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Math 4567. Homework Set # 4 Solutions

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Chapter 2 (page 42, problem 8), (page 54, problems 1,5,6,7), Chapter 3 (page 63, problem 3), (page 71, problems 1,2,8), (page 76, problem 1).

Chapter 2, page 42, Problem 8 From the Fourier series

$$f(x) = \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} (a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi n}{c})$$

derive the complex series

$$f(x) = \lim_{n\to\infty} \sum_{n=-N}^{N} A_n \exp(i\frac{n\pi x}{c}),$$

where $A_0 = \frac{a_0}{2}$, $A_n = \frac{a_n - ib_n}{2}$, $A_{-n} = \frac{a_n + ib_n}{2}$ for $n = 1, 2, \cdots$. Derive the formula

$$A_k = \frac{1}{2c}\int_{-c}^{c} f(t) \exp(-i\frac{k\pi t}{c})dt, \quad k = 0, \pm 1, \pm 2, \cdots.$$

Solution:

$$\sum_{n=-N}^{N} A_N \exp(i\frac{n\pi x}{c}) = A_0 + \sum_{n=1}^{N}\left(A_n \exp(i\frac{n\pi x}{c}) + A_{-n}\exp(-i\frac{n\pi x}{c})\right) = \frac{a_0}{2} +$$

$$\frac{1}{2}\sum_{n=1}^{N}\left[(a_n - ib_n)(\cos\frac{n\pi x}{c} + i\sin\frac{n\pi x}{c}) + (a_n + ib_n)(\cos\frac{n\pi x}{c} - i\sin\frac{n\pi x}{c})\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{N}\left[a_n\cos(\frac{n\pi x}{c}) + b_n \sin(\frac{n\pi x}{c})\right],$$

because the cross terms cancel out in the last expansion. Furthermore,

$$A_0 = \frac{a_0}{2} = \frac{1}{2c}\int_{-c}^{c} f(t)dt,$$

for $k > 0$,

$$A_k = \frac{a_k - ib_k}{2} = \frac{1}{2c}\int_{-c}^{c} f(t)\left(\cos\frac{k\pi t}{c} - i\sin\frac{k\pi t}{c}\right)dt = \frac{1}{2c}\int_{-c}^{c} f(t)\exp(-i\frac{k\pi t}{c})dt,$$

and for $k < 0$,

$$A_k = \frac{a_{-k} + ib_{-k}}{2} = \frac{1}{2c}\int_{-c}^{c} f(t)\left(\cos\frac{k\pi t}{c} + i\sin(-\frac{k\pi t}{c})\right)dt = \frac{1}{2c}\int_{-c}^{c} f(t)\exp(-i\frac{k\pi t}{c})dt.$$

Chapter 2, page 54, Problem 1 a. Show that the function

$$f(x) = \begin{cases} 0 & \text{when } -\pi \le x \le 0, \\ \sin x & \text{when } 0 < x \le \pi, \end{cases}$$

satisfies all conditions for uniform convergence on $[-\pi, \pi]$.

b. Verify that the Fourier series

$$f \sim \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}, \qquad -\pi < x < \pi$$

converges pointwise uniformly to f on $[-\pi, \pi]$.

c. State why the series can be differentiated on $(-\pi, \pi)$ and describe the function that is represented by the differentiated series for all x.

Solution:

a. f is continuously differentiable on the open intervals $0 < x < \pi$ and $-\pi < x < 0$. We have $f(-\pi) = f(0) = f(\pi) = 0$, so it is continuous on $[-\pi, \pi]$. $f'(0{+}0) = 1$, $f'(0{-}0) = 0$, $f'(\pi{-}0) = -1$, $f'(-\pi + 0) = 0$ so f is piecewise smooth.

b. By part [a.] the series for f converges poinwise uniformly to f on $[-\pi, \pi]$.

c. Since f satisfies the conditions for uniform convergence and since $f''(x)$ is piecewise continuous on $(-\pi, \pi)$ the Fourier series can be differentiated term-by-term. The differentiated series converges to 0 for $-\pi < x < 0$, to $\cos x$ for $0 < x < \pi$, to $\frac{1}{2}$ for $x = 0$ and to $-\frac{1}{2}$ for $x = \pm\pi$.

Chapter 2, page 54, Problem 5 Integrate from $s = 0$ to $s = x$, $(-\pi \leq x \leq \pi)$ the Fourier series

$$s = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin ns$$

and the Fourier series

$$f(s) = 2\sum_{n=1}^{\infty} \frac{\sin(2n-1)s}{2n-1}$$

for

$$f(s) = \begin{cases} -\pi/2 & \text{when } -\pi < s < 0, \\ \pi/2 & \text{when } 0 < s < \pi \end{cases}$$

In each case describe graphically the function represented by the series.

