Homework Problem Set #3

Math 5467

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As we have seen, one problem in the employment of the Fourier transform, the Fourier series and the DFT in analysis of a signal $f(t)$ is the relative difficulty in determining the local behavior of the signal, i.e., the behavior near any $t = t_0$ from the Fourier coefficients. One tool for tackling this problem is the use of a "window" or probe. Typically this is a smooth function $w_{t_0}(t)$ that is equal to 1 in a small interval containing t_0 and quickly drops to 0 (or at least a very small value) outside of the interval. Then the Fourier transform of the function $g_{t_0}(t) = w_{t_0}(t)f(t)$ necessarily contains information about f that is restricted to a neighborhood of t_0. Furthermore $g_{t_0}(t) = f(t)$ for t close to t_0. By moving the window to other choices of t_0 we can study the function f near any point. Other types of window functions w, such as those localized in frequency space rather than signal space, are also useful. These ideas are part of the motivation for the windowed Fourier transform, and the discrete and continuous wavelet tranforms that we shall take up later. The first two problems will give you some understanding of the utility of windows.

Exercise 1 *(Adapted from Klammerer, page 228) Let N be even, let $-\infty < \alpha < \infty$, let f be the N-vector with elements*

$$f[n, \alpha] = e^{2\pi i \alpha n/N}, \quad n = 0, 1, \cdots, N-1,$$

and let $F[k, \alpha]$ be the corresponding DFT of f. Suppose we have a discrete signal that we know has the form $s[n] = \sum_j c_j f[n, \alpha_j]$ where the α_j are some set of real numbers, and there may be many α_j. The problem is to identify the frequencies α_j that occur in the signal from the data $s[n]$. (In astronomy, *orbits of planets and satellites can be represented in this form, the so-called epicycles.) If we take a DFT of the data, we will see peaks in $S[k]$ corresponding to $k = \alpha_j$ but if there are many α_j it may be difficult to resolve the distinct values. The last part of this problem shows how a particular window function (the Hanning window) can make it easier to locate these frequencies with accuracy*

a. *Explain why the samples $f[n, \alpha]$ always appear to come from sinusoids having a frequency parameter in the interval $-N/2 \leq \alpha < N/2$. In parts b.-d./ we will assume that this is the case. Hint: Consider $f[n, \alpha + mN]$, $m = \pm 1, \pm 2, \cdots$.*

b. *Show that*

$$F[k, \alpha] = N e^{-\pi i(N-1)(k-\alpha)/N} \frac{\mathrm{sinc}(k-\alpha)}{\mathrm{sinc}([k-\alpha]/N)}.$$

c. *Let*

$$g[n, \alpha] = w[n] f[n, \alpha],$$

so

$$NG[k, \alpha] = (W * F)[k, \alpha],$$

the convolution of the DFT W of w and the DFT F of f. Here the nonnegative function $w[n]$ is a window that is used to smoothly turn on and turn off the samples of the complex exponential f. What happens in the two extreme cases where $w[n] = 1$ for $n = 0, 1, \cdots, N-1$, and where $w[n] = \delta[n - (N/2)]$, $n = 01, \cdots, N-1$ with δ the Kronecker delta function?

d. *Let w be the Hanning window (or raised cosine window)*

$$w[n] = \frac{1}{2}\left(1 - \cos[\frac{\pi(2n+1)}{N}]\right).$$

Show that

$$\frac{1}{N} W[k] = -\frac{1}{4} e^{i\pi/N} \delta[k-1] + \frac{1}{2}\delta[k] - \frac{1}{4} e^{-i\pi/N} \delta[k+1],$$

$$G[k, \alpha] = -\frac{1}{4} e^{i\pi/N} F[k-1, \alpha] + \frac{1}{2} F[k, \alpha] - \frac{1}{4} e^{-i\pi/N} F[k+1, \alpha].$$

Note: A computer-generated plot of $|F[k,\alpha]|$ versus k will have a peak at $\lfloor\alpha\rfloor$, (the greatest integer $\le$ alpha) or at $\lfloor\alpha\rfloor+1$, but as k moves away from α the graph will decay like $|k-\alpha|^{-1}$ when $\alpha \neq 0, \pm1, \pm2, \cdots$. In contrast, the graph of $|G[k,\alpha]|$ decays like $|k-\alpha|^{-3}$ which is much sharper. Indeed, $\sin(\pi(k\pm1-\alpha)) = -\sin(\pi(k-\alpha))$ so for N very large with respect to $k-\alpha$, we have

$$|G[k,\alpha]| \approx \frac{1}{2}\left|\frac{1}{(k-\alpha)[(k-\alpha)^2-1]}\right| \approx |k-\alpha|^{-3}.$$

Thus the Hanning window helps us to resolve closely spaced peaks in the spectrum corresponding to a linear combination of complex exponentials.

