0.1 Hyperbolic trajectories

As we have shown, in the standard perifocal frame (i.e., special coordinate system with the line through the perigee and the focus as the x-axis [called the apse axis by rocket scientists], and the focus as the origin) the differential equation for a hyperbolic trajectory solution of Newton's gravitational equation takes the form

$$\frac{d\theta}{dt} = \beta(1 - e\cos\theta)^2$$

where $\beta = k^2/\ell^3$, and ℓ is the angular momentum of the trajectory. Also recall that the polar coordinates describing the trajectory are $[r, \theta]$ where

$$r = \frac{ep}{1 - e\cos\theta}$$

and $ep = \ell^2/k$. The constant of the motion $\mathcal{E}$ is related to the eccentricity $e > 1$ via $\mathcal{E} = ek$. The origin of coordinates is at the focus closest to perihelion (which occurs for $\theta = \pi$). For trajectory plotting it will be simpler to introduce the angle $\phi = \theta - \pi$ so that perihelion occurs at $\phi = 0$. Since

$$\cos\theta = \cos(\phi + \pi) = -\cos\phi, \quad \sin\theta = -\sin\phi$$

we have $x = -r\cos\phi,$, $y = -r\sin\phi$ and

$$r = \frac{\ell^2/k}{1 + e\cos\phi}$$

Recall that the constants of the motion always satisfy the relation $e^2k^2 = 2\ell^2 E + k^2$, so the energy is $E = k^2(e^2 - 1)/2\ell^2$ which is positive for a hyperbolic trajectory. On the other hand, at any point on the trajectory $E = v^2/2 - k/r$ where v is the speed at that point. Thus as $r \to \infty$, $v \to v_{exc}$ where

$$v_{exc} = \frac{k\sqrt{e^2 - 1}}{\ell}$$

is the *hyperbolic excess velocity*, i.e., the speed in excess of the minimum required to escape the gravitational pull of the more massive body.

A very special point on a hyperbolic trajectory is the perigee (about the Earth), perihelion (about the Sun), or periapsis (the general term), the point on the trajectory closest to the focus. The perigee distance is r_p, and it occurs for true anomaly $\phi = 0$, so

$$r_p = \ell^2/k(1+e).$$

Perigee is the only point on the trajectory where the velocity is perpendicular to the radius (or position) vector: $\mathbf{r}_p \cdot \mathbf{r}'_p = 0$. Thus the angular momentum is given by $\ell = r_p v_p$ where v_p is the speed at perigee. Note also that $E = v_p^2 - k/r_p$.

Problem 1 *A meteoroid is first observed approaching the Earth when it is 402,000 km. from the center of the Earth with a true anomaly of $\phi = 150^o$. If the speed of the meteoroid at that time is 2.23 km./s., find the following:*

a. *The eccentricity of the trajectory. (ans. $e = 1.086$)*

b. *The altitude at perigee. (ans. 5,088 km.)*

c *The speed at perigee. (ans. 8.516 km./s.)*

0.2 The Kepler equation

As in the elliptic orbit case, for astronomers and rocket scientists it is critical to know the function $\phi(t)$. To find this function we need to solve the differential equation (noting that $\phi = \theta - \pi$)

$$\frac{d\phi}{dt} = \frac{k^2}{\ell^3}(1 + e\cos\phi)^2, \quad \phi(0) = 0 \tag{1}$$

where we have chosen the initial position to be at perihelion.

Separating variables in the trajectory equation we obtain the formal solution

$$\frac{k^2 t}{\ell^3} = \int_0^\phi \frac{d\phi}{(1+e\cos\phi)^2}. \tag{2}$$

Thus our problem reduces to evaluating the integral

$$I = \int \frac{d\phi}{(1+e\cos\phi)^2}, \quad e > 1$$

We have earlier evaluated this integral, but for a different range of values of e. We can use some of our original calculation that uses the fact that the integrand is a rational function of $\cos\phi$. However, at crucial points we will need to modify the argument to take into account the new range of values for e. We make the substitution

