mathcal{H},\mathcal{K}\} = 0$. Here the third order Killing tensor a^{kji} is symmetric in the indices k,j,i. We are interested in such third order symmetries that could possibly arise as commutators of second order symmetries. Thus we require that the Killing tensor terms be independent of the four independent parameters in V. However, the b^ℓ must depend on these parameters. We set

$$b^\ell(x,y,z) = \sum_{j=1}^{3} f^{\ell,j}(x,y,z)V_j(x,y,z). \tag{15}$$

In [20] the following result is obtained.

Theorem 9 *Let $\mathcal{K}$ be a third order constant of the motion for a conformally flat superintegrable system with nondegenerate potential V. Then $f^{\ell,j} + f^{j,\ell} = 0$, $\quad 1 \le \ell, j \le 3$. The a^{ijk}, b^ℓ are uniquely determined by the four numbers $f^{1,2}$, $f^{1,3}$, $f^{2,3}$, $f^{1,2}_3$ at any regular point (x_0,y_0,z_0) of V.*

Let

$$\mathcal{S}_1 = \sum a^{kj}_{(1)}p_k p_j + W_{(1)}, \quad \mathcal{S}_2 = \sum a^{kj}_{(2)}p_k p_j + W_{(2)}$$

be second order constants of the the motion for a superintegrable system with nondegenerate potential and let $\mathcal{A}_{(i)}(x,y,z) = \{a^{kj}_{(i)}(x,y,z)\}$, $i = 1,2$ be 3×3 matrix functions. Then the Poisson bracket of these symmetries is given by

$$\{\mathcal{S}_1,\mathcal{S}_2\} = \sum_{k,j,i=1}^{3} a^{kji}(x,y,z)p_k p_j p_i + b^\ell(x,y,z)p_\ell$$

where

$$f^{k,\ell} = 2\lambda \sum_j (a^{kj}_{(2)}a^{j\ell}_{(1)} - a^{kj}_{(1)}a^{j\ell}_{(2)}).$$

Differentiating, we find

$$f^{k,\ell}_i = 2\lambda \sum_j (\partial_i a^{kj}_{(2)}a^{j\ell}_{(1)} + a^{kj}_{(2)}\partial_i a^{j\ell}_{(1)} - \partial_i a^{kj}_{(1)}a^{j\ell}_{(2)} - a^{kj}_{(1)}\partial_i a^{j\ell}_{(2)}) + G_i f^{k,\ell}. \tag{16}$$

Thus, there is a standard structure allowing the identification of the space of second order constants of the motion with the space S_3 of 3×3 symmetric matrices, as well as identification of the space of third order constants of the motion with a subspace of the space $K_3\times F$ of 3×3 skew-symmetric matrices K_3. crossed with the line. $F=\{\mathcal{F}(\mathbf{x}_0)\}$. A consequence of these results is [20]

Corollary 4 *Let V be a superintegrable nondegenerate potential on a conformally flat space. Then the space of third order constants of the motion is 4-dimensional and is spanned by Poisson brackets of the second order constants of the motion.*

Let $\mathcal{E}^{ij}$ be the 3×3 matrix with a 1 in row i, column j and 0 for every other matrix element. Then the symmetric matrices

$$\mathcal{A}^{(ij)}=\frac{1}{2}(\mathcal{E}^{ij}+\mathcal{E}^{ji})=\mathcal{A}^{(ji)},\quad i,j=1,2,3 \tag{17}$$

form a basis for the 6-dimensional space of symmetric matrices.

Corollary 5 *We can define a standard set of 6 second order basis symmetries*

$$\mathcal{S}^{(jk)}=\sum a^{hs}_{(jk)}(\mathbf{x})p_hp_s+W^{(jk)}(\mathbf{x})$$

corresponding to a regular point $\mathbf{x}_0$ *by* $(a_{(jk)}))(\mathbf{x}_0)=\mathcal{A}^{(jk)},\quad W^{(jk)}(\mathbf{x}_0)=0$.

In [20] we proved the following.

Theorem 10 *The dimension of the space of fourth order symmetries for a nondegenerate 3D potential is* 21*. The dimension of the space of sixth order symmetries is* 56*.*

Theorem 11 *The 21 distinct standard monomials $\mathcal{S}^{(ij)}\mathcal{S}^{(jk)}$, defined with respect to a regular point $\mathbf{x}_0$, form a basis for the space of fourth order symmetries.*

Theorem 12 *The 56 distinct standard monomials $\mathcal{S}^{(hi)}\mathcal{S}^{(jk)}\mathcal{S}^{(\ell m)}$, defined with respect to a regular $\mathbf{x}_0$, form a basis for the space of sixth order symmetries.*

We conclude that the quadratic algebra closes.

