Nondegenerate superintegrable systems in n dimensional spaces of constant curvature

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Abstract

We analyze the concept of a nondegenerate superintegrable system in n dimensional Euclidean space. Attached to this idea is the notion that every such system affords a separation of variables in one of the various types of generic elliptical coordinates that are possible in complex Euclidean space. An analysis of how these coordinates are arrived at in terms of their expression in terms of Cartesian coordinates is presented in detail. The use of well defined limiting processes illustrates just how all these systems can be obtained from the most general nondegenerate superintegrable system in n dimensional Euclidean space. Two examples help with the understanding of how the general results are obtained.

1 Nondegenerate superintegrable systems in n dimensional complex Euclidean space

The subject of our discussion is a type of second order superintegrable systems in classical mechanics which we call nondegenerate. By this we mean a Hamiltonian system of the form $H = g^{ij}p_ip_j + V(x_1, \cdots, x_n)$ where we restrict ourselves to metrics g^{ij} corresponding to complex Euclidean spaces in n dimensions. Nondegenerate second order superintegrability means that the Hamiltonian H admit 2n-1 functionally independent constants of the motion of the form $L_i = a^{ij}p_ip_j + \ell_i$, $i = 1, \cdots, 2n-1$ (where we take $L_1 = H$) such that the Potential $V(x_1, .., x_n)$ depends on $n+2$ arbitrary constants one of which can be an additive constant. (A precise definition is given in [1]. Essentially this means that at any point where the potential is defined there is a unique choice of parameters such that the potential, its n first order derivatives and one second order derivative V_{11} take arbitrarily prescribed values.) Thus $\{L_1, L_j\} = 0$, $j = 2, \cdots, 2n-1$ where $\{\ ,\ \}$ denotes the Poisson bracket. Here we give an idea of the variety of these systems in n dimensions and indicate how they are related via well defined limiting processes. In so doing we lay the ground work for a more precise analysis of this subject. The prototype superintegrable system which is nondegenerate in n dimensional flat space has the Hamiltonian

$$H = \sum_{i=1}^{n}(p_i^2 + \alpha x_i^2 + \frac{\beta_i}{x_i^2}) + \delta, \quad p_i = p_{x_i} \tag{1}$$

This Hamiltonian is nondegenerate superintegrable with a basis of $n(n+1)/2$ constants of the motion given by

$$P_i = p_i^2 + \alpha x_i^2 + \frac{\beta_i}{x_i^2}, \quad J_{ij} = (x_ip_j - x_jp_i)^2 + \beta_i\frac{x_j^2}{x_i^2} + \beta_j\frac{x_i^2}{x_j^2}, \quad i \neq j.$$

We see immediately that there appear to be "too many" constants of the motion. This is resolved by imposing sufficiently many functional relations. A crucial observation is that the corresponding Hamilton-Jacobi equation admits additive separation in n dimensional elliptical coordinates. $x_i^2 = c^2\Pi_{j=1}^n(u_j - e_i)/\Pi_{k\neq i}(e_k - e_i)$ where $e_i \neq e_j$ if $i \neq j$ and $i, j = 1, \cdots, n$. If we were dealing with real Euclidean space then we could assume that the inequalities $e_1 < e_2 < \cdots < e_n$ and $e_1 < u_1 < e_2 < \cdots < e_n < u_n$ hold. Additive separation of variables of the Hamilton-Jacobi equation is then possible in the usual way. We discuss this in Appendix A. Here, however, We allow all variables to be complex. The infinitesimal distance in elliptical coordinates has the form

$$ds^2 = -\frac{c^2}{4}\sum_{i=1}^{n}\frac{\Pi_{j\neq i}(u_i-u_j)}{\Pi_{k=1}^{n}(u_i-e_k)}du_i^2 = -\frac{c^2}{4}\sum_{i=1}^{n}\frac{\Pi_{j\neq i}(u_i-u_j)}{P(u_i)}du_i^2,$$

where $P(\lambda) = \Pi_{k=1}^{n}(\lambda - e_k)$. From this form of the infinitesimal distance we might ask the question "Is there a nondegenerate superintegrable system that affords a separation of variables in elliptic type coordinates and for which the corresponding polynomial $P(\lambda)$ has degree $\leq n$ and may admit multiple roots?" This question can be asked because the restriction that the infinitesimal distance ds^2 be flat is only that the degree of the polynomial $P(\lambda)$ be $\leq n$. The answer to our question is "yes."

In fact there are definite methods for calculating these systems for the various types of polynomial $P_n(\lambda)$. We will illustrate these methods with examples and then move on to discuss the general class of nondegenerate superintegrable systems. A comprehensive treatment of these matters will be given elsewhere. For purposes of illustration we restrict ourselves to dimension $n = 3$ where $x_1 = x$, $y_1 = y$ and $z_1 = z$. The corresponding Hamiltonian is

$$H = p_x^2 + p_y^2 + p_z^2 + \alpha(x^2+y^2+z^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2} + \mu \qquad (2)$$

The six functionally related constants, quadratic in the momenta, can be taken as

$$P_1 = p_x^2 + \alpha x^2 + \frac{\beta}{x^2}, \quad P_2 = p_y^2 + \alpha y^2 + \frac{\gamma}{y^2}, \quad P_3 = p_z^2 + \alpha z^2 + \frac{\delta}{z^2},$$

$$J_1 = (yp_z - zp_y)^2 + \gamma\frac{z^2}{y^2} + \delta\frac{y^2}{z^2}, \quad J_2 = (zp_x - xp_z)^2 + \delta\frac{x^2}{z^2} + \beta\frac{z^2}{x^2},$$

$$J_3 = (yp_x - xp_y)^2 + \gamma\frac{x^2}{y^2} + \beta\frac{y^2}{x^2}.$$

The relation which demonstrates this functional dependence is

$$P_1^2J_1^2+P_2^2J_2^2+P_3^2J_3^2-2P_1P_2J_1J_2-2P_1P_3J_1J_3-2P_2P_3J_2J_3+4\alpha J_1J_2J_3+4\beta P_2P_3J_1$$

$$+4\gamma P_1P_3J_2+4\delta P_1P_2J_3-4\alpha\beta J_1^2-4\alpha\gamma J_2^2-4\alpha\delta J_3^2-4\gamma\delta P_1^2-4\beta\delta P_2^2-4\beta\gamma P_3^2+16\alpha\beta\gamma\delta = 0.$$

