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Math 4567. Homework Set # VII

April 23, 2010

Chapter 8, (page 201, problems 1,2,3), (page 209, problems 2,4), (page 215, problem 3), (page 221, problem 2), (page 228, problem 1), Chapter 6 (page 157, problem 2). (page 162, problem 1)

Chapter 8 page 201, Problem 1 **(a)** Consider the Sturm - Liouville problem

$$[xX'(x)]' + \frac{\lambda}{x}X(x) = 0, \quad 1 < x < b,$$
$$X(1) = 0,\ X(b) = 0,$$

and use the substitution $x = \exp s$ to convert the problem to

$$\frac{d^2X}{ds^2} + \lambda X = 0, \quad 0 < s < \ln b,$$
$$X|_{s=0} = 0,\ X|_{s=\ln b} = 0.$$

Show that the eigenvalues and eigenfunctions of the original problem are

$$\lambda_n = \alpha_n^2,\ X_n(x) = \sin(\alpha_n \ln x), \quad n = 1, 2, \cdots,$$

where $\alpha_n = n\pi/\ln b$.

(b) By making the substitution

$$s = \pi\frac{\ln x}{\ln b}$$

give a direct verification that the eigenfunctions $X_n(x)$ of part (a) are orthogonal on the interval $1 < x < b$, with weight function $p(x) = 1/x$.

Solution:

(a) We have $\frac{d}{ds} = \frac{dx}{ds}\frac{d}{dx} = x\frac{d}{dx}$, so

$$x[xX'(x)]' + \lambda X(x) = 0 \leftrightarrow \frac{d^2X}{ds^2} + \lambda X = 0, \quad o < s < \ln b,$$

since $s = \ln x$. Thus, in the new coordinates the boundary conditions are

$$X|_{s=0} = 0, \ X|_{s=\ln b} = 0.$$

For the original problem we solve the eigenvalue problem.

Case 1: $\lambda = \alpha^2$, $\alpha > 0$. The solution of the differential equation is

$$X = A\cos\alpha s + B\sin\alpha s = A\cos(\alpha\ln x) + B\sin(\alpha\ln x).$$

Then $X(1) = 0 = A$, and $X(b) = 0 = B\sin(\alpha\ln b)$, so we can have a nonzero solution only for $\alpha\ln b = n\pi$, or $\alpha = \alpha_n = n\pi/\ln b$, with $X_n(x) = \sin(\frac{n\pi\ln x}{\ln b})$, $n = 1, 2, \cdots$.

(b) Since $s = \pi\frac{\ln x}{\ln b}$, it follows that $ds = \pi dx/x\ln b$. We have for $m \neq n$,

$$\begin{aligned}\int_1^b X_n(x)X_m(x)\frac{dx}{x} &= \int_1^b \sin(\frac{n\pi\ln x}{\ln b})\sin(\frac{m\pi\ln x}{\ln b})\frac{dx}{x} \\ &= \frac{\ln b}{\pi}\int_0^\pi \sin ns\sin ms \ ds \\ &= 0,\end{aligned}$$

if $m \neq n$.

Chapter 8, page 201, Problem 2 Let

$$\mathcal{L}[X] = (rX')' + qX$$

so that the Sturm-Liouville differential equation can be written as

$$\mathcal{L}[X] + \lambda pX = 0.$$

Derive Lagrange's identity

$$X\mathcal{L}[Y] - Y\mathcal{L}[X] = \frac{d}{dx}[r(XY' - YX')].$$

Solution:

$$X\mathcal{L}[Y] - Y\mathcal{L}[X] = X(rY')' + qXY - Y(rX')' - qYX$$

$$= Xr'Y' + XrY'' - Yr'X' - YX'' = r'(XY' - YX') + r(XY'' - YX'').$$

Since

$$\frac{d}{dx}[r(XY' - YX')] = r'(XY' - YX') + r(XY'' - YX''),$$

this establishes the identity.