$$\cos\phi = \frac{1-u^2}{1+u^2}, \quad \sin\phi = \frac{2u}{1+u^2}$$

with inverse $u = \tan(\phi/2)$. Then we find

$$I = \int \frac{d\phi}{(1+e\cos\phi)^2} = 2\int \frac{1+u^2}{(1+e+(1-e)u^2)^2} du$$
$$= \frac{2}{(e+1)^2}\int \frac{1+u^2}{(1-\gamma u^2)^2} du$$

where $\gamma = (e-1)/(e+1)$. We simplify the problem slightly by rewriting the integral in the form

$$I = \frac{2}{(e+1)^2}\int \left[\frac{-\frac{1}{\gamma}}{1-\gamma u^2} + \frac{1+\frac{1}{\gamma}}{(1-\gamma u^2)^2}\right] du$$

or, with the substitution $v = \gamma^{1/2}u$,

$$I = \frac{2}{\gamma^{1/2}(e+1)^2}\int \left[\frac{\frac{-1}{\gamma}}{1-v^2} + \frac{1+\frac{1}{\gamma}}{(1-v^2)^2}\right] dv. \tag{3}$$

To evaluate this integral and obtain the time development of the trajectories it is very convenient (though not necessary) to make use of the definitions and properties of the hyperbolic functions.

0.2.1 The hyperbolic functions

We will review some basic properties of the hyperbolic functions that are directly useful in rocket science. The most important functions are

$$\cosh x = \frac{\exp(x)+\exp(-x)}{2}, \quad \sinh x = \frac{\exp(x)-\exp(-x)}{2},$$
$$\tanh x = \frac{\exp(x)-\exp(-x)}{\exp(x)+\exp(-x)} = \frac{\sinh x}{\cosh x}.$$

Occasionally we use $\coth x = 1/\tanh x$, $\operatorname{sech} x = 1/\cosh x$, $\coth x = 1/\tanh x$. Two basic identities are

$$\cosh^2 x - \sinh^2 x = 1, \quad 1 - \tanh^2 x = \frac{1}{\cosh^2 x} \equiv \operatorname{sech}^2 x.$$

Furthermore

$$\frac{d}{dx}\sinh x = \cosh x, \quad \frac{d}{dx}\cosh x = \sinh x.$$

Problem 2 *Verify that*

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

Problem 3 *Verify that*

$$\sinh 2x = 2\sinh x \cosh x.$$

It is easy to check that the function $y = \tanh x$ is defined for all x and monotone increasing with range $-1 < y < 1$. Thus it has a unique inverse $x = \operatorname{arctanh}\, y$, defined by

$$x = \operatorname{arctanh}\, y \longleftrightarrow y = \tanh x, \quad -\infty < x < \infty,\ -1 < y < 1.$$

Problem 4 *Show that*

$$\operatorname{arctanh}\, y = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right).$$

We can use these results to find alternate evaluations of some familiar integrals:

1. $$I_1 = \int \frac{dv}{1-v^2}.$$

   We make the substitution $v = \tanh x$. Then $dv = \operatorname{sech}^2 x\ dx$ and substitution gives

   $$I_1 = \int \frac{\operatorname{sech}^2 x\ dx}{\operatorname{sech}^2 x} = \int\ dx = x + C = \operatorname{arctanh}\, v + C.$$

2.

$$I_2 = \int \frac{dv}{(1-v^2)^2}.$$

Using the same substitution as in part 1, we find

$$I_2 = \int \frac{\text{sech}^2 x\ dx}{\text{sech}^4 x} = \int \cosh^2 x\ dx = \int (\frac{1}{4}\exp(2x) + \frac{1}{4}\exp(-2x) + \frac{1}{2})dx$$

$$= \frac{1}{4}\sinh 2x + \frac{x}{2} + C = \frac{1}{2}\sinh x \cosh x + \frac{x}{2} + C.$$

Now note that $\cosh x = 1/\sqrt{1-v^2}$ and $\sinh x = v/\sqrt{1-v^2}$. Hence

$$I_2 = \frac{1}{2}\left(\frac{v}{1-v^2}\right) + \frac{1}{2}\text{arctanh}\ v + C.$$

0.3 Evaluation of the integral for $t(\theta)$.

Now we return to the evaluation of the integral (3). Using the integrals I_1, I_2 of the last section, we find

$$I = \frac{1}{\gamma^{1/2}(e+1)^2}\left[(1-\frac{1}{\gamma})\text{arctanh}\ v + (1+\frac{1}{\gamma})\frac{v}{1-v^2}\right] + C. \tag{4}$$