The standard additive separation ansatz for Cartesian coordinates for the Hamilton-Jacobi equation $H = E$ is to write $p_x = \frac{\partial S}{\partial x}$, $p_y = \frac{\partial S}{\partial y}$ and $p_z = \frac{\partial S}{\partial z}$. We then look for solutions which give complete integrals of the form $S = S_1(x) + S_2(y) + S_3(z)$. The separation equations are

$$(\frac{\partial S}{\partial x})^2+\alpha x^2+\frac{\beta}{x^2}+\nu_1 = 0, \quad (\frac{\partial S}{\partial y})^2+\alpha y^2+\frac{\gamma}{y^2}+\nu_2 = 0, \quad (\frac{\partial S}{\partial z})^2+\alpha z^2+\frac{\delta}{z^2}+\nu_3 = 0,$$

where $\nu_1+\nu_2+\nu_3=E$. If we do the same in elliptical coordinates then the Hamilton-Jacobi equation assumes the form

$$[\frac{1}{(u_1-u_2)(u_1-u_3)}[-\frac{4}{c^2}(u_1-e_1)(u_1-e_2)(u_1-e_1)p_{u_1}^2+\alpha c^2\Pi_{i=1}^3(u_1-e_i)$$
$$+\frac{\beta}{c^2}\frac{(u_1-e_2)(u_1-e_3)}{(u_1-e_1)}+\frac{\gamma}{c^2}\frac{(u_1-e_1)(u_1-e_3)}{(u_1-e_2)}+\frac{\delta}{c^2}\frac{(u_1-e_2)(u_1-e_1)}{(u_1-e_3)}]$$
$$+\frac{1}{(u_2-u_1)(u_2-u_3)}[-\frac{4}{c^2}(u_2-e_1)(u_2-e_2)(u_2-e_1)p_{u_2}^2+\alpha c^2\Pi_{i=1}^3(u_2-e_i)$$
$$+\frac{\beta}{c^2}\frac{(u_2-e_2)(u_2-e_3)}{(u_2-e_1)}+\frac{\gamma}{c^2}\frac{(u_2-e_1)(u_2-e_3)}{(u_2-e_2)}+\frac{\delta}{c^2}\frac{(u_2-e_2)(u_2-e_1)}{(u_2-e_3)}]$$
$$+\frac{1}{(u_3-u_2)(u_3-u_1)}[-\frac{4}{c^2}(u_3-e_1)(u_3-e_2)(u_3-e_1)p_{u_3}^2+\alpha c^2\Pi_{i=1}^3(u_3-e_i)$$
$$+\frac{\beta}{c^2}\frac{(u_3-e_2)(u_3-e_3)}{(u_3-e_1)}+\frac{\gamma}{c^2}\frac{(u_3-e_1)(u_3-e_3)}{(u_3-e_2)}+\frac{\delta}{c^2}\frac{(u_3-e_2)(u_3-e_1)}{(u_3-e_3)}]]=E$$

where the elliptic coordinates have been taken to be

$$x^2=c^2\frac{(u_1-e_1)(u_2-e_1)(u_3-e_1)}{(e_1-e_2)(e_1-e_3)},\quad y^2=c^2\frac{(u_1-e_2)(u_2-e_2)(u_3-e_2)}{(e_2-e_1)(e_2-e_3)},$$

$$z^2=c^2\frac{(u_1-e_3)(u_2-e_3)(u_3-e_3)}{(e_3-e_2)(e_3-e_1)},$$

and we have made use of the identities

$$x^2+y^2+z^2=c^2[\frac{\Pi_{i=1}^3(u_1-e_i)}{(u_1-u_2)(u_1-u_3)}+\frac{\Pi_{i=1}^3(u_2-e_i)}{(u_2-u_1)(u_2-u_3)}+\frac{\Pi_{i=1}^3(u_3-e_i)}{(u_3-u_2)(u_3-u_1)}],$$

$$\frac{1}{x^2}=\frac{1}{c^2}\left[\frac{(u_1-e_2)(u_1-e_3)}{(u_1-u_2)(u_1-u_3)(u_1-e_1)}+\frac{(u_2-e_2)(u_2-e_3)}{(u_2-u_1)(u_2-u_3)(u_2-e_1)}+\right.$$
$$\left.\frac{(u_3-e_2)(u_3-e_3)}{(u_3-u_2)(u_3-u_1)(u_3-e_1)}\right],$$

and similar related identities for $1/y^2$, $1/z^2$. This is a convenient juncture to note that the solution of the functional equation

$$\frac{f(u_1)}{(u_1-u_2)(u_1-u_3)}+\frac{f(u_2)}{(u_2-u_1)(u_2-u_3)}+\frac{f(u_3)}{(u_3-u_2)(u_3-u_1)}=E$$

is $f(u_i) = Eu_i^2 + \lambda_1 u_i + \lambda_2$. This follows from the general discussion of appendix A. The infinitesimal distance in elliptical coordinates is

$$ds^2 = -\frac{c^2}{4}\left[\frac{(u_1-u_2)(u_1-u_3)}{(u_1-e_1)(u_1-e_2)(u_1-e_3)}du_1^2 + \frac{(u_2-u_1)(u_2-u_3)}{(u_2-e_1)(u_2-e_2)(u_2-e_3)}du_2^2 \right.$$
$$\left. + \frac{(u_3-u_1)(u_3-u_2)}{(u_3-e_1)(u_3-e_2)(u_3-e_3)}du_3^2\right].$$

