0.1 Epicycles

Two and a half millennia ago Aristotle claimed that the Earth was the center of the universe and that the planets (and the Sun) occupied circular orbits about the Earth. Thus in the plane of a planetary orbit with the origin at the center of the Earth and axes fixed in the heavens (so that the rotation of the earth was ignored), the location of a planet at time t would be

$$(x(t), y(t)) = (r\cos\omega t, r\sin\omega t).$$

The planetary orbit would be a circle of radius r and the planet would move with angular velocity ω. In terms of vectors, we would describe the position of the planet by the position vector

$$\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = r\cos\omega t\ \mathbf{i} + r\sin\omega t\ \mathbf{j} = r(\cos\omega t\ \mathbf{i} + \sin\omega t\ \mathbf{j}).$$

By two millennia ago it had become apparent (to astronomers who actually made observations) that this theory was untenable. It simply did not fit the observed facts. For example it couldn't explain retrograde motion where a planet is observed to pause and then move backward in the sky, before resuming forward motion. The attempt to maintain the concept that uniform circular motion was primary while getting a better fit to actual observations lead to the theory of epicycles, or circles rolling on circles.

Epicycles are easily understood in terms of vector addition. A planet is assumed to follow a regular circular path

$$\mathbf{R}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} = r_1\cos\omega_1 t\ \mathbf{i} + r_1\sin\omega_1 t\ \mathbf{j}$$

with large radius r_1, except that motion along a circular path with smaller radius r_2 and center $\mathbf{R}_1(t)$ is added:

$$\mathbf{R}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} = r_2\cos\omega_2 t\ \mathbf{i} + r_2\sin\omega_2 t\ \mathbf{j}$$

The motion of the planet is described by the vector sum

$$\mathbf{R}(t) = \mathbf{R}_1(t) + \mathbf{R}_2(t)$$

or

$$(x(t), y(t)) = (x_1(t) + x_2(t), y_1(t) + y_2(t)) =$$
$$(r_1\cos(\omega_1 t) + r_2\cos(\omega_2 t), r_1\sin(\omega_1 t) + r_2\sin(\omega_2 t).$$

By varying r_1, r_2, ω_1, and ω_2 a great variety of smooth trajectories can be represented. [See the accompanying Maple plots.] If the ratio ω_1/ω_2 is a rational number then the trajectories become closed orbits, i.e., they repeat. Otherwise we have a nonperiodic curve.

As observational data became more available and more accurate it became obvious that the simple epicycles we have described were themselves not adequate to describe that date. One modification was to describe the trajectories by additional circles rolling on circles, e.g.,

$$\mathbf{R}(t) = \mathbf{R}_1(t) + \mathbf{R}_2(t) + \cdots + \mathbf{R}_N(t),$$

where

$$\mathbf{R}_\ell(t) = x_(t)_\ell \mathbf{i} + y_\ell(t)\mathbf{j} = r_\ell \cos\omega_\ell t\ \mathbf{i} + r_\ell \sin\omega_\ell t\ \mathbf{j}, \quad \ell = 1, \cdots, N.$$

Another modification was to offset the smaller circle so that its center was not on the perimeter of the larger circle. In terms of vectors, this amounts to setting

$$\mathbf{R}(t) = \mathbf{R}_1(t) + \mathbf{R}_2(t) + \mathbf{A}$$

where $\mathbf{A} = a_1\ \mathbf{i} + a_2\ \mathbf{j}$ is a constant vector.

The epicycle approach was not without merit. Indeed from the modern theory of Fourier analysis we know that any smooth trajectory can be approximated with arbitrarily high precision over a fixed time interval $0 \le t \le T$ by suitably chosen epicycles of the form

$$\mathbf{R}(t) = \mathbf{R}_1(t) + \mathbf{R}_2(t) + \cdots + \mathbf{R}_N(t) + \mathbf{A}.$$

However, there were severe drawbacks

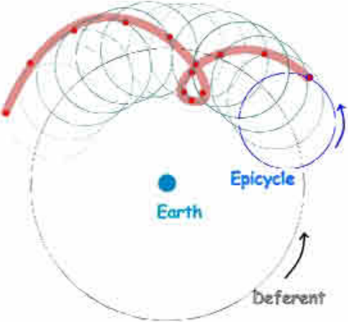
1. The procedure was extremely complicated and had to be revised frequently as new and more accurate data became available. By the early 1600s the Copernican notion that the planets orbited in epicycles about the Sun rather than the Earth had become popular, and this simplified the computations and improved accuracy somewhat. However, the much more accurate observations of Brahe showed that severe problems remained.