Exercise 2 *(Lowering the Nyquist rate) In this example we see how a window function w with a sharp frequency can be used to lower the Nyquist rate at which a frequency-bounded signal $f(t)$ can be sampled . Here we are referring to the Shannon-Whittaker sampling theorem in the notes. Suppose f satisfies the conditions of the theorem, and that the support of $\hat{f}(\lambda)$ is contained in the symmetric interval $[-\Omega,\Omega]$. Thus we can reconstruct f exactly from the samples $f(j\pi/\Omega)$ as j runs over the integers. In general, the support of $\hat{f}$ will be contained in an interval (α,β), so that $\Omega = \max\{|\alpha|,|\beta|\}$. If $|\alpha| \neq |\beta|$, i.e., if the interval of support is not symmetric about 0, then we can decrease the sampling rate.*

a. *Let $w_\gamma(t) = e^{i\gamma t}$ be the probe function and set $f_\gamma(t) = w_\gamma(t)f(t)$. Show that the support of $\hat{f}_\gamma(\lambda)$ is contained in the interval $(\alpha+\gamma, \beta+\gamma)$.*

b. *Show that with the choice $\gamma = -(\alpha+\beta)/2$, f_γ is frequency-bounded in the symmetric interval $[-\tilde{\Omega},\tilde{\Omega}]$, where $\tilde{\Omega} = (\beta-\alpha)/2$. Show that $\tilde{\Omega} < \Omega$.*

c. *Compute the series expansion for $f_\gamma(t)$ in terms of the more widely spaced sample points $j\pi/\tilde{\Omega}$. from this result obtain a series expansion for $f(t)$ in terms of the values $f(j\pi/\tilde{\Omega})$.*

Exercise 3 *Suppose the only nonzero components of the input vector $\mathbf{x}$ and the impulse response vector $\mathbf{h}$ are $\mathbf{x}(0) = 2$, $\mathbf{x}(1) = 1$, $\mathbf{x}(2) = -1$, and $\mathbf{h}(0) = \frac{1}{4}$, $\mathbf{h}(1) = \frac{1}{2}$ and $\mathbf{h}(2) = \frac{1}{4}$. Compute the outputs $\mathbf{y}(n) = \mathbf{x} * \mathbf{h}(n)$. Verify in the frequency domain that $Y(\omega) = H(\omega)X(\omega)$.*

Exercise 4 *Let f be a piecewise smooth function on the interval $-\infty < t < \infty$, i.e., f' and f'' exist and are piecewise continuous on the real line, and suppose this function vanishes outside of some finite interval. Show that*

$$\sum_{n=-\infty}^{\infty} f(t - 2\pi n) = 1,$$

i.e., we can partition unity with the 2π-translates of f, if and only if

$$\hat{f}(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k = \pm 1, \pm 2, \cdots \end{cases}$$

This property (usually scaled to 1-translates) is characteristic of Daubechies' scaling functions.

Exercise 5 *(Filtering with the FFT) Let*

$$f(t) = e^{-\frac{t}{11}} \left(2 \sin t - \cos 2t + 0.4 \sin t \cos 60t\right).$$

Discretize f by setting $y_k = f(2k\pi/256), k = 1, \cdots, 256$. Use MATLAB's FFT to compute $\hat{y}_k$ for $0 \le k \le 255$. (Note that $y_{n-k} = \overline{y_k}$. Thus the low frequency coefficients are $\hat{y}_0, \cdots, \hat{y}_m$ and $\hat{y}_{256-m}, \cdots, \hat{y}_{256}$ for some small integer m. Filter out the high-frequency terms by setting $\hat{y}_k = 0$ for $m \le k \le 255 - m$ with $m = 6$. Then apply the inverse FFT to these filtered $\hat{y}_k$ to compute the filtered y_k. Plot the results and compare with the original unfiltered signal. Experiment with several different values of m.

Exercise 6 *(Compression with the FFT) Consider the signal $f(t)$ as given in the previous problem. Let $tol = 1.0$ In the previous problem compress the transformed signal by setting $\hat{y}_k = 0$ whenever $|\hat{y}_k| < tol$. Apply the inverse FFT to the compressed transfirmed signal to get a compressed signal y_k. Plot the results and compare with the original uncompressed signal. Experiment with several different values of tol. Keep track of the percentage of Fourier coefficients that have been filtered out.*

The following MATLAB routines should be useful:

You can discretize the interval $[0, 2\pi]$ and read in the signal as a vector by using the commands

```
t=linspace(0,2*pi,2^8);
f=exp(-t./11).*(2*sin(t)-cos(2*t)+0.4*sin(t).*cos(60*t));
```

If $hatf$ is the FFT of f, i.e., $hatf = fft(f)$, you can filter out high frequency components from $hatf$ with a command such as

```
filterhatf=[ hatf(1: m) zeros(1, 2^8-2*m) hatf(2^8-m+1:2^8)]
```

```
function fc=compress( f, r)
% Input the vector f and ratio r: 0<= r <=1.
% The output is the vector fc in which the smallest
% 100r% of the terms f_k, in absolute value, are set
% equal to zero.
if (r<0) | (r>1)
   error ('r should be between 0 and 1')
end;
N=length(f); Nr=floor(N*r);
ff=sort(abs(w));
tol=abs(ww(Nr+1);
fc=(abs(w)>=tol).*w;
```

```
function L2error =fftcomp(t,f,r)
% Input: time vector t, signal vector f, compression rate r, (between
% 0 and 1)
Output: graph of f, graph of the compression of f, and the relative L2
error
if (r<0) | (r>1)
   error ('r should be between 0 and 1')
end;
hatf=fft(f);
hatfc=compress(hatf,r);
fc=ifft(hatfc);
plot(t,f,t,fc)
L2error=norm(f-fc,2)/norm(f)
```