If we make the usual substitution $p_{u_i} = \frac{\partial S}{\partial u_i}$ together with the ansatz $S = \hat{S}_1(u_1) + \hat{S}_2(u_2) + \hat{S}_3(u_3)$, separation is achieved via the equations

$$-\frac{4}{c^2}(\lambda-e_1)(\lambda-e_2)(\lambda-e_1)(\frac{\partial S}{\partial \lambda})^2 + \alpha c^2 \Pi_{i=1}^3(\lambda-e_i) + \frac{\beta}{c^2}\frac{(\lambda-e_2)(\lambda-e_3)}{(\lambda-e_1)}$$
$$+\frac{\gamma}{c^2}\frac{(\lambda-e_1)(\lambda-e_3)}{(\lambda-e_2)} + \frac{\delta}{c^2}\frac{(\lambda-e_2)(\lambda-e_1)}{(\lambda-e_3)} + (\mu-E)\lambda^2 + \nu_2\lambda + \nu_3 = 0$$

for $\lambda = u_1, u_2, u_3$. It is clear that H affords a separation of variables in elliptic coordinates and therefore that H separates in more than one coordinate system, as it already separates in Cartesian coordinates, i.e., it is multiseparable.

What if the polynomial $P(\lambda)$ has multiple zeros? If we make the substitutions $e_1 = E_1 + \epsilon_1$, $e_2 = E_1 + \epsilon_2$, $e_3 = E_2$ in the formulas for elliptic coordinates, the expressions for the Cartesian coordinates are

$$x = \frac{c}{\sqrt{\epsilon_1-\epsilon_2}}[y_1 + \epsilon_1 y_2 + O(\epsilon_1^2)], \quad y = \frac{c}{\sqrt{\epsilon_2-\epsilon_1}}[y_1 + \epsilon_2 y_2 + O(\epsilon_2^2)],$$

$$z = y_3 + O(\epsilon_1,\epsilon_2) = \frac{c}{(E_1-E_2)}\sqrt{(u_1-E_2)(u_2-E_2)(u_3-E_2)} + O(\epsilon_1,\epsilon_2).$$

If we regard ϵ_1 and ϵ_2 as small quantities it is readily verified that

$$ds^2 = dx^2 + dy^2 + dz^2 = 2dy_1dy_2 + dy_3^2 + O(\epsilon_1,\epsilon_2)$$
$$= -\frac{c^2}{4}[\frac{(u_1-u_2)(u_1-u_3)}{(u_1-E_1)^2(u_1-E_2)}du_1^2 + \frac{(u_2-u_3)(u_2-u_1)}{(u_2-E_1)^2(u_2-E_2)}du_2^2$$
$$+\frac{(u_3-u_2)(u_3-u_1)}{(u_1-E_3)^2(u_3-E_2)}du_3^2] + O(\epsilon_1,\epsilon_2).$$

If we choose, for example, $y_1 = \frac{1}{\sqrt{2}}(\hat{x}+i\hat{y})$, $y_2 = \frac{1}{\sqrt{2}}(\hat{x}-i\hat{y})$, $y_3 = \hat{z}$, then $d\hat{s}^2 = d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2 =$

$$-\frac{c^2}{4}\left[\frac{(u_1-u_2)(u_1-u_3)}{(u_1-E_1)^2(u_1-E_2)}du_1^2 + \frac{(u_2-u_3)(u_2-u_1)}{(u_2-E_1)^2(u_2-E_2)}du_2^2 + \frac{(u_3-u_2)(u_3-u_1)}{(u_1-E_3)^2(u_3-E_2)}du_3^2\right].$$

Thus we have constructed a new set of Cartesian coordinates for which $P = (\lambda - E_1)^2(\lambda - E_2)$. Indeed if we use the coordinates y_i, $i = 1, 2, 3$ then we can readily deduce the relations

$$y_1 = c\sqrt{\frac{(u_1 - E_1)(u_2 - E_1)(u_3 - E_1)}{(E_1 - E_2)}},\ y_3 = \frac{c}{(E_1 - E_2)}\sqrt{(u_1 - E_2)(u_2 - E_2)(u_3 - E_2)}$$

where $y_2 = \frac{\partial}{\partial e_1} y_1$, or in alternate form

$$y_1^2 = c^2\frac{(u_1 - E_1)(u_2 - E_1)(u_3 - E_1)}{(E_1 - E_2)}, \quad 2y_1y_2 = \frac{\partial}{\partial e_1}[c^2\frac{(u_1 - E_1)(u_2 - E_1)(u_3 - E_1)}{(E_1 - E_2)}],$$

$$y_3^2 = c^2\frac{(u_1 - E_2)(u_2 - E_2)(u_3 - E_2)}{(E_1 - E_2)^2}.$$

There are generalizations of this procedure but for the moment we stay with our example. If we take the Hamiltonian H for $n = 3$, viz.

$$H = p_x^2 + p_y^2 + p_z^2 + \alpha(x^2 + y^2 + z^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2} + \mu \qquad (3)$$

and make the substitutions

$$\beta \longrightarrow -\frac{\beta\ \epsilon_2 + \frac{1}{2}\gamma}{(\epsilon_1 - \epsilon_2)^2}, \quad \alpha \longrightarrow \alpha, \quad \gamma \longrightarrow -\frac{\beta\ \epsilon_1 + \frac{1}{2}\gamma}{(\epsilon_1 - \epsilon_2)^2}, \quad \mu \longrightarrow \mu.$$