Chapter 8, page 201, Problem 3 **(a)** Let $\mathcal{L}$ be the operator of the previous problem, defined on a space of functions on $a < x < b$, satisfying the conditions

$$a_1X(a)+a_2X'(a) = 0,\ b_1X(b)+b_2X'(b) = 0, \quad |a_1|+|a_2| > 0,\ |b_1|+|b_2| > 0,$$

and with inner product with weight function $p(x) = 1$. Show that

$$(X, \mathcal{L}[Y]) = (\mathcal{L}[X], Y).$$

(b) Let $\lambda_m \neq \lambda_n$ be eigenvalues of the problem $\mathcal{L}[X] + \lambda pX = 0$ with boundary conditions

$$a_1X(a)+a_2X'(a) = 0,\ b_1X(b)+b_2X'(b) = 0, \quad |a_1|+|a_2| > 0,\ |b_1|+|b_2| > 0.$$

Show that if X_m, X_n are the corresponding eigenfunctions, then

$$(pX_m, X_n) = 0.$$

Solution:

(a)

$$(X, \mathcal{L}[Y]) - (\mathcal{L}[X], Y) = \int_a^b \frac{d}{dx}[r(XY' - YX')]\, dx = [r(XY' - YX')]_a^b =$$

$$r(b)(X(b)Y'(b) - Y(b)X'(b)) - r(a)(X(a)Y'(a) - Y(a)X'(a)).$$

Now suppose $a_1 \neq 0$. Then

$$X(a) = -\frac{a_2 X'(a)}{a_1},\ Y(a) = -\frac{a_2 Y'(a)}{a_1}$$

$$\longrightarrow X(a)Y'(a) - Y(a)X'(a) = -\frac{a_2 X'(a)Y'(a)}{a_1} + \frac{a_2 X'(a)Y'(a)}{a_1} = 0.$$

If $a_2 \neq 0$ then

$$X'(a) = -\frac{a_1 X(a)}{a_2},\ Y'(a) = -\frac{a_1 Y(a)}{a_2}$$

$$\longrightarrow X(a)Y'(a) - Y(a)X'(a) = -\frac{a_1 X(a)Y(a)}{a_2} + \frac{a_1 X(a)Y(a)}{a_2} = 0.$$

Thus always $X(a)Y'(a) - Y(a)X'(a) = 0$. A similar argument applied to the endpoint b gives $X(b)Y'(b) - Y(b)X'(b) = 0$. Thus, $(X, \mathcal{L}[Y]) - (\mathcal{L}[X], Y) = 0$.

(b) We have

$$\mathcal{L}[X_m] + \lambda_m p X_m = 0,\ \mathcal{L}[X_n] + \lambda_n p X_n = 0.$$

Thus

$$(X_m, \mathcal{L}[X_n]) - (\mathcal{L}[X_m], X_n) = -(X_m, \lambda_n p X_n) + (\lambda_m p X_m, X_n) = [\lambda_m - \lambda_n](pX_m, X_n)$$

However, from part (a) we have $(X_m, \mathcal{L}[X_n]) - (\mathcal{L}[X_m], X_n) = 0$, so $[\lambda_m - \lambda_n](pX_m, X_n) = 0$. Since $\lambda_m \neq \lambda_n$ it follows that $(pX_m, X_n) = 0$.

Chapter 8, page 209, Problem 2 Find the eigenvalues and eigenfunctions:

$$X'' + \lambda X = 0,\ X(0) = 0,\ hX(1) + X'(1) = 0,\ h > 0.$$

Solution: If $\lambda = 0$ then $X(x) = Ax + B$ and $X'(x) = A$. Thus the boundary conditions are $B = 0$, $A(h + 1) = 0$, so $A = 0$ and $\lambda = 0$ is not an eigenvalue.

If $\lambda = -\alpha^2$, $\alpha > 0$ then $X(x) = Ae^{\alpha x} + Be^{-\alpha x}$, $X'(x) = \alpha(Ae^{\alpha x} - Be^{-\alpha x})$. Thus the boundary conditions are $A + B = 0$ and $h(Ae^{\alpha} + Be^{-\alpha}) + \alpha(Ae^{\alpha} - Be^{-\alpha}) = 0$, or

$$A[h \sinh \alpha + \alpha \cosh \alpha] = 0.$$

Since $h \sinh \alpha + \alpha \cosh \alpha > 0$, we have $A = B = 0$ and $\lambda = -\alpha^2$ is not an eigenvalue.