We make use of the same idea for simplification as used for elliptic orbits, i.e., we set

$$\frac{F}{2} = \text{arctanh}\ v = \text{arctanh}\left(\sqrt{\frac{e-1}{e+1}}\tan\frac{\phi}{2}\right), \tag{5}$$

so

$$\sqrt{\frac{e-1}{e+1}}\tan\frac{\phi}{2} = \tanh\frac{F}{2}.$$

Then

$$I = \frac{1}{(e^2-1)^{3/2}}[-F + e\sinh F] + C. \tag{6}$$

Problem 5 *Show how (6) follows from (4).*

From (2) we have

$$\frac{k^2 t}{\ell^3} = \int_0^\phi \frac{d\phi}{(1+e\cos\phi)^2} = \frac{1}{(e^2-1)^{3/2}}(-F + e\sinh F). \tag{7}$$

Thus, the analogy of the Kepler equation for hyperbolic trajectories is

$$M_h = -F + e\ \sinh F, \text{ where } M_h = \frac{k^2(e^2-1)^{3/2}t}{\ell^3}. \tag{8}$$

Problem 6 *To derive the Kepler equation for the time behavior of solutions to the Newton gravitational equation we have to evaluate the integral*

$$I = \int \frac{d\phi}{(1+e\cos\phi)^2}$$

where e is the eccentricity of the trajectory. As we have seen, for elliptical ($0 < e < 1$) or hyperbolic ($1 < e$) trajectories this is fairly difficult and requires use of the general substitution $u = \tan(\phi/2)$. However, for parabolic trajectories ($e = 1$) it is easy and can be done using the half angle formula

$$\cos^2\frac{\phi}{2} = \frac{1}{2}(1+\cos\phi).$$

Verify this by evaluating

$$\int \frac{d\phi}{(1+\cos\phi)^2}.$$

Solution:

$$\int \frac{d\phi}{(1+\cos\phi)^2} = \frac{1}{6}\tan^3(\phi/2) + \frac{1}{2}\tan(\phi/2) + C$$

Problem 7 *The Kepler equation for parabolic trajectories is*

$$M_p \equiv \frac{k^2}{\ell^3}t = \int_0^\phi \frac{d\phi}{(1+\cos\phi)^2} = \frac{1}{6}\tan^3(\phi/2) + \frac{1}{2}\tan(\phi/2) \tag{9}$$

This is a third order polynomial in $\tan(\phi/2)$ and, unlike the elliptic and hyperbolic cases, can be solved explicitly by Cardano's formula to express $\tan(\phi/2)$ as a function of t. Verify that

$$\tan(\phi/2) = \left(3M_p + \sqrt{9M_p^2+1}\right)^{1/3} - \left(3M_p + \sqrt{9M_p^2+1}\right)^{-1/3}$$

is the solution of equation (9).

0.4 Inversion of the Kepler equation for hyperbolic trajectories

Equation (8) for hyperbolic trajectories gives M_h (hence t) as a function of F (hence ϕ or θ). We want to invert this expression to get $F(M_h)$. Numerically, the best approach is to use Newton's method.

Problem 8 *Show that* $\cosh x \geq 1$ *for all* x *and* $\cosh x = 1$ *if and only if* $x = 0$.

Since the constant $e > 1$ we have

$$\frac{dM_h}{dF} = -1 + e \ \cosh F > 0$$

always, so M_h is a monotone increasing function of F. Thus this function is uniquely invertible. To apply Newton's method we note that the value of F that corresponds to a given number M_h is the single root of the equation $f_{M_h}(F) = 0$ where

$$f_{M_h}(F) = e \ \sinh F - F - M_h.$$

If we make the guess F_1 for the root then the Newton algorithm determines update numbers given by

$$F_{n+1} = F_n - \frac{f_{M_h}(F_n)}{f'_{M_h}(F_n)} = F_n - \frac{e \sinh F_n - F_n - M_h}{e \cosh F_n - 1}, \quad n = 1, 2, \cdots.$$

The theory for this algorithm guarantees that the sequence $\{F_n\}$ will converge to the root as $n \to \infty$ if the initial guess and the root lie in a connected interval such that

$$\left| \frac{(e \sinh F - F - M_h) e \sinh F}{(e \cosh F - 1)^2} \right| < K,$$

where K is some positive constant with $K < 1$. Such an interval can always be found.