Then we obtain a nondegenerate superintegrable Hamiltonian

$$H \longrightarrow H_1 = 2p_{y_1}p_{y_2} + p_{y_3}^2 + \alpha(2y_1y_2 + y_3^2) + \frac{\beta}{y_1^2} + \gamma\frac{y_2}{y_1^3} + \frac{\delta}{y_3^2} + \mu,$$

with six constants of the motion given by

$$M = (y_1p_{y_3} - y_3p_{y_2})^2 + \delta\frac{y_1^2}{y_3^2} - \frac{1}{2}\gamma\frac{y_3^2}{y_1^2}, \quad N = (y_1p_{y_1} - y_2p_{y_2})^2 - 2\beta\frac{y_2}{y_1} - 2\gamma\frac{y_2^2}{y_1^2},$$

$$L = 2(y_1p_{y_3} - y_3p_{y_2})(y_2p_{y_3} - y_3p_{y_1}) + y_3^2(\frac{\beta}{y_1^2} + \gamma\frac{y_2}{y_1^3}) + 2\delta\frac{y_1y_2}{y_3^2},$$

$$P_1 = 2p_{y_1}p_{y_2} + 2\alpha y_1y_2 + \frac{\beta}{y_1^2} + \gamma\frac{y_2}{y_1^3}, \quad P_2 = p_{y_2}^2 + \alpha y_1^2 - \frac{1}{2}\frac{\gamma}{y^2}, \quad P_3 = p_{y_3}^2 + \alpha y_3^2 + \frac{\delta}{y_3^2}.$$

The functional relation between these six constants is

$$P_2^2L^2 - 2P_2P_1LN - 4P_2P_3MN + P_2^2N^2 + 4\alpha MN^2 - 2\beta P_2P_3L - 2\gamma P_1P_3L + 4\delta MN^2 - 2\gamma P_1P_3L$$

$$-2\beta P_1P_2N+\beta^2P_1^2+4\beta\delta P_2P_3+4\gamma\delta P_2^2+2\alpha\gamma L^2+4\alpha\beta LN+8\alpha\delta\gamma M-4\alpha\delta\beta^2=0.$$

Using the identity

$$\frac{1}{y_1^2}=G(E_1)=-\frac{1}{c^2}\Big[\frac{(u_1-E_2)}{(u_1-u_2)(u_1-u_3)(u_1-E_1)}+\frac{(u_2-E_2)}{(u_2-u_1)(u_2-u_3)(u_2-E_1)}$$
$$+\frac{(u_3-E_2)}{(u_3-u_2)(u_3-u_1)(u_3-E_1)}\Big]$$

we see that separation of variables can be achieved in the coordinates u_i, $i=1,2,3$ for the term $1/y_1^2$. In addition it can be seen that $\partial/\partial E_1(y_1^{-2})=-2y_1y_2y_1^{-4}$ also has this property. Indeed we observe that the function $G(E_1+\epsilon)$ is such that the first two coefficients of its Taylor expansion with respect to ϵ give the two independent terms of the nondegenerate superintegrable Hamiltonian H_1. The separation equations have the form

$$\Big[\frac{4}{c^2}(\lambda-E_1)^2(\lambda-E_2)S_\lambda^2+\frac{\beta}{c^2}\frac{(\lambda-E_2)}{(\lambda-E_1)}-\frac{\gamma}{2c^2}\frac{(\lambda-E_2)}{(\lambda-E_1)^2}+\frac{\delta}{c^2}\frac{(\lambda-E_1)^2}{(\lambda-E_2)^2}+$$
$$\alpha c^2(\lambda-E_1)^2(\lambda-E_2)+(\mu-E)\lambda^2+\nu_2\lambda+\nu_3\Big]=0$$

for $\lambda=u_1,u_2,u_3$.

What happens if $P=(\lambda-B_1)(\lambda-B_2)$? In order to see this we make the substitutions $e_1=1/\epsilon$, $e_2=B_1$, $e_3=B_2$ and $c\to c/\sqrt{\epsilon}$ in our expressions for elliptic coordinates. The infinitesimal distance $d\hat{s}^2$ has the form

$$-\frac{c^2}{4}\left[\frac{(u_1-u_2)(u_1-u_3)}{(u_1-B_1)(u_1-B_2)}du_1^2+\frac{(u_2-u_1)(u_2-u_3)}{(u_2-B_1)(u_2-B_2)}du_2^2+\frac{(u_3-u_2)(u_3-u_1)}{(u_3-B_1)(u_3-B_2)}du_3^2\right].$$

For the corresponding Cartesian coordinates we can write

$$x=\frac{1}{\epsilon}[y_0+\epsilon y_1+O(\epsilon^2)],\quad y=y_2+O(\epsilon),\quad z=y_3+O(\epsilon),$$

where $y_0=i/c$. From this we conclude that $d\hat{s}^2=dy_1^2+dy_2^2+dy_3^2$. The explicit form of the coordinates y_i is

$$y_1=-\frac{i}{2}c(u_1+u_2+u_3-B_1-B_2),\quad y_2^2=-\frac{c^2(u_1-B_1)(u_2-B_1)(u_3-B_1)}{(B_1-B_2)},$$
$$y_3^2=-\quad c^2(u_1-B_2)(u_2-B_2)(u_3-B_2)(B_2-B_1).$$

If we now make the transformations

$$\alpha\longrightarrow\alpha-\frac{i}{2}\frac{\beta}{c}\epsilon,\quad\gamma\longrightarrow\gamma,\quad\beta\longrightarrow\alpha\frac{c^4}{\epsilon^4},\quad\delta\longrightarrow\delta,\quad\mu\longrightarrow\mu+2\alpha\frac{c^2}{\epsilon^2}-\frac{i}{2}\frac{\beta}{\epsilon}\frac{c}{\epsilon},$$

then the Hamiltonian H undergoes the transformation

$$H \longrightarrow H_2 = p_{y_1}^2 + p_{y_2}^2 + p_{y_3}^2 + \alpha(4y_1^2 + y_2^2 + y_3^2) + \beta y_1 + \frac{\gamma}{y_2^2} + \frac{\delta}{y_3^2} + \mu.$$