If $\lambda = \alpha^2$, $\alpha > 0$ then $X(x) = A\cos\alpha x + B\sin\alpha x$, $X'(x) = \alpha(-A\sin\alpha x + B\cos\alpha x)$, and the boundary conditions can be read as

$$A = 0, \quad hB\sin\alpha + \alpha B\cos\alpha = 0,$$

or $h\sin\alpha + \alpha\cos\alpha = 0$, so $\lambda_n = \alpha_n^2$ where

$$\tan\alpha_n = \frac{-\alpha_n}{h}, \;\; X_n(x) = \sin\alpha_n x \quad n = 1, 2, \cdots.$$

As follows from the text and simple geometry, there is exactly one solution α_n in the interval

$$\frac{\pi}{2}(2n-1) < \alpha_n < \pi n.$$

Since

$$\int_0^1 X_n^2(x)dx = \frac{1}{2}\int_0^1(1-\cos 2\alpha_n x)dx = \frac{1}{2} - \frac{\sin 2\alpha_n}{4\alpha_n} = \frac{1}{2} - \frac{\tan\alpha_n}{2\alpha_n}\cos^2\alpha_n$$

$$= \frac{1}{2} + \frac{\cos^2\alpha_n}{h} = \frac{h + \cos^2\alpha_n}{2h},$$

the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2h}{h + \cos^2\alpha_n}}\sin\alpha_n x.$$

Chapter 8, page 209, Problem 4 Solve the S-L problem

$$X'' + \lambda X = 0, \;\; X(0) = 0, \;\; X(1) - X'(1) = 0.$$

Solution:

Case 1: $\lambda = \alpha^2 > 0, \;\; \alpha > 0$. Then

$$X(x) = A\cos\alpha x) + B\sin\alpha x \;\; X'(x) = -\alpha A\sin\alpha x + \alpha B\cos\alpha x.$$

The conditions

$$X(0) = 0 = A,\ X(1) - X'(1) = 0 = B\sin\alpha - \alpha B\cos\alpha,$$

imply $\alpha = \tan\alpha$. Similar to what is shown in the book, the solutions are α_n, $n = 1, 2, \cdots$ such that $(n-1)\pi < \alpha_n < (2n-1)\frac{\pi}{2}$. The eigenvalues are $\lambda_n = \alpha_n^2$ Here $X_n(x) = \sin\alpha_n x$, so

$$||X_n||^2 = (X_n, X_n) = \int_0^1 \sin^2(\alpha_n x)dx = \frac{1}{2}\int_0^1 (1 - \cos 2\alpha_n x)dx$$

$$= \frac{1}{2}(1 - \frac{1}{2\alpha_n}\sin 2\alpha_n) = \frac{1}{2}(1 - \cos^2\alpha_n),$$

since $\sin\alpha_n = \alpha\cos\alpha_n$. But

$$\cos^2\alpha_n = \frac{1}{1 + \tan^2\alpha_n} = \frac{1}{1 + \alpha_n^2},$$

so

$$||X_n||^2 = \frac{1}{2}(1 - \frac{1}{1 + \alpha_n^2}) = \frac{1}{2}\frac{\alpha_n^2}{1 + \alpha_n^2}$$

and the normalized eigenfunctions are

$$\phi_n(x) = \frac{\sqrt{2(\alpha_n^2 + 1)}}{\alpha_n}\sin\alpha_n x.$$

Case 2: $\lambda = 0$. Then $X(x) = Ax + B$. The conditions

$$X(0) = 0 = B,\ X(1) - X'(1) = 0 = A - A$$

imply $\lambda_0 = 0$, $X_0(x) = x$. We have

$$||X_0||^2 = (X_0, X_0) = \int_0^1 x^2 dx = \frac{1}{3},$$

so the normalized eigenfunction is $\phi_0(x) = \sqrt{3}x$.