Solution:

a. Integrating both sides of the Fourier series term-by term from 0 to x we get

$$\frac{x^2}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(\cos nx - 1), \quad -\pi \leq x \leq \pi.$$

The series is representing the parabola $F(x) = x^2/2$ in the interval $[-\pi, \pi]$.

b. Integrating both sides of the Fourier series term-by term from 0 to x we get

$$F(x) = \int_0^x f(s)ds = 2\sum_{n=1}^{\infty} \frac{(-1)}{(2n-1)^2}(\cos(2n-1)x - 1), \quad -\pi \leq x \leq \pi,$$

where

$$F(x) = \begin{cases} \frac{\pi x}{2} & \text{when } 0 \leq x \leq \pi \\ -\frac{\pi x}{2} & \text{when } -\pi \leq x < 0. \end{cases}$$

Thus $F(x) = \frac{\pi |x|}{2}$ for $-\pi \leq x \leq \pi$.

Chapter 2, page 54, Problem 6 Let p_n, q_n $n = 1, \cdots, N$ be real numbers where at least one of the p_n is nonzero. By considering the quadratic equation

$$\sum_{n=1}^{N} (p_n x + q_n)^2 = 0,$$

derive the Cauchy inequality

$$\left(\sum_{n=1}^{N} p_n q_n\right)^2 \leq \left(\sum_{n=1}^{N} p_n^2\right) \left(\sum_{n=1}^{N} q_n^2\right).$$

Solution: Write the quadratic equation as

$$(P, P)x^2 + 2(P, Q)x + (Q, Q) = 0,$$

where

$$(P, P) = \sum_{n=1}^{N} p_n^2, \ (P, Q) = \sum_{n=1}^{N} p_n q_n, \ (Q, Q) = \sum_{n=1}^{N} q_n^2.$$

By assumption, $(P, P) > 0$. Since the original form of the quadratic equation is as a sum of squares, this equation has at most one real solution x, which would be such that $p_n x + q_n = 0$ for all n. The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $D = b^2 - 4ac$ and it has the property that $D > 0$ for the case that there are 2 distinct real roots, $D = 0$ if there is exactly one real root, and $D < 0$ when there are no real roots. In this case $D = 4(P, Q)^2 - 4(P, P)(Q, Q)$ and there is at most one real root. Hence we must have $D \leq 0$ or $(P, Q)^2 \leq (P, P)(Q, Q)$.

Chapter 2, page 54, Problem 7 Let $S_N(x)$ be the Nth partial sum of the Fourier series

$$f(x) = 2\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

for

$$f(x) = \begin{cases} -\pi/2 & \text{when } -\pi < x < 0, \\ \pi/2 & \text{when } 0 < x < \pi \end{cases}$$

1. By writing $A = x$, $B = (2n-1)x$ in the identity,
2. $$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

   and then summing from $n=1$ to $n=N$ derive

   $$2\sum_{n=1}^{N} \cos(2n-1)x = \frac{\sin 2Nx}{\sin x}, \quad x \neq 0, \pm\pi, \pm 2\pi, \cdots.$$

   Verify that

   $$S_N'(x) = \frac{\sin 2Nx}{\sin x}, \quad 0 < x < \pi.$$
3. Show that the first extremum of $S_N(x)$ in $0 < x < \pi$ is a relative maximum occuring when $x = \pi/(2N)$.
4. Show that

   $$S_N(\frac{\pi}{2N}) = I_1 + I_2, \quad I_1 = \int_0^{\pi/(2N)} \frac{x - \sin x}{x \sin x} \sin 2Nx \; dx,$$

   $$I_2 = \int_0^{\pi/(2N)} \frac{\sin 2Nx}{x} dx.$$

   Verify that the integrands are piecewise continuous on $0 < x < \pi/(2N)$. so that the integrals converge.
5. Show that $I_1 \to 0$ as $N \to \infty$ so that

   $$\lim_{N\to\infty} S_N(\frac{\pi}{2N}) = \int_0^{\pi} \frac{\sin t}{t} dt.$$

Solution:

1. We have

   $$2\sin x \cos(2n-1)x = \sin 2nx - \sin 2(n-1)x.$$

   Thus by truncation

   $$2\sin x \sum_{n=1}^{N} \cos(2n-1) = \sum_{n=1}^{N} (\sin 2nx - \sin 2(n-1)x) = \sin 2Nx,$$

   so, dividing by $\sin x$ we have

   $$2\sum_{n=1}^{N} \cos(2n-1)x = \frac{\sin 2Nx}{\sin x}, \quad x \neq 0, \pm\pi, \pm 2\pi, \cdots.$$