This Hamiltonian is nondegenerate superintegrable with five functionally independent constants and six constants quadratic in the canonical momenta p_{u_i} given by

$$P_1 = p_{y_1}^2 + 4\alpha y_1^2 + \beta y_1, \quad P_2 = p_{y_2}^2 + \alpha y_2^2 + \frac{\gamma}{y_2^2}, \quad P_3 = p_{y_3}^2 + \alpha y_3^2 + \frac{\delta}{y_3^2},$$

$$J_1 = (y_2 p_{y_3} - y_3 p_{y_2})^2 + \gamma\frac{y_3^2}{y_2^2} + \delta\frac{y_2^2}{y_3^2}, \quad M_3 = (y_3 p_{y_1} - y_1 p_{y_3})p_{y_3} + \frac{1}{4}\beta y_3^2 + \alpha y_1 y_3^2 - \gamma\frac{y_1}{y_3^2},$$

$$M_2 = (y_2 p_{y_1} - y_1 p_{y_2})p_{y_2} + \frac{1}{4}\beta y_2^2 + \alpha y_1 y_2^2 - \gamma\frac{y_1}{y_2^2}.$$

The relation which demonstrates the functional dependence of these constants is

$$2P_1P_3M_3M_2 + P_1P_2P_3J_1 - P_2^2M_3^2 - P_3^2M_2^2 - \delta P_1P_2^2 - \gamma P_1P_z^2 - \alpha P_1J_1^2 + \frac{1}{2}\beta J_1(P_2M_3 + P_3M_2)$$

$$-4\alpha J_1M_2M_3 - \frac{1}{16}\beta^2P_1^2 - \gamma\beta P_3M_3 + 4\alpha\gamma M_3^2 - \beta\delta P_2M_2 + 4\alpha\delta M_2^2 + 4\alpha\gamma\delta P_1 + \frac{1}{4}\gamma\delta\beta^2 = 0.$$

If we consider the function

$$G(\epsilon) = \frac{1}{c^2}\frac{(\epsilon B_1 - 1)(\epsilon B_2 - 1)}{(\epsilon u_1 - 1)(\epsilon u_2 - 1)(\epsilon u_3 - 1)} = \frac{1}{c^2}\left[\frac{(u_1 - B_1)(u_1 - B_2)}{(u_1 - u_2)(u_1 - u_3)(\epsilon u_1 - 1)}\right.$$

$$\left. + \frac{(u_2 - B_1)(u_2 - B_2)}{(u_2 - u_1)(u_2 - u_3)(\epsilon u_2 - 1)} + \frac{(u_3 - B_1)(u_3 - B_2)}{(u_3 - u_2)(u_3 - u_1)(\epsilon u_3 - 1)}\right]$$

we see that all the coefficients of powers of ϵ are Stäckel multipliers and enable a separation in the coordinates u_i. In fact we can determine that

$$G(\epsilon)\Big|_{\epsilon=0} = y_0^2\ , \quad \frac{d}{d\epsilon}G(\epsilon)\Big|_{\epsilon=0} = -\frac{2}{c^2}y_0y_1\ , \quad \frac{d^2}{d\epsilon^2}G(\epsilon)\Big|_{\epsilon=0} = \frac{2}{c^4}(4y_1^2 + y_2^2 + y_3^2)\ ,$$

and these are essentially the two terms that occur in H_2. It is easy to compute these in the coordinate representation. The separation equations have the form

$$[-\frac{4}{c^2}(\lambda - B_1)(\lambda - B_2)S_\lambda^2 + (-\alpha c^2\lambda^2 - i\frac{\beta}{2}c\lambda)(\lambda - B_1)(\lambda - B_2) - \gamma(\frac{\lambda - B_2}{\lambda - B_1})$$

$$-\delta(\frac{\lambda - B_1}{\lambda - B_2}) + (\mu - E)\lambda^2 + \nu_1\lambda + \nu_2)] = 0$$

for $\lambda = u_1, u_2, u_3$.

Our examples in three dimensions illustrate what happens quite generally in n dimensional complex Euclidean space. These methods can also be applied to the n dimensional complex sphere. In general it is possible to associate a separable coordinate system of elliptic type corresponding to a general polynomial $P(\lambda)$ having arbitrary zero structure. Indeed if we are in n dimensional Euclidean space then we can divide our Cartesian coordinates according to x_i^I where $I = 0, \cdots, r$, $i = 1, \cdots, p_I$. The corresponding polynomial P has the form $P = \Pi_{I=1}^r(\lambda - e_I)^{p_I}$ with the constraint $\sum_{I=0}^r p_I = n$. We can obtain this coordinate system from generic elliptical coordinates by relabeling the e_i as e_i^I in accordance with our relabeling of the Cartesian coordinates. We then make the substitutions

$$e_i^0 \longrightarrow \frac{1}{\epsilon_i^0}, \quad e_j^J \longrightarrow E_J + \epsilon_j^J, \; J \neq 0, \quad c \longrightarrow \frac{c}{(\Pi_{i=1}^{p_0}\epsilon_i^0)^{\frac{1}{2}}}$$

and find that

$$(x_i^0)^2 = \frac{c^2\Pi_{k=1}^n(\epsilon_i^0 u_k - 1)}{(\epsilon_i^0)^2\Pi_{j\neq i}(\epsilon_j^0 - \epsilon_i^0)\Pi_{J=1}^r\Pi_{j=1}^{p_J}[1 - \epsilon_i^0(E_J + \epsilon_j^J)]}$$

for $i = 1, \cdots, p_0$, and

$$(x_j^J)^2 = \frac{c^2\Pi_{k=1}^n(u_k - E_J - \epsilon_j^J)}{\Pi_{k\neq j}(\epsilon_j^J - \epsilon_k^J)\Pi_{\ell=1}^{p_0}[\epsilon_\ell^0(E_J + \epsilon_j^J) - 1]\left[\Pi_{L\neq J, L\neq 0}\Pi_{\ell=1}^{P_L}[E_J - E_I + \epsilon_j^J - \epsilon_\ell^L]\right]}$$