Case 3: $\lambda = -\alpha^2 < 0$, $\alpha > 0$. From the left hand boundary condition, we must have $X(x) = \sinh\alpha x$. The remaining boundary condition is then $\sinh\alpha - \alpha\cosh\alpha = 0$ or $\alpha = \tanh\alpha$. The issue is then the points of intersection of the curves $y = \alpha$ and $y = \tanh\alpha$. These curves clearly intersect at $\alpha = 0$. If they intersect again at some $\alpha_0 > 0$ then the function $g(x) = \alpha - \tanh\alpha$ is continuous on the closed interval $0 \le \alpha \le \alpha_0$ and differentiable on the open interval $(0, \alpha_0)$. Furthermore $g(0) = g(\alpha_0) = 0$. By the Mean Value Theorem of calculus, there must be a value $c \in (0, \alpha_0)$ such that $g'(c) = 0$ But $g'(\alpha) = \tanh^2\alpha > 0$ for all $\alpha > 0$. Thus no such c can exist, so there is no negative eigenvalue $-\alpha_0^2$.

Chapter 8, page 215, Problem 3 Use the normalized eigenfunctions of Problem 2, page 209, namely

$$X'' + \lambda X = 0,\ X(0) = 0,\ hX(1) + X'(1) = 0,\ h > 0,$$

$$\lambda_n = \alpha_n^2,\ \tan\alpha_n = \frac{-\alpha_n}{h},\ \phi_n(x) = \sqrt{\frac{2h}{h + \cos^2\alpha_n}}\sin\alpha_n x. \quad n = 1, 2, \cdots,$$

to derive

$$1 = 2h\sum_{n=1}^{\infty}\frac{1 - \cos\alpha_n}{\alpha_n(h + \cos^2\alpha_n)}\sin\alpha_n, \quad 0 < x < 1.$$

Solution: We have

$$1 = \sum_{n=1}^{\infty} c_n\phi_n(x),\ c_n = \int_0^1 1 \cdot \phi_n(s)ds, \quad 0 < x < 1.$$

Now

$$c_n = \sqrt{\frac{2h}{h + \cos^2\alpha_n}}\int_0^1 \sin\alpha_n s\ ds = -\sqrt{\frac{2h}{h + \cos^2\alpha_n}}\frac{\cos\alpha_n - 1}{\alpha_n}.$$

Thus

$$1 = \sum_{n=1}^{\infty}\frac{2h}{\alpha_n(h + \cos^2\alpha_n)}(1 - \cos\alpha_n).$$

Chapter 8, page 221, Problem 2 Use the normalized eigenfunctions of the S-L problem

$$X'' + \lambda X = 0,\ X(0) = 0,\ X'(\pi) = 0$$

to solve the boundary value problem

$$u_t(x,t) = ku_{xx}(x,t), \quad 0 < x < \pi,\ t > 0,$$

$$u(0,t) = 0,\ u_x(\pi,t) = 0,\ u(x,0) = f(x).$$

Solution: The normalized eigenfunctions are a renormalization of those in the previous problem:

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin\alpha_n x,\ \alpha_n = \frac{(2n-1)}{2}, \quad n = 1, 2, \cdots.$$

The corresponding separated functions $T_n(t)$ satisfy $T' + \alpha_n^2 kT = 0$, so $T_n(t) = \exp\left(-\alpha_n^2 kt\right)$. Thus

$$u(x,t) = \sum_{n=1}^{\infty} B_{2n-1}\exp(-\alpha_n^2 kt)\phi_n(x),\ B_{2n-1}\exp(-\alpha_n^2 kt) = \int_0^\pi u(x,t)\phi_n(x)dx.$$

Since $u(x,0) = f(x)$, we have

$$B_{2n-1} = \int_0^\pi f(x)\phi_n(x)dx = \sqrt{\frac{2}{\pi}}\int_0^\pi f(x)\sin\frac{(2n-1)x}{2}\ dx$$

for $n = 1, 2, \cdots$.

Chapter 8, page 228, Problem 1 Use the expansion of x,

$$x = \frac{2}{c}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\alpha_n^2}\sin\alpha_n x, \quad 0 < x < c$$

in terms of the eigenfunctions of the S-L problem

$$X'' + \lambda X = 0,\ X(0) = 0,\ X'(c) = 0,$$

$$\lambda_n = \alpha_n^2,\ \phi_n(x) = \sqrt{\frac{2}{c}}\sin\alpha_n, \quad n = 1, 2, \cdots,$$

where

$$\alpha_n = \frac{(2n-1)\pi}{2c},$$

to show that the temperature function

$$u(x,t) = \frac{A}{K}\left[x + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha_n^2}\exp(-\alpha_n^2 kt)\sin\alpha_n x\right], \quad 0 < x < 1,\ t > 0$$

with $\alpha_n = \frac{(2n-1)\pi}{2}$, can be written as

$$u(x,t) = \frac{2A}{K}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\alpha_n^2}[1 - \exp(-\alpha_n^2 kt)]\sin\alpha_n x, \quad 0 < x < 1,\ t > 0.$$