2.
$$S_N'(x) = \frac{d}{dx}\left(2\sum_{n=1}^{N}\frac{\sin(2n-1)x}{2n-1}\right) = 2\sum_{n=1}^{N}\cos(2n-1)x$$
$$= \frac{\sin 2Nx}{\sin x}, \quad 0 < x < \pi.$$

3. $S_N'(0) = 2\sum_{n=1}^{N}\cos(2n-1)0 = 2N > 0$ so $S_N(x)$ is initially increasing from $S_N(0) = 0$ for x increasing from 0. The first maximum of $S_N(x)$ is at the first positive x_N such that $S_N'(x_N) = 0$. Thus $x_N = \pi/(2N)$.

4.
$$S_N(\frac{\pi}{2N}) = S_N(0) + \int_0^{\pi/(2N)} \frac{\sin 2Nx}{\sin x}dx = \int_0^{\pi/(2N)} \frac{\sin 2Nx}{\sin x}dx$$
$$\int_0^{\pi/(2N)} \left[\frac{x-\sin x}{x\sin x}\sin 2Nx + \frac{\sin 2Nx}{x}\right]dx$$
$$= I_1 + I_2,$$

where
$$I_1 = \int_0^{\pi/(2N)} \frac{x-\sin x}{x\sin x}\sin 2Nx\ dx,\ I_2 = \int_0^{\pi/(2N)} \frac{\sin 2Nx}{x}dx.$$

The integrand of I_1 is piecewise continuous over $[0,\pi/(2N)]$ except perhaps at $x=0$. However, by the l'Hôpital rules
$$\lim_{x\to 0+}\frac{x-\sin x}{x\sin x} \overset{l'H}{=} \lim_{x\to 0+}\frac{1-\cos x}{\sin x + x\cos x} \overset{l'H}{=} \lim_{x\to 0+}\frac{\sin x}{2\cos x - x\sin x} = 0$$
so the integrand goes to 0 as $x \to 0+$. Thus the integrand of I_1 is continuous over $[0,\pi/(2N)]$, hence bounded above in absolute value by a positive constant M. Similarly the integrand of I_2 is piecewise continuous over $[0,\pi/(2N)]$ except perhaps at $x=0$. However, by the l'Hôpital rule
$$\lim_{x\to 0+}\frac{\sin 2Nx}{x} \overset{l'H}{=} \lim_{x\to 0+}\frac{2N\cos 2Nx}{1} = 2N$$
so the integrand is piecewise continuous on $[0,\pi/(2N)]$.

5. We have
$$|I_1| \le \int_0^{\pi/(2N)} M\ dx = \frac{M\pi}{2N} \to 0$$
as $N \to \infty$. Thus
$$\lim_{N\to\infty} S_N(\frac{\pi}{2N}) = \lim_{N\to\infty} I_2 = \lim_{N\to\infty} \int_0^{\pi/(2N)} \frac{\sin 2Nx}{x} dx$$
$$\lim_{N\to\infty} \int_0^{\pi} \frac{\sin u}{u} du = \int_0^{\pi} \frac{\sin u}{u} du = \sigma = 1.85\cdots$$
Since $\pi/2 = 1.57\cdots$, this shows that the partial sums overshoot the function values $f(x_N)$ for $x_N \to 0+$ by the difference $\sigma - \pi/2$.

Chapter 3, page 63, problem 3 Show that the substitution $\tau = kt$ can be used to write the equation
$$u_t = k(u_{xx} + u_{yy}),$$
in the form
$$u_\tau = u_{xx} + u_{yy}.$$

Solution: Since $\tau = kt$ we have
$$\partial_t u = \frac{\partial \tau}{\partial t}\ u_\tau = k u_\tau.$$
Thus we can cancel the common factor k from both sides of the first equation to obtain the desired result.

Chapter 3, page 71, Problem 1 Let $u(x)$ be the steady-state temperature in a slab bounded by planes $x = 0$ $x = c$ when those faces are kept at fixed temperatures $u = 0$, $u = u_0$, respectively. Solve the boundary value problem for $u(x)$ to show that
$$u(x) = \frac{u_0}{c} x, \quad \Phi_0 = K \frac{u_0}{c},$$
where Φ_0 is the flux of heat to the left across each plane. $x = x_0$.

Solution: The boundary value problem for this system is $u = u(x)$ where u is continuous on $[0, c]$ and 2 times differentiable on $(0, c)$, with
$$u_{xx} = 0,\ 0 < x < c \quad \text{where } u(0) = 0,\ u(c) = u_0.$$

The general solution of the differential equation is $u(x) = ax + b$. The boundary conditions give $u(0) = b = 0$ and $u(c) = ac = u_0$ so the unique solution is $u(x) = \frac{u_0}{c}x$. The flux of heat to the left across each plane $x = x_0$ is $\Phi_0 = K\frac{du(x)}{dx} = \frac{Ku_0}{c}$.