for $j = 1, \cdots, p_J$, $J = 1, \cdots, r$. From each of these expressions we can deduce the coordinates that give rise to an infinitesimal distance with our choice of $P(\lambda)$. Indeed, if we write

$$dx_i^0 = \frac{1}{\sqrt{\Pi_{j\neq i}(\epsilon_j^0 - \epsilon_i^0)}}\left[d\hat{y}_1^0 + \cdots + (\epsilon_i^0)^{p_0 - 1}d\hat{y}_{p_0}^0 + O((\epsilon_i^0)^{p_0})\right]$$

it follows that

$$\sum_{i=1}^{p_0}(dx_i^0)^2 = \sum_{r+s=p_0+1} d\hat{y}_r^0 d\hat{y}_s^0 + O(\epsilon_i^0).$$

From this observation we determine that we can put $\epsilon_j^J = 0, J \neq 0$ in the expressions for y_i^0 that are of interest to us as coordinates. This follows because each $\hat{y}_i^0$ is a function of $E_J + \epsilon_j^J$ which is finite at $\epsilon_j^J = 0$, i.e., $\hat{y}_i^0 = \hat{y}_i^0(E_J + \epsilon_j^J)$. Consequently we can define real coordinates $y_i^0 = \hat{y}_i^0(E_J)$.

Thus the limits $\epsilon_j^J \to 0$ have no effect on the determination of the coordinates y_i^0 and we can, with justification, place $\epsilon_j^J = 0$ in the expressions for y_i^0 when determining them. Therefore we write $\sum_{i=1}^{p_0}(dx_i^0)^2 = \sum_{r+s=p_0+1} dy_r^0 dy_s^0$. We can proceed further to write

$$dx_i^I = \frac{1}{\sqrt{\Pi_{j\neq i}(\epsilon_j^I - \epsilon_i^I)}}\left[d\hat{y}_1^I + \cdots + (\epsilon_i^I)^{p_0-1} d\hat{y}_{p_0}^0 + O((\epsilon_i^I)^{p_I})\right], \quad I \neq 0,$$

and then determine that $\sum_{i=1}^{p_I}(dx_i^I)^2 = \sum_{r+s=p_I+1} d\hat{y}_r^I d\hat{y}_s^I + O(\epsilon_i^I)$. The expressions for the infinitesimal distances $\sum_{i=1}^{p_I}(dx_i^I)^2, I = 0, \cdots, r$ follow from the use of the identities given in Appendix A for $n = p_I$ and $u_i = \epsilon_i^I$. From this observation we can set $\epsilon_j^J = 0, J \neq I$ in the expressions for y_i^0 that are of interest, for exactly the same reasons that were outlined in the case $I = 0$. Thus, we can write the infinitesimal distance as

$$ds^2 = \sum_{J=0}^{r}\sum_{j=1}^{p_r}(dx_j^J)^2 = \sum_{J=0}^{r}\sum_{r+s=p_J+1} dy_r^J dy_s^J$$

where the coordinates y_i^I are given by

$$y_r^0 = \frac{c}{r!}\left(\frac{\partial}{\partial\epsilon}\right)^r\left[\frac{\Pi_{s=1}^n(\epsilon u_s - 1)}{\Pi_{J=1}^r(1-\epsilon E_J)^{p_J}}\right]^{\frac{1}{2}}|_{\epsilon=0}, \quad r = 1, \cdots, p_0\ ,$$

$$y_r^J = \frac{c}{r!}\left(\frac{\partial}{\partial E_J}\right)^r\left[\frac{\Pi_{s=1}^n(u_s - E_J)}{\Pi_{I\neq J}^r(E_J - E_I)^{p_I}}\right]^{\frac{1}{2}}, \quad r = 1, \cdots, p_J,\ J \neq 0,$$

or in quadratic rational form

$$\sum_{r+s=t} y_r^0 y_s^0 = \frac{1}{t!}\left(\frac{\partial}{\partial\epsilon}\right)^t\left[\frac{\Pi_{s=1}^n(\epsilon u_s - 1)}{\Pi_{J=1}^r(1-\epsilon E_J)^{p_J}}\right]|_{\epsilon=0}\ ,$$

$$\sum_{r+s=t} y_s^J y_r^J = \frac{c}{t!}\left(\frac{\partial}{\partial E_J}\right)^t\left[\frac{\Pi_{s=1}^n(u_s - E_J)}{\Pi_{I\neq J}^r(E_J - E_I)^{p_I}}\right].$$

Associated with each of these coordinate systems is a nondegenerate potential in n dimensions. The terms in the potential can be determined according to the formulas that follow. For $J \neq 0$ then we can immediately determine the partial fraction decomposition

$$\frac{1}{(y_1^J)^2} = \Pi_{K\neq J}(E_J - E_K)^{p_K}\left[\sum_{i=1}^{n}\frac{1}{\Pi_{j\neq i}(u_j - u_i)(u_i - E_J)}\right].$$