0.5 Examples of hyperbolic trajectory calculations

We will consider a problem involving the hyperbolic trajectory of a satellite about the Earth. (Again this problem and its solution are adapted from the book, *Orbital Mechanics for Engineering Students* by Howard D. Curtis, Elsevier, Amsterdam, 2005.) Recall that for Earth, Newton's equation is $\mathbf{r}'' = k\hat{\mathbf{r}}/r^3$ with $k = 398,600\ \mathrm{km}^3/\mathrm{s}^2$ and the radius of the Earth is about 6,378 km. In our standard coordinate system with the Earth at the origin and the apse line (x-axis) passing through the perigee, the equation of the trajectory is

$$r = \frac{\ell^2/k}{1 + e\cos\phi},$$

where ℓ is the angular momentum. Clearly $r_p = \ell^2/k/(1+e)$ (in kilometers) where r_p is the distance from the satellite to the center of the Earth at perigee. The angular momentum is related to the speed v_p at perigee via $\ell = r_p v_p$ The time-dependent trajectory equation is

$$M_h = -F + e\sinh F$$

where $M_h = k^2(e^2-1)^{3/2}t/\ell^3$ and

$$\sqrt{\frac{e-1}{e+1}}\tan\frac{\phi}{2} = \tanh\frac{F}{2}. \tag{10}$$

We can assume $t = 0$ at perigee.

Problem 9 *The geocentric trajectory of a satellite is $r_p = 6678$ km. from the center of the Earth and has velocity $v_p = 15$ km./s. at perigee.*

1. *Calculate the distance r to the Earth when the true anomaly is $\phi = 100^o$.*

2. *Find the position and speed of the satellite 3 hours after the position in part 1.*

Solution of part 1. From the data given $\ell = r_p v_p = 100{,}170\ \mathrm{km}^2/\mathrm{s}$. From the perigee equation $r_p = \ell^2/(k(1+e)$ we calculate that $e = 2.7696$ so the trajectory is a hyperbola. Substituting these values and $\phi = 100^o$ into the (time-independent) trajectory equation we have $r \approx 48{,}497$ km.

Solution of part 2. We need first to compute the time t at which $\phi = 100^o$. From (10) we have $\tanh(F/2) \approx 0.81652$, so $F \approx 2.2927$ radians. Thus, $M_h = -2.2927 + 2.7696\sinh 2.2827 \approx 11.279$ radians. Now

$$t = \frac{\ell^3}{k^2}\frac{M_h}{(e^2-1)^{3/2}} \approx 4141.4\ \mathrm{s}.$$

Three hours later $t = 4141.4+3(3600) \approx 14{,}941$ s. or 4.15 hr. The corresponding value of M_h is $M_h \approx 40.690$ radians. We now use Newton's algorithm to find F. Graphing the right-hand side of the equation $M_h = -F + e\sinh F$ we see that F must be between 3 and 4. Thus we try $F_1 = 3$ in the algorithm with an error tolerance of 10^{-6}. After 3 steps we find $F \approx 3.4631$. From (10) we obtain $\phi \approx 107.78^o$. From the trajectory equation we find $r \approx 163{,}180$ km. To obtain the speed v at this point on the trajectory we use the fact that the energy is

$$E = \frac{1}{2}v_p^2 - k/r_p = \frac{1}{2}v^2 - k/r$$

Then $E \approx 52.81$ and the final result is $v \approx 10.51$ km./s.

Problem 10 *Recall that the hyperbolic excess velocity v_{exc} of a object with a hyperbolic trajectory is the limiting speed as the radius of the trajectory goes to infinity. Suppose a spacecraft on a hyperbolic trajectory around the Earth has a perigee radius of 7500 km. and a perigee speed of 1.1 v_{exc}. Answer the following:*

a. *How long does it take to fly from $\phi = -90^o$ to $\phi = +90^o$? (ans. 1.14 hr.)*

b. *How far is the spacecraft from the center of the Earth 24 hours after passing through perigee? (ans. 456,000 km.)*