Chapter 3, page 71, Problem 2 A slab occupies the region $0 \le x \le c$. There is a constant flux of heat Φ_0 into the slab through the face $x = 0$. The face $x = c$ is kept at temperature $u = 0$. Solve the boundary value problem for the steady-state temperatures $u(x)$ in the slab.

Solution: The boundary value problem for this system is $u = u(x)$ with u continuous on $[0, c]$, left differentiable at $x = 0$ and 2 times differentiable on $(0, c)$, with

$$u_{xx} = 0,\ 0 < x < c, \quad \text{where } \Phi_0 = -Ku_x(0),\ u(c) = u_0.$$

The general solution of the differential equation is $u(x) = ax + b$. The boundary conditions give $\Phi_0 = -Ka$ and $u(c) = ac + b = u_0$ so the unique solution is $a = -\Phi_0/K$, $b = u_0 + c\Phi_0/K$ or

$$u(x) = -\frac{\Phi_0}{K}(x - c) + u_0.$$

Chapter 3, page 71, Problem 8 Derive expressions for $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ in cylindrical coordinates.

Solution: Cylindrical coordinates are defined by relations

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z.$$

or

$$\rho = \sqrt{x^2 + y^2}, \quad \tan\phi = \frac{y}{x}, \quad z = z.$$

Thus,

$$\rho_y = \frac{y}{\sqrt{x^2 + y^2}} = \sin\phi,\ \sec^2\phi\ \phi_y = \frac{1}{x}, \quad z_y = 0.$$

Since $\sec^2\phi = \tan^2\phi + 1 = \frac{y^2}{x^2} + 1$ we have

$$\phi_y = \frac{x}{x^2 + y^2} = \frac{\cos\phi}{\rho}.$$

By the chain rule:

$$\partial_y = \rho_y \partial_\rho + \phi_y \partial_\phi + z_y \partial_z = \sin\phi \partial_\rho + \frac{\cos\phi}{\rho}\partial_\phi.$$

Thus

$$u_y = \sin\phi u_\rho + \frac{\cos\phi}{\rho} u_\phi.$$

and

$$u_{yy} = \left(\sin\phi\partial_\rho + \frac{\cos\phi}{\rho}\partial_\phi\right)\left(\sin\phi u_\rho + \frac{\cos\phi}{\rho}u_\phi\right)$$
$$= \sin^2\phi u_{\rho\rho} - \frac{\sin\phi\cos\phi}{\rho^2}u_\phi + \frac{\sin\phi\cos\phi}{\rho}u_{\phi\rho} + \frac{\cos^2\phi}{\rho}u_\rho$$
$$+ \frac{\sin\phi\cos\phi}{\rho}u_{\phi\rho} - \frac{\sin\phi\cos\phi}{\rho^2}u_\phi + \frac{\cos^2\phi}{\rho^2}u_{\phi\phi}$$
$$= \sin^2\phi u_{\rho\rho} + 2\frac{\sin\phi\cos\phi}{\rho}u_{\phi\rho} - 2\frac{\sin\phi\cos\phi}{\rho^2}u_\phi + \frac{\cos^2\phi}{\rho}u_\rho + \frac{\cos^2\phi}{\rho^2}u_{\phi\phi}$$

Chapter 3, page 76, Problem 1 A stretched string with ends fixed at $x = 0$, $x = 2c$ hangs at rest under its own weight. Show how it follows from equation

$$y_{tt}(x,t) = a^2 y_{xx} - g$$

that the static $y(x)$ must satisfy the equation $a^2 y''(x) = g$, where $a^2 = H/\delta$.

Solution: If the solution y is static then $y_t \equiv 0$, hence $y_{tt} = 0$ for all t and $y_x(x) = y'(x)$. The general solution of equation $a^2y''(x) = g$ is $y(x) = \frac{gx^2}{2a^2} + Ax + B$ where A, B are constants. Since $y(0) = 0$, we have $B = 0$. Since $y(2c) = 0$ we have $A = -\frac{gc}{a^2}$. Thus

$$y(x) = \frac{gx}{2a^2}(x - 2c), \quad \text{and } (x-c)^2 = \frac{2a^2}{g}(y + \frac{gc^2}{2a^2}).$$

This is an inverted parabola with vertex at $x = c$ and depth $|y(c)| = \frac{gc^2}{2a^2} = \frac{g\delta c^2}{2H}$.