Using this expression we immediately see that in the coordinate system $u_i, i = 1, \cdots, n$ the addition of such a term as a potential will afford a separation of variables. The additional terms that arise in the nondegenerate potential for $J \neq 0$ have the form

$$Y_r^J = (\frac{\partial}{\partial E_J})^r \left[\frac{1}{(y_1^J)^2}\right],$$

for $r = 0, \cdots, p_J - 1$. It is then straightforward to show that these expressions can be determined via the recursion relation

$$\frac{\partial}{\partial E_J}(y_j^J) = (j+1) y_{j+1}^J.$$

The case $J = 0$ requires a little more care. If we define $(\tilde{y}_0^0)^2 = \Pi_{I=1}^r (1 - \epsilon E_I)^{p_I} / \Pi_{i=1}^n (\epsilon u_i - 1)$ then

$$\frac{1}{(\tilde{y}_0^0)^2} = \sum_{i=1}^n \frac{u_i^{p_0-1} \Pi_{J=1}^r (u_i - E_J)^{p_J}}{\Pi_{j\neq i}(u_j - u_i)(\epsilon u_i - 1)}$$

is clearly a Stäckel multiplier in the coordinates u_i. The remaining terms of contribution to the potential by coordinates y_i^0 are

$$Y_t^0 = (\frac{\partial}{\partial \epsilon})^t \left[\frac{1}{(\tilde{y}_0^0)^2}\right] |_{\epsilon=0}, \quad t = 0, \cdots, p_0 ,$$

where we can use the recurrence formula $\frac{\partial}{\partial \epsilon}(j+1)\tilde{y}_{j+1}^0$, and once all calculations have been done we put $\epsilon \to 0$, in which case $\tilde{y}_j^0 \to y_j^0$. We note that this prescription includes a constant term. The Hamiltonian has the form

$$H = \sum_{J=0}^r \sum_{r+s=p_J+1} p_{y_s} p_{y_r} + \sum_{J=1}^r \sum_{j=0}^{p_J-1} (\hat{\beta}_j^J Y_j^J + \alpha \sum_{r+s=p_J+1} y_s^J y_r^J) + \sum_{\ell=0}^{p_0} \hat{\beta}_\ell^0 Y_\ell^0$$

where $\hat{\beta}_\ell^0 = \alpha$. This Hamiltonian can be obtained via a limiting procedure from our original prototype in n dimensions as follows. If we relabel the β_i by β_i^I in obvious notation then the transformations of these constants is achieved by solving the equations

$$\sum_{i=1}^{p_I} \tilde{\beta}_i^I (\epsilon_i^I)^s = \hat{\beta}_s^I, \quad s = 0, \cdots, p_I - 1, \quad I \neq 0,$$

where $\beta_i^I = \tilde{\beta}_i^I / \Pi_{j\neq i}(\epsilon_i^I - \epsilon_j^I)$. For the case $I = 0$ we solve the equations

$$\sum_{i=0}^{p_0} \tilde{\beta}_i^0 (\epsilon_i^0)^s = \hat{\beta}_s^0, \ s = 2, \cdots, p_0 + 1, \quad \sum_{i=0}^{p_0} \tilde{\beta}_i^0 \epsilon_i^I + \frac{\alpha}{\Pi_{r=0}^{p_0} \epsilon_r^0} = \hat{\beta}_1^0$$

where $\beta_i^0 = \tilde{\beta}_i^0 / [(\epsilon_i^0)^2 \Pi_{j\neq i}(\epsilon_i^0 - \epsilon_j^0)]$, and we make the identification $\beta_0^0 = \alpha$.

2 Summary

In this article we have indicated the key ideas and formulas that enable us to obtain, via a definite limiting process, a very general class of nondegenerate superintegrable systems in n dimensional complex Euclidean space. In dimension 3 these methods have given an almost comprehensive list. The two exceptions are given in Appendix B. The complete analysis of these systems will be given elsewhere. Much the same analysis can be extended for the n dimensional sphere. This is part of an ongoing program which studies the nature of the notion of superintegrability. Superintegrability has its origins in the observation that classical systems like the Kepler problem (planetary motion) have an extra constants of the motion as exemplified by the Runge Lenz vector. This extra constant, additional to the usual components of angular momenta, enables the corresponding Hamilton-Jacobi equation to be solved in more then one coordinate system. This phenomenon has been revived in recent years by Winternitz [2, 3] and Evans [4]. Now a theory of such systems is evolving, [5, 6]. Indeed the study of such systems from the point of view of both classical and quantum mechanics has recently been completed in two complex dimensions, [7, 8]. These ideas have crucial relevance for the notion of quasiexact solvability and this will be discussed elsewhere, [9]. What remains as part of our ongoing program is to determine the extent to which the n dimensional nondegenerate systems found here are exhaustive and what are the exceptions. Integral to this study is the extension of these ideas to the n dimensional complex sphere.

3 Appendix A. Identities in n variables

The key identities that are made use of in generic separation in n dimensions are

$$\sum_{i=1}^{n} \frac{u_i^s}{\Pi_{j\neq i}(u_i-u_j)} = \delta_{sn}$$

where $s = 0, 1, \cdots, n-1$. If we have an equation

$$\sum_{i=1}^{n} \frac{f(u_i)}{\Pi_{j\neq i}(u_i-u_j)} = E$$

then a general solution can be found in the separable form $f(u_i) = Eu_i^{n-1} + \nu_{n-2}u_i^{n-2} + \cdots + \nu_0$ for arbitrary parameters λ_i, $i = 0, \cdots, n-2$. Writing the Hamilton-Jacobi equation in generic ellipsoidal coordinates, we obtain the equation

$$\sum_{i=1}^{n}\frac{1}{\Pi_{j\neq i}(u_i-u_j)}[-\frac{4}{c^2}\Pi_{j=1}^{n}(u_i-e_j)(\frac{\partial S}{\partial u_i})^2+\alpha c^2\Pi_{\ell=1}^{n}(u_i-e_\ell)+\frac{\beta_j}{c^2}\frac{\Pi_{k\neq j}(u_i-e_k)}{u_i-e_j}]=E$$

which naturally gives rise to the separation equations

$$-\frac{4}{c^2}\Pi_{j=1}^{n}(\lambda-e_j)(\frac{\partial S}{\partial \lambda})^2+\alpha c^2\Pi_{\ell=1}^{n}(\lambda-e_\ell)+\frac{\beta_j}{c^2}\frac{\Pi_{k\neq j}(\lambda-e_k)}{\lambda-e_j}$$

$$-E\lambda^{n-1}+\nu_{n-2}\lambda^{n-2}+\cdots+\nu_0,\quad \lambda=u_i,\ i=1,\cdots,n.$$