From the general theory of variable separation for Hamilton-Jacobi equations [16, 17] we know that second order symmetries $\mathcal{S}_1,\mathcal{S}_2$ define a separable coordinate system for the equation

$$H=\frac{p_x^2+p_y^2+p_z^2}{\lambda(x,y,z)}+V(x,y,z)=E$$

if and only if

1. The symmetries $\mathcal{H}, \mathcal{S}_1, \mathcal{S}_2$ form a linearly independent set as quadratic forms.

2. $\{\mathcal{S}_1, \mathcal{S}_2\} = 0$.

3. The three quadratic forms have a common eigenbasis of differential forms.

This last requirement means that, expressed in coordinates x, y, z, at least one of the matrices $\mathcal{A}_{(j)}(\mathbf{x})$ can be diagonalized by conjugacy transforms in a neighborhood of a regular point and that $[\mathcal{A}_{(2)}(\mathbf{x}), \mathcal{A}_{(1)}(\mathbf{x})] = 0$. However, for nondegenerate superintegrable potentials in a conformally flat space we see that

$$\{\mathcal{S}_1, \mathcal{S}_2\} = 0 \Longleftrightarrow [\mathcal{A}_{(2)}(\mathbf{x}_0), \mathcal{A}_{(1)}(\mathbf{x}_0)] = 0, \quad \text{and} \quad \mathcal{F}(\mathbf{x}_0) = 0$$

so that the intrinsic conditions for the existence of a separable coordinate system are simplified.

Theorem 13 *Let V be a superintegrable nondegenerate potential in a 3D conformally flat space. Then V defines a multiseparable system.*

The Stäckel transform for 3D systems [42], or coupling constant metamorphosis [43], can be constructed in exact analogy with the 2D case.

Theorem 14 *Every superintegrable system with nondegenerate potential on a 3D conformally flat space is Stäckel equivalent to a superintegrable system on either 3D flat space or the 3-sphere.*

See [21] for the details of the proofs. In this same reference we exhibited 9 superintegrable systems in Euclidean space and another on the sphere that was not Stäckel equivalent to any of these. Eight of these systems are "generic" in the sense that they correspond to generic Jacobi separable coordinates and are uniquely determined by this correspondence.

The quantization of 3D results is carried out in [22]. For a manifold with metric $ds^2 = \lambda(x, y, z)(dx^2 + dy^2 + dz^2)$ the Hamiltonian system $\mathcal{H} = (p_1^2 + p_2^2 + p_3^2)/\lambda(x, y, z) + V(x, y, z)$ is replaced by the Hamiltonian (Schrödinger) operator with potential

$$H = \sum_{i=1}^{3} \frac{1}{\lambda^{\frac{3}{2}}} \partial_i (\lambda^{\frac{1}{2}} \partial_i) - \frac{1}{8} R + V,$$

where R is the scalar curvature. Similarly a second order constant of the motion is replaced by the formally self-adjoint symmetry operator

$$S = \sum_{i,j=1}^{3} \left(\frac{1}{\lambda^{\frac{3}{2}}} \partial_i (a^{ij} \lambda^{\frac{3}{2}} \partial_j) + a^{ij} (\mathcal{R}_{ij} + 5\mathcal{R}_i \mathcal{R}_j) + a_i^{ij} \mathcal{R}_j \right) + W,$$

where $\mathcal{R} = \frac{1}{4} \ln \lambda$.Both of these expressions can be written in covariant form.

Through use of the self-adjointness of even order operator symmetries and the skew adjointness of odd order symmetries, it follows again that the main classical results for symmetries corresponding to a nondegenerate potential can be taken over with relatively little change, except that the potential is modified by bits of the metric. In particular there is closure of the quadratic algebra for 3D quantum superintegrable potentials: All fourth order and sixth order symmetry operators can be expressed as symmetric polynomials in the second order symmetry operators.

4 Fine structure of superintegrable systems

For fine structure of superintegrable systems we drop the requirement of nondegeneracy and study the various possibilities for systems with potentials depending on fewer parameters. For 2D systems the structure is very simple.