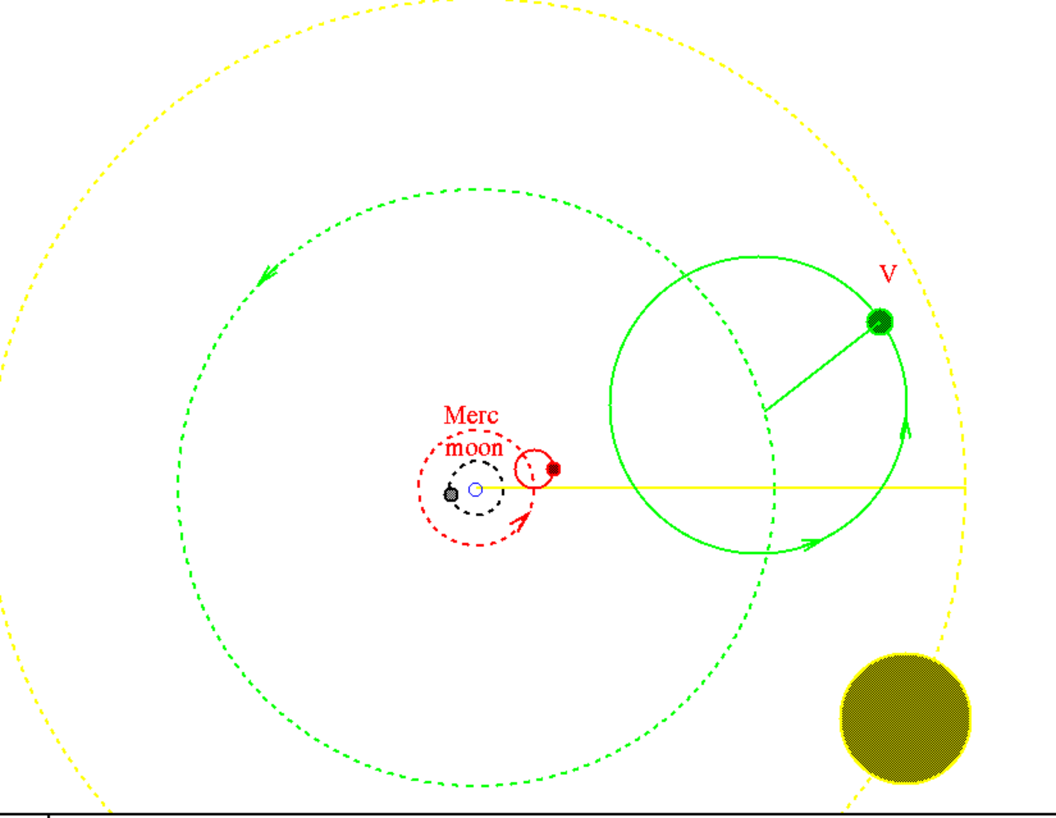
2. The greatest deficiency in the method was that it was merely descriptive; it provided no insight into the underlying natural processes that accounted for the motion.

With the work of Kepler and the publication of his three laws of planetary motion, roughly 1615-1630, the epicycle edifice came crashing down. Kepler showed that the assumption that the planets followed elliptical orbits with the Sun at one focus gave a much simpler and more accurate means of fitting the observational data. Newton, 50 years later, used this breakthrough in his formulation of mechanics to determine the gravitational force, and to derive the planetary orbits from physical principles.

We will follow the path from Kepler to Newton to modern day rocket science as a special project in this calculus course, emphasizing the importance of calculus in the development. Kepler's breakthrough came as a result of his detailed study of the orbit of Mars, so it is fitting that at the end of our project we will plan a rocket trip to Mars. We will sit on the Martian equator, look up at a satellite in a nearly stationary orbit above us — and find an epicycle!

Retrograde Motion in Ptolemy's Geocentric View

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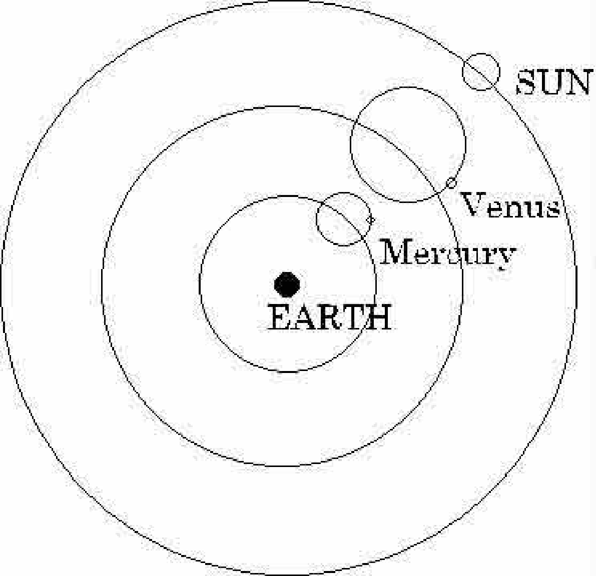
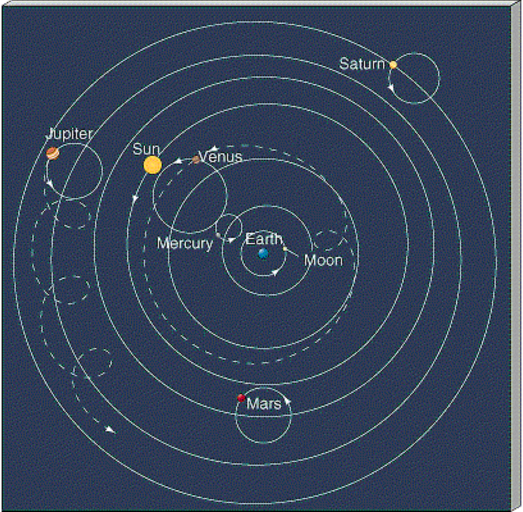

SUN
Venus
Mercury
EARTH


Saturn
Jupiter
Sun
Venus
Mercury
Earth
Moon
Mars