4 Appendix B. Unusual nondegenerate potentials in three dimensions

In addition to the nondegenerate families in n dimensions that we have outlined there are exceptions that have different properties, such as not being multiseparable in distinct coordinates. In three dimensions there are two such systems with corresponding Hamiltonians

1. $H=p_x^2+p_y^2+p_z^2+\delta(x^2+y^2+z^2)+\alpha x+\beta y+\gamma z$
2. $H=p_x^2+p_y^2+p_z^2+\delta(4x^2+4y^2+z^2)+\beta x+\gamma y+\frac{\alpha}{z^2}$

For the first Hamiltonian the space of second order constants of the motion has a basis

$$P_1=p_x^2+\delta x^2+\alpha x,\quad P_2=p_y^2+\delta y^2+\beta y,\quad P_3=p_z^2+\delta z^2+\gamma z,$$

$$L_1=p_xp_y+\delta xy+\frac{1}{2}\beta x+\frac{1}{2}\alpha y,\quad L_2=p_xp_z+\delta xz+\frac{1}{2}\gamma x+\frac{1}{2}\alpha z,$$

$$L_3=p_zp_y+\delta zy+\frac{1}{2}\beta z+\frac{1}{2}\gamma y.$$

In addition to these quadratic constants there are three linear (parameter dependent) constants: $M_1=\delta(xp_z-zp_x)-\frac{1}{2}\gamma p_x+\frac{1}{2}\alpha p_z$, $M_2=\delta(yp_x-xp_y)+\frac{1}{2}\beta p_x-\frac{1}{2}\alpha p_y$, $M_3=\delta(zp_y-yp_z)+\frac{1}{2}\gamma p_y-\frac{1}{2}\beta p_z$. Only five of these nine are functionally independent. Indeed we can demonstrate five identities which confirm this:

$$\delta(P_1P_2-S_1^2)-M_2^2+\frac{1}{4}\beta^2P_1+\frac{1}{4}\alpha^2P_2-\frac{1}{2}\alpha\beta S_1=0,$$

$$\delta(P_1P_3-S_2^2)-M_1^2+\frac{1}{4}\gamma^2P_1+\frac{1}{4}\alpha^2P_3-\frac{1}{2}\alpha\gamma S_2=0,$$

$$\delta(P_2P_3 - S_3^2) - M_3^2 + \frac{1}{4}\beta^2 P_3 + \frac{1}{4}\gamma^2 P_2 - \frac{1}{2}\gamma\beta S_3 = 0,$$

$$\delta(P_1M_3^2 + P_2M_1^2 + P_3M_2^2 + 2M_1M_3S_1 + 2M_2M_3S_2 + 2M_1M_2S_3) +$$

$$\frac{1}{4}(\beta^2M_1^2 + \gamma^2M_2^2 + \alpha^2M_3^2) + \frac{1}{2}(\gamma\beta M_1M_2 + \alpha\beta M_1M_3 + \alpha\gamma M_2M_3) = 0.$$

Technically speaking, these potentials are multiseparable, but in variants of Cartesian coordinates. What sets them apart is that the 5 constants of the motion are not functionally independent if the potential is switched off, [1].

5 Appendix C.

We give a complete list of potentials which give rise to nondegenerate (generic) superintegrable systems in three dimensional complex Euclidean space. Associated with each of these is indicated the algebraic structure of the polynomial of the associated generic elliptical coordinate system in which separation occurs.

[1]. $V = \alpha_1(x^2+y^2+z^2) + \frac{\alpha_2}{x^2} + \frac{\alpha_3}{y^2} + \frac{\alpha_4}{z^2}, \quad P(\lambda) = (\lambda - e_1)(\lambda - e_2)(\lambda - e_3).$

[2]. $V = \alpha_1(x^2+y^2+z^2) + \alpha_2\frac{x-iy}{(x+iy)^3} + \frac{\alpha_3}{(x+iy)^2} + \frac{\alpha_4}{z^2}, \ P(\lambda) = (\lambda - e_1)^2(\lambda - e_2).$

[3]. $V = \alpha_1(x^2+y^2+z^2) + \frac{\alpha_2}{(x+iy)} + \frac{\alpha_3 z}{(x+iy)^3} + \alpha_4\frac{x^2+y^2-3z^2}{(x+iy)^4}, \ P(\lambda) = (\lambda - e_1)^3.$

[4]. $V = \alpha_1(4x^2+y^2+z^2) + \alpha_2 x + \frac{\alpha_3}{y^2} + \frac{\alpha_4}{z^2}, \qquad P(\lambda) = (\lambda - e_1)(\lambda - e_2).$

[5]. $V = \alpha_1(4x^2+y^2+z^2) + \alpha_2 x + \frac{\alpha_3}{(y+iz)^2} + \frac{\alpha_4}{(y+iz)^3}, \quad P(\lambda) = (\lambda - e_1)^2.$

[6]. $V = \alpha_1(z^2 - 2(x-iy)^3 + 4(x^2+y^2)) + \alpha_2(2(x+iy) - 3(x-iy)^2) + \alpha_3(x+iy) + \frac{\alpha_4}{z^2},$

with $P(\lambda) = (\lambda - e_1)$.

[7]. $V = \alpha_1(x+iy) + \alpha_2(\frac{3}{4}(x+iy)^2 + \frac{1}{4}z) + \alpha_3((x+iy)^3 + \frac{1}{16}(x-iy) + \frac{3}{4}(x+iy)z)$

$$+\alpha_4(\frac{5}{16}(x+iy)^4 + \frac{1}{16}z^2 + \frac{1}{16}(x^2+y^2)^2 + \frac{3}{8}(x+iy)^2 z), \quad P(\lambda) = 1.$$

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