Theorem 15 *Every 2D system with a two-parameter potential and 3 functionally linearly independent second-order symmetries is the restriction of some nondegenerate (three-parameter) potential. Every 2D system with a one-parameter potential and 3 functionally linearly independent second-order symmetries is the restriction of some nondegenerate to a single parameter, such that the restricted potential is annihilated by some Killing vector of the underlying space.*

For 3D systems the results are much more complicated and have not yet been fully determined. We announce some new results here whose detailed proofs will appear subsequently. We first consider those systems that just fail to be nondegenerate in the sense that the four functions $\mathcal{S}_h$ together with $\mathcal{H}$ are functionally linearly independent in the six-dimensional phase space but that the associated potential functions V span only a 3 dimensional subspace of the 4 dimensional space of solutions of equations (7), ignoring the trivial added constant. In particular, we stipulate that we can arbitrarily prescribe V_1, V_2, V_3 at a regular point, but not V_{11} independently of these. This circumstance can occur in only two ways: either the potential is a 3-parameter restriction of a nondegenerate potential, or the integrability conditions for the system (7) are not satisfied identically and an additional condition is imposed. In either case equations (7) are replaced by the 6 equations

$$V_{ij} = \tilde{A}^{ij} V_1 + \tilde{B}^{ij} V_2 + \tilde{C}^{ij} V_3, \qquad i \leq j, \tag{18}$$

whose integrability conditions are satisfied identically. Equations (7) still hold, but with the identifications

$$D^{ij} = \tilde{D}^{ij}, \ 1 \leq i < j \leq 3, \qquad D^{kk} = \tilde{D}^{kk} - \tilde{D}^{11}, \ k = 2, 3,$$

where $D = A, B, C$. For short, we will call the solutions of (18) **3-parameter potentials**. In analogy to the nondegenerate potential case we can compute the full set of integrability conditions satisfied by the potential, and we can use the 10 second order Killing tensor equations and the $3 \times 3 = 9$ conditions for the derivatives $a_h^{\ell m}$ that result from substituting relations (18) into the 3 B-D equations and equating coefficients of V_1, V_2, V_3, respectively. Thus there are 19 conditions for the 18 derivatives $a_h^{\ell m}$. We get exactly equations (12) and the remaining condition

$$\begin{gathered} a^{11}(\tilde{C}^{12} - \tilde{B}^{13}) + a^{22}(\tilde{A}^{23} - \tilde{C}^{12}) + a^{33}(\tilde{B}^{13} - \tilde{A}^{23}) + a^{12}(\tilde{A}^{13} + \tilde{C}^{22} - \tilde{C}^{11} - \tilde{B}^{23}) \\ + a^{13}(\tilde{C}^{23} + \tilde{B}^{11} - \tilde{B}^{33} - \tilde{A}^{12}) + a^{23}(\tilde{B}^{12} + \tilde{A}^{33} - \tilde{A}^{22} - \tilde{C}^{13}) = 0, \end{gathered} \tag{19}$$

which we can regard as an obstruction to extending the assumed 5 dimensional space of second order symmetries to the full 6 dimensional space. Note that the analogous obstruction equation appears for the nondegenerate potential case, but there the linear integrability conditions (9) for the nondegenerate potential cause the obstruction to vanish identically. By exploitation of the integrability conditions for the potential and for equations (12) we have obtained the following results:

Theorem 16 *A 3D 3-parameter potential is a restriction of a nondegenerate potential if and only if the obstruction (19) vanishes identically. If the obstruction doesn't vanish then the space of second order symmetries is 5 dimensional and the system is uniquely determined by the values of $\tilde{D}^{ij}, i \leq j$, $D = A, B, C$ at a single regular point.*

The extended Kepler-Coulomb system (1) is an example of a 3-parameter potential with obstruction, as are two other real Euclidean space potentials in Evans' list [2]. Another example is defined by the potential

$$V = \frac{\alpha}{\sqrt{x^2 + y^2 + z^2}} + \frac{\beta}{(x + iy)^2} + \frac{\gamma(x - iy)}{(x + iy)^3}.$$

These are true 3-parameter potentials in the sense that they cannot be extended to nondegenerate potentials.

Third order constants of the motion for a true 3-parameter potential superintegrable system again take the form (14), (15) where $f^{\ell,j} + f^{j,\ell} = 0, \quad 1 \leq \ell, j \leq 3$.

