Name: ______________________________

**Math 4567. Midterm Exam I (Solutions)**

**February 19, 2010**

There are a total of 100 points on this 55 minute exam. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. A standard calculator and ONE $8.5 \times 11$ inch sheet of notes are allowed, but no books, other notes, cell phones or other elecronic devices are allowed. Do all of your calculations on this test paper.

| Problem | Score |
|---|---|
| 1. | ________ |
| 2. | ________ |
| 3. | ________ |
| Total: | ________ |

**Problem 1** *Let* $S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ *be the Fourier series for the function*

$$f(x) = \begin{cases} 0 & -\pi < x \le -\frac{\pi}{2} \\ 1 & -\frac{\pi}{2} < x \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x \le \pi. \end{cases}$$

**a) (10 points)** *State the precise numerical value of* $S(x)$ *for each* $x$ *in the interval* $-\pi \le x \le \pi$.

**Solution:** $f(x)$ *is piecewise continuous on* $(-\pi, \pi)$ *with piecewise continuous derivative. The discontinuities are at* $x = \pm\pi/2$ *where the midpoints of the disconinuity are* $1/2$. *Thus*

$$S(x) = \begin{cases} 0 & \text{if } -\pi \le x < -\frac{\pi}{2} \\ \frac{1}{2} & \text{if } x = -\frac{\pi}{2} \\ 1 & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \frac{1}{2} & \text{if } x = \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases}$$

**b) (15 points)** *Compute the Fourier coefficients* $a_j$, $b_n$ *for* $f(x)$.

**Solution:**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dt = 1,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\ dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nt\ dt = \frac{1}{\pi}\frac{\sin nt}{n}\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{2\sin(n\pi/2)}{n\pi} = \begin{cases} \frac{2(-1)^{k+1}}{(2k-1)\pi} & n = 2k-1 \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\ dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nt\ dt = 0.$$

*Thus,*

$$S(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cos(2k-1)x}{2k-1}, \quad -\pi < x < \pi.$$

**c) (10 points)** *Using the fact that* $\int_0^x f(t)\,dt = x$ *for* $-\frac{\pi}{2} < x < \frac{\pi}{2}$ *integrate the Fourier series for* $f(x)$ *term - by - term to obtain the expansion*

$$\frac{\pi x}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \sin(2k-1)x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

*Why is this justified?*

**Solution:** *Since* $f$ *and* $f'$ *are piecewise continuous on* $(-\pi/2, \pi/2)$ *and extended by* $\pi$*-periodicity then term-by-term integration is OK. Integrating* $S(t)$ *from* $0$ *to* $x$ *we get*

$$x = \frac{x}{2} + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{(2k-1)\pi} \int_0^x \cos(2k-1)t\,dt = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{\pi(2k-1)^2} \sin(2k-1)t|_0^x$$

$$= \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{\pi(2k-1)^2} \sin(2k-1),$$

*so*

$$\frac{\pi x}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \sin(2k-1)x.$$

**Problem 2** *The Fourier cosine series for the function* $f(x) = 1 - x^2$ *on the domain* $0 \le x << 1$ *is*

$$f(x) \sim S(x) = \frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(n\pi x).$$

**a) (10 points)** *State the precise numerical value of* $S(x)$ *for each* $x$ *in the closed interval* $0 \le x \le 1$*. Justify your statement.*

**Solution:** $f(x)$ *is continuous on* $0 < x \le 1$ *and since it is even, we can extend it to* $-1 \le x \le 1$ *with* $f(-1) = f(1) = 0$ *and make it* $1-$ *periodic to get a function contiuous everywhere, Obviously* $f'$ *is piecewise continuous. Therefore* $S(x) = 1 - x^2$ *for* $0 \le x \le 1$.

**b) (20 points)** *Is it permissible the differentiate the series for* $f(x)$ *term - by - term to derive the Fourier sine series* $S'(x)$ *for* $f'(x) = -2x$ *on the interval* $0 < x < 1$*? If so, do it and state the precise numerical value of* $S'(x)$ *for each* $x$ *in the closed interval* $0 \le x \le 1$.

**Solution:** *Since the extended function $f(x)$ is continuous everywhere and has a piecewise continuous derivative on $(0,1)$ it is permissible to differentiate term - by- term. Thus we obtain*

$$-2x \sim \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x).,\quad 0<x<1$$

*Since this is a Fourier sine series, it converges to the odd extension of $-2x$ to the domain $[-1,1]$ and made $2\pi$-periodic. Since $f'(1-)=-2$ and $f'(1+)=+2$, the Fourier series converges to $-2x$ for $0\le x<1$ and to $0$ for $x=1$.*

**Problem 3** *The Fourier series for the triangular wave $f(x) = |x|$, $-\pi\le x\le\pi$, is*

$$f(x)\sim S(x)=\frac{a_0}{2}+\sum_{n=0}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)x.$$

**a) (10 points)** *Does the series $S(x)$ converge uniformly on the interval $-\pi\le x\le\pi$? Justfy your answer.*

**Solution:** *$f(x)$ is continuous on $[-\pi,\pi]$ and $f(-\pi)=f(\pi)=\pi$ Also $f'(x)$ is piecewise continuous on this interval. Hence the Fouier series of $f$ converges uniformly on the interval.*

**b) (10 points)** *Without actually performing the operation, justify that term - by -term differentiation of the series $S(x)$ permitted. To what numerical value does $S'(x)$ converge for each $x$ in the closed interval $-\pi\le x\le\pi$?*

**Solution:** *Since the extended function $f(x)$ is continuous everywhere and has a piecewise continuous derivative on $(-\pi,\pi)$ it is permissible the defferentiate term - by- term. The derivative is*

$$f'(x)=\begin{cases}1 & \text{if } 0<x<\pi\\ -1 & \text{if } -\pi<x<0.\end{cases}$$

*This function is piecewise continuous with a piecewise continuous derivative on $(-\pi,\pi)$. The discontinuities are at $x=0,\pm\pi$. and the midpoints of these dicontinuities is $0$ Thus the differentiated series converges to $1$ for $0<x<\pi$, to $-1$ for $-\pi<x<0$ and to $0$ for $x=0,\pm\pi$.*

**c) (15 points)** *Evaluate the sum*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

*of the Fourier coefficients from part a) with a minimum of computation. Justify your answer.*

**Solution:** *Since the series for $f(x)$ converges uniformly, Parseval's theorem holds. Thus*

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\ dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

*Evaluating the integral we have*

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\ dx = \frac{2\pi^2}{3}.$$