Solution: Set $c = 1$ in the expansion for x, substitute this in the expansion for $u(x,t)$ and write the sum of two infinite series as a single series to get

$$u(x,t) = \frac{A}{K}\left[x + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha_n^2}\exp(-\alpha_n^2 kt)\sin\alpha_n x\right] =$$

$$\frac{2A}{K}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\alpha_n^2}\sin\alpha_n x + \sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha_n^2}\exp(-\alpha_n^2 kt)\sin\alpha_n x\right]$$

$$= \frac{2A}{K}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\alpha_n^2}[1 - \exp(-\alpha_n^2 kt)]\sin\alpha_n x, \quad 0 < x < 1,\ t > 0.$$

Chapter 6, page 157, Problem 2 Show that the function

$$f(x) = \begin{cases} 1 & \text{when } |x| < 1, \\ 0 & \text{when } |x| > 1, \\ \frac{1}{2} & \text{when } x = \pm 1, \end{cases}$$

satisfies the conditions of the Fourier integral pointwise convergence theorem. Establish

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{\sin\alpha(1+x) + \sin\alpha(1-x)}{\alpha}\,d\alpha = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\alpha\cos\alpha x}{\alpha}\,d\alpha.$$

Solution: f is piecewise continuous on every bounded interval and

$$\int_{-\infty}^{\infty}|f(x|\,dx = \int_{-1}^{1} 1\,dx = 2 < \infty,$$

so
$$\frac{f(x+)+f(x-)}{2} = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(s)\cos\alpha(s-x)\ ds\ d\alpha,$$
at each x such that $f'_R(x)$ and $f'_L(x)$ exist, and these derivatives exist at all x. Further, this function satisfies
$$\frac{f(x+)+f(x-)}{2} = f(x)$$
for all x. Now
$$\int_{-\infty}^\infty f(s)\cos\alpha(s-x)\ ds = \int_{-1}^1 \cos\alpha(s-x)\ ds = \left[\frac{\sin\alpha(s-x)}{\alpha}\right]_{-1}^1$$
$$= \frac{\sin\alpha(1-x)+\sin\alpha(1+x)}{\alpha},$$
so
$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\sin\alpha(1-x)+\sin\alpha(1+x)}{\alpha}d\alpha.$$
From the addition formulas for $\sin x$ we have
$$\sin\alpha(1-x)+\sin\alpha(1+x) = \sin\alpha\cos\alpha x-\cos\alpha\sin\alpha x+\sin\alpha\cos\alpha x+\cos\alpha\sin\alpha x$$
$$= 2\sin\alpha\cos\alpha x,$$
so
$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha\cos\alpha x}{\alpha}d\alpha.$$

Chapter 6, page 162, Problem 1 Show that the function
$$f(x) = \begin{cases} 1 & \text{when } 0 < x < b, \\ 0 & \text{when } x > b, \\ \frac{1}{2} & \text{when } x = b, \end{cases}$$
satisfies the conditions of the Fourier sine integral pointwise convergence theorem. Establish
$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos b\alpha}{\alpha}\sin\alpha x\ d\alpha, \quad x > 0.$$

Solution: f is piecewise smooth on every bounded interval over the positive x axis and is absolutely integrable. For every $x > 0$ f satisfies

$$\frac{f(x+) + f(x-)}{2} = f(x)$$

Thus

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \alpha x \int_0^\infty f(s) \sin \alpha s \; ds \; d\alpha, \quad x > 0.$$

Now

$$\int_0^\infty f(s) \sin \alpha s \; ds = \int_0^b \sin \alpha s \; ds = -\frac{\cos \alpha s}{\alpha}|_0^b = \frac{1 - \cos \alpha b}{\alpha},$$

so

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \alpha x \frac{1 - \cos \alpha b}{\alpha} d\alpha, \quad x > 0.$$