Theorem 17 *For a true 3-parameter system the a^{ijk}, b^ℓ are uniquely determined by the three numbers $f^{1,2}$, $f^{1,3}$, $f^{2,3}$, at any regular point (x_0, y_0, z_0) of V.*

Corollary 6 *Let V be a superintegrable true 3-parameter potential on a conformally flat space. Then the space of third order constants of the motion is 3-dimensional and is spanned by Poisson brackets of the second order constants of the motion. The Poisson bracket of two second order constants of the motion is uniquely determined by the matrix commutator of the second order constants at a regular point.*

Theorem 18 *Let V be a superintegrable true 3-parameter potential in a 3D conformally flat space. Then V defines a multiseparable system.*

Theorem 19 *Every superintegrable system with true 3-parameter potential on a 3D conformally flat space is Stäckel equivalent to a superintegrable system on either 3D flat space or the 3-sphere.*

Although the spaces of higher order symmetries for true 3-parameter systems have an interesting structure, the quadratic algebra doesn't close.

Theorem 20 *For a superintegrable system with true 3-parameter potential on a 3D conformally flat space there exist two second order constants of the motion $\mathcal{S}_1, \mathcal{S}_2$ such that $\{\mathcal{S}_1, \mathcal{S}_2\}^2$ is not expressible as a cubic polynomial in the second order constants of the motion.*

5 Polynomial ideals

In this section we introduce a very different way of studying and classifying superintegrable systems, through polynomial ideals. Here we confine our analysis to 3D Euclidean superintegrable systems with nondegenerate potentials, though the approach is also effective in 2D and for spheres. The equations for the second order symmetries in this case are just (12) with $G \equiv 0$. Due to the linear conditions (9) all of the functions A^{ij}, B^{ij}, C^{ij} can be expressed in terms of the 10 basic terms

$$(A^{12}, A^{13}, A^{22}, A^{23}, A^{33}, B^{12}, B^{22}, B^{23}, B^{33}, C^{33}). \tag{20}$$

Since the equations (12) admit 6 linearly independent solutions a^{hk} the integrability conditions $\partial_i a^{hk}_\ell = \partial_\ell a^{hk}_i$ for these equations must be satisfied identically. As follows from [20], these conditions plus the integrability conditions (11) for the potential allow us to compute the 30 derivatives $\partial_\ell D^{ij}$ of the 10 basic terms. Each is a quadratic polynomial in the 10 terms. In addition there are 5 quadratic conditions remaining [20]:

$$\begin{array}{lrl}
a) & -A^{23}B^{33} - A^{12}A^{23} + A^{13}B^{12} + B^{22}A^{23} + B^{23}A^{33} & \\
 & -A^{22}B^{23} & = 0, \\
b) & (A^{33})^2 + B^{12}A^{33} - A^{33}A^{22} - B^{33}A^{12} - C^{33}A^{13} + B^{22}A^{12} & \\
 & -B^{12}A^{22} + A^{13}B^{23} - (A^{12})^2 + & \\
c) & -(B^{33})^2 - B^{33}A^{12} + B^{33}B^{22} + B^{12}A^{33} + B^{23}C^{33} - (B^{23})^2 & \\
 & +(B^{12})^2 & = 0, \\
d) & -B^{12}A^{23} - A^{33}A^{23} + A^{13}B^{33} + A^{12}B^{23} & = 0, \\
e) & A^{12}B^{12} + C^{33}A^{23} - A^{23}B^{23} + B^{33}A^{22} - B^{33}A^{33} & = 0.
\end{array} \tag{21}$$

These 5 polynomials determine an ideal Σ'. Already we see that the values of the 10 terms at a fixed regular point must uniquely determine a superintegrable system. However, choosing those values such that the 5 conditions (21) are satisfied will not guarantee the existence of a solution, because the conditions may be violated for values of (x, y, z) away from the chosen regular point. To test this we compute the derivatives $\partial_i\Sigma'$ and obtain a single new condition, the square of the quadratic expression

$$\begin{array}{ll}
f) & A^{13}C^{33} + 2A^{13}B^{23} + B^{22}B^{33} - (B^{33})^2 + A^{33}A^{22} - (A^{33})^2 + 2A^{12}B^{22} \\
 & +(A^{12})^2 - 2B^{12}A^{22} + (B^{12})^2 + B^{23}C^{33} - (B^{23})^2 - 3(A^{23})^2 = 0.
\end{array} \tag{22}$$

The polynomial (22) extends the ideal. Let Σ be the ideal generated by the 6 quadratic polynomials. It can be verified that $\partial_i\Sigma \subseteq \Sigma$, so that the system is closed under differentiation! This leads us to a fundamental result.

Theorem 21 *Choose the 10-tuple (20) at a regular point, such that the 6 polynomial identities (21), (22) are satisfied. Then there exists one and only one Euclidean superintegrable system with nondegenerate potential that takes on these values at a point.*

We see that all possible nondegenerate 3D Euclidean superintegrable systems are encoded into the 6 quadratic polynomial identities. These identities define an algebraic variety that generically has dimension 6, though there are singular points, such as the origin $(0, \cdots, 0)$, where the dimension of the tangent space is greater. This result gives us the means to classify all superintegrable systems.

An issue is that many different 10-tuples correspond to the same superintegrable system. How do we sort this out? The key is that the Euclidean group E(3,C) acts as a transformation group on the variety and gives rise to a foliation. The action of the translation subgroup is determined by the derivatives $\partial_k D^{ij}$ that we have already determined. The action of the rotation subgroup on the D^{ij} can be determined from the behavior of the canonical equations (7) under rotations. The local action on a 10-tuple is then given by 6 Lie derivatives that are a basis for the Euclidean Lie algebra $e(3,C)$. For "most" 10-tuples $\mathbf{D}_0$ on the 6 dimensional variety the action of the Euclidean group is locally transitive with isotropy subgroup only the identity element. Thus the group action on such points sweeps out a solution surface homeomorphic to the 6 parameter $E(3,C)$ itself. This occurs for the generic Jacobi elliptic system with potential

$$V = \alpha(x^2+y^2+z^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2}.$$

At the other extreme the isotropy subgroup of the origin $(0,\cdots,0)$ is $E(3,C)$ itself, i.e., the point is fixed under the group action. This corresponds to the isotropic oscillator with potential

$$V = \alpha(x^2+y^2+z^2) + \beta x + \gamma y + \delta z.$$

More generally, the isotropy subgroup at $\mathbf{D}_0$ will be H and the Euclidean group action will sweep out a solution surface homeomorphic to the homogeneous space $E(3,C)/H$ and define a unique superintegrable system. For example, the isotropy subalgebra formed by the translation and rotation generators $\{P_1, P_2, P_3, J_1 + iJ_2\}$ determines a system with potential

$$\alpha\left((x-iy)^3 + 6(x^2+y^2+z^2)\right) + \beta\left((x-iy)^2 + 2(x+iy)\right) + \gamma(x-iy) + \delta z.$$

Indeed, each class of Stäckel equivalent Euclidean superintegrable systems is associated with a unique isotropy subalgebra of $e(3,C)$, although not all subalgebras occur. (Indeed, there is no isotropy subalgebra conjugate to $\{P_1, P_2, P_3\}$.) Thus to find all superintegrable systems we need to determine a list of all subalgebras of $e(3,C)$, defined up to conjugacy, and then for each subalgebra to determine if it occurs as an isotropy subalgebra. Further we must resolve the degeneracy problem in which more than one superintegrable system may correspond to a single isotropy subalgebra.

6 Outlook

We have given an overview of some of the tools used and results obtained in the study of second order superintegrable systems. The basic problems for

2D systems have been solved, and the extension of these methods to complete the fine structure analysis for 3D systems appears relatively straightforward. The 3D fine structure analysis can be extended to analyze 2 parameter and 1 parameter potentials with 5 functionally linearly independent second order symmetries. Here first order PDEs for the potential appear as well as second order, and Killing vectors may occur. The other class of 3D superintegrable systems is that for which the 5 functionally independent symmetries are functionally linearly dependent. This class contains the Calogero potential [44, 45, 46] and necessarily leads to first order PDEs for the potential, as well as second order [22]. However, the integrability methods discussed here should be able to handle this class with no special difficulties. On a deeper level, we think that the algebraic geometry methods of the last section can be extended to classify the possible superintegrable systems in all these cases.

Whereas 2D superintegrable systems are very special, the 3D systems seem to be good guides to the structure of general nD systems, and we intend to proceed with this analysis. The ultimate aim is to understand the structure of superintegrable systems in general. We have started with second order systems because of their historical connection to the Kepler-Coulomb problem and to separation of variables. However, most of the methods that we have developed make use of integrability conditions alone, not separation of variables (a purely second order phenomenon) and show promise of being extendable to higher order superintegrable systems.

Finally, the algebraic geometry related results that we have sketched in the last section suggest strongly that there is an underlying geometric structure to superintegrable systems that is not apparent from the usual presentations